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Semilinear Elliptic Problems with Nonlinear Boundary Conditions in Unbounded Domains

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Abstract. We study a semilinear elliptic boundary value problem in an unbounded domain of $\mathbb{R}^n$ $(n \geq 3)$ which arises for example in electromagnetic wave propagation in fibres. The boundary condition is nonlinear and has the form $\partial_{\mathrm{n}} u = |u|^{p-1}u$. A Mountain Pass Lemma approach is used to construct a weak solution of this problem.

Keywords: *Nonlinear elliptic boundary value problems, unbounded domains, variational methods*

AMS subject classification: 35 J 65, 35 J 20

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary Γ. In this paper we study the problem of finding solutions of the equation

$$-\Delta u + a(x)u = g(x,u) \qquad \text{in } \Omega,$$

which satisfy the nonlinear boundary condition

$$\partial_{\mathrm{n}} u = \varphi(\xi, u) \qquad \text{on } \Gamma,$$

where ∂_{n} denotes the outer normal derivative on Γ. It is assumed that g and φ are of subcritical growth in the second variable. Problems of this kind arise for example in electromagnetic wave propagation in fibres (where $\Omega = Q \times \mathbb{R}$ is an infinite cylinder in $\mathbb{R}^3$). In particular, we consider problems where

$$g(x,u) = P(x)|u|^{p-1}u \qquad \text{and} \qquad \varphi(\xi,u) = Q(\xi)|u|^{p-1}u \qquad (p > 1). \tag{1}$$

For bounded domains such problems were considered previously for example in [7] and [10]. The present paper is a modified version of a part of the author's thesis [13].

To be more precise, we consider the following

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Problem 1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be an open domain, $0 \in \Omega$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\varphi : \Gamma \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions, a be an L^∞-function satisfying $a(x) \geq A > 0$ for almost every $x \in \Omega$. Then find a function u, $\lim_{|x|\to\infty} u(x) = 0$, which is a solution of the equations*

$$-\Delta u + au = g(x,u) \qquad in \quad \Omega \tag{2}$$

$$\partial_n u = \varphi(\xi, u) \qquad on \quad \Gamma. \tag{3}$$

Equations (2) - (3) are the Euler-Lagrange equations of the functional

$$F(u) = \int_\Omega \Big(\frac{1}{2}(|\nabla u|^2 + a(x)u^2) - G(x,u)\Big)dx - \int_\Gamma \Phi(\xi,u)d\Gamma, \tag{4}$$

where G and Φ are the primitive functions of g and φ, respectively, i.e. $G(x,u) = \int_0^u g(x,t)\,dt$ and $\Phi(\xi,u) = \int_0^u \varphi(\xi,t)\,dt$. Let $\mathcal{H}$ be the completion of the set

$$\Big\{\eta \in C^\infty(\Omega)\Big| \operatorname{supp}\eta \text{ is compact in } \mathbb{R}^n,\ \|\eta\|_{1,2} < \infty\Big\}$$

in the $H^1(\Omega)$-norm $\|\cdot\|_{1,2}$ which will be simply denoted by $\|\cdot\|$ in the sequel. Obviously $\mathcal{H}$ is a subspace of $H^1(\Omega)$, and a critical point of the functional (4) in $\mathcal{H}$ is a weak solution to Problem 1. Here and everywhere in the paper $\Phi(\xi,u)$ should be read as $\Phi(\xi,\gamma u)$, where γ is the trace operator $\gamma : H^1(\Omega) \to L^q(\Gamma)$.

The critical Sobolev exponents for the embedding $H^1(\Omega) \to L^p(\Omega)$ and the trace operator $H^1(\Omega) \to L^q(\Gamma)$ are denoted by $n^* = \frac{2n}{n-2}$ and $n_* = \frac{2(n-1)}{n-2}$, respectively.

Assumptions 1.1. The functions g and φ are assumed to satisfy the following conditions:

1° $\lim\limits_{u\to 0} \dfrac{g(x,u)}{u} = 0$ uniformly in $x \in \Omega$ and there exist an open, non-empty subset $O \subset \Omega$ and a number $R > 0$ such that $G(x,u) > 0$ for every $u \geq R$ and $x \in O$.

2° There exists a constant $C > 0$ such that $|g(x,u)| \leq C(1 + |u|^p)$ for every $(x,u) \in \Omega \times \mathbb{R}$ and $g(x,u) \geq 0$ if $u \geq 0$, where $1 < p < n^* - 1 = \frac{n+2}{n-2}$.

3° There is a Carathéodory function $\widetilde{\varphi}$ and a non-negative function $\alpha \in L^\infty(\Gamma)$ such that $\varphi(\xi,u) = \widetilde{\varphi}(\xi,u) - \alpha(\xi)u$.

4° $\lim\limits_{u\to 0} \dfrac{\widetilde{\varphi}(\xi,u)}{u} = 0$ uniformly in $\xi \in \Gamma$.

5° There exists a constant $C > 0$ such that $|\widetilde{\varphi}(\xi,u)| \leq C(1 + |u|^q)$ for every $(\xi,u) \in \Gamma \times \mathbb{R}$ and $\widetilde{\varphi}(\xi,u) \geq 0$ if $u \geq 0$, where $1 < q < n_* - 1 = \frac{n}{n-2}$.

6° For almost every $x \in \Omega$ and $\xi \in \Gamma$ we have $g(x,0) = 0$ and $\varphi(\xi,0) = 0$.

Furthermore, we assume that there is a $\theta \in \big[0, \frac{1}{2}\big)$ such that

7° $\widetilde{\Phi}(\xi,u) \leq \theta\,\widetilde{\varphi}(\xi,u)u$ for every $\xi \in \Gamma$ and $u \in \mathbb{R}$, where $\widetilde{\Phi}$ is the primitive of $\widetilde{\varphi}$.

8° $G(x,u) \leq \theta\, g(x,u)u$ for every $x \in \Omega$ and $u \in \mathbb{R}$.

Remark. It would be sufficient to assume that conditions $7^o - 8^o$ are satisfied for $|u| \geq R$ with some positive constant R, but for simplicity we take $R = 0$ here.

In the course of the paper, these assumptions are completed by other conditions, which are needed in the different steps to obtain a solution of Problem 1. The reader should keep in mind that all conditions on the functions g and φ which are formulated in this paper are satisfied by functions of the form (1). However, the results in Sections 2 - 4 are valid for more general non-linearities.

Conditions 2^o and 5^o now imply that the functional (4) is Fréchet differentiable and its derivative is given by the formula

$$\langle F'(u), v\rangle = \int\limits_\Omega (\nabla u \nabla v + a(x)uv)\, dx - \int\limits_\Omega g(x,u)v\, dx - \int\limits_\Gamma \varphi(\xi, u)v\, d\Gamma. \tag{5}$$

To prove the existence of critical points of F via the Mountain Pass Lemma, we have to investigate the following *Palais-Smale condition*

(PS) Any Palais-Smale sequence $\{u_k\}_{k\in I\!N}$ in $\mathcal{H}$ (i.e. a sequence satisfying $|J(u_k)| \leq M$ and $\lim_{k\to\infty} J'(u_k) = 0$ in $\mathcal{H}'$ with some constant M) has a convergent subsequence in $\mathcal{H}$.

Since for unbounded domains Ω the embedding $H^1(\Omega) \to L^p(\Omega)$ is in general not compact for any p, we cannot expect the Palais-Smale condition to hold for the functional (4) on $\mathcal{H}$. Therefore in the next section, a sequence of solutions is constructed for bounded domains. In Section 3 the limit of this sequence is investigated and in Section 4 a comparison theorem is proved. This theorem is then used in Section 5 to prove the existence of a solution to Problem 1 for some special functions g and φ; in particular the coefficient functions P and Q in (1) must satisfy a certain relation. The main results of this paper are Theorems 5.1 and 5.3.

2. Approximation by bounded domains

For $k \in I\!N$ let $B_k \subset I\!R^n$ be the open ball of radius k, $\Omega_k = \Omega \cap B_k$, $\Gamma_k = \partial\Omega \cap B_k$, and $\Sigma_k = \partial B_k \cap \Omega$. The truncated problem reads as follows.

Problem 2. *Find a function u, which satisfies the equations*

$$-\Delta u + a(x)u = g(x,u) \qquad in \quad \Omega_k \tag{6}$$

$$\partial_n u = \varphi(\xi, u) \qquad on \quad \Gamma_k \tag{7}$$

$$u = 0 \qquad on \quad \Sigma_k. \tag{8}$$

Let $\mathcal{H}_k$ be the closure of $\{\eta \in C^\infty(\Omega) | \, \eta$ has compact support in $B_k\}$ in the $H^1(\Omega)$-norm. Obviously $\bigcup_{k\in I\!N} \mathcal{H}_k$ is dense in $\mathcal{H}$. The elements of $\mathcal{H}_k$ may be interpreted as functions $u \in H^1(\Omega_k)$, which are continued by zero on $\Omega \setminus \Omega_k$. The trace operator

$$\mathcal{H}_k \longrightarrow H^1(\Omega_k) \longrightarrow L^q(\Gamma_k \cup \Sigma_k)$$

is continuous if $2 \le q \le n_*$, and compact if $2 \le q < n_*$. Functions $u \in C^\infty(\overline{\Omega}_k) \cap \mathcal{H}_k$ satisfy $u\big|_{\Sigma_k} = 0$, thus the boundary condition (8) is contained in the definition of $\mathcal{H}_k$. Let F_k be the corresponding functional on $\mathcal{H}_k$:

$$F_k(u) := \int\limits_{\Omega_k} \Big(\frac{1}{2}(|\nabla u|^2 + a(x)u^2) - G(x,u)\Big)\,dx - \int\limits_{\Gamma_k} \Phi(\xi, u)\,d\Gamma. \tag{9}$$

Now we can prove the following

Lemma 2.1. *Every Palais-Smale sequence for the functional* (4) *in* $\mathcal{H}$ *is bounded.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a Palais-Smale sequence for F (see (4)). Because of $F'(u_k) \to 0$ there exists to every $\varepsilon > 0$ a j_ε such that, for every $j \ge j_\varepsilon$ and every $v \in \mathcal{H}$, we have $|\langle F'(u_j), v/\|v\|\rangle| \le \varepsilon$. Inserting $v = u_j$ we get the inequality

$$\left| \int\limits_\Omega \big(|\nabla u_j|^2 + a(x)u_j^2 - g(x,u_j)u_j\big)\,dx - \int\limits_\Gamma \varphi(\xi, u_j)u_j\,d\Gamma \right| \le \varepsilon\,\|u_j\|. \tag{10}$$

If we set $\varepsilon = 1$, from (10) it follows that

$$\int\limits_\Omega \Big(-|\nabla u_j|^2 - au_j^2 + g(u_j)u_j\Big)dx + \int\limits_\Gamma \varphi(u_j)u_j\,d\Gamma - \|u_j\| \le 0. \tag{11}$$

Since $|F(u_j)|$ is bounded by M, we get

$$\frac{1}{2}\int\limits_\Omega \big(|\nabla u_j|^2 + au_j^2\big)dx - \int\limits_\Omega G(u_j)\,dx - \int\limits_\Gamma \Phi(u_j)\,d\Gamma \le M\,. \tag{12}$$

Multiplying (11) by $\theta \in \left[0, \frac{1}{2}\right)$ (defined in Assumptions 1.1) and adding this to (12), it follows that

$$\begin{aligned} &\Big(\frac{1}{2} - \theta\Big)\int\limits_\Omega \big(|\nabla u_j|^2 + au_j^2\big)\,dx - \int\limits_\Omega \big(G(u_j) - \theta g(u_j)u_j\big)dx \\ &\qquad - \theta\|u_j\| - \int\limits_\Gamma \Phi(u_j)\,d\Gamma + \theta\int\limits_\Gamma \varphi(u_j)u_j\,d\Gamma \le M. \end{aligned} \tag{13}$$

Now Assumptions 1.1/7° – 8° imply $\left(\frac{1}{2} - \theta\right)\int_\Omega \big(|\nabla u_j|^2 + a|u_j|^2\big)dx - \theta\|u_j\| \le M$ and with $\delta = \min\{1, A\}$ we get the estimate $\left(\frac{1}{2} - \theta\right)\delta\|u_j\|^2 - \theta\|u_j\| \le M$. Consequently, the sequence $\{u_j\}$ is bounded in $\mathcal{H}$ ∎

By standard arguments, the compact embeddings $H^1(\Omega_k) \to L^p(\Omega_k)$ and $H^1(\Omega_k) \to L^q(\Gamma_k)$ can now be used to show the following

Lemma 2.2. *For every* $k \in \mathbb{N}$, *the functional* F_k *in* (9) *satisfies the Palais-Smale condition* (PS) *on* $\mathcal{H}_k$.

Now we shall prove the existence of a non-trivial critical point of F_k by using the Mountain Pass Lemma of Ambrosetti and Rabinowitz [3] in its "classical" form.

Theorem 2.3. *Let $F : V \to \mathbb{R}$ be a C^1-functional satisfying the Palais-Smale condition* (PS) *on V. Assume that the following conditions hold:*

1° $F(0) = 0$.

2° *There are real numbers $r, \delta > 0$ such that $F(u) \geq \delta$ whenever $\|u\|_V = r$.*

3° *There exists some $v \in V$, $\|v\| > r$, satisfying $F(v) < \delta$.*

Then $\beta := \inf_{w \in W} \max_{u \in w} F(u)$ is a critical value of F, where $W := \{w : [0,1] \to V |\, w$ is continuous $, w(0) = 0,\, w(1) = v\}$.

In order to apply this theorem to the functional (4) (resp. (9)), we have to show the validity of conditions 2° and 3° (observe that $F(0) = 0$ was assumed in Assumption 1.1/6°).

Condition 2° for F. By Assumptions 1.1/1° – 2° it follows that to every $\varepsilon > 0$ there is a C_ε such that $|G(x,u)| \leq \varepsilon u^2 + C_\varepsilon |u|^{n^*}$ uniformly in x, and by Assumptions 1.1/3° – 5° it follows that to every $\varepsilon > 0$ there is a C'_ε such that $|\widetilde{\Phi}(\xi, u)| \leq \varepsilon u^2 + C'_\varepsilon |u|^{n_*}$ uniformly in ξ. This leads to

$$
\begin{aligned}
F(u) &\geq \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + (a - \varepsilon) u^2\right) dx - C_\varepsilon \int_\Omega |u|^{n^*} dx + \int_\Gamma \left((\alpha - \varepsilon) u^2 - C'_\varepsilon |u|^{n_*}\right) d\Gamma \\
&\geq \frac{1}{2} \min\{1, A - \varepsilon\} \|u\|^2 - C\|u\|^{n^*} - \varepsilon C_\Gamma \|u\|^2 - C\|u\|^{n_*}
\end{aligned}
$$

where the constants C_Γ and C come from the trace and embedding operators, respectively. Now we can choose ε so small that

$$
F(u) \geq \delta' \|u\|^2 - C\|u\|^{n^*} - C\|u\|^{n_*}
$$

with some $\delta' > 0$. Consequently, if $r > 0$ is small enough, we find some $\delta > 0$ such that $F(u) \geq \delta$ if $\|u\| = r$. Clearly, this estimate is valid for every F_k $(k \in \mathbb{N})$, and δ and r are independent of k.

Condition 3° for F. It is sufficient to choose some fixed, positive $v \in \mathcal{H}$ with compact support in Ω such that $\|v\| > 0$ and the set $\{x \in \Omega |\, G(x,v) > 0\}$ has positive Lebesgue measure (such a v exists by Assumption 1.1/1°). Let $R > 0$ be such that $G(x,R) > 0$. Assumption 1.1/8° for g implies $G(x,y) \leq \theta\, g(x,y) y = \theta\, y \frac{d}{dy} G(x,y)$. With $p = \frac{1}{\theta} > 2$ it follows for $y > R > 0$ that

$$
0 \leq y \frac{d}{dy} G(x,y) - p\, G(x,y) = y^{p+1} \frac{d}{dy} \left(y^{-p} G(x,y)\right).
$$

Integration over $[R, u]$ shows that

$$
0 \leq \int_R^u \frac{d}{dy} \left(y^{-p} G(x,y)\right) dy = u^{-p} G(x,u) - R^{-p} G(x,R).
$$

Therefore, for every $u > R$ we have $G(x,u) \geq h(x)u^p$, where $h(x) = R^{-p}G(x,R) > 0$. Consequently for real $\lambda > 0$ we have

$$\begin{aligned} F(\lambda v) &= \frac{1}{2}\int\limits_\Omega \left(|\nabla v|^2 + av^2\right) dx - \int\limits_\Omega G(x,\lambda v)\, dx \\ &\leq \frac{1}{2}\lambda^2 \int\limits_\Omega \left(|\nabla v|^2 + av^2\right) dx - \lambda^p \int\limits_{\{|\lambda v| \geq R\}} h(x)|v|^p dx - \int\limits_{\{|\lambda v| < R\}} G(x,\lambda v)\, dx \\ &\leq \lambda^2 \max\left\{1, \|a\|_{L^\infty}\right\} \|v\|^2 - C(R) - \lambda^p C(h)\, \|v\|^p_{L^p(\Omega)}, \end{aligned}$$

where the constant $C(h) > 0$ only depends on h, and $C(R)$ does not depend on λ. If $\lambda \to \infty$ we see that $F(\lambda v) \leq 0 < \delta$ and $\|\lambda v\| > r$.

Without loss of generality we may assume that v chosen above, lies in $\mathcal{H}_1$ and $F(tv) < 0$ is valid for every $t > 1$. Furthermore the conditions 2^o and 3^o of Theorem 2.3 are obviously satisfied by the truncated functionals F_k on $\mathcal{H}_k$ for every $k \in \mathbb{N}$. Therefore we have proved the following

Theorem 2.4. *For every $k \in \mathbb{N}$ there exists a critical point u_k of the functional F_k (see (9)) in $\mathcal{H}_k$, corresponding to the critical value*

$$\beta_k := \inf_{w \in W_k} \max_{u \in w} F_k(u)$$

where $W_k := \{w : [0,1] \to \mathcal{H}_k | \, w$ is continuous $, w(0) = 0,\, w(1) = v\}$.

3. Passage to the limit

Corresponding to β_k define $\beta := \inf_{w \in W} \max_{u \in w} F(u)$. For $\|u\| = r$ we always have $F(u) \geq \delta > 0$ and $W_k \subset W_{k+1} \subset \ldots \subset W$, so that

$$\beta_k \geq \beta_{k+1} \geq \ \ldots \ \geq \beta \geq \delta > 0 \qquad \text{for every } \ k \in \mathbb{N}.$$

In the sequel let $\|\cdot\|_E$ denote the norm $\|u\|_E = \left(\int_\Omega (|\nabla u|^2 + au^2)\, dx\right)^{1/2}$, which is equivalent to the norm $\|\cdot\|$. Let $u_k \in \mathcal{H}_k$ be a critical point corresponding to the value β_k, i.e. $F_k(u_k) = \beta_k$ and $F_k'(u_k) = 0$. We have

$$\langle F_k'(u_k), u_k \rangle = \|u_k\|_E^2 - \int\limits_\Omega u_k g(x,u_k)\, dx - \int\limits_\Gamma \varphi(\xi, u_k) u_k \, d\Gamma = 0 \tag{14}$$

$$F_k(u_k) = \frac{1}{2}\|u_k\|_E^2 - \int\limits_\Omega G(x,u_k)\, dx - \int\limits_\Gamma \Phi(\xi, u_k)\, d\Gamma = \beta_k. \tag{15}$$

Using Assumptions 1.1/7° – 8°, we see from (14) that

$$\begin{aligned}\theta\|u_k\|_E^2 &= \theta\int_\Omega u_k\, g(x,u_k)dx + \theta\int_\Gamma \varphi(\xi,u_k)u_k\, d\Gamma\\ &\geq \int_\Omega G(x,u_k)\, dx - \theta\int_\Gamma \alpha u_k^2\, d\Gamma + \int_\Gamma \theta\widetilde{\varphi}(\xi,u_k)u_k\, d\Gamma\\ &\geq \int_\Omega G(x,u_k)\, dx - \theta\int_\Gamma \alpha u_k^2\, d\Gamma + \int_\Gamma \widetilde{\Phi}(\xi,u_k)\, d\Gamma.\end{aligned}$$

Inserting the last estimate into (15), it follows that

$$\begin{aligned}\left(\frac{1}{2}-\theta\right)\|u_k\|_E^2 &= \beta_k + \int_\Omega G(x,u_k)\, dx - \int_\Gamma \frac{\alpha}{2}u_k^2\, d\Gamma + \int_\Gamma \widetilde{\Phi}(\xi,u_k)\, d\Gamma - \theta\|u_k\|_E^2\\ &\leq \beta_k \left(\theta-\frac{1}{2}\right)\int_\Gamma \alpha u_k^2\, d\Gamma\\ &\leq \beta_k.\end{aligned}$$

This implies

$$\|u_k\|_E^2 \leq \frac{\beta_k}{\left(\frac{1}{2}-\theta\right)} \leq \frac{\beta_1}{\left(\frac{1}{2}-\theta\right)}.$$

Because of the equivalence of the norms $\|\cdot\|_E$ and $\|\cdot\|$, the sequence of critical points $\{u_k\}_k$ in $\mathcal{H}$ is bounded and there is a subsequence (again denoted by $\{u_k\}_k$), weakly converging to a limit $\bar{u} = w\lim_{k\to\infty} u_k$ and $\bar{u}$ is a critical point of F. However, it is not clear whether $\bar{u} \neq 0$. This question is treated in the next two sections, but first we shall prove the following

Lemma 3.1. *The sequence of critical values β_k of the functional* (9) *satisfies* $\lim_{k\to\infty}\beta_k = \beta$.

Proof. Since $\beta = \inf_{w\in W}\max_{u\in w} F(u)$, for every $\delta > 0$ there is a path $\widehat{w}$ in

$$W = \left\{ w : [0,1] \to \mathcal{H} \,\middle|\, w(0) = 0 \text{ and } w(1) = v \right\}$$

such that

$$\kappa := \max_{u\in\widehat{w}} F(u) \geq \beta \qquad \text{und} \qquad |\kappa-\beta| < \frac{\delta}{2}. \tag{16}$$

Since $\widehat{w}$ is compact, there is a $\widehat{u} \in \widehat{w}$ such that $F(\widehat{u}) = \kappa$.

If $\varepsilon > 0$ is arbitrary, we find for every $u \in \widehat{w}$ a $k_{\varepsilon,u} \in I\!N$ such that for the open ball $B(\frac{\varepsilon}{6},u) \subset \mathcal{H}$ we have $B(\frac{\varepsilon}{6},u) \cap \mathcal{H}_k \neq \emptyset$ for every $k \geq k_{\varepsilon,u}$, since $\bigcup_{k\in I\!N}\mathcal{H}_k$ is dense in $\mathcal{H}$. The set of all these balls $\left\{B(\frac{\varepsilon}{6},u)\right\}_{u\in\widehat{w}}$ forms an open cover of $\widehat{w}$, which possesses a finite subcover $\left\{B(\frac{\varepsilon}{6},u_j)\right\}_{j=1}^m$ because $\widehat{w}$ is compact. Therefore there exists

$k_0 = \max_{j\in\{1,\dots,m\}} k_{\varepsilon,u_j}$ such that to every $u \in \widehat{w}$ there is a $u_{k_0} \in \mathcal{H}_{k_0}$ satisfying $\|u - u_{k_0}\| < \frac{\varepsilon}{6}$.

Now we can construct a path $w_\varepsilon \in W_{k_0}$ such that $\operatorname{dist}(w_\varepsilon, \widehat{w}) < \varepsilon$. For that purpose let $B_j = B(\frac{\varepsilon}{6}, u_j)$ be chosen in such a way that $B_j \cap B_{j+1} \neq \emptyset$ $(j = 1,\dots,m-1)$. In each B_j choose some $u_{k_0,j} \in \mathcal{H}_{k_0}$, and set $u_{k_0,0} = 0$ and $u_{k_0,m+1} = v$. By

$$w_{\varepsilon,j}(t) = u_{k_0,j} + t(u_{k_0,j+1} - u_{k_0,j}) \qquad (0 \le t \le 1;\ j = 0,\dots,m)$$

a path $w_\varepsilon \in W_{k_0}$ is defined piecewise. Further, since to every j there is a $v_j \in \widehat{w}$, $v_j \in B_j \cap B_{j+1}$, satisfying the inequalities

$$\|u_{k_0,j} - v_j\| < \frac{\varepsilon}{3} \qquad \text{and} \qquad \|u_{k_0,j+1} - v_j\| < \frac{\varepsilon}{3},$$

it follows for every $u \in w_{\varepsilon,j}$, $u = u_{k_0,j} + t(u_{k_0,j+1} - u_{k_0,j})$ (with some $t \in [0,1]$) the estimate

$$\begin{aligned}\|u - v_i\| &\le \|u - u_{k_0,j}\| + \|u_{k_0,j} - v_j\| \\ &= t\,\|u_{k_0,j+1} - u_{k_0,j}\| + \|u_{k_0,j} - v_j\| \\ &\le \|u_{k_0,j+1} - v_j\| + \|u_{k_0,j} - v_j\| + \|u_{k_0,j} - v_j\| \\ &< \varepsilon,\end{aligned}$$

which shows $\operatorname{dist}(w_\varepsilon, \widehat{w}) < \varepsilon$.

In this way, for any sequence $\varepsilon_i \to 0$, a sequence of paths $w_i \in W_{k_i}$ can be constructed, such that $\operatorname{dist}(w_i, \widehat{w}) < \varepsilon_i$. Let κ_i be the corresponding maximum of the functional F on w_i, attained at the point u_i, i.e. $\kappa_i = \max_{u\in w_i} F(u) = F(u_i)$. Clearly, $\kappa_i \ge \beta_{k_i}$.

Now we prove that there is a subsequence of $\{u_i\}_{i\in\mathbb{N}}$ converging strongly to some $\tilde{u} \in \widehat{w}$. For, suppose this is not true. Then to every $u \in \widehat{w}$ we could find a $\delta_u > 0$ such that the ball $B(\delta_u, u)$ contains at most a finite number of these u_i's. By compactness there is a finite number of such balls, denoted by B_j $(j = 1,\dots,m)$, covering $\widehat{w}$ and containing at most a finite number of points u_i. Let $\tilde{\delta} = \min\{\delta_{u_j} \mid j = 1,\dots,m\}$. Then for almost every u_i it follows $\operatorname{dist}(u_i, \widehat{w}) \ge \tilde{\delta} > 0$ which is a contradiction to the construction of the sequence $\{w_j\}$.

Therefore there exists a subsequence (again denoted by $\{u_i\}$), satisfying $\lim_{i\to\infty} u_i = \tilde{u} \in \widehat{w}$. Since F is continuous we have $F(\tilde{u}) = \lim_{i\to\infty} F(u_i) = \lim_{i\to\infty} \kappa_i$. Consequently there is a $\iota \in \mathbb{N}$ such that $|F(\tilde{u}) - \kappa_i| < \frac{\delta}{2}$ for every $i \ge \iota$.

If $\beta \le \beta_{k_i} \le \kappa$ (κ from (16)), then $|\beta - \beta_{k_i}| < \frac{\delta}{2}$. Otherwise, if $\beta \le \kappa \le \beta_{k_i}$, then the inequalities $F(\tilde{u}) \le \kappa \le \beta_{k_i} \le \kappa_i$ lead to the estimate

$$|\beta - \beta_{k_i}| \le |\beta - \kappa| + |\kappa - \beta_{k_i}| \le |\beta - \kappa| + |F(\tilde{u}) - \kappa_i| < \delta$$

for every $i \ge \iota$. Since $\{\beta_k\}_{k\in\mathbb{N}}$ was monotone decreasing and bounded from below, it follows that $\lim_{k\to\infty} \beta_k = \beta$ ∎

4. A comparison argument

In this section a comparison functional will be defined and a necessary condition for $\bar{u} = 0$ will be proved. This condition will be used in the next section to prove that for some special functions g and φ there exists a solution $\bar{u} \neq 0$ of Problem 1. The methods of proof used in these sections are based in part on ideas of W.-Y. Ding and W.-M. Ni [6].

For $0 \leq \alpha \in L^\infty(\Gamma)$

$$\|u\|_L = \left(\int_\Omega (|\nabla u|^2 + a(x)u^2)dx + \int_\Gamma \alpha(\xi)u^2 d\Gamma \right)^{1/2}$$

defines a norm on $\mathcal{H}$, equivalent to $\|\cdot\|$ und $\|\cdot\|_E$. With $\varphi(\xi,u) = \widetilde{\varphi}(\xi,u) - \alpha(\xi)u$ (see Assumption 1.1/3°) we have the representations

$$\langle F'(u), u\rangle = \|u\|_L^2 - \int_\Omega g(x,u)u\,dx - \int_\Gamma \widetilde{\varphi}(\xi,u)u\,d\Gamma$$

$$F(u) = \frac{1}{2}\|u\|_L^2 - \int_\Omega G(x,u)\,dx - \int_\Gamma \widetilde{\Phi}(\xi,u)\,d\Gamma.$$

We require some additional conditions for the functions g and $\widetilde{\varphi}$.

Assumptions 4.1. The functions g and $\widetilde{\varphi}$ are assumed to satisfy the following conditions:

1° g and $\widetilde{\varphi}$ are assumed to be odd functions in u, i.e. $g(\cdot,-u) = -g(\cdot,u)$ and $\widetilde{\varphi}(\cdot,-u) = -\widetilde{\varphi}(\cdot,u)$.

2° $\dfrac{g(x,u)}{u}$ and $\dfrac{\widetilde{\varphi}(\xi,u)}{u}$ are non-decreasing in $u > 0$ for all $x \in \Omega$ and $\xi \in \Gamma$, respectively.

From Assumption 4.1/1° and Assumptions 1.1/2°, 5° it follows that (for $u \neq 0$) the functions

$$g(x,u)u, \qquad \frac{g(x,u)}{u}, \qquad \widetilde{\varphi}(\xi,u)u, \qquad \frac{\widetilde{\varphi}(\xi,u)}{u}$$

are positive for all $x \in \Omega$ and $\xi \in \Gamma$, respectively.

Under these conditions we can prove for the functionals F in (4) and F_k in (9) the following

Lemma 4.2. *For $u \in \mathcal{H}$ set $\Lambda_u = \{tu \mid 0 \leq t \in \mathbb{R}\}$. The the following statements are true.*

(i) *If $\bar{u}$ is a critical point of F, then $F(\bar{u})$ is the absolute maximum of F in $\Lambda_{\bar{u}}$.*

(ii) *If u_k is a critical point of F_k, then $F(u_k)$ is the absolute maximum of F in Λ_{u_k}.*

Proof. Let $\bar{u}$ be a critical point of F, i.e. $\langle F'(\bar{u}), \bar{u}\rangle = 0$. Consequently

$$\|\bar{u}\|_L^2 = \int_\Omega \bar{u} g(x,\bar{u})\,dx + \int_\Gamma \bar{u}\widetilde{\varphi}(\xi,\bar{u})\,d\Gamma \tag{17}$$

$$F(\bar{u}) = \int_\Omega \Big(\frac{1}{2}\bar{u} g(x,\bar{u}) - G(x,\bar{u})\Big)\,dx + \int_\Gamma \Big(\frac{1}{2}\bar{u}\widetilde{\varphi}(\xi,\bar{u}) - \widetilde{\Phi}(\xi,\bar{u})\Big)\,d\Gamma. \tag{18}$$

Analogously we have for the critical points u_k of F_k

$$F(u_k) = \int_\Omega \Big(\frac{1}{2}u_k g(x,u_k) - G(x,u_k)\Big)\,dx + \int_\Gamma \Big(\frac{1}{2}u_k\widetilde{\varphi}(\xi,u_k) - \widetilde{\Phi}(\xi,u_k)\Big)\,d\Gamma. \tag{19}$$

For $t \geq 0$ we set

$$\mu(t) = F(t\bar{u}) = \frac{1}{2}t^2\|\bar{u}\|_L^2 - \int_\Omega G(x,t\bar{u})\,dx - \int_\Gamma \widetilde{\Phi}(\xi,t\bar{u})\,d\Gamma. \tag{20}$$

Since F is differentiable, μ can be differentiated with respect to t and with (17) we obtain

$$\begin{aligned}
\mu'(t) &= t\|\bar{u}\|_L^2 - \int_\Omega \bar{u} g(x,t\bar{u})\,dx - \int_\Gamma \bar{u}\widetilde{\varphi}(\xi,t\bar{u})\,d\Gamma \\
&= \int_\Omega \Big(t\bar{u} g(x,\bar{u}) - \bar{u} g(x,t\bar{u})\Big)dx + \int_\Gamma \Big(t\bar{u}\widetilde{\varphi}(\xi,\bar{u}) - \bar{u}\widetilde{\varphi}(\xi,t\bar{u})\Big)d\Gamma \\
&= \int_\Omega t\bar{u}^2\left(\frac{g(x,\bar{u})}{\bar{u}} - \frac{g(x,t\bar{u})}{t\bar{u}}\right)dx + \int_\Gamma t\bar{u}^2\left(\frac{\widetilde{\varphi}(\xi,\bar{u})}{\bar{u}} - \frac{\widetilde{\varphi}(\xi,t\bar{u})}{t\bar{u}}\right)d\Gamma.
\end{aligned}$$

Since g und $\widetilde{\varphi}$ are odd in u and $\frac{g(x,u)}{u}$ and $\frac{\widetilde{\varphi}(\xi,u)}{u}$ are non-decreasing in $u > 0$ (by Assumption 4.1/2°), it follows that

$$\mu'(t) \geq 0 \quad \text{if } 0 < t < 1 \qquad \text{and} \qquad \mu'(t) \leq 0 \quad \text{if } t \geq 1.$$

Therefore $\mu(1) = F(\bar{u})$ is the absolute maximum of F in $\Lambda_{\bar{u}}$. The same arguments can be repeated for u_k and the proof is complete ∎

To define a comparison functional, let h be a Carathéodory function, differentiable in the second variable, and satisfying the following conditions (such a function will be defined explicitely in the next section):

(H1) For every $x \in \Omega$ and $u \geq 0$ we have $h(x,u) \geq 0$ and h is odd in u. Furthermore there is an $R > 0$ such that

$$h(x,u) > 0 \qquad \text{for every } x \in \Omega,\ u \geq R.$$

(H2) There is an $\varepsilon > 0$ such that

$$u\frac{dh}{du}(x,u) \geq (1+\varepsilon)\,h(x,u) \qquad \text{for every } \; x \in \Omega,\; u \geq 0. \tag{21}$$

(H3) For every $x \in \Omega$ and $u \in I\!R$,

$$|h(x,u)| \leq C(1+|u|^p) \qquad \left(1 < p < n^* - 1 = \frac{n+2}{n-2}\right). \tag{22}$$

The corresponding primitive function is $H(x,u) = \int_0^u h(x,y)\,dy$. The comparison functional is now defined as

$$F_h(u) = \frac{1}{2}\|u\|_L^2 - \int_\Omega H(x,u)\,dx - \int_\Gamma \widetilde{\Phi}(x,u)\,d\Gamma.$$

From Assumption (H2) it follows in particular that, for $u \geq 0$,

$$\int_0^u y\frac{dh}{dy}(x,y)\,dy \geq (1+\varepsilon)\int_0^u h(x,y)\,dy.$$

Integration by parts shows that $uh(x,u) \geq (2+\varepsilon)H(x,u)$. Since h is odd, this is true for all u, i.e. h satisfies the Assumption 1.1/8°. Together with Assumptions (H1) and (H3) it can now be proved, just as in the verification of condition 3° of the Mountain Pass Lemma for the functional F in (4), that there exists a $\tilde{v} \in \mathcal{H}_1$ which satisfies $F_h(t\tilde{v}) < 0$ for $t > 1$. Without loss of generality it can be assumed that v, fixed in Section 2, satisfies the inequalities $F(tv) < 0$ and $F_h(tv) < 0$ for every $t > 1$.

Corresponding to $\beta = \inf_{w\in W} \max_{u\in w} F(u)$ we define

$$\beta_h = \inf_{w\in W}\max_{u\in w} F_h(u).$$

Furthermore set

$$M_g = \left\{ u \in \mathcal{H}\setminus\{0\} \;\middle|\; \|u\|_L^2 = \int_\Omega g(x,u)u\,dx + \int_\Gamma \widetilde{\varphi}(\xi,u)u\,d\Gamma \right\}$$

$$M_h = \left\{ u \in \mathcal{H}\setminus\{0\} \;\middle|\; \|u\|_L^2 = \int_\Omega h(x,u)u\,dx + \int_\Gamma \widetilde{\varphi}(\xi,u)u\,d\Gamma \right\}$$

Lemma 4.3. *Let $u \in \mathcal{H}\setminus\{0\}$. Then there is a real number $\tau > 0$ such that $\tau u \in M_h$, i.e. Λ_u intersects M_h at one point.*

Proof. As in the verification of conditions 2° and 3° of the Mountain Pass Lemma for the functional F in (4) (see Section 2) it can be shown that there exist $\delta > 0$ and $\tau_\delta > 0$ such that

$$\nu(\tau_\delta) := \|\tau_\delta u\|_L^2 - \int_\Omega h(x,\tau_\delta u)\tau_\delta u\,dx - \int_\Gamma \widetilde{\varphi}(\xi,\tau_\delta u)\tau_\delta u\,d\Gamma \geq \delta > 0$$

(observe that $h(\cdot,u)u$ and $\widetilde{\varphi}(\cdot,u)u$ are both positive and satisfy the same growth conditions in u as $G(\cdot,u)$ and $\widetilde{\Phi}(\cdot,u)$, respectively.) On the other hand there is a $\tau_\infty > 0$ with $\nu(\tau_\infty) \le 0$. A comparison with the arguments in Section 2 (verification of condition 3^o) shows that the existence of such a τ_∞ requires that $\{x \in \Omega | H(x,u) > 0\}$ is not a zero set. Since in Section 2 only one v satisfying $F(v) \le 0$ had to be found, Assumption $1.1/1^o$ on g was sufficient. In the present case the existence of a τ_∞ is needed for every $u \ne 0$, which is guaranteed by the stronger condition (H1) for h. Since ν is continuous it follows that there is a τ such that $\nu(\tau) = 0$ ∎

Lemma 4.4. *Set $\beta^* = \inf_{u\in M_g} F(u)$ and $\beta_h^* = \inf_{u \in M_h} F_h(u)$. Then $\beta \le \beta^*$ and $\beta_h \le \beta_h^*$.*

Proof. To show $\beta \le \beta^*$ it suffices to construct to every $\tilde{u} \in M_g$ a path $w \in W$ such that $F(\tilde{u}) = \max_{u\in w} F(u)$. Because of the definition of β it then follows at once that $\inf_{\tilde{u}\in M_g} F(\tilde{u}) \ge \beta$.

Let $\tilde{u} \in M_g$ be arbitrary. Using the same arguments as in the proof of Lemma 4.2, it follows that $F(\tilde{u})$ is the absolute maximum of F in $\Lambda_{\tilde{u}} = \{t\tilde{u} | t \ge 0\}$. Namely, for the function μ defined as in (20) we have again $\mu'(t) \ge 0$ for $0 < t < 1$ and $\mu'(t) \le 0$ for $t \ge 1$.

Now let $v \in \mathcal{H}$ from the proof of condition 3^o of Theorem 2.4 be fixed, i.e. $F(tv) \le 0$ for all $t \ge 1$. As in the verification of condition 3^o of the Mountain Pass Lemma in Section 2 it follows again that $F(\tilde{t}\tilde{u}) \le 0$ if $\tilde{t} > 1$ is large enough. Let V denote the two-dimensional subspace of $\mathcal{H}$, spanned by $\{v, \tilde{u}\}$, and let $R > \max\{\|\tilde{t}\tilde{u}\|, \|v\|\}$ be so large that for S_R, the sphere of radius R in $\mathcal{H}$, we have $F|_{V\cap S_R} \le 0$. Such an R exists, since for fixed R_0 the functional $\|\cdot\|_L^2$ attains its maximum (in $u_{\max}$) and $\int_\Omega G(x,\cdot) + \int_\Gamma \widetilde{\Phi}(\xi,\cdot)$ attains its minimum (in $u_{\min}$) on the (compact) set $S_{R_0} \cap V$. For $\lambda > 1$ we have $F(\lambda u) \le \lambda^2 \|u_{\max}\|_L^2 - \lambda^p C \|u_{\min}\|_{L^p}^p$ (compare with Section 2). If λ is large enough, it follows that $F(u) \le 0$ for every $u \in S_{\lambda R_0} \cap V$.

Let $\tilde{u}_R = \Lambda_{\tilde{u}} \cap S_R$, $v_R = \Lambda_v \cap S_R$ and w be a path connecting 0, $\tilde{u}$, $\tilde{u}_R$, v_R and v and lying in $\Lambda_{\tilde{u}} \cup (S_R \cap V) \cup \Lambda_v$. Obviously $w \in W$ and $F(\tilde{u}) = \max_{u\in w} F(u)$.

The same arguments show likewise $\beta_h \le \beta_h^*$ ∎

Now the following theorem can be proved.

Theorem 4.5. *Let h satisfy Assumptions* (H1) - (H3) *and assume that $\widetilde{\varphi}$ satisfies*

$$\widetilde{\varphi}(\xi, tu) \ge t^{1+\epsilon}\widetilde{\varphi}(\xi,u) \qquad \text{for every} \quad t \ge 1,\ u \ge 0. \tag{23}$$

For an open domain $D \subset\subset \Omega$ with compact closure assume that $g(x,u) \le h(x,u)$ for all $x \in \Omega \setminus D$ and all $u \ge 0$. Let $\bar{u}$ be the weak limit of the sequence of critical points u_k of the functional F_k in (9)*. Then $\bar{u} \equiv 0$ implies $\beta \ge \beta_h^*$.*

Proof. Assume $\bar{u} \equiv 0$. According to Lemma 3.1, $\beta = \lim_{k\to\infty} \beta_k$. Let u_k be a critical point of F_k and $F_k(u_k) = \beta_k$. By standard regularity arguments it can be shown that $u_k \in C^{1,\nu}(\overline{D})$ for every domain D with compact closure in Ω and that there is a subsequence of $\{u_k\}$, converging to 0 uniformly in $\overline{D}$. For this subsequence we have

$$0 \le \varepsilon_k := \int_D u_k g(x,u_k)\,dx \to 0 \qquad \text{as} \quad k \to \infty.$$

To every u_k there exists $t_k > 0$ with $t_k u_k \in M_h$ (Lemma 4.3), i.e.

$$t_k^2 \|u_k\|_L^2 = t_k \int_\Omega h(x, t_k u_k) u_k \, dx + t_k \int_\Gamma \widetilde{\varphi}(\xi, t_k u_k) u_k \, d\Gamma. \tag{24}$$

Since u_k is a critical point, we also have

$$\begin{aligned}
\|u_k\|_L^2 &= \int_\Omega g(x, u_k) u_k \, dx + \int_\Gamma \widetilde{\varphi}(\xi, u_k) u_k \, d\Gamma \\
&= \varepsilon_k + \int_{\Omega \setminus D} g(x, u_k) u_k \, dx + \int_\Gamma \widetilde{\varphi}(\xi, u_k) u_k \, d\Gamma \\
&\leq \varepsilon_k + \int_{\Omega \setminus D} h(x, u_k) u_k \, dx + \int_\Gamma \widetilde{\varphi}(\xi, u_k) u_k \, d\Gamma.
\end{aligned}$$

In the last inequality the fact was used that from $g(x,u) \leq h(x,u)$ for all $u \geq 0$, g and h odd, it follows that $g(x,u)u \leq h(x,u)u$ for all u.

First of all it will be shown now that the sequence $\{t_k\}$ is bounded. Therefore assume $t_k \geq 1$ for a subsequence (if there is no such subsequence, then $t_k < 1$ for almost all $k \in \mathbb{N}$ and the boundedness follows at once). If $t_k \geq 1$, we see from the last inequality and (24)

$$\begin{aligned}
&t_k^2 \varepsilon_k + t_k^2 \int_{\Omega \setminus D} h(x, u_k) u_k \, dx + t_k^2 \int_\Gamma \widetilde{\varphi}(\xi, u_k) u_k \, d\Gamma \\
&\geq t_k^2 \|u_k\|_L^2 \\
&= \int_\Omega t_k h(x, t_k u_k) u_k \, dx + \int_\Gamma t_k \widetilde{\varphi}(\xi, t_k u_k) u_k \, d\Gamma \\
&\geq \int_{\Omega \setminus D} t_k^{2+\varepsilon} h(x, u_k) u_k \, dx + \int_\Gamma t_k^{2+\varepsilon} \widetilde{\varphi}(\xi, u_k) u_k \, d\Gamma.
\end{aligned}$$

In the last line the estimates

$$\widetilde{\varphi}(\xi, tu) \geq t^{1+\varepsilon} \widetilde{\varphi}(\xi, u) u \qquad (t \geq 1) \tag{25}$$

$$h(x, tu) \geq t^{1+\varepsilon} h(x, u) u \qquad (t \geq 1) \tag{26}$$

were used for arbitrary u. Inequality (25) follows directly from (23) and the fact that $\widetilde{\varphi}$ is odd. On the other hand, (26) follows from condition (H2) if this is again (for u resp. $y > 0$) reformulated as a differential inequality:

$$y^{2+\varepsilon} \frac{d}{dy} \left(y^{-(1+\varepsilon)} h(x, y) \right) = y \frac{dh}{dy}(x, y) - (1 + \varepsilon) h(x, y) \geq 0.$$

Integration over $[u, tu]$ shows that $(tu)^{-(1+\varepsilon)}h(x,tu) - u^{-(1+\varepsilon)}h(x,u) \geq 0$ which implies $h(x,tu) \geq t^{1+\varepsilon}h(x,u)$ for every $u \geq 0$. Since h is odd in u, now (26) follows for every u. Therefore we get

$$\begin{aligned} t_k^2 \varepsilon_k &\geq (t_k^{2+\varepsilon} - t_k^2) \int\limits_{\Omega\setminus D} h(x,u_k)u_k\,dx + (t_k^{2+\varepsilon} - t_k^2) \int\limits_{\Gamma} \widetilde{\varphi}(\xi,u_k)u_k\,d\Gamma \\ &\geq (t_k^{2+\varepsilon} - t_k^2) \left(\int\limits_{\Omega} g(x,u_k)u_k dx - \varepsilon_k \right) + (t_k^{2+\varepsilon} - t_k^2) \int\limits_{\Gamma} \widetilde{\varphi}(\xi,u_k)u_k\,d\Gamma. \end{aligned}$$

Since u_k is a critical point of F_k, we have further

$$\begin{aligned} t_k^2 \varepsilon_k \geq (t_k^{2+\varepsilon} - t_k^2) &\left(\|u_k\|_L^2 - \int\limits_{\Gamma} \widetilde{\varphi}(\xi,u_k)u_k\,d\Gamma - \varepsilon_k \right) \\ &+ (t_k^{2+\varepsilon} - t_k^2) \int\limits_{\Gamma} \widetilde{\varphi}(\xi,u_k)u_k\,d\Gamma \\ &= (t_k^{2+\varepsilon} - t_k^2) \left(\|u_k\|_L^2 - \varepsilon_k \right). \end{aligned} \tag{27}$$

Since g and $\widetilde{\varphi}$ are odd, and positive for $u \geq 0$, G and $\widetilde{\Phi}$ are positive for all u and it follows for every k that

$$\|u_k\|_L^2 \geq 2F_k(u_k) \geq 2\beta_k \geq 2\beta.$$

Now $\varepsilon_k \to 0$, so that we can choose k_β such that $\varepsilon_k \leq \beta$ for every $k \geq k_\beta$, therefore $\left(\|u_k\|_L^2 - \varepsilon_k\right) \geq (2\beta - \beta) = \beta$. Using this in (27) we get $\varepsilon_k \geq (t_k^\varepsilon - 1)\beta$. This shows $\lim_{k\to\infty} t_k = 1$. In particular $\lim_{k\to\infty} t_k u_k = 0$.

According to Lemma 4.2, $F(u_k) = \max_{v\in\Lambda_{u_k}} F(v) = \max_{t\geq 0} F(tu_k)$, which shows that

$$\begin{aligned} \beta_k &= F(u_k) \\ &\geq F(t_k u_k) \\ &= \frac{1}{2} t_k^2 \|u_k\|_L^2 - \int\limits_{\Omega\setminus D} G(x,t_k u_k)\,dx - \int\limits_{D} G(x,t_k u_k)\,dx - \int\limits_{\Gamma} \widetilde{\Phi}(\xi,t_k u_k)\,d\Gamma \\ &\geq \frac{1}{2} t_k^2 \|u_k\|_L^2 - \int\limits_{\Omega} H(x,t_k u_k)\,dx - \int\limits_{\Gamma} \widetilde{\Phi}(\xi,t_k u_k)\,d\Gamma - \int\limits_{D} G(x,t_k u_k)\,dx \\ &= F_h(t_k u_k) - \int\limits_{D} G(x,t_k u_k)\,dx \\ &\geq \beta_h^* - \int\limits_{D} G(x,t_k u_k)\,dx. \end{aligned}$$

The last inequality follows directly from the definition of β_h^* and the fact that t_k was chosen in such a way that $t_k u_k \in M_h$. From $t_k u_k \to 0$ it follows again $\int_D G(x,t_k u_k)\,dx \to 0$, i.e. $\beta = \lim_{k\to\infty} \beta_k \geq \beta_h^*$ which proves the theorem ∎

From this theorem we have immediately the following

Corollary 4.6. *If $\beta < \beta_h^*$, then $\bar{u}$ is a non-trivial solution to Problem* 1.

According to the inequality $\beta \leq \beta^*$, proved in Lemma 4.4, it follows now

Corollary 4.7. *If $\beta^* < \beta_h^*$, then $\bar{u}$ is a non-trivial solution to Problem* 1.

This corollary will be used in the next section to prove the existence of non-trivial solutions to some special cases of Problem 1.

5. Existence theorems for some special cases

In this section we consider as special cases non-linearities of the form

$$g(x,u) = P(x)|u|^{p-1}u, \qquad h(x,u) = K(x)|u|^{p-1}u$$
$$\varphi(\xi,u) = -\alpha(\xi)u + Q(\xi)|u|^{p-1}u,$$

where $1 < p < n_* - 1$ and P, α, Q, K are positive L^∞-functions, P not a constant, $P(x) > 0$ everywhere and $K(x) = P(x)$ outside some bounded subdomain of Ω (for the precise definition see below).

The methods we use here to show the existence of a non-trivial solution to Problem 1 require the same exponent p in the non-linearities g and φ. As was shown in [13], this is quite natural from a physical point of view. However, this leads to a stronger restriction on p, since the critical Sobolev exponent n_* for the trace operator is smaller than n^*. In the three-dimensional case, we have $n_* = 4$, and consequently $1 < p < 3$.

Clearly Assumptions 1.1 and 4.1, those required in Theorem 4.5 and conditions (H1) - (H3) from Section 4 are satisfied for these functions g, φ and h.

The functional F now has the form

$$F(u) = \frac{1}{2}\|u\|_L^2 - \frac{1}{p+1}\int_\Omega P(x)|u|^{p+1}dx - \frac{1}{p+1}\int_\Gamma Q(\xi)|u|^{p+1}d\Gamma.$$

Correspondingly we define

$$M_g = \left\{ u \in \mathcal{H}\setminus\{0\} \;\middle|\; \|u\|_L^2 = \int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}$$

$$M_h = \left\{ u \in \mathcal{H}\setminus\{0\} \;\middle|\; \|u\|_L^2 = \int_\Omega K(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}.$$

In Lemma 4.3 it was shown that for every $u \in \mathcal{H}\setminus\{0\}$ there exists a $\tau > 0$ such that $\tau u \in M_h$. In the proof condition (H1) was used, which is slightly stronger than Assumption 1.1/1°. Because of the special choice of the function g in this section, we see that g also satisfies condition (H1). Therefore the same arguments used in the proof of Lemma 4.3 can now be applied to the set M_g to show that for every $u \in \mathcal{H} \setminus \{0\}$ there exists a $t > 0$ such that $tu \in M_g$.

Consequently we get the following representations for β^* and β_h^*:

$$\begin{aligned}\beta^* &= \inf_{u \in M_g} F(u) \\ &= \inf_{\substack{u \neq 0 \\ tu \in M_g}} \left\{ \left(\frac{1}{2} - \frac{1}{p+1}\right) t^{p+1} \left(\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right) \right\} \\ \beta_h^* &= \inf_{u \in M_h} F_h(u) \\ &= \inf_{\substack{u \neq 0 \\ \tau u \in M_h}} \left\{ \left(\frac{1}{2} - \frac{1}{p+1}\right) \tau^{p+1} \left(\int_\Omega K(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right) \right\}.\end{aligned}$$

In view of Corollary 4.7, to prove the existence of a non-trivial solution $\bar{u}$ of Problem 1, it must be shown that $\beta^* < \beta_h^*$ for an appropriate comparison function h (resp. K).

Let $\{D_j\}_{j \in I\!N}$ be a sequence of open subdomains in Ω, $D_1 \subset D_2 \subset \ldots$, such that the closure of each D_j is compact in $I\!R^n$, $\bigcup_{j \in I\!N} D_j = \Omega$ and

$$\operatorname{dist}(D_j, \Gamma) = \inf_{x \in D_j, \xi \in \Gamma} \{|x - \xi|\} > 0.$$

Theorem 5.1. *Let $P \neq$ const, $\inf_{x \in \Omega} P(x) = m > 0$ and assume that there is a sequence of open subdomains $\{D_j\}$ of Ω with the properties described above and a sequence of positive real numbers $\varepsilon_j \to 0$ such that $P(x) \leq m + \varepsilon_j$ for almost every $x \in \Omega \setminus D_j$. Furthermore let $1 < p < n_* - 1$ and assume that the inequality*

$$\begin{aligned}\sigma := \sup_{\|u\|_L = 1} &\left\{ \int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right\} \\ > \sup_{\|u\|_L = 1} &\left\{ \int_\Omega m|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right\} =: \sigma_m\end{aligned} \tag{28}$$

holds. Then there exists a non-trivial weak solution to Problem 1.

Remark. A sufficient condition on the function P such that inequality (28) holds is given below, see Corollary 5.2.

Proof of Theorem 5.1. For some fixed $j_0 \in I\!N$ set

$$K(x) = \begin{cases} P(x) & \text{for } x \in \Omega \setminus D_{j_0} \\ m & \text{for } x \in D_{j_0}. \end{cases} \tag{29}$$

Then the function $h(x, u) = K(x)|u|^{p-1}u$ satisfies the condition $g(x, u) \leq h(x, u)$ outside the subset D_{j_0} of Ω, i.e. Theorem 4.5 applies to this function.

Now we have to show $\beta^* < \beta_h^*$. Let

$$\tilde{\beta} := \inf_{\substack{u \neq 0 \\ tu \in M_g}} \left\{ t^{p+1} \left(\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right) \right\}$$

$$\tilde{\beta}_h := \inf_{\substack{u \neq 0 \\ \tau u \in M_h}} \left\{ \tau^{p+1} \left(\int_\Omega K(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{u+1} \right) \right\}.$$

From the definition of the positive real numbers t and τ it follows that

$$t^2 \|u\|_L^2 = t^{p+1} \int_\Omega P(x)|u|^{p+1} dx + t^{p+1} \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma$$

$$\tau^2 \|u\|_L^2 = \tau^{p+1} \int_\Omega K(x)|u|^{p+1} dx + \tau^{p+1} \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma.$$

Consequently we have

$$t^{p+1} = \left(\frac{\|u\|_L^2}{\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma} \right)^{(p+1)/(p-1)}$$

$$\tau^{p+1} = \left(\frac{\|u\|_L^2}{\int_\Omega K(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma} \right)^{(p+1)/(p-1)}.$$

Inserting this into the definition of $\tilde{\beta}$ one gets

$$\tilde{\beta} = \inf_{u \neq 0} \left\{ \frac{\left(\|u\|_L^2\right)^{(p+1)/(p-1)} \left(\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right)}{\left(\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right)^{(p+1)/(p-1)}} \right\}$$

$$= \inf_{u \neq 0} \left\{ \left(\frac{\|u\|_L}{\left(\int_\Omega P(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right)^{1/(p+1)}} \right)^{2(p+1)/(p-1)} \right\}.$$

Correspondingly

$$\tilde{\beta}_h = \inf_{u \neq 0} \left\{ \left(\frac{\|u\|_L}{\left(\int_\Omega K(x)|u|^{p+1} dx + \int_\Gamma Q(\xi)|u|^{p+1} d\Gamma \right)^{1/(p+1)}} \right)^{2(p+1)/(p-1)} \right\}.$$

Now $\beta^* < \beta_h^*$ if and only if $\tilde{\beta} < \tilde{\beta}_h$, and in order to apply Corollary 4.7 it must be shown that

$$\begin{aligned}&\inf_{u\neq 0}\left\{\frac{\|u\|_L}{\left(\int\limits_\Omega P(x)|u|^{p+1}dx+\int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}}\right\}\\&\quad<\inf_{u\neq 0}\left\{\frac{\|u\|_L}{\left(\int\limits_\Omega K(x)|u|^{p+1}dx+\int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}}\right\}.\end{aligned}\tag{30}$$

For that purpose the following inequality will be proved first:

$$\begin{aligned}\sigma &= \sup_{\|u\|_L=1}\left\{\int\limits_\Omega P(x)|u|^{p+1}dx+\int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right\}\\&> \lim_{j\to\infty}\left(\sup_{\|u\|_L=1}\left\{\int\limits_{\Omega\setminus D_j} P(x)|u|^{p+1}dx+\int\limits_{D_j} m|u|^{p+1}dx+\int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right\}\right)\\&=:\lim_{j\to\infty}\sigma_j.\end{aligned}\tag{31}$$

Let

$$\sigma_m' := \sup_{\|u\|_L=1}\left\{\int\limits_\Omega m|u|^{p+1}dx\right\}.$$

By assumption, there is a sequence $\varepsilon_j \to 0$ such that

$$\begin{aligned}\sigma_j' &:= \sup_{\|u\|_L=1}\left\{\int\limits_{\Omega\setminus D_j} P(x)|u|^{p+1}dx+\int\limits_{D_j} m|u|^{p+1}dx\right\}\\&\le \sup_{\|u\|_L=1}\left\{(m+\varepsilon_j)\int\limits_{\Omega\setminus D_j}|u|^{p+1}dx+m\int\limits_{D_j}|u|^{p+1}dx\right\}\\&\le (m+\varepsilon_j)\frac{\sigma_m'}{m}.\end{aligned}$$

This inequality remains true if the boundary integral is added on both sides. It follows that

$$\sigma_j = \sup_{\|u\|_L=1} \left\{ \int_{\Omega\setminus D_j} P(x)|u|^{p+1}dx + \int_{D_j} m|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}$$
$$\le \sup_{\|u\|_L=1} \left\{ (m+\varepsilon_j) \int_\Omega |u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}$$
$$\le \frac{m+\varepsilon_j}{m} \sup_{\|u\|_L=1} \left\{ \int_\Omega m|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}$$
$$= \frac{m+\varepsilon_j}{m}\sigma_m.$$

Using (28) we get (because of $\varepsilon_j \to 0$) in the limit $\lim_{j\to\infty}\sigma_j \le \sigma_m < \sigma$, which proves (31). Consequently it follows that

$$\inf_{\|u\|_L=1} \left\{ \frac{1}{\int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma} \right\}$$
$$< \lim_{j\to\infty} \inf_{\|u\|_L=1} \left\{ \frac{1}{\int_{\Omega\setminus D_j} P(x)|u|^{p+1}dx + \int_{D_j} m|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma} \right\}.$$

If $\lambda > 0$, we have $\|\lambda u\|_L = \lambda\|u\|_L$ and

$$\left(\int_\Omega P(x)|\lambda u|^{p+1}dx + \int_\Gamma Q(x)|\lambda u|^{p+1}d\Gamma \right)^{1/(p+1)}$$
$$= \lambda \left(\int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(x)|u|^{p+1}d\Gamma \right)^{1/(p+1)},$$

so that

$$\inf_{u\neq 0} \left\{ \frac{\|u\|_L}{\left(\int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}} \right\}$$
$$< \lim_{j\to\infty} \inf_{u\neq 0} \left\{ \frac{\|u\|_L}{\left(\int_{\Omega\setminus D_j} P(x)|u|^{p+1}dx + \int_{D_j} m|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}} \right\}.$$

Now in (29), the definition of K, we can choose some j_0 large enough to conclude from the last inequality the inequality (30), i.e. $\beta^* < \beta_h^*$ ∎

For arbitrary given functions P and Q it is not easy to decide, wether condition (28) is satisfied, since the suprema are in general not attained by some function in $H^1(\Omega)$. Of course, if P is assumed to be "large enough" (in some compact region) compared to Q and m, estimate (28) can be shown. First of all observe that

$$\begin{aligned}\sigma_m &= \sup_{\|u\|_L=1}\left\{\int\limits_\Omega m|u|^{p+1}dx + \int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right\}\\ &\le \sup_{\|u\|_L=1}\left\{m\|u\|^{p+1}_{L^{p+1}} + \|Q\|_{L^\infty}\|u\|^{p+1}_{L^{p+1}(\Gamma)}\right\}\\ &\le mC_1 + \|Q\|_{L^\infty}C_2\\ &=: C_{m,Q},\end{aligned}$$

where, once again, the Sobolev imbedding resp. trace theorem was used and the constant $C_{m,Q}$ only depends on m, Q and the Sobolev constants (with respect to the norm $\|\cdot\|_L$).

Now, given $u \in H^1(\Omega)$ with support in some D_j, $\|u\|_L = 1$ and an arbitrary constant $C > 0$, one can easily find a function P such that

$$\inf_{x\in\Omega} P(x) = m \qquad \text{and} \qquad \int_\Omega P(x)|u|^{p+1}dx \ge C.$$

This observation leads to the following

Corollary 5.2. *Let $P \ne$ const, $\inf_{x\in\Omega} P(x) = m > 0$ and assume that there is a sequence $\varepsilon_j \to 0$ such that $P(x) \le m+\varepsilon_j$ for almost every $x \in \Omega\setminus D_j$. Let $1 < p < n_*-1$ and assume that there exists a subdomain $B \subset D_j$ for some j, $0 < meas\, B < 1$, such that*

$$P(x) > \frac{C_{m,Q}}{meas\, B} \qquad \text{for every } \ x \in B.$$

Then there exists a non-trivial solution to Problem 1.

Proof. Choose $u_c \in \mathcal{H}$, $\|u_c\|_L = 1$, such that $u_c \ge 1$ in B. Then

$$\begin{aligned}\sigma &= \sup_{\|u\|_L=1}\left\{\int\limits_\Omega P(x)|u|^{p+1}dx + \int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right\}\\ &\ge \int\limits_B P(x)|u_c|^{p+1}dx\\ &> C_{m,Q}\\ &\ge \sup_{\|u\|_L=1}\left\{\int\limits_\Omega m|u|^{p+1}dx + \int\limits_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right\}\\ &= \sigma_m.\end{aligned}$$

So (28) is valid and Theorem 5.1 can be applied ∎

The condition $\inf_{x\in\Omega} P(x) = m > 0$ is not essential for the existence of solutions of Problem 1, but was needed only for technical reasons. The case $m = 0$ will be considered now, where the arguments are somewhat different – due to the fact that $h(x,u) = K(x)|u|^{p-1}u$ does not satisfy condition (H1) if in (29), the definition of K, we set $m = 0$.

Theorem 5.3. *With the same notations as in Theorem* 5.1 *let* $\inf_{x\in\Omega} P(x) = 0, P \neq$ const *and* $P(x) \leq \varepsilon_j$ *for almost every* $x \in \Omega \setminus D_j$. *Let* $1 < p < n_* - 1$ *and assume that the inequality*

$$\begin{aligned}\sigma := \sup_{\|u\|_L=1} &\left\{ \int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} \\ &> \sup_{\|u\|_L=1} \left\{ \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} =: \sigma_0\end{aligned}$$

holds. Then there exists a non-trivial weak solution to Problem 1.

Proof. As in (29) let

$$K(x) := \begin{cases} P(x) & \text{if } x \in \Omega \setminus D_{j_0} \\ 0 & \text{if } x \in D_{j_0} \end{cases} \qquad \text{for some } j_0 \in I\!N.$$

Since for functions $u \in \mathcal{H}$ with $\operatorname{supp} u \subset D_{j_0}$ we have $h(x,u) = K(x)|u|^{p-1}u = 0$ for every $x \in \Omega$, h does not satisfy condition (H1). Consequently, given a critical point u_k of F_k, it is a priori not clear whether there is a $t_k > 0$ such that $t_k u_k \in M_h$. But this fact was used to prove $\beta \geq \beta_h^*$ (in Theorem 4.5) if the weak limit of $\{u_k\}$ were zero.

Now we have to distinguish two cases. For $\{u_k\}$, the sequence of critical points of F_k

(i) there exists $k_0 \in I\!N$ such that for every $k \geq k_0$, $\operatorname{supp} u_k \cap \operatorname{supp} K$ is a set of measure zero; or

(ii) there is a subsequence of $\{u_k\}$ such that $\operatorname{supp} u_k \cap \operatorname{supp} K$ has non-vanishing Lebesgue measure for every $k \in I$ for some infinite index set $I \subset I\!N$.

Assume (i). According to the definition of K there is a $j_0 \in I\!N$ such that $\operatorname{supp} u_k \subset D_{j_0}$ for every $k \geq k_0$. Since $\{u_k\}$ is bounded in H^1-norm, there is a subsequence (again denoted $\{u_k\}$) converging in $L^{p+1}(D_{j_0})$-norm to some function $\bar{u}$. Since u_k and $\bar{u}$ are critical points we know that

$$\begin{aligned} F(u_k) &= \int_{D_{j_0}} \left(\frac{1}{2} - \frac{1}{p+1}\right) P(x)|u_k|^{p+1}dx \\ F(\bar{u}) &= \int_{D_{j_0}} \left(\frac{1}{2} - \frac{1}{p+1}\right) P(x)|\bar{u}|^{p+1}dx. \end{aligned} \tag{32}$$

(The boundary integral is zero because $\operatorname{supp} u_k$ lies in the interior of Ω.) Since the functionals in (32) are continuous on $L^{p+1}(D_{j_0})$, we get

$$\beta = \lim_{k\to\infty} \beta_k = \lim_{k\to\infty} F(u_k) = F(\bar{u}),$$

and therefore $\bar{u} \neq 0$.

Assume (ii). For each u_k of this subsequence there is a $t_k > 0$ such that $t_k u_k \in M_h$. Now the arguments in the proof of Theorem 4.5 can be repeated to show that, if the weak limit of $\{u_k\}$ is zero, then $\beta \geq \beta_h^*$. To prove $\beta^* < \beta_h^*$ the assumption $\sigma > \sigma_0$ is used. Since $P(x) \leq \varepsilon_j$ in $\Omega \setminus D_j$, we have

$$\begin{aligned}
\sigma_j &:= \sup_{\|u\|_L=1} \left\{ \int_{\Omega\setminus D_j} P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} \\
&\leq \sup_{\|u\|_L=1} \left\{ \int_{\Omega\setminus D_j} \varepsilon_j |u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} \\
&\leq \varepsilon_j C + \sup_{\|u\|_L=1} \left\{ \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} \\
&= \varepsilon_j C + \sigma_0
\end{aligned}$$

where the constant is due to the Sobolev embedding. Now $\varepsilon_j \to 0$ and therefore $\lim_{j\to\infty} \sigma_j \leq \sigma_0 < \sigma$, i.e.

$$\begin{aligned}
&\sup_{\|u\|_L=1} \left\{ \int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\} \\
&\quad > \lim_{j\to\infty} \sup_{\|u\|_L=1} \left\{ \int_{\Omega\setminus D_j} P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma \right\}.
\end{aligned}$$

If we now choose j_0 in the definition of K large enough we get

$$\begin{aligned}
&\inf_{u\neq 0} \left\{ \frac{\|u\|_L}{\left(\int_\Omega P(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}} \right\} \\
&\quad < \inf_{u\neq 0} \left\{ \frac{\|u\|_L}{\left(\int_\Omega K(x)|u|^{p+1}dx + \int_\Gamma Q(\xi)|u|^{p+1}d\Gamma\right)^{1/(p+1)}} \right\}
\end{aligned}$$

and consequently $\beta^* < \beta_h^*$ ∎

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