# **On a Class of Nonlinear Elliptic Problems with Neumann Boundary Conditions Growing Like a Power**

M. **Chipot and F. Voirol** 

Abstract. One investigates the issue of existence and number of solutions for the problem

$$
\Delta u = au^p \quad \text{in} \quad \Omega
$$
  
 
$$
u = 0 \quad \text{on} \quad \Gamma_0, \qquad \frac{\partial u}{\partial n} = u^q \quad \text{on} \quad \Gamma_1.
$$

where  $\Gamma_0$  and  $\Gamma_1$  are two parts of the boundary of the open set  $\Omega$ . In dimension one we are able to find all the solutions to the problem. In higher dimension we give for different solutions depending on  $p, q$  and  $\Omega$  existence and non-existence results.

Keywords: *Neumann boundary conditions, nonlinear elliptic equations* 

AMS subject classification: 34B15,35A15, 35JXX

#### **1. Introduction**

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  with boundary  $\Gamma$ . This paper is concerned with the problem of finding a positive solution *u* to the problem

et of R<sup>n</sup> with boundary 
$$
\Gamma
$$
. This paper is concerned with  
we solution *u* to the problem  

$$
\Delta u = au^p \qquad \text{in} \quad \Omega
$$

$$
u = 0 \qquad \text{on} \quad \Gamma_0
$$

$$
\frac{\partial u}{\partial n} = u^q \qquad \text{on} \quad \Gamma_1
$$

$$
(1.1)
$$

where  $a$  and  $p,~q$  are positive constants such that  $p,q>1,~\Gamma_0$  and  $\Gamma_1$  are two portions of the boundary  $\Gamma$  that we will assume to be disjoint and covering  $\Gamma$ , and *n* is the outward unit normal to  $\Gamma$ . Moreover, we will assume that  $\Gamma_0$  has a positive superficial measure. where a and p, q are positive constants such that p, q<br>the boundary  $\Gamma$  that we will assume to be disjoint an<br>unit normal to  $\Gamma$ . Moreover, we will assume that  $\Gamma_0$ <br>We refer the reader to [1] for the case where  $\Gamma_1 =$ 

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The above problem models the equilibrium of the temperature  $u$  in a domain  $\Omega$ . It is assumed that cooling is provided at a rate proportional to *u'* inside the body and a flux of heat is entering the boundary through  $\Gamma_1$  at a rate  $u^q$ . The other part of the boundary is maintained at a constant temperature. The question is then to determine if an equilibrium can be reached by the temperature inside the body.

#### 2. The one-dimensional case

In this section we consider the problem of finding  $u > 0$ ,  $u \in C^2(0, L) \cap C^1([0, L])$ , such that

boundary integral 
$$
I_1
$$
 at a late  $u^2$ . The other part of the a constant temperature. The question is then to determine the body.

\nOn all case

\nthe problem of finding  $u > 0$ ,  $u \in C^2(0, L) \cap C^1([0, L])$ , such

\n
$$
u'' = au^p \quad \text{on } (0, L)
$$

\n
$$
u(0) = 0 \quad \text{and} \quad u'(L) = u^q(L)
$$

\n(2.1)

where  $a > 0$  and  $p, q > 1$ . In this case the situation is complete and we have the following result.

**Theorem 2.1.** *The problem (2.1) can be described through the following cases.* 

(1) If  $2q > p + 1$ , then the problem (2.1) has for any  $L > 0$  a unique non-trivial *solution.*

*(2) If 2q=p+1, then* 

*for*  $a < q$  *the problem* (2.1) has for any  $L > 0$  a unique non-trivial solution *for*  $a \geq q$  *the problem* (2.1) has no non-trivial solution.

(3) If  $2q < p+1$ , then there exists  $L^* > 0$  such that *for*  $L < L^*$  *the problem* (2.1) has no non-trivial solution *for*  $L = L^*$  *the problem* (2.1) has a unique non-trivial solution *for*  $L > L^*$  *the problem* (2.1) has two non-trivial solutions. for any  $L > 0$  a un<br>no non-trivial solution<br>e exists  $L^* > 0$  such<br>no non-trivial solution<br>is a unique non-trivial so<br>is a unique non-trivial so<br> $(3)$  will be given if<br> $(1)$  Theorem 2.1.  $u''_m = au^p_m$ <br> $(0) = 0$  and  $u'_m(0)$ 

The proof of assertions  $(1)$  -  $(3)$  will be given in separate parts.

**Proof of assertion (1) of Theorem 2.1.** We introduce *Urn* the solution of the Cauchy problem

has two non-trivial solutions.  
\n1) - (3) will be given in separate parts.  
\n**of Theorem 2.1.** We introduce 
$$
u_m
$$
 the solution of the  
\n $u''_m = au^p_m$   
\n $u_m(0) = 0$  and  $u'_m(0) = m$   
\n $u_m(0) = 0$  and  $u'_m(0) = m$   
\n $\frac{u(r)}{u(r)}$  for all  $(m,r) \in (0, +\infty) \times (0, l_m)$ .  
\n**the function**  $r \rightarrow b(m,r)$  is decreasing on  $(0, l_m)$ . Indeed,  
\n**so** of (2.2) by  $u'_{m}$  we obtain

where  $m$  is a positive constant and we denote by  $[0, l_m)$  the interval where the solution exists. Then we set

$$
u_m \n\begin{cases}\n & m \quad \text{if } m \quad \
$$

We claim that for any  $m > 0$  the function  $r \to b(m, r)$  is decreasing on  $(0, l_m)$ . Indeed, if we mutiply the first equation of  $(2.2)$  by  $u'_m$  we obtain

$$
\frac{1}{2}(u_m'^2)'=\frac{a}{p+1}(u_m^{p+1})'.
$$

Integrating between 0 and *r* we get

$$
\frac{1}{2}u_m'^2(r)-\frac{1}{2}m^2=\frac{a}{p+1}u_m^{p+1}(r)
$$

hence

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\nwe get\n
$$
\frac{1}{2}u_m'^2(r) - \frac{1}{2}m^2 = \frac{a}{p+1}u_m^{p+1}(r)
$$
\n
$$
u'_m(r) = \sqrt{m^2 + \frac{2a}{p+1}u_m^{p+1}(r)}.
$$
\n(2.4)

So, we deduce

On a Class of Nonlinear Elliptic Problems 855  
\n
$$
\lim_{2} u_{m}^{2}(r) - \frac{1}{2}m^{2} = \frac{a}{p+1}u_{m}^{p+1}(r)
$$
\n
$$
u_{m}'(r) = \sqrt{m^{2} + \frac{2a}{p+1}u_{m}^{p+1}(r)}.
$$
\n
$$
b(m,r) = \frac{u_{m}'(r)}{u_{m}^{2}(r)} = \sqrt{m^{2}u_{m}^{-2q}(r) + \frac{2a}{p+1}u_{m}^{p+1-2q}(r)}.
$$
\n
$$
b(m,r) = \frac{u_{m}'(r)}{u_{m}^{2}(r)} = \sqrt{m^{2}u_{m}^{-2q}(r) + \frac{2a}{p+1}u_{m}^{p+1-2q}(r)}.
$$
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$$
c(2.5)
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c(2.5)
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c(2.1)
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\n
$$
c(2.2)
$$

From  $u_m$  being clearly increasing on  $(0, l_m)$  there follows since  $p + 1 - 2q < 0$  that  $r \rightarrow b(m, r)$  is decreasing on  $(0, l_m)$ . Next, let us establish the following  $\frac{S}{s} = \sqrt{m^2 u_m^{-2q}}(r) + \frac{1}{p+1} u_m^{p+1-2q}(r).$  (2.5)<br>
ng on  $(0, l_m)$  there follows since  $p + 1 - 2q < 0$  that<br>  $l_m$ ). Next, let us establish the following<br>  $d(m,r) \in (0, +\infty) \times (0, l_m)$ . Then<br>  $x^{-(p-1)/2} \in (0, +\infty) \times (0, l_{m\alpha^{(p+$ 

**Lemma 2.1.** *Let*  $\alpha > 0$  *and*  $(m, r) \in (0, +\infty) \times (0, l_m)$ *. Then* Next, let us establis<br> *a*, *r*)  $\in (0, +\infty) \times (0, 0, 0)$ <br>  $\in (1, +\infty) \times (0, 0)$ <br>  $\in (1, +\infty) \times (0, 0)$ <br>  $\in (1, +\infty)$ <br>  $\in (1, +$ 

$$
(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) \in (0, +\infty) \times (0, l_{m\alpha^{(p+1)/2}})
$$
  
\n
$$
b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m, r).
$$
  
\nder  
\n
$$
s(t) = \alpha u_m(\alpha^{(p-1)/2}t).
$$
  
\n
$$
s'(t) = \alpha^{(p+1)/2}u'_m(\alpha^{(p-1)/2}t)
$$
  
\n
$$
s'(t) = \alpha^p u''_m(\alpha^{(p-1)/2}t) = \alpha^p a u''_m(\alpha^{(p-1)/2}t) = a s(t)
$$

*and one has*

$$
o(m\alpha^{(p+1)/2},r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m,r).
$$
 (2.6)

**Proof.** Consider

$$
s(t) = \alpha u_m(\alpha^{(p-1)/2}t). \tag{2.7}
$$

One has

k,

sider  
\n
$$
s(t) = \alpha u_m(\alpha^{(p-1)/2}t).
$$
\n(2.7)  
\n
$$
s'(t) = \alpha^{(p+1)/2} u'_m(\alpha^{(p-1)/2}t)
$$
\n(2.8)

$$
tan(\pi, t) = (0, +\infty) \times (0, t_m). Then
$$
  
\n
$$
(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) \in (0, +\infty) \times (0, t_m\alpha^{(p+1)/2})
$$
  
\n
$$
b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m, r).
$$
 (2.6)  
\nsider  
\n
$$
s(t) = \alpha u_m(\alpha^{(p-1)/2}t).
$$
 (2.7)  
\n
$$
s'(t) = \alpha^{(p+1)/2}u'_m(\alpha^{(p-1)/2}t)
$$
 (2.8)  
\n
$$
s''(t) = \alpha^p u''_m(\alpha^{(p-1)/2}t) = \alpha^p au''_m(\alpha^{(p-1)/2}t) = as(t)^p.
$$
 (2.9)  
\nsfies  
\n
$$
s'' = as^p
$$

So,  $s = s(t)$  satisfies

$$
b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m,r).
$$
 (2.6)  
\nsider  
\n
$$
s(t) = \alpha u_m(\alpha^{(p-1)/2}t).
$$
 (2.7)  
\n
$$
s'(t) = \alpha^{(p+1)/2}u'_m(\alpha^{(p-1)/2}t)
$$
 (2.8)  
\n
$$
s''(t) = \alpha^p u''_m(\alpha^{(p-1)/2}t) = \alpha^p au_m^p(\alpha^{(p-1)/2}t) = as(t)^p.
$$
 (2.9)  
\n
$$
s'' = as^p
$$
  
\n
$$
s(0) = 0 \text{ and } s'(0) = m\alpha^{(p+1)/2}
$$
 (2.10)  
\n
$$
s = u_{m\alpha^{(p+1)/2}}
$$
 and  $l_{m\alpha^{(p+1)/2}} = \alpha^{-(p-1)/2}l_m.$   
\n
$$
s = \frac{(p+1)^2}{2}, r\alpha^{-(p-1)/2} = \frac{s'(r\alpha^{-(p-1)/2})}{s^q(r\alpha^{-(p-1)/2})} = \frac{\alpha^{(p+1-2q)/2} \frac{u'_m(r)}{u_m^q(r)}}{u_m^q(r)}
$$
  
\nFrom (2.6) we deduce easily that for r fixed the function  $m \to b(m,r)$  rst note that for  $m' > m$  one has, since the trajectories of the system

and by the uniqueness of the solution of the Cauchy problem

$$
s = u_{m\alpha^{(p+1)/2}}
$$
 and  $l_{m\alpha^{(p+1)/2}} = \alpha^{-(p-1)/2} l_m$ 

Next, we have

$$
b\big(m\alpha^{(p+1)/2},r\alpha^{-(p-1)/2}\big) = \frac{s'(r\alpha^{-(p-1)/2})}{s^q(r\alpha^{-(p-1)/2})} = \alpha^{(p+1-2q)/2} \frac{u'_m(r)}{u^q_m(r)}
$$

which gives (2.6). From (2.6) we deduce easily that for r fixed the function  $m \to b(m,r)$ is decreasing. First note that for  $m' > m$  one has, since the trajectories of the system (2.2) cannot cross,  $\begin{aligned} u^{(r)}, r\alpha^{(r)}, \alpha^{(r)}, \cdots^{(r)} \end{aligned} = \frac{1}{s^q(r\alpha^{-(p-1)/2})} = \alpha^{(r+1)/2}$ <br>
om (2.6) we deduce easily that for r fixed the function of that for  $m' > m$  one has, since the traje<br>  $u_m(r) \le u_{m'}(r)$  and  $u'_m(r) \le u'_{m'}(r)$ 

$$
u_m(r) \le u_{m'}(r) \quad \text{and} \quad u'_m(r) \le u'_{m'}(r)
$$

and thus

$$
l_m \ge l_{m'}.\tag{2.11}
$$

 $I_m \geq I_{m'}$ . (2.11)<br>
on some interval  $(0, m_r)$ . Next,  $m' > m$  can be<br>  $> 1$ . Then, since *b* is decreasing in *r* and by (2.6) So, for  $r > 0$  fixed,  $b(m, r)$  is defined on some interval  $(0, m_r)$ . Next,  $m' > m$  can be written as  $m' = m\alpha^{(p+1)/2}$  for some  $\alpha > 1$ . Then, since *b* is decreasing in *r* and by (2.6)

$$
b(m',r) < b(m',r\alpha^{-(p-1)/2})
$$
\n
$$
= b(m\alpha^{(p+1)/2},r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m,r)
$$
\n
$$
< b(m,r)
$$

and the result follows  $\blacksquare$ 

We are now in a position to establish (1). First remark that, by (2.11),  $\lim_{m\to 0} l_m$ exists. We claim that this limit is  $+\infty$ . This follows clearly from the continuous dependence in *m* of the solution to the Cauchy problem (2.2) and from the fact that, for  $m = 0$ , the solution is 0 and defined on the whole real line. Thus, given an  $L$ , one can find  $m > 0$  such that  $L < l_m$ . If  $b(m, L) = 1$ , then  $u_m$  provides us with a solution to our problem. If  $b(m, L) > 1$ , then one can select  $\alpha > 1$  such that

$$
b(m\alpha^{(p+1)/2}, L\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2}b(m, L) = 1.
$$

Then, due to the fact that *b* is decreasing in *r,* 

$$
b(m\alpha^{(p+1)/2}, L) < b(m\alpha^{(p+1)/2}, L\alpha^{-(p-1)/2}) = 1 < b(m, L).
$$

But due to the continuity of the map  $m \to b(m,L)$  one can find  $m_0 \in (m,m\alpha^{(p+1)/2})$ such that  $b(m_0,L) = 1$  and  $u_{m_0}$  is solution to our problem. In the case where  $b(m,L) < 1$ one proceeds the same way selecting  $\alpha < 1$ .

To see that uniqueness holds, assume that we have two distinct solutions  $u_1$  and  $u_2$ to (2.1). Then,  $m_1 = u_1'(0) \neq u_2'(0) = m_2$  and we cannot have  $b(m_1, L) = b(m_2, L)$ . Thus, uniqueness follows and assertion (1) of Theorem 2.1 is proved  $\blacksquare$ 

**Proof of assertion (2) of Theorem 2.1.** So, we assume  $2q = p + 1$  and as above we introduce  $u_m$ , the solution to problem  $(2.1)$ . In this case  $(2.5)$  reads

$$
0) \neq u'_2(0) = m_2
$$
 and we cannot have  $b(m_1, L) = b(m_2, L)$ .  
and assertion (1) of Theorem 2.1 is proved **1**  
**2**) of Theorem 2.1. So, we assume  $2q = p + 1$  and as above  
tion to problem (2.1). In this case (2.5) reads  

$$
b(m,r) = \frac{u'_m(r)}{u_m^q(r)} = \sqrt{\frac{m^2}{u_m^2(r)} + \frac{a}{q}}.
$$
 (2.12)  

$$
\rightarrow b(m,r)
$$
 is decreasing on  $(0, l_m)$ . Moreover, we claim that  

$$
\lim_{r \to l_m} u_m(r) = +\infty.
$$
 (2.13)  
learly exists. Moreover,  $(u, v) = (u_m, u'_m)$  is solution to the

Since  $u_m$  is increasing,  $r \to b(m, r)$  is decreasing on  $(0, l_m)$ . Moreover, we claim that

$$
\lim_{r \to l_m} u_m(r) = +\infty. \tag{2.13}
$$

Indeed, the above limit clearly exists. Moreover,  $(u, v) = (u_m, u'_m)$  is solution to the system

$$
\lim_{r \to l_m} u_m(r) = +\infty.
$$
\n(2.13)\nly exists. Moreover,  $(u, v) = (u_m, u'_m)$  is solution to the

\n
$$
u' = v \text{ and } v' = au^p
$$
\n
$$
u(0) = 0 \text{ and } v(0) = m.
$$
\n(2.14)

The functions *u* and *v* are both increasing and have a limit. If  $l_m < +\infty$  and  $\lim_{r \to l_m}$  $u_m(r)$  <  $+\infty$ , then, due to the first equation of (2.14),  $\lim_{r\to l_m} u''_m(r)$  <  $+\infty$  and so does  $\lim_{r \to l_m} u'_m(r)$  which is impossible. If now  $l_m = +\infty$  and  $\lim_{r \to l_m} u_m(r) < +\infty$ , then  $u''_m(r)$  and thus  $u'_m(r)$  are unbounded which contradicts the fact that  $u_m$  is. So, in all cases we have (2.13). It follows from (2.12) that for any  $r < l_m$ 

are unbounded which contrad't follows from (2.12) that for 
$$
b(m,r) > \lim_{r \to l_m} b(m,r) = \sqrt{\frac{a}{q}}
$$
.

Thus, when  $a \geq q$ , then the problem (2.1) cannot have a solution. The case  $a = q$ gives rise to no solution due to the fact that  $u_m(r)$  is unbounded when  $r \to l_m$ . When  $a < q$ , then, clearly, for any *m* we can find a unique  $L_m$  such that  $b(m, L_m) = 1$ . Now, it is easy to check that if  $u_1$  denotes the s  $a < q$ , then, clearly, for any m we can find a unique  $L_m$  such that  $b(m, L_m) = 1$ . Now, it is easy to check that if  $u_1$  denotes the solution to (2.2) corresponding to  $m = 1$ , then

$$
v'' = av^p \quad \text{on} \quad (0, \alpha^{-(p-1)/2} L_1)
$$
\n
$$
v(0) = 0 \quad \text{and} \quad v'(\alpha^{-(p-1)/2} L_1) = \alpha^q.
$$
\nThus,

\n
$$
u_{\alpha^q}(t) = \alpha u_1(\alpha^{(p-1)/2}t) \qquad \text{and} \qquad L_{\alpha^q} = \alpha^{-(p-1)/2} L_1.
$$
\nIt follows that for any  $L > 0$  there exists a unique  $\alpha$  such that  $L = \alpha^{-(p-1)/2} L_1$  and

Thus,

$$
u_{\alpha^q}(t) = \alpha u_1(\alpha^{(p-1)/2}t) \quad \text{and} \quad L_{\alpha^q} = \alpha^{-(p-1)/2}L_1.
$$

 $u_{\alpha}$ <sup>*i*</sup> is the unique solution to problem (2.1). This completes the proof of assertion (2) of Theorem **2.11**  =  $av^p$  on  $(0, \alpha^{-(p-1)/2}L_1)$ <br>
= 0 and  $v'(\alpha^{-(p-1)/2}L_1) = \alpha^q$ .<br>  $(\alpha^{(p-1)/2}t)$  and  $L_{\alpha^q} = \alpha^{-(p-1)/2}L_1$ .<br>
0 there exists a unique  $\alpha$  such that  $L = \alpha^{-(p-1)/2}L_1$  and<br>
0 problem (2.1). This completes the proof of ass *F*( $\alpha^{(p-1)/2}t$ ) and  $L_{\alpha^q} = \alpha^{-(p-1)/2}L_1$ .<br>  $> 0$  there exists a unique  $\alpha$  such that  $L = \alpha^{-(p-1)/2}L_1$  and<br>  $\alpha$  to problem (2.1). This completes the proof of assertion (2)<br> **(3) of Theorem 2.1.** So, we assume thro

**Proof of assertion (3) of Theorem 2.1.** So, we assume throughout this part that  $2q < p + 1$ . We introduce as before the solution  $u_m$  to problem (2.2). Recall that we have  $(see (2.4))$ 

$$
u'_{m}(r) = \sqrt{m^{2} + \frac{2a}{p+1} u_{m}^{p+1}(r)}.
$$
 (2.15)

So, in order for  $u_m(L)$  to solve  $u'_m(L) = u_m(L)$  it needs to be a root of the equation

$$
F(u) = u^{2q} - \frac{2a}{p+1}u^{p+1} - m^2 = 0.
$$
 (2.16)

We have  $F'(u) = 2qu^{2q-1} - 2au^p$  hence  $F'(u) = 0$  if and only if u is equal to  $\tau =$ ( $\left(\frac{q}{p}\right)^{1/(p+1-2q)}$ . Thus, *F* is increasing between 0 and *r* starting from the value  $-m^2$  and decreasing after *r* going to  $-\infty$  when  $u \to +\infty$ . So, in order for the equation (2.16) to have a root we need to have decreasing after  $\tau$  going to  $-\infty$  when  $u \to +\infty$ . So, in order for the equation (2.16) to have a root we need to have  $F(\tau) \geq 0$  which reads after an easy computation So, in<br>So, in<br> $\frac{1}{2}$ <br> $\frac{2q}{+1}$  $u(r) = \sqrt{m^2}$ <br> *2*  $u'_m(L) = 2$ <br>  $u = u^2$   $\frac{2}{p}$ <br> *2au<sup>p</sup>* hence<br> *2au<sup>p</sup>* hence<br> *2au<sup>p</sup>* hence<br> *2au<sup>p</sup>* hence<br> *2au<sup>p</sup>* hence<br> *2au<sup>p</sup>* hence  $p+1$ <br>hence  $F'(u) =$ <br> $y$  between 0 and<br> $y = 0$  which reads<br> $y = 1-2q$ <br> $\left\{1 - \frac{2q}{p+1}\right\}$ <br>to problem  $\left(2 - \left(\frac{q}{a}\right)^{q/(p+1-2q)}\right\}$ 

$$
\left(\frac{q}{a}\right)^{2q/(p+1-2q)}\left\{1-\frac{2q}{p+1}\right\}\geq m^2.
$$

So, in order for 
$$
u_m
$$
 to be a solution to problem (2.1) we have to restrict  $m$  to satisfy\n
$$
0 < m \leq M = \left(\frac{q}{a}\right)^{q/(p+1-2q)} \left\{1 - \frac{2q}{p+1}\right\}^{1/2}.
$$

In this case (2.16) has two roots  $R_1(m) < r < R_2(m)$  which coincide with  $\tau$  in the case where  $m = M$ . Going back to (2.15) we have

s 
$$
R_1(m) < \tau < R_2(m)
$$
wl  
(2.15) we have  

$$
\frac{u'_m(r)}{\sqrt{\frac{2a}{p+1}u_m^{p+1}(r)+m^2}} = 1
$$

hence integrating (recall that  $u'_m > 0$ )

$$
\frac{u'_m(r)}{\sqrt{\frac{2a}{p+1}u_m^{p+1}(r)+m^2}} = 1
$$
  
\n
$$
u'_m > 0
$$
  
\n
$$
\int_0^{u_m(r)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}+m^2}} = r.
$$
  
\nsolution to problem (2.1) for  
\n
$$
R_i(m)
$$
  
\n
$$
\int_0^{R_i(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}+m^2}}
$$
  
\n
$$
m (2.1) is a solution to per
$$

Then it is clear that  $u_m$  is a solution to problem (2.1) for  $L = L_1(m)$  and  $L = L_2(m)$ where *R, (m)* 

$$
\frac{u'_m(r)}{\sqrt{\frac{2a}{p+1}}u_m^{p+1}(r)+m^2} = 1
$$
\n  
\nrecall that  $u'_m > 0$ )\n  
\n
$$
\int_{0}^{u_m(r)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}+m^2}} = r.
$$
\n  
\n $u_m$  is a solution to problem (2.1) for  $L = L_1(m)$  and  $L = L_2(m)$   
\n
$$
L_i(m) = \int_{0}^{R_i(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}+m^2}} \qquad (i = 1, 2).
$$
\n(2.17)\n  
\nto problem (2.1) is a solution to problem (2.2) for some *m*, when  
\nrs  $L_i(m)$  are going to run over all the possible values for *L*. So, we  
\nnctions  $L_1(m)$  and  $L_2(m)$ . Let us start with  $L_2(m)$ .  
\n $u_2$  is a decreasing function of *m* on (0, *M*]. Moreover,  
\n $m) = +\infty$  and  $L_2(M) = \int_{0}^{r} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}+M^2}}.$   
\nance when *m* increases the graph of *F* goes down one has  
\n $R_1(m)$  is increasing with *m*

Since every solution to problem  $(2.1)$  is a solution to problem  $(2.2)$  for some  $m$ , when *m* varies the numbers  $L_i(m)$  are going to run over all the possible values for  $L$ . So, we need to study the functions  $L_1(m)$  and  $L_2(m)$ . Let us start with  $L_2(m)$ .

Lemma 2.2.  $L_2$  is a decreasing function of m on  $(0, M]$ . Moreover,

clear that 
$$
u_m
$$
 is a solution to problem (2.1) for  $L = L_1(m)$  and  
\n
$$
L_i(m) = \int_0^{R_i(m)} \frac{ds}{\sqrt{\frac{2a}{p+1} s^{p+1} + m^2}}
$$
 (*i* = 1,2).  
\n<sup>7</sup> solution to problem (2.1) is a solution to problem (2.2) for so  
\nthe numbers  $L_i(m)$  are going to run over all the possible values for  
\ndy the functions  $L_1(m)$  and  $L_2(m)$ . Let us start with  $L_2(m)$ .  
\na 2.2.  $L_2$  is a decreasing function of m on (0, M]. Moreover,  
\n
$$
\lim_{m\to 0} L_2(m) = +\infty \qquad and \qquad L_2(M) = \int_0^{\frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + M^2}}} \frac{ds}{\sqrt{\frac{2a}{p+1} s^{p+1} + M^2}}
$$
\nFirst, since when m increases the graph of F goes down one h:  
\n $R_1(m)$  is increasing with m  
\n $R_2(m)$  is decreasing with m.  
\n
$$
m'
$$
 one has  
\n
$$
\frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m^2}} < \frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m'^2}}
$$
\n
$$
\frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m'^2}}
$$
\n
$$
\frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m'^2}}
$$

Proof. First, since when *m* increases the graph of *F* goes down one has

*R1 (m)* is increasing with *<sup>m</sup>*

 $R_2(m)$  is decreasing with  $m$ .

So, if  $m > m'$  one has

$$
\frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m^2}} < \frac{1}{\sqrt{\frac{2a}{p+1} s^{p+1} + m'^2}}
$$

and, integrating,

one has  
\n
$$
\frac{1}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}} < \frac{1}{\sqrt{\frac{2a}{p+1}s^{p+1} + m'^2}}
$$
\nng,  
\n
$$
L_2(m) = \int_0^R \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}}
$$
\n
$$
< \int_0^{R_2(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m'^2}} < \int_0^{R_2(m')} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m'^2}}
$$
\n
$$
= L_2(m').
$$

 $\mathbf{I}$ 

On a Class of Nonlinear

\nThus, 
$$
L_2
$$
 is decreasing. On the other hand,  $R_2(m) \geq \tau$  so that

\n
$$
L_2(m) \geq \int_0^{\tau} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}}.
$$

\nLetting  $m \to 0$  one obtains  $\lim_{m \to 0} L_2(m) = +\infty$  since

\n
$$
\int_0^{\tau} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1}}}
$$

Letting  $m \to 0$  one obtains  $\lim_{m \to 0} L_2(m) = +\infty$  since

$$
\int_0 \frac{ds}{\sqrt{\frac{2a}{p+1} s^{p+1}}}
$$

diverges  $\left(\frac{p+1}{2} > 1\right)$ . This concludes the proof of Lemma 2.2

Next we turn to the study of *L1 .* We have

**Lemma 2.3.** When  $m \to 0$ , then  $R_1(m) \sim m^{1/q}$  and  $L_1(m) \sim m^{\frac{1}{q}-1}$ . In particular  $\lim_{m\to 0} L_1(m) = +\infty$ .

**Proof.** First, note that when  $m \to 0$ , then any limit value of  $R_1(m)$  must satisfy diverges  $(\frac{p+1}{2} > 1)$ . This concludes the proof of Lemma 2.2 **I**<br>
Next we turn to the study of  $L_1$ . We have<br>
Lemma 2.3. When  $m \to 0$ , then  $R_1(m) \sim m^{1/q}$  and  $L_1(m) \sim m^{\frac{1}{q}-1}$ . In periodic Im<sub>m $\to 0$ </sub>  $L_1(m) = +\infty$ Next we t<br>
Lemma 2<br>  $\lim_{m\to 0} L_1(m)$ <br>
Proof. F<br>  $u^{2q} - \frac{2a}{p+1} u^{p+1}$ <br>
and the only p<br>
to (2.16) we h<br>
or<br>
or<br>
Since,  $\lim_{m\to 0}$ 

**P + 1)p+1-29**  >( q ) *p+1-2q - =r>Ri(rn)* 2a *a* 

and the only possible limit value for  $R_1(m)$  is 0 so that  $\lim_{m\to 0} R_1(m) = 0$ . Going back<br>to (2.16) we have<br> $R_1(m)^{2q} - \frac{2a}{m} R_1(m)^{p+1} - m^2 = 0$ to (2.16) we have

$$
R_1(m)^{2q}-\frac{2a}{p+1}R_1(m)^{p+1}-m^2=0
$$

$$
R_1(m)^{2q}\left\{1-\frac{2a}{p+1}R_1(m)^{p+1-2q}\right\}=m^2.
$$

Since,  $\lim_{m\to 0} R_1(m) = 0$  we deduce that, when  $m \to 0$ ,  $R_1(m)^{2q} \sim m^2$  and thus or<br>Since,  $\lim_{m}$ <br> $R_1(m) \sim m$  $h^{1/q}$ . Going back to the definition (2.17) we have

$$
[(m)^{2q} - \frac{2a}{p+1} R_1(m)^{p+1} - m^2 =
$$
  
\n
$$
[n]^{\frac{2q}{q}} \left\{ 1 - \frac{2a}{p+1} R_1(m)^{p+1-2q} \right\} =
$$
  
\nwe deduce that, when  $m \to 0$   
\nto the definition (2.17) we have  
\n
$$
R_1(m)
$$
  
\n
$$
L_1(m) = \int_0^{\frac{2a}{p+1} \cdot 1} \frac{ds}{s^{p+1} + m^2}
$$
  
\netting

Changing of variable, i.e. setting

$$
m) = 0
$$
 we deduce that, when  $m \to 0$ ,  $R_1(m)$ ;  
ing back to the definition (2.17) we have  

$$
R_1(m)
$$

$$
L_1(m) = \int_0^1 \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}}.
$$
  
le, i.e. setting  

$$
u = \left(\frac{2a}{p+1}\right)^{1/(p+1)} m^{-2/(p+1)} s = C m^{-2/(p+1)} s
$$

**we obtain**

Since, 
$$
\lim_{m\to 0} R_1(m) = 0
$$
 we deduce that, when  $m \to 0$ ,  $R_1(m)^{2q} \sim m^2$  and thus  
\n $R_1(m) \sim m^{1/q}$ . Going back to the definition (2.17) we have  
\n
$$
L_1(m) = \int_0^{R_1(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}}.
$$
\nChanging of variable, i.e. setting  
\n
$$
u = \left(\frac{2a}{p+1}\right)^{1/(p+1)} m^{-2/(p+1)} s = C m^{-2/(p+1)} s
$$
\nwe obtain  
\n
$$
L_1(m) = \frac{1}{C m^{1-2/(p+1)}} \int_0^{C m^{-2/(p+1)} R_1(m)} \frac{du}{\sqrt{u^{p+1}+1}}
$$
\n
$$
= \frac{R_1(m)}{m} \frac{1}{C m^{-2/(p+1)} R_1(m)} \int_0^{C m^{-2/(p+1)} R_1(m)} \frac{du}{\sqrt{u^{p+1}+1}}.
$$
\nSince  $C m^{-2/(p+1)} R_1(m) \sim C m^{1/q-2/(p+1)} \to 0$  when  $m \to 0$  we obtain  $L_1(m) \sim \frac{R_1(m)}{m} \sim m^{\frac{1}{q}-1}$  which concludes the proof

 $1/q-2/(p+1) \rightarrow 0$  when  $m \rightarrow 0$  we obtain  $L_1(m)$  $m^{\frac{1}{q}-1}$  which concludes the proof  $\blacksquare$ 

Next let us show

Lemma 2.4. Let us denote by  $L_i'(m)$  the derivative of  $L_i$  with respect to  $m$ . Then *one has*

M. Chipot and F. Voirol  
\nt let us show  
\n
$$
\mathbf{n} \mathbf{m} \mathbf{a} \quad \mathbf{2.4.} \quad Let \text{ us denote by } L'_i(m) \text{ the derivative of } L_i \text{ with respect to } m. \text{ Then}
$$
\n
$$
\left(1 - \frac{2}{p+1}\right) L_i(m) + mL'_i(m) = \left(1 - \frac{2q}{p+1}\right) \left\{ \frac{1}{qR_i^{q-1} - aR_i^{p-q}} \right\} \qquad (2.19)
$$
\n1,2. Moreover,  
\n
$$
\lim_{m \to M} L'_1(m) = +\infty \qquad \text{and} \qquad \lim_{m \to M} L'_2(m) = -\infty. \qquad (2.20)
$$
\n
$$
\text{of. Going back to (2.18) one has}
$$

 $for i = 1, 2. Moreover,$ 

$$
\lim_{m \to M} L'_1(m) = +\infty \qquad \text{and} \qquad \lim_{m \to M} L'_2(m) = -\infty. \tag{2.20}
$$

**Proof.** Going back to *(2.18)* one has

$$
\int L_i(m) + mL'_i(m) = \left(1 - \frac{2q}{p+1}\right) \left\{\frac{1}{qR_i^{q-1} - aR}\right\}
$$
  
ver,  

$$
\lim_{n \to M} L'_1(m) = +\infty \qquad and \qquad \lim_{m \to M} L'_2(m) = -\infty
$$
  
back to (2.18) one has  

$$
Cm^{1-2/(p+1)}L_i(m) = \int_0^{2m-2/(p+1)} \frac{du}{\sqrt{u^{p+1}+1}}.
$$
  
ng with respect to  $m$  we get

Hence differentiating with respect to *m* we get

$$
\lim_{m \to M} L'_{1}(m) = +\infty \quad \text{and} \quad \lim_{m \to M} L'_{2}(m) = -\infty. \tag{2.20}
$$
\nProof. Going back to (2.18) one has\n
$$
Cm^{-2/(p+1)}L_{i}(m) = \int_{0}^{2M} \frac{du}{\sqrt{u^{p+1}+1}}.
$$
\n
$$
Cm^{1-2/(p+1)}L_{i}(m) = \int_{0}^{2M} \frac{du}{\sqrt{u^{p+1}+1}}.
$$
\n
$$
C\left(1 - \frac{2}{p+1}\right) m^{-2/(p+1)}L_{i}(m) + Cm^{1-2/(p+1)}L'_{i}(m)
$$
\n
$$
= \frac{1}{\sqrt{(Cm^{-2/(p+1)}R_{i}(m))^{p+1}+1}} (Cm^{-2/(p+1)}R_{i}(m))' \qquad (2.21)
$$
\n
$$
= \frac{m}{\sqrt{\frac{2a}{p+1}R_{i}(m)^{p+1}+m^{2}}} C\left(-\frac{2}{p+1}m^{-1-2/(p+1)}R_{i}(m) + m^{-2/(p+1)}R'_{i}(m)\right).
$$
\n
$$
Cm^{2}
$$
\n
$$
R_{i}(m)^{2q} = \frac{2a}{p+1}R_{i}(m)^{p+1} + m^{2} \qquad (2.22)
$$
\n
$$
Cm^{2}
$$
\n
$$
Cm^{2}/(p+1)
$$
\n
$$
\left(1 - \frac{2}{p+1}\right) L_{i}(m) + mL'_{i}(m) = \frac{m}{R_{i}(m)^{q}} \left\{ R'_{i}(m) - \frac{2}{p+1} \frac{R_{i}(m)}{m} \right\}
$$
\n
$$
= \left\{ mR'_{i}(m)R_{i}(m)^{-q} - \frac{2}{R_{i}(m)^{1-q}} \right\}.
$$
\n
$$
(2.23)
$$

Since

$$
R_i(m)^{2q} = \frac{2a}{p+1}R_i(m)^{p+1} + m^2
$$
 (2.22)

relation (2.21) reads after pulling out  $Cm^{-2/(p+1)}$ 

$$
R_i(m)^{2q} = \frac{2a}{p+1} R_i(m)^{p+1} + m^2
$$
(2.22)  
tion (2.21) reads after pulling out  $Cm^{-2/(p+1)}$   

$$
\left(1 - \frac{2}{p+1}\right) L_i(m) + mL'_i(m) = \frac{m}{R_i(m)^q} \left\{ R'_i(m) - \frac{2}{p+1} \frac{R_i(m)}{m} \right\}
$$

$$
= \left\{ mR'_i(m)R_i(m)^{-q} - \frac{2}{p+1}R_i(m)^{1-q} \right\}.
$$
(2.23)  
erentiating (2.22) we obtain
$$
R'_i(m) = \frac{m}{qR_i(m)^{2q-1} - aR_i(m)^p}
$$
hat
$$
mR'_i(m) = \frac{m^2}{qR_i(m)^{2q-1} - aR_i(m)^p} = \frac{R_i(m)^{2q} - \frac{2a}{p+1}R_i(m)^{p+1}}{qR_i(m)^{2q-1} - aR_i(m)^p}.
$$
lacing into (2.23) we obtain (2.19). Now when  $m \to \tau$ , then  $R_i(m) \to \tau$  where  $n = -a$ <sup>p-1</sup> vanishes. Since  $R_1(m) < \tau < R_2(m)$  we see passing to the limit in (2.19)

Differentiating *(2.22)* we obtain

$$
R_i'(m) = \frac{m}{qR_i(m)^{2q-1} - aR_i(m)^p}
$$

so that

$$
R_i'(m) = \frac{m}{qR_i(m)^{2q-1} - aR_i(m)^p}
$$
\nso that

\n
$$
mR_i'(m) = \frac{m^2}{qR_i(m)^{2q-1} - aR_i(m)^p} = \frac{R_i(m)^{2q} - \frac{2a}{p+1}R_i(m)^{p+1}}{qR_i(m)^{2q-1} - aR_i(m)^p}.
$$
\nReplacing into (2.23) we obtain (2.19). Now when  $m \to \tau$ , then  $R_i(m) \to \tau$  where

 $qu^{q-1} - au^{p-1}$  vanishes. Since  $R_1(m) < r < R_2(m)$  we see passing to the limit in (2.19) that *(2.20)* holds I

Next we have

Lemma 2.5. On  $(0, M)$  the function  $L_1$  is decreasing until a value  $m_0 \in (0, M)$ *and then increasing until M.* 

Proof. Due to Lemmas *2.3* and *2.4* it is enough to show that at a point where  $L'_1(m) = 0$ , then  $L''_1(m) > 0$  so that *m* could only be a minimum. For that, differentiating *(2.19)* we obtain

Next we have  
\nLemma 2.5. On (0, M) the function 
$$
L_1
$$
 is decreasing until a value  $m_0 \in (0, M)$   
\nd then increasing until M.  
\nProof. Due to Lemmas 2.3 and 2.4 it is enough to show that at a point where  
\n $(m) = 0$ , then  $L_1''(m) > 0$  so that m could only be a minimum. For that, differenti-  
\ning (2.19) we obtain  
\n
$$
\left(1 - \frac{2}{p+1}\right) L_1'(m) + L_1'(m) + mL_1''(m)
$$
\n
$$
= \left(1 - \frac{2q}{p+1}\right) \left\{ \frac{1}{qR_1^{q-1} - aR_1^{p-q}} \right\}' \qquad (2.24)
$$
\n
$$
= -\left(1 - \frac{2q}{p+1}\right) \frac{1}{(qR_1^{q-1} - aR_1^{p-q})^2} \left\{ q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1} \right\} R_1'.
$$
\nis clear from (2.21) that at a point where  $L_1'(m) = 0$  one must have  $R_1'(m) >$   
\nthen (see (2.24)) at a point where  $L_1'(m) = 0$  the sign of  $L_1''(m)$  is given by the  
\npositive sign of  $\{q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1}\}$  so that  $L_1''(m) > 0$  if and only if  
\n $q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1} < 0$  or  
\n
$$
\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q} \qquad (2.25)
$$
\n
$$
\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q} \qquad (2.25)
$$
\n
$$
\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q} \qquad (2.25)
$$

It is clear from (2.21) that at a point where  $L'_1(m) = 0$  one must have  $R'_1(m)$ 0, then (see (2.24)) at a point where  $L'_1(m) = 0$  the sign of  $L''_1(m)$  is given by the opposite sign of  $\{q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1}\}$  so that  $L_1''(m) > 0$  if and only if  $q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1} < 0$  or *q* $R_1^{q-1} - aR_1^{p-q}$ ?  $q(q-1)R_1^r - a(p-q)R_1^r$ <br>
1) that at a point where  $L'_1(m) = 0$  one must<br>
at a point where  $L'_1(m) = 0$  the sign of  $L''_1(m)$ <br>  $q \cdot p-1$ ,  $R_1^{q-2} - a(p-q)R_1^{p-q-1}$  so that  $L''_1(m) > 0$ <br>  $q \cdot q$ ,  $R_1^{p-q-1} < 0$ 

$$
\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q} \tag{2.25}
$$

(note that  $q < p$  since  $2q < p + 1 < 2p$ ). Next, going back to (2.19), at a point where  $L'_1(m) = 0$  we have

*p* since 
$$
2q < p + 1 < 2p
$$
). Next, going back to (2.19), at  
\nhave  
\n
$$
\left(1 - \frac{2}{p+1}\right) L_1(m) = \left(1 - \frac{2q}{p+1}\right) \left\{\frac{1}{qR_1^{q-1} - aR_1^{p-q}}\right\}.
$$
\n
$$
= \int_{0}^{R_1(m)} \frac{ds}{\sqrt{\frac{2a}{\sqrt{16} + 1} + m^2}} > \frac{R_1(m)}{\sqrt{\frac{2a}{\sqrt{16} + 1} + m^2}} = R_1(m)
$$

Clearly

$$
\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q}
$$
\nnat  $q < p$  since  $2q < p+1 < 2p$ ). Next, going back to (2.19), at a point  $q < p$  since

\n
$$
\left(1 - \frac{2}{p+1}\right) L_1(m) = \left(1 - \frac{2q}{p+1}\right) \left\{\frac{1}{qR_1^{q-1} - aR_1^{p-q}}\right\}.
$$
\n
$$
L_1(m) = \int_0^{R_1(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}s^{p+1} + m^2}} > \frac{R_1(m)}{\sqrt{\frac{2a}{p+1}R_1(m)^{p+1} + m^2}} = R_1(m)^{1-q}.
$$
\npoint where  $L_1'(m) = 0$  we have

So at a point where  $L'_1(m) = 0$  we have

here 
$$
L'_1(m) = 0
$$
 we have  
\n
$$
\left(1 - \frac{2}{p+1}\right) R_1^{1-q} < \left(1 - \frac{2q}{p+1}\right) \left\{\frac{1}{q R_1^{q-1} - a R_1^{p-q}}\right\}
$$
\n
$$
\left(1 - \frac{2}{p+1}\right) \left\{q - a R_1^{p+1-2q}\right\} < \left(1 - \frac{2q}{p+1}\right)
$$

which reads also

$$
\left(1-\frac{2}{p+1}\right)\left\{q-aR_1^{p+1-2q}\right\} < \left(1-\frac{2q}{p+1}\right)
$$

or

Voirol  
\n
$$
\frac{p-1}{p+1} \{q - aR_1^{p+1-2q}\} < \frac{p+1-2q}{p+1}
$$
\ninequality

This is equivalent to the inequality

equality  
\n
$$
\{q - aR_1^{p+1-2q}\} < \frac{p+1-2q}{p-1}
$$
\n
$$
q - \frac{p+1-2q}{p-1} < aR_1^{p+1-2q}.
$$
\n
$$
p - q < q - \frac{p+1-2q}{p-1}
$$
\n
$$
\frac{q(q-1)}{p-q} < q - \frac{p+1-2q}{p-1}
$$

and the last to the inequality

$$
q-\frac{p+1-2q}{p-1}
$$

So we will be done thanks to *(2.25) if* 

$$
-\frac{p+1-2q}{p-1} < aR_1^{p+1-2q}
$$
\n2.25) if

\n
$$
\frac{q(q-1)}{p-q} < q - \frac{p+1-2q}{p-1}
$$

or equivalently

$$
q - \frac{p+1-2q}{p-1} < aR_1^{p+1-2q}.
$$
\nbe done thanks to (2.25) if

\n
$$
\frac{q(q-1)}{p-q} < q - \frac{p+1-2q}{p-1}
$$
\nthey

\n
$$
\frac{p+1-2q}{p-1} < q - \frac{q(q-1)}{p-q} = q \left\{ 1 - \frac{q-1}{p-q} \right\} = q \left\{ \frac{p+1-2q}{p-q} \right\}
$$
\nThus, the following equation is:\n
$$
P = \frac{1}{p-1} \left( \frac{q}{p-q} \right)
$$
\nwhich is true since  $q > 1$ .

or equivalently<br> $\frac{p+1-2q}{p-1}$ <br>This will be true if  $\frac{1}{p-1}$ <br>Combining the inform  $< \frac{q}{p-q}$  which is true since  $q > 1$ 

Combining the information of the different lemmas we see that the curves  $L_1$  and *L2* look as below.



Set  $L^* =$ solution, h  $\inf_{(0,M)} L_1$ . Then for  $L < L^*$ ,  $L = L^*$  and  $L > L^*$  problem (2.1) has no solution, has a unique solution and has exactly two solutions, respectively.

Remark 2.1. The method used in proving assertion (3) of Theorem *2.1* could also have been used to establish assertions (1) and (2).

## **3. The higher dimensional case**

In this section we assume that *u* is a weak solution to (1.1) such that  $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$ .

In the case where  $2q = p + 1$ , we have a similar result to the one-dimensional case:

**Theorem 3.1.** Assume that  $2q = p + 1$ . Then, if a is large enough, the problem *(1.1) cannot have a non-trivial solution.*  On a Class of Nonlinear Elliptic Problems 863<br> **nsional case**<br>
that u is a weak solution to (1.1) such that  $u \in H^1(\Omega) \cap$ <br>  $p + 1$ , we have a similar result to the one-dimensional case:<br>
the that  $2q = p + 1$ . Then, if a is l

**Proof.** Let us denote by  $\nu$  a smooth vector field such that

$$
\nu = n \quad \text{on } \Gamma_1 \qquad \text{and} \qquad |\nu| \le 1. \tag{3.1}
$$

Multiplying the first equation of (1.1) by  $u$  and integrating over  $\Omega$  we get

this section we assume that *u* is a weak solution to (1.1) such that 
$$
u \,\epsilon H^1(\Omega) \cap
$$
  
\n<sup>11</sup>( $\Omega$ ).  
\nIn the case where  $2q = p + 1$ , we have a similar result to the one-dimensional case:  
\n**Theorem 3.1.** Assume that  $2q = p + 1$ . Then, if *a* is large enough, the problem  
\n1) cannot have a non-trivial solution.  
\n**Proof.** Let us denote by *v* a smooth vector field such that  
\n $v = n$  on  $\Gamma_1$  and  $|v| \le 1$ . (3.1)  
\nItiplying the first equation of (1.1) by *u* and integrating over  $\Omega$  we get  
\n
$$
a \int_{\Omega} u^{p+1} dx = \int_{\Omega} \Delta u u dx = \int_{\Omega} \nabla \cdot (\nabla u u) dx - \int_{\Omega} |\nabla u|^2 dx
$$
\n
$$
= \int_{\Gamma_1} \frac{\partial u}{\partial n} u d\sigma(x) - \int_{\Omega} |\nabla u|^2 dx = \int_{\Gamma_1} u^{q+1} d\sigma(x) - \int_{\Omega} |\nabla u|^2 dx
$$
\n
$$
= d\sigma(x)
$$
 denotes the superficial measure on  $\Gamma$ . Hence  
\n
$$
\int_{\Omega} |\nabla u|^2 dx + a \int_{\Omega} u^{p+1} dx = \int_{\Gamma_1} u^{q+1} d\sigma(x).
$$
\n(3.3)  
\n
$$
= \int_{\Gamma_1} u^{q+1} d\sigma(x) = \int_{\Omega} \nabla \cdot (u^{q+1} v) dx
$$
\n
$$
= (q+1) \int_{\Omega} u^q \nabla u \cdot v dx + \int_{\Omega} u^{q+1} \nabla \cdot v dx.
$$
\n(3.4)

were  $d\sigma(x)$  denotes the superficial measure on  $\Gamma$ . Hence

$$
\int_{\Omega} |\nabla u|^2 \, dx + a \int_{\Omega} u^{p+1} \, dx = \int_{\Gamma_1} u^{q+1} \, d\sigma(x). \tag{3.3}
$$

Next, remark that

$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) = \int_{\Omega} \nabla \cdot (u^{q+1} \nu) dx
$$
\n
$$
= (q+1) \int_{\Omega} u^q \nabla u \cdot \nu dx + \int_{\Omega} u^{q+1} \nabla \cdot \nu dx.
$$
\nHence,\n
$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le (q+1) \int_{\Omega} u^q |\nabla u| dx + C \int_{\Omega} u^{q+1} dx \qquad (3.5)
$$
\nwhere *C* denotes the  $L^{\infty}(\Omega)$ -norm of  $\nabla \cdot \nu$ , i.e.  $C = |\nabla \cdot \nu|_{\infty}$ . Using the Young inequality\n
$$
ab \le \frac{\epsilon^2}{2} a^2 + \frac{1}{2\epsilon^2} b^2 \text{ we obtain}
$$
\n
$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le \frac{(q+1)\epsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{q+1}{2\epsilon^2} \int_{\Omega} u^{2q} dx
$$
\n
$$
+ \frac{C\epsilon^2}{2} \int_{\Omega} u^2 dx + \frac{C}{2\epsilon^2} \int_{\Omega} u^{2q} dx.
$$

Hence,

$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le (q+1) \int_{\Omega} u^q |\nabla u| dx + C \int_{\Omega} u^{q+1} dx \tag{3.5}
$$

$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le (q+1) \int_{\Omega} u^q |\nabla u| dx + C \int_{\Omega} u^{q+1} dx
$$
  
as the  $L^{\infty}(\Omega)$ -norm of  $\nabla \cdot \nu$ , i.e.  $C = |\nabla \cdot \nu|_{\infty}$ . Using the Y  

$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le \frac{(q+1)\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{q+1}{2\varepsilon^2} \int_{\Omega} u^{2q} dx
$$

$$
+ \frac{C\varepsilon^2}{2} \int_{\Omega} u^2 dx + \frac{C}{2\varepsilon^2} \int_{\Omega} u^{2q} dx.
$$

Due to the Poincaré inequality one has for some constant *K* 

$$
\int_{\Omega} u^2 dx \leq K \int_{\Omega} |\nabla u|^2 dx
$$

so that we derive

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\nnat we derive  
\n
$$
\int_{\Gamma_1} u^{q+1} d\sigma(x) \le \frac{\varepsilon^2}{2} \{(q+1) + CK\} \int_{\Omega} |\nabla u|^2 dx + \left\{ \frac{q+1}{2\varepsilon^2} + \frac{C}{2\varepsilon^2} \right\} \int_{\Omega} u^{p+1} dx
$$
\n
$$
e^2 Q = p+1.
$$
 Combining this with (3.3) and selecting  $\varepsilon$  such that  $\frac{\varepsilon^2}{2} \{(q+1) + CK\}$    
\nbtain for some constant C  
\n
$$
a \int_{\Omega} u^{p+1} dx \le C \int_{\Omega} u^{p+1} dx
$$

since  $2q = p+1$ . Combining this with (3.3) and selecting  $\varepsilon$  such that  $\frac{\varepsilon^2}{2}$  {(q+1)+CK} = 1 we obtain for some constant *<sup>C</sup>*

$$
a\int_{\Omega}u^{p+1}dx\leq C\int_{\Omega}u^{p+1}dx
$$

hence a contradiction when *a* is large enough,  $u \neq 0$ .

In the case where  $2q < p+1$ , then, as in the one-dimensional case we can show that the problem (1.1) can fail to have a solution when the size of  $\Omega$  is too small. More precisely let us show Theorem 3.2. *Assume that*  $2q < p + 1$ , then, as in the one-dimensional case we can show<br>the problem (1.1) can fail to have a solution when the size of  $\Omega$  is too small. More<br>isely let us show<br>**Theorem 3.2.** *Assume that*

of  $\Omega$  is small enough the problem  $(1.1)$  cannot have a non-trivial solution. **Theorem 3.2.** Assume that  $2q < p+1$  and  $p \leq \frac{n+2}{n-2}$  when  $n \geq 3$ . Then, if the size

**Proof.** Consider for instance for  $\varepsilon \in (0, 1]$ 

$$
\Omega_{\varepsilon} = (-1,1)^{n-1} \times (0,\varepsilon) \quad \text{and} \quad \Gamma_1 = (-1,1)^{n-1} \times \{0\}
$$

and denote by  $u = u_{\varepsilon}$  the solution to problem (1.1) corresponding to  $\Omega = \Omega_{\varepsilon}$ . Recall that by (3.2) one has

Assume that 
$$
2q < p+1
$$
 and  $p \leq \frac{n+2}{n-2}$  when  $n \geq 3$ . Then, if the size the problem (1.1) cannot have a non-trivial solution.

\nfor instance for  $\varepsilon \in (0,1]$ 

\n1,  $1)^{n-1} \times (0,\varepsilon)$  and  $\Gamma_1 = (-1,1)^{n-1} \times \{0\}$ 

\n, the solution to problem (1.1) corresponding to  $\Omega = \Omega_{\varepsilon}$ . Recall

\n
$$
\int_{\Omega} |\nabla u|^2 dx + a \int_{\Omega} u^{p+1} dx = \int_{\Gamma_1} u^{q+1} d\sigma(x).
$$
 (3.6)

\nLet the Young Inequality

Next, remark that due to the Young Inequality

$$
\Omega_{\epsilon} = (-1,1)^{n-1} \times (0,\epsilon) \quad \text{and} \quad \Gamma_{1} = (-1,1)^{n-1} \times \{0\}
$$
\nand denote by  $u = u_{\epsilon}$  the solution to problem (1.1) corresponding to  $\Omega = \Omega_{\epsilon}$ . Recall  
\nthat by (3.2) one has\n
$$
\int_{\Omega} |\nabla u|^{2} dx + a \int_{\Omega} u^{p+1} dx = \int_{\Gamma_{1}} u^{q+1} d\sigma(x). \qquad (3.6)
$$
\nNext, remark that due to the Young Inequality\n
$$
\int_{\Gamma_{1}} u^{q+1} d\sigma(x) = -\int_{\Omega} \frac{\partial}{\partial x_{n}} u^{q+1} dx
$$
\n
$$
= -(q+1) \int_{\Omega} u^{q} \frac{\partial u}{\partial x_{n}} dx
$$
\n
$$
\leq (q+1) \left\{ \frac{\delta^{2}}{2} \int_{\Omega} |\nabla u|^{2} dx + \frac{1}{2\delta^{2}} \int_{\Omega} u^{2q} dx \right\}
$$
\n
$$
\leq (q+1) \left\{ \frac{\delta^{2}}{2} \int_{\Omega} |\nabla u|^{2} dx + \frac{1}{2\delta^{2}} \left( \int_{\Omega} u^{p+1} dx \right)^{2q/(p+1)} |\Omega|^{1-2q/(p+1)} \right\}.
$$
\nCombining with (3.6) and selecting  $(q+1)\frac{\delta^{2}}{2} = \frac{1}{2}$  we obtain for some constant  $C$ \n
$$
\frac{1}{2} \int_{\Omega} |\nabla u|^{2} dx + a \int_{\Omega} u^{p+1} dx \leq C \left( \int_{\Omega} u^{p+1} dx \right)^{2q/(p+1)} |\Omega|^{1-2q/(p+1)}.
$$

Combining with (3.6) and selecting  $(q + 1)\frac{\delta^2}{2} = \frac{1}{2}$  we obtain for some constant *C* 

$$
\frac{1}{2}\int_{\Omega}|\nabla u|^2dx+a\int_{\Omega}u^{p+1}dx\leq C\left(\int_{\Omega}u^{p+1}dx\right)^{2q/(p+1)}|\Omega|^{1-2q/(p+1)}.
$$

On a Class of Nonlinear Elliptic Problems<br>Thus, if we denote by  $|u|_{p+1}$  the usual  $L^{p+1}(\Omega)$ -norm we get for some constants<br> $|u|_{n+1}^{p+1} \leq C|\Omega|$ 

$$
|u|_{p+1}^{p+1} \le C|\Omega| \tag{3.7}
$$

On a Class of Nonlinear Elliptic Problems  
\n
$$
865
$$
\n $+1$  the usual  $L^{p+1}(\Omega)$ -norm we get for some constants\n
$$
|u|_{p+1}^{p+1} \leq C|\Omega|
$$
\n $\int_{\Omega} |\nabla u|^2 dx \leq C |u|_{p+1}^2 |\Omega|^{1-2q/(p+1)}$ \n(3.8)\nnbedding Theorem (see [2: p. 148]) we know that there exists

Next, from the Sobolev embedding Theorem (see *[2: p.* 148]) we know that there exists a constant *C* such that

$$
|v|_{p+1}^2 \leq C \int_{\Omega_1} |\nabla v|^2 dx
$$

for any  $v \in H^1(\Omega_1)$  vanishing on  $\partial \Omega_1 \setminus \Gamma_1$ . So, extending  $u = u_{\epsilon}$  by 0 outside of  $\Omega = \Omega_{\epsilon}$ we derive  $|v|^2_{p+1} \leq C \int_{\Omega_1} |\nabla v|^2 dx$ <br>
i vanishing on  $\partial \Omega_1 \setminus \Gamma_1$ . So, extending  $u = u_{\epsilon}$  by<br>  $|u|^2_{p+1} \leq C \int_{\Omega} |\nabla u|^2 dx$ .<br>
and (3.9) we obtain<br>  $|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)}$  and (if  $u \neq 0$ )  $1 \leq C |u|_{p+1}^{2q-2}$ <br>  $1 \leq C |\Omega$ 

$$
|u|_{p+1}^{p+1} \le C|\Omega|
$$
\n
$$
\int_{\Omega} |\nabla u|^2 dx \le C|u|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)}.
$$
\n(3.8)  
\nthe Sobolev embedding Theorem (see [2: p. 148]) we know that there exists  
\nsuch that\n
$$
|v|_{p+1}^2 \le C \int_{\Omega_1} |\nabla v|^2 dx
$$
\n
$$
I^1(\Omega_1)
$$
\nvanishing on  $\partial\Omega_1 \setminus \Gamma_1$ . So, extending  $u = u_{\epsilon}$  by 0 outside of  $\Omega = \Omega_{\epsilon}$   
\n
$$
|u|_{p+1}^2 \le C \int_{\Omega} |\nabla u|^2 dx.
$$
\n(3.9)  
\n3.8) and (3.9) we obtain\n
$$
\le C|u|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)} \quad \text{and (if } u \ne 0) \quad 1 \le C|u|_{p+1}^{2q-2} |\Omega|^{1-2q/(p+1)}.
$$

Combining (3.8) and (3.9) we obtain

$$
|u|_{p+1}^2 \le C|u|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)} \quad \text{and (if } u \neq 0) \quad 1 \le C|u|_{p+1}^{2q-2} |\Omega|^{1-2q/(p+1)}.
$$

Hence by (3.7)

$$
1 \leq C |\Omega|^{2(q-1)/(p+1)} |\Omega|^{1-2q/(p+1)} = C |\Omega|^{1-2/(p+1)}
$$

and a contradiction when  $|\Omega| = |\Omega_{\epsilon}|$  is small enough

In fact, as we are going to see, what is important is the size of  $|\Gamma_1|$  with respect to the one of  $|\Omega|$ . So, we would like to conclude this paper by an existence result referring the reader to forthcoming works for more on this topic. In what follows we will assume that *p*  $f(p+1)$  and (if  $u \neq 0$ )  $1 \leq C |u|_{p+1}^{2q-2} |\Omega|^{1-2q/(p+1)}$ .<br>  $g(t) = C |\Omega|^{1-2q/(p+1)} = C |\Omega|^{1-2/(p+1)}$ <br>  $= |\Omega_{\epsilon}|$  is small enough **if**  $\epsilon$  see, what is important is the size of  $|\Gamma_1|$  with respect to the to conclude this pontradiction when  $|\Omega| = |\Omega_{\epsilon}|$  is stand, as we are going to see, what<br>of  $|\Omega|$ . So, we would like to conclear to forthcoming works for more<br> $q < \frac{n}{n-2}$ <br>the trace operator is compact fro<br> $E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{a}{p$ is important is the size of  $|\Gamma_1|$  with respect to<br>
ude this paper by an existence result referring<br>  $e$  on this topic. In what follows we will assume<br>
when  $n \ge 3$  (3.10)<br>
om  $H^1(\Omega)$  into  $L^{q+1}(\Gamma)$ . We define<br>  $|v|^{p$  $\frac{1}{2}$  ion when  $|\Omega| =$ <br>*2 re* are going to<br> $\frac{1}{2}$  o, we would lilt<br> $\frac{1}{2}$  or  $\frac{1}{2}$   $\int_{\Omega} |\nabla v|^2 dx +$  $|\Omega_{\epsilon}|$  is small enough <br>
ee, what is important is the<br>
to conclude this paper by a<br>
s for more on this topic. In w<br>  $<\frac{n}{n-2}$  when  $n \geq 3$ <br>
mpact from  $H^1(\Omega)$  into  $L^{q+1}$ <br>  $\frac{a}{p+1} \int_{\Omega} |v|^{p+1} dx - \frac{1}{q+1} \int_{\Gamma}$ 

$$
q < \frac{n}{n-2} \qquad \text{when} \ \ n \geq 3 \tag{3.10}
$$

so that the trace operator is compact from  $H^1(\Omega)$  into  $L^{q+1}(\Gamma)$ . We define

$$
E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx + \frac{a}{p+1} \int_{\Omega} |v|^{p+1} dx - \frac{1}{q+1} \int_{\Gamma_1} |v|^{q+1} d\sigma(x).
$$
 (3.11)  
we have  
correm 3.3. Assume that  $2q < p+1$  and that (3.10) holds. Set  

$$
V_0 = \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\} \quad \text{and} \quad V = V_0 \cap L^{p+1}(\Omega).
$$

Then we have

**Theorem 3.3.** Assume that  $2q < p+1$  and that  $(3.10)$  holds. Set

$$
V_0 = \left\{ v \in H^1(\Omega) \middle| v = 0 \text{ on } \Gamma_0 \right\} \qquad \text{and} \qquad V = V_0 \cap L^{p+1}(\Omega).
$$

*Then there exists*  $u \in V$  *such that*  $E(u) \leq E(v)$  *for all*  $v \in V$ *.* 

**Proof.** First remark that arguing as in (3.4) and (3.5) one has for  $v \in V$ 

2.30 
$$
p+1.3\Omega
$$
  $q+1.3\Gamma_1$   
\n3.3. Assume that  $2q < p+1$  and that (3.10) holds. Set  
\n
$$
= \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\} \quad \text{and} \quad V = V_0 \cap L^{p+1}(\Omega).
$$
\n
$$
= V_0 \cap L^{p+1}(\Omega).
$$

866 M. Chipot and F. Voirol<br>Then, since  $q < \frac{p+1}{2} < p$ , by Hölder's inequality

Let 
$$
|x| < \frac{p+1}{2} < p
$$
, by Hölder's inequality

\n
$$
\int_{\Omega} |v|^{q+1} dx \leq \left( \int_{\Omega} |v|^{p+1} dx \right)^{(q+1)/(p+1)} |\Omega|^{1-(q+1)/(p+1)} = |\Omega|^{1-(q+1)/(p+1)} |v|_{p+1}^{q+1}.
$$

\nLet  $|v| < 0$  and  $|v| < 0$  and  $|v| < 0$  and  $|v| < 0$ .

\nLet  $|v| < 0$  and  $|v| < 0$  and  $|v| < 0$  are a constant  $|v| < 0$ .

Moreover, using the Young inequality one has for some  $\varepsilon$  and some constant  $C$ 

Since 
$$
q < \frac{p+1}{2} < p
$$
, by Hölder's inequality  
\n
$$
\int_{\Omega} |v|^{q+1} dx \leq \left( \int_{\Omega} |v|^{p+1} dx \right)^{(q+1)/(p+1)} | \Omega|^{1-(q+1)/(p+1)}
$$
\n
$$
= |\Omega|^{1-(q+1)/(p+1)} |v|_{p+1}^{q+1}.
$$
\n
$$
= |\Omega|^{1-(q+1)/(p+1)} |v|_{p+1}^{q+1}.
$$
\n
$$
(q+1) \int_{\Omega} |v|^q |\nabla u| dx \leq \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} \int_{\Omega} |v|^{2q} dx
$$
\n
$$
\leq \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} \left( \int_{\Omega} |v|^{p+1} dx \right)^{2q/(p+1)} \qquad (3.14)
$$
\n
$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} \left( \int_{\Omega} |v|^{p+1} dx \right)^{2q/(p+1)} \qquad (3.14)
$$
\n
$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} |v|_{p+1}^{2q}.
$$
\n
$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} |v|_{p+1}^{2q+1}.
$$
\n
$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + \frac{a}{p+1} |v|_{p+1}^{p+1} - C_1 |v|_{p+1}^{q+1} - C_2 |v|_{p+1}^{2q}.
$$
\n
$$
= \varepsilon \text{ such that } \varepsilon < \frac{1}{2} \text{ and denoting}
$$
\n
$$
|v| = |\nabla v|_{2} + |v|_{p+1}
$$
\n
$$
= |\Omega v|_{2} + |v|_{p+1}
$$
\n
$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + \frac{a}{p+1} |v|_{p+1}^{p+1} - C_1 |v|_{p+1}^{q+1}.
$$
\n
$$
= \varepsilon \int_{\Omega}
$$

Thus, collecting  $(3.11)$  -  $(3.14)$  we obtain for some constants  $C_1$  and  $C_2$  depending eventually of  $\varepsilon$ 

$$
= \varepsilon \int_{\Omega} |\nabla v|^2 dx + C_{\varepsilon} |v|_{p+1}^{24}.
$$
\nThus, collecting (3.11) - (3.14) we obtain for some constants  $C_1$  and  $C_2$  depending  
eventually of  $\varepsilon$   
\n
$$
E(v) \ge \left(\frac{1}{2} - \varepsilon\right) \int_{\Omega} |\nabla v|^2 dx + \frac{a}{p+1} |v|_{p+1}^{p+1} - C_1 |v|_{p+1}^{q+1} - C_2 |v|_{p+1}^{2q}.
$$
\nSelecting  $\varepsilon$  such that  $\varepsilon < \frac{1}{2}$  and denoting  
\n
$$
|v| = |\nabla v|_2 + |v|_{p+1}
$$
\nis clear since  $q + 1 < p + 1$  and  $2q < p + 1$  that

\n
$$
\lim_{|v| \to +\infty} E(v) = +\infty.
$$
\n(3.16)

\nLet us denote by  $\{v_k\}$  a minimizing sequence of  $E$  on  $V$ , i.e. a sequence  $\{v_k\}$  satisfying  
\n
$$
\lim_{k \to +\infty} E(v_k) = \inf_{v \in V} E(v).
$$
\n(3.17)

\nBy (3.15) and (3.16) one has, for some constant  $C$ ,  $|\nabla v_k|_2 \leq C$  and  $|v_k|_{p+1} \leq C$ . So, one can extract a subsequence that for convenience we will still denote by  $v_k$  such that

$$
|v| = |\nabla v|_2 + |v|_{p+1} \tag{3.15}
$$

it is clear since  $q + 1 < p + 1$  and  $2q < p + 1$  that

$$
\lim_{|v| \to +\infty} E(v) = +\infty. \tag{3.16}
$$

Let us denote by  $\{v_k\}$  a minimizing sequence of *E* on *V*, i.e. a sequence  $\{v_k\}$  satisfying

$$
\lim_{k \to +\infty} E(v_k) = \inf_{v \in V} E(v). \tag{3.17}
$$

By (3.15) and (3.16) one has, for some constant *C*,  $|\nabla v_k|_2 \leq C$  and  $|v_k|_{p+1} \leq C$ . So, one can extract a subsequence that for convenience we will still denote by  $v_k$  such that for some  $u \in V$  one has  $\lim_{k \to +\infty} E(v_k) = \inf_{v \in V}$ <br>
for some constant<br>  $v_k \to u$  in  $V_0$ <br>  $v_k \to u$  in  $L^p$ 

$$
\lim_{k \to +\infty} E(v_k) = \inf_{v \in V} E(v)
$$
  
for some constant C,  $|\nabla$   
that for convenience we  
 $v_k \to u$  in  $V_0$   
 $v_k \to u$  in  $L^{p+1}(\Omega)$   
 $v_k \to u$  in  $L^{q+1}(\Gamma)$   
lower semicontinuity of

(recall (3.10)). Using now the lower semicontinuity of the maps  $v \to |\nabla v|_2^2$  and v  $|v|_{p+1}^{p+1}$  one deduces

$$
v_k \to u \qquad \text{in} \quad L^{q+1}(\Gamma)
$$
\nall (3.10)). Using now the lower semicontinuity of the maps  $v \to |\nabla v|_2^2$  and  
\n $v_1$  one deduces\n
$$
\inf_{v \in V} E(v) = \lim_{k \to +\infty} E(v_k)
$$
\n
$$
\geq \frac{1}{2} \liminf_{k} |\nabla v_k|_2^2 + \frac{a}{p+1} \liminf_{k} |v_k|_{p+1}^{p+1} - \frac{1}{q+1} \lim_{k} \int_{\Gamma_1} |v_k|^{q+1} d\sigma(x)
$$
\n
$$
\geq E(u).
$$

So,  $u$  is a minimizer of  $E$  and the result follows

**Remark 3.1.** At this stage, nothing prevents the solution *u* to be equal to 0. As we will see this happens for instance under the assumptions of Theorem 3.2. Note also that the proof of Theorem 3.3 holds when  $|\Gamma_0| = 0$ . *IDi* II is the assumptions of Theorem 3.2. Note also<br> *I* 3.3 holds when  $|\Gamma_0| = 0$ .<br> *I* rexistence result.<br> *Ine that*  $2q < p + 1$  *and that* (3.10) *holds. Set*<br>  $= \inf_{y \in \Gamma_0} |x - y|$  *and*  $D_1 = \{x \in \mathbb{R}^n | d(x) \le 1\},$ <br>

Let us now turn to our existence result.

**Theorem 3.4.** Assume that  $2q < p + 1$  and that  $(3.10)$  holds. Set

By the proof of Theorem 3.3 holds when 
$$
|1 \ 0| = 0
$$
.

\nAs now turn to our existence result.

\nFor example,  $3.4$ . Assume that  $2q < p + 1$  and that (3.10) holds. Set

\n
$$
d(x) = \text{dist}(x, \Gamma_0) = \inf_{y \in \Gamma_0} |x - y|
$$
 and  $D_1 = \{x \in \mathbb{R}^n | d(x) \le 1\},$ 

where  $\lvert \cdot \rvert$  denotes either the Lebesgue measure, either the superficial measure on  $\Gamma$ . Then *if*

$$
D_0) = \inf_{y \in \Gamma_0} |x - y| \quad and \quad D_1 = \{x \in \mathbb{R}^n | d(x) \le 1\},
$$
  
\nthe Lebesgue measure, either the superficial measure on  $\Gamma$ . Then  
\n
$$
\frac{1}{2}|D_1| + \frac{a}{p+1}|\Omega| - \frac{1}{q+1}|\Gamma_1 \setminus D_1| < 0,
$$
\n*isolution u to problem* (1.1).

*there exists a non-trivial solution u to problem (1.1).* 

**Proof.** Consider the function  $v = d \wedge 1$  where  $\wedge$  denotes the minimum of two tions. It is clear that  $v \in V$ . Moreover, since *d* is a Lipschitz continuous function a Lipschitz constant less than 1,  $|\nabla d(x)| \le 1$  for a functions. It is clear that  $v \in V$ . Moreover, since *d* is a Lipschitz continuous function with a Lipschitz constant less than 1,  $|\nabla d(x)| \leq 1$  for a.e.  $x \in \Omega$ . So, we have

denotes either the Lebesgue measure, either the superficial measure of 
$$
\frac{1}{2}|D_1| + \frac{a}{p+1}|\Omega| - \frac{1}{q+1}|\Gamma_1 \setminus D_1| < 0,
$$
\nis a non-trivial solution u to problem (1.1).  
\n6. Consider the function  $v = d \wedge 1$  where  $\wedge$  denotes the minimum  
\nIt is clear that  $v \in V$ . Moreover, since d is a Lipschitz continuous  
\nschitz constant less than 1,  $|\nabla d(x)| \leq 1$  for a.e.  $x \in \Omega$ . So, we have  
\n
$$
E(v) = \frac{1}{2} \int_{D_1} |\nabla d|^2 dx + \frac{a}{p+1} \int_{\Omega} |v|^{p+1} dx - \frac{1}{q+1} \int_{\Gamma_1} |v|^{q+1} d\sigma(x)
$$
\n
$$
\leq \frac{1}{2}|D_1| + \frac{a}{p+1}|\Omega| - \frac{1}{q+1}|\Gamma_1 \setminus D_1|
$$
\n
$$
< 0.
$$

Thus, the infimum (3.17) is negative and achieved for a non-zero function *u.* Noting that  $|u| \in V$  and  $E(u) = E(|u|)$ , there is no loss of generality in assuming  $u \geq 0$ . But then, it is easy to see that  $u$  is solution to problem  $(1.1)$ . This completes the proof of the theorem  $\blacksquare$ gative and achieved<br>
), there is no loss of<br>
solution to problem<br>
is very easy to find a<br>
xed domain it is enou<br>
where  $|\Gamma_0| = 0$  one re<br>  $\frac{a}{p+1} |\Omega| \varepsilon^{p+1} - \frac{1}{q+1}$ <br>
case problem (1.1) has

**Remark 3.2.** Note that it is very easy to find an open set  $\Omega$  for which (3.18) holds. Assuming  $\Omega$  included in some fixed domain it is enough to choose  $|\Gamma_1 \setminus D_1|$  large enough.

**Remark 3.3.** In the case where  $|\Gamma_0| = 0$  one remarks that since  $q < p$ ,

$$
E(\varepsilon) = \frac{a}{p+1} |\Omega| \varepsilon^{p+1} - \frac{1}{q+1} |\Gamma| \varepsilon^{q+1} < 0
$$

for *e* small enough. So, in this case problem (1.1) has always a solution (compare with [3]).

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