Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 14 (1995), No. 4, 853–868

On a Class of Nonlinear Elliptic Problems with Neumann Boundary Conditions Growing Like a Power

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Abstract. One investigates the issue of existence and number of solutions for the problem

$$\Delta u = au^p \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \Gamma_0, \qquad \frac{\partial u}{\partial n} = u^q \quad \text{on} \quad \Gamma_1$$

where Γ_0 and Γ_1 are two parts of the boundary of the open set Ω. In dimension one we are able to find all the solutions to the problem. In higher dimension we give for different solutions depending on p, q and Ω existence and non-existence results.

Keywords: *Neumann boundary conditions, nonlinear elliptic equations*

AMS subject classification: 34 B 15, 35 A 15, 35 J XX

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary Γ. This paper is concerned with the problem of finding a positive solution u to the problem

$$\left.\begin{aligned} \Delta u &= au^p && \text{in} \quad \Omega \\ u &= 0 && \text{on} \quad \Gamma_0 \\ \frac{\partial u}{\partial n} &= u^q && \text{on} \quad \Gamma_1 \end{aligned}\right\} \tag{1.1}$$

where a and p, q are positive constants such that $p, q > 1$, Γ_0 and Γ_1 are two portions of the boundary Γ that we will assume to be disjoint and covering Γ, and n is the outward unit normal to Γ. Moreover, we will assume that Γ_0 has a positive superficial measure. We refer the reader to [1] for the case where $\Gamma_1 = \Gamma$.

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ISSN 0232-2064 / $ 2.50

The above problem models the equilibrium of the temperature u in a domain Ω. It is assumed that cooling is provided at a rate proportional to u^p inside the body and a flux of heat is entering the boundary through Γ_1 at a rate u^q. The other part of the boundary is maintained at a constant temperature. The question is then to determine if an equilibrium can be reached by the temperature inside the body.

2. The one-dimensional case

In this section we consider the problem of finding $u > 0$, $u \in C^2(0,L) \cap C^1([0,L])$, such that

$$\left.\begin{aligned} u'' &= au^p \quad \text{on} \quad (0,L) \\ u(0) &= 0 \quad \text{and} \quad u'(L) = u^q(L) \end{aligned}\right\} \tag{2.1}$$

where $a > 0$ and $p, q > 1$. In this case the situation is complete and we have the following result.

Theorem 2.1. *The problem* (2.1) *can be described through the following cases.*

(1) *If $2q > p+1$, then the problem* (2.1) *has for any $L > 0$ a unique non-trivial solution.*

(2) *If $2q = p+1$, then*
for $a < q$ the problem (2.1) *has for any $L > 0$ a unique non-trivial solution*
for $a \geq q$ the problem (2.1) *has no non-trivial solution.*

(3) *If $2q < p+1$, then there exists $L^* > 0$ such that*
for $L < L^*$ the problem (2.1) *has no non-trivial solution*
for $L = L^*$ the problem (2.1) *has a unique non-trivial solution*
for $L > L^*$ the problem (2.1) *has two non-trivial solutions.*

The proof of assertions (1) - (3) will be given in separate parts.

Proof of assertion (1) of Theorem 2.1. We introduce u_m the solution of the Cauchy problem

$$\left.\begin{aligned} u_m'' &= au_m^p \\ u_m(0) &= 0 \quad \text{and} \quad u_m'(0) = m \end{aligned}\right\} \tag{2.2}$$

where m is a positive constant and we denote by $[0, l_m)$ the interval where the solution exists. Then we set

$$b(m,r) = \frac{u_m'(r)}{u_m^q(r)} \qquad \text{for all} \quad (m,r) \in (0,+\infty) \times (0, l_m). \tag{2.3}$$

We claim that for any $m > 0$ the function $r \to b(m,r)$ is decreasing on $(0, l_m)$. Indeed, if we mutiply the first equation of (2.2) by u_m' we obtain

$$\frac{1}{2}(u_m'^2)' = \frac{a}{p+1}(u_m^{p+1})'.$$

Integrating between 0 and r we get

$$\frac{1}{2}u_m'^2(r) - \frac{1}{2}m^2 = \frac{a}{p+1}u_m^{p+1}(r)$$

hence

$$u_m'(r) = \sqrt{m^2 + \frac{2a}{p+1}u_m^{p+1}(r)}. \tag{2.4}$$

So, we deduce

$$b(m,r) = \frac{u_m'(r)}{u_m^q(r)} = \sqrt{m^2 u_m^{-2q}(r) + \frac{2a}{p+1}u_m^{p+1-2q}(r)}. \tag{2.5}$$

From u_m being clearly increasing on $(0, l_m)$ there follows since $p+1-2q < 0$ that $r \to b(m,r)$ is decreasing on $(0, l_m)$. Next, let us establish the following

Lemma 2.1. *Let $\alpha > 0$ and $(m,r) \in (0,+\infty) \times (0, l_m)$. Then*

$$(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) \in (0,+\infty) \times (0, l_{m\alpha^{(p+1)/2}})$$

and one has

$$b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2} b(m,r). \tag{2.6}$$

Proof. Consider

$$s(t) = \alpha u_m(\alpha^{(p-1)/2}t). \tag{2.7}$$

One has

$$s'(t) = \alpha^{(p+1)/2}u_m'(\alpha^{(p-1)/2}t) \tag{2.8}$$

$$s''(t) = \alpha^p u_m''(\alpha^{(p-1)/2}t) = \alpha^p a u_m^p(\alpha^{(p-1)/2}t) = as(t)^p. \tag{2.9}$$

So, $s = s(t)$ satisfies

$$\left.\begin{aligned} &s'' = as^p \\ &s(0) = 0 \text{ and } s'(0) = m\alpha^{(p+1)/2} \end{aligned}\right\}. \tag{2.10}$$

and by the uniqueness of the solution of the Cauchy problem

$$s = u_{m\alpha^{(p+1)/2}} \qquad \text{and} \qquad l_{m\alpha^{(p+1)/2}} = \alpha^{-(p-1)/2} l_m.$$

Next, we have

$$b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \frac{s'(r\alpha^{-(p-1)/2})}{s^q(r\alpha^{-(p-1)/2})} = \alpha^{(p+1-2q)/2}\frac{u_m'(r)}{u_m^q(r)}$$

which gives (2.6). From (2.6) we deduce easily that for r fixed the function $m \to b(m,r)$ is decreasing. First note that for $m' > m$ one has, since the trajectories of the system (2.2) cannot cross,

$$u_m(r) \le u_{m'}(r) \qquad \text{and} \qquad u_m'(r) \le u_{m'}'(r)$$

and thus

$$l_m \geq l_{m'}. \tag{2.11}$$

So, for $r > 0$ fixed, $b(m, r)$ is defined on some interval $(0, m_r)$. Next, $m' > m$ can be written as $m' = m\alpha^{(p+1)/2}$ for some $\alpha > 1$. Then, since b is decreasing in r and by (2.6)

$$\begin{aligned} b(m', r) &< b(m', r\alpha^{-(p-1)/2}) \\ &= b(m\alpha^{(p+1)/2}, r\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2} b(m, r) \\ &< b(m, r) \end{aligned}$$

and the result follows ∎

We are now in a position to establish (1). First remark that, by (2.11), $\lim_{m \to 0} l_m$ exists. We claim that this limit is $+\infty$. This follows clearly from the continuous dependence in m of the solution to the Cauchy problem (2.2) and from the fact that, for $m = 0$, the solution is 0 and defined on the whole real line. Thus, given an L, one can find $m > 0$ such that $L < l_m$. If $b(m, L) = 1$, then u_m provides us with a solution to our problem. If $b(m, L) > 1$, then one can select $\alpha > 1$ such that

$$b(m\alpha^{(p+1)/2}, L\alpha^{-(p-1)/2}) = \alpha^{(p+1-2q)/2} b(m, L) = 1.$$

Then, due to the fact that b is decreasing in r,

$$b(m\alpha^{(p+1)/2}, L) < b(m\alpha^{(p+1)/2}, L\alpha^{-(p-1)/2}) = 1 < b(m, L).$$

But due to the continuity of the map $m \to b(m, L)$ one can find $m_0 \in (m, m\alpha^{(p+1)/2})$ such that $b(m_0, L) = 1$ and u_{m_0} is solution to our problem. In the case where $b(m, L) < 1$ one proceeds the same way selecting $\alpha < 1$.

To see that uniqueness holds, assume that we have two distinct solutions u_1 and u_2 to (2.1). Then, $m_1 = u_1'(0) \neq u_2'(0) = m_2$ and we cannot have $b(m_1, L) = b(m_2, L)$. Thus, uniqueness follows and assertion (1) of Theorem 2.1 is proved ∎

Proof of assertion (2) of Theorem 2.1. So, we assume $2q = p + 1$ and as above we introduce u_m, the solution to problem (2.1). In this case (2.5) reads

$$b(m, r) = \frac{u_m'(r)}{u_m^q(r)} = \sqrt{\frac{m^2}{u_m^{2q}(r)} + \frac{a}{q}}. \tag{2.12}$$

Since u_m is increasing, $r \to b(m, r)$ is decreasing on $(0, l_m)$. Moreover, we claim that

$$\lim_{r \to l_m} u_m(r) = +\infty. \tag{2.13}$$

Indeed, the above limit clearly exists. Moreover, $(u, v) = (u_m, u_m')$ is solution to the system

$$\left.\begin{aligned} &u' = v \ \text{ and } \ v' = au^p \\ &u(0) = 0 \ \text{ and } \ v(0) = m. \end{aligned}\right\} \tag{2.14}$$

The functions u and v are both increasing and have a limit. If $l_m < +\infty$ and $\lim_{r\to l_m} u_m(r) < +\infty$, then, due to the first equation of (2.14), $\lim_{r\to l_m} u''_m(r) < +\infty$ and so does $\lim_{r\to l_m} u'_m(r)$ which is impossible. If now $l_m = +\infty$ and $\lim_{r\to l_m} u_m(r) < +\infty$, then $u''_m(r)$ and thus $u'_m(r)$ are unbounded which contradicts the fact that u_m is. So, in all cases we have (2.13). It follows from (2.12) that for any $r < l_m$

$$b(m,r) > \lim_{r\to l_m} b(m,r) = \sqrt{\frac{a}{q}}.$$

Thus, when $a \geq q$, then the problem (2.1) cannot have a solution. The case $a = q$ gives rise to no solution due to the fact that $u_m(r)$ is unbounded when $r \to l_m$. When $a < q$, then, clearly, for any m we can find a unique L_m such that $b(m, L_m) = 1$. Now, it is easy to check that if u_1 denotes the solution to (2.2) corresponding to $m = 1$, then (compare to (2.7) - (2.10)) $v(t) = \alpha u_1(\alpha^{(p-1)/2}t)$ satisfies

$$\left.\begin{aligned} &v'' = av^p \quad \text{on} \quad (0, \alpha^{-(p-1)/2}L_1) \\ &v(0) = 0 \quad \text{and} \quad v'(\alpha^{-(p-1)/2}L_1) = \alpha^q. \end{aligned}\right\}$$

Thus,

$$u_{\alpha^q}(t) = \alpha u_1(\alpha^{(p-1)/2}t) \qquad \text{and} \qquad L_{\alpha^q} = \alpha^{-(p-1)/2}L_1.$$

It follows that for any $L > 0$ there exists a unique α such that $L = \alpha^{-(p-1)/2}L_1$ and u_{α^q} is the unique solution to problem (2.1). This completes the proof of assertion (2) of Theorem 2.1 ∎

Proof of assertion (3) of Theorem 2.1. So, we assume throughout this part that $2q < p + 1$. We introduce as before the solution u_m to problem (2.2). Recall that we have (see (2.4))

$$u'_m(r) = \sqrt{m^2 + \frac{2a}{p+1}\, u_m^{p+1}(r)}. \tag{2.15}$$

So, in order for $u_m(L)$ to solve $u'_m(L) = u_m^q(L)$ it needs to be a root of the equation

$$F(u) = u^{2q} - \frac{2a}{p+1}\, u^{p+1} - m^2 = 0. \tag{2.16}$$

We have $F'(u) = 2qu^{2q-1} - 2au^p$ hence $F'(u) = 0$ if and only if u is equal to $\tau = (\frac{q}{a})^{1/(p+1-2q)}$. Thus, F is increasing between 0 and τ starting from the value $-m^2$ and decreasing after τ going to $-\infty$ when $u \to +\infty$. So, in order for the equation (2.16) to have a root we need to have $F(\tau) \geq 0$ which reads after an easy computation

$$\left(\frac{q}{a}\right)^{2q/(p+1-2q)} \left\{1 - \frac{2q}{p+1}\right\} \geq m^2.$$

So, in order for u_m to be a solution to problem (2.1) we have to restrict m to satisfy

$$0 < m \leq M = \left(\frac{q}{a}\right)^{q/(p+1-2q)} \left\{1 - \frac{2q}{p+1}\right\}^{1/2}.$$

In this case (2.16) has two roots $R_1(m) < \tau < R_2(m)$ which coincide with τ in the case where $m = M$. Going back to (2.15) we have

$$\frac{u_m'(r)}{\sqrt{\frac{2a}{p+1}\, u_m^{p+1}(r) + m^2}} = 1$$

hence integrating (recall that $u_m' > 0$)

$$\int_0^{u_m(r)} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}} = r.$$

Then it is clear that u_m is a solution to problem (2.1) for $L = L_1(m)$ and $L = L_2(m)$ where

$$L_i(m) = \int_0^{R_i(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}} \qquad (i = 1, 2). \tag{2.17}$$

Since every solution to problem (2.1) is a solution to problem (2.2) for some m, when m varies the numbers $L_i(m)$ are going to run over all the possible values for L. So, we need to study the functions $L_1(m)$ and $L_2(m)$. Let us start with $L_2(m)$.

Lemma 2.2. *L_2 is a decreasing function of m on $(0, M]$. Moreover,*

$$\lim_{m \to 0} L_2(m) = +\infty \qquad \textit{and} \qquad L_2(M) = \int_0^{\tau} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + M^2}}.$$

Proof. First, since when m increases the graph of F goes down one has

$R_1(m)$ is increasing with m

$R_2(m)$ is decreasing with m.

So, if $m > m'$ one has

$$\frac{1}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}} < \frac{1}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m'^2}}$$

and, integrating,

$$\begin{aligned} L_2(m) &= \int_0^{R_2(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}} \\ &< \int_0^{R_2(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m'^2}} < \int_0^{R_2(m')} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m'^2}} \\ &= L_2(m'). \end{aligned}$$

Thus, L_2 is decreasing. On the other hand $R_2(m) \geq \tau$ so that

$$L_2(m) \geq \int_0^\tau \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}}.$$

Letting $m \to 0$ one obtains $\lim_{m\to 0} L_2(m) = +\infty$ since

$$\int_0 \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1}}}$$

diverges ($\frac{p+1}{2} > 1$). This concludes the proof of Lemma 2.2 ∎

Next we turn to the study of L_1. We have

Lemma 2.3. *When $m \to 0$, then $R_1(m) \sim m^{1/q}$ and $L_1(m) \sim m^{\frac{1}{q}-1}$. In particular* $\lim_{m\to 0} L_1(m) = +\infty$.

Proof. First, note that when $m \to 0$, then any limit value of $R_1(m)$ must satisfy $u^{2q} - \frac{2a}{p+1}\, u^{p+1} = 0$ so that $u = 0$ or $u = \left(\frac{p+1}{2a}\right)^{p+1-2q}$. Since $\frac{p+1}{2a} > \frac{q}{a}$, we have

$$\left(\frac{p+1}{2a}\right)^{p+1-2q} > \left(\frac{q}{a}\right)^{p+1-2q} = \tau > R_1(m)$$

and the only possible limit value for $R_1(m)$ is 0 so that $\lim_{m\to 0} R_1(m) = 0$. Going back to (2.16) we have

$$R_1(m)^{2q} - \frac{2a}{p+1}\, R_1(m)^{p+1} - m^2 = 0$$

or

$$R_1(m)^{2q} \left\{1 - \frac{2a}{p+1}\, R_1(m)^{p+1-2q}\right\} = m^2.$$

Since, $\lim_{m\to 0} R_1(m) = 0$ we deduce that, when $m \to 0$, $R_1(m)^{2q} \sim m^2$ and thus $R_1(m) \sim m^{1/q}$. Going back to the definition (2.17) we have

$$L_1(m) = \int_0^{R_1(m)} \frac{ds}{\sqrt{\frac{2a}{p+1}\, s^{p+1} + m^2}}.$$

Changing of variable, i.e. setting

$$u = \left(\frac{2a}{p+1}\right)^{1/(p+1)} m^{-2/(p+1)} s = C m^{-2/(p+1)} s$$

we obtain

$$\begin{aligned} L_1(m) &= \frac{1}{C m^{1-2/(p+1)}} \int_0^{C m^{-2/(p+1)} R_1(m)} \frac{du}{\sqrt{u^{p+1}+1}} \\ &= \frac{R_1(m)}{m} \frac{1}{C m^{-2/(p+1)} R_1(m)} \int_0^{C m^{-2/(p+1)} R_1(m)} \frac{du}{\sqrt{u^{p+1}+1}}. \end{aligned} \tag{2.18}$$

Since $C m^{-2/(p+1)} R_1(m) \sim C m^{1/q - 2/(p+1)} \to 0$ when $m \to 0$ we obtain $L_1(m) \sim \frac{R_1(m)}{m} \sim m^{\frac{1}{q}-1}$ which concludes the proof ∎

Next let us show

Lemma 2.4. *Let us denote by $L_i'(m)$ the derivative of L_i with respect to m. Then one has*

$$\left(1-\frac{2}{p+1}\right)L_i(m)+mL_i'(m)=\left(1-\frac{2q}{p+1}\right)\left\{\frac{1}{qR_i^{q-1}-aR_i^{p-q}}\right\} \tag{2.19}$$

for $i=1,2$. Moreover,

$$\lim_{m\to M} L_1'(m)=+\infty \qquad \textit{and} \qquad \lim_{m\to M} L_2'(m)=-\infty. \tag{2.20}$$

Proof. Going back to (2.18) one has

$$Cm^{1-2/(p+1)}L_i(m)=\int_0^{Cm^{-2/(p+1)}R_i(m)}\frac{du}{\sqrt{u^{p+1}+1}}.$$

Hence differentiating with respect to m we get

$$\begin{aligned}C&\left(1-\frac{2}{p+1}\right)m^{-2/(p+1)}L_i(m)+Cm^{1-2/(p+1)}L_i'(m)\\&=\frac{1}{\sqrt{\left(Cm^{-2/(p+1)}R_i(m)\right)^{p+1}+1}}\left(Cm^{-2/(p+1)}R_i(m)\right)'\\&=\frac{m}{\sqrt{\frac{2a}{p+1}R_i(m)^{p+1}+m^2}}C\left(-\frac{2}{p+1}m^{-1-2/(p+1)}R_i(m)+m^{-2/(p+1)}R_i'(m)\right).\end{aligned} \tag{2.21}$$

Since

$$R_i(m)^{2q}=\frac{2a}{p+1}R_i(m)^{p+1}+m^2 \tag{2.22}$$

relation (2.21) reads after pulling out $Cm^{-2/(p+1)}$

$$\begin{aligned}\left(1-\frac{2}{p+1}\right)L_i(m)+mL_i'(m)&=\frac{m}{R_i(m)^q}\left\{R_i'(m)-\frac{2}{p+1}\frac{R_i(m)}{m}\right\}\\&=\left\{mR_i'(m)R_i(m)^{-q}-\frac{2}{p+1}R_i(m)^{1-q}\right\}.\end{aligned} \tag{2.23}$$

Differentiating (2.22) we obtain

$$R_i'(m)=\frac{m}{qR_i(m)^{2q-1}-aR_i(m)^p}$$

so that

$$mR_i'(m)=\frac{m^2}{qR_i(m)^{2q-1}-aR_i(m)^p}=\frac{R_i(m)^{2q}-\frac{2a}{p+1}R_i(m)^{p+1}}{qR_i(m)^{2q-1}-aR_i(m)^p}.$$

Replacing into (2.23) we obtain (2.19). Now when $m\to\tau$, then $R_i(m)\to\tau$ where $qu^{q-1}-au^{p-1}$ vanishes. Since $R_1(m)<\tau<R_2(m)$ we see passing to the limit in (2.19) that (2.20) holds ∎

Next we have

Lemma 2.5. *On* $(0, M)$ *the function* L_1 *is decreasing until a value* $m_0 \in (0, M)$ *and then increasing until* M.

Proof. Due to Lemmas 2.3 and 2.4 it is enough to show that at a point where $L_1'(m) = 0$, then $L_1''(m) > 0$ so that m could only be a minimum. For that, differentiating (2.19) we obtain

$$\begin{aligned}
&\left(1 - \frac{2}{p+1}\right) L_1'(m) + L_1'(m) + mL_1''(m) \\
&= \left(1 - \frac{2q}{p+1}\right) \left\{ \frac{1}{qR_1^{q-1} - aR_1^{p-q}} \right\}' \qquad (2.24)\\
&= -\left(1 - \frac{2q}{p+1}\right) \frac{1}{(qR_1^{q-1} - aR_1^{p-q})^2} \left\{ q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1} \right\} R_1'.
\end{aligned}$$

It is clear from (2.21) that at a point where $L_1'(m) = 0$ one must have $R_1'(m) > 0$, then (see (2.24)) at a point where $L_1'(m) = 0$ the sign of $L_1''(m)$ is given by the opposite sign of $\{q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1}\}$ so that $L_1''(m) > 0$ if and only if $q(q-1)R_1^{q-2} - a(p-q)R_1^{p-q-1} < 0$ or

$$\frac{q(q-1)}{a(p-q)} < R_1^{p+1-2q} \qquad (2.25)$$

(note that $q < p$ since $2q < p + 1 < 2p$). Next, going back to (2.19), at a point where $L_1'(m) = 0$ we have

$$\left(1 - \frac{2}{p+1}\right) L_1(m) = \left(1 - \frac{2q}{p+1}\right) \left\{ \frac{1}{qR_1^{q-1} - aR_1^{p-q}} \right\}.$$

Clearly

$$L_1(m) = \int_0^{R_1(m)} \frac{ds}{\sqrt{\frac{2a}{p+1} s^{p+1} + m^2}} > \frac{R_1(m)}{\sqrt{\frac{2a}{p+1} R_1(m)^{p+1} + m^2}} = R_1(m)^{1-q}.$$

So at a point where $L_1'(m) = 0$ we have

$$\left(1 - \frac{2}{p+1}\right) R_1^{1-q} < \left(1 - \frac{2q}{p+1}\right) \left\{ \frac{1}{qR_1^{q-1} - aR_1^{p-q}} \right\}$$

which reads also

$$\left(1 - \frac{2}{p+1}\right) \left\{ q - aR_1^{p+1-2q} \right\} < \left(1 - \frac{2q}{p+1}\right)$$

or

$$\frac{p-1}{p+1}\{q - aR_1^{p+1-2q}\} < \frac{p+1-2q}{p+1}.$$

This is equivalent to the inequality

$$\{q - aR_1^{p+1-2q}\} < \frac{p+1-2q}{p-1}$$

and the last to the inequality

$$q - \frac{p+1-2q}{p-1} < aR_1^{p+1-2q}.$$

So we will be done thanks to (2.25) if

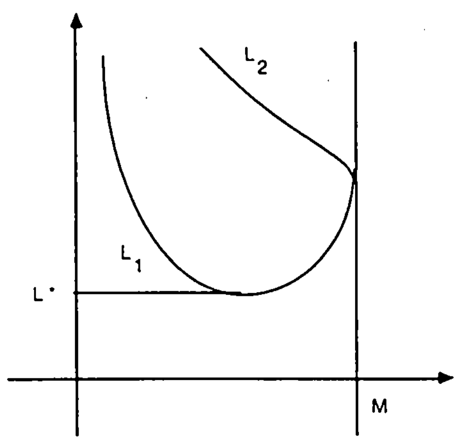
$$\frac{q(q-1)}{p-q} < q - \frac{p+1-2q}{p-1}$$

or equivalently

$$\frac{p+1-2q}{p-1} < q - \frac{q(q-1)}{p-q} = q\left\{1 - \frac{q-1}{p-q}\right\} = q\left\{\frac{p+1-2q}{p-q}\right\}.$$

This will be true if $\frac{1}{p-1} < \frac{q}{p-q}$ which is true since $q > 1$ ∎

Combining the information of the different lemmas we see that the curves L_1 and L_2 look as below.



Set $L^* = \inf_{(0,M)} L_1$. Then for $L < L^*$, $L = L^*$ and $L > L^*$ problem (2.1) has no solution, has a unique solution and has exactly two solutions, respectively.

Remark 2.1. The method used in proving assertion (3) of Theorem 2.1 could also have been used to establish assertions (1) and (2).

3. The higher dimensional case

In this section we assume that u is a weak solution to (1.1) such that $u \in H^1(\Omega) \cap L^{p+1}(\Omega)$.

In the case where $2q = p + 1$, we have a similar result to the one-dimensional case:

Theorem 3.1. *Assume that $2q = p + 1$. Then, if a is large enough, the problem (1.1) cannot have a non-trivial solution.*

Proof. Let us denote by ν a smooth vector field such that

$$\nu = n \quad \text{on } \Gamma_1 \qquad \text{and} \qquad |\nu| \le 1. \tag{3.1}$$

Multiplying the first equation of (1.1) by u and integrating over Ω we get

$$\begin{aligned} a\int_\Omega u^{p+1}dx &= \int_\Omega \Delta u u\,dx = \int_\Omega \nabla\cdot(\nabla u u)\,dx - \int_\Omega |\nabla u|^2\,dx \\ &= \int_{\Gamma_1} \frac{\partial u}{\partial n} u\,d\sigma(x) - \int_\Omega |\nabla u|^2\,dx = \int_{\Gamma_1} u^{q+1}\,d\sigma(x) - \int_\Omega |\nabla u|^2\,dx \end{aligned} \tag{3.2}$$

were $d\sigma(x)$ denotes the superficial measure on Γ. Hence

$$\int_\Omega |\nabla u|^2\,dx + a\int_\Omega u^{p+1}\,dx = \int_{\Gamma_1} u^{q+1}\,d\sigma(x). \tag{3.3}$$

Next, remark that

$$\begin{aligned} \int_{\Gamma_1} u^{q+1}\,d\sigma(x) &= \int_\Omega \nabla\cdot(u^{q+1}\nu)\,dx \\ &= (q+1)\int_\Omega u^q\nabla u\cdot\nu\,dx + \int_\Omega u^{q+1}\nabla\cdot\nu\,dx. \end{aligned} \tag{3.4}$$

Hence,

$$\int_{\Gamma_1} u^{q+1}\,d\sigma(x) \le (q+1)\int_\Omega u^q|\nabla u|\,dx + C\int_\Omega u^{q+1}\,dx \tag{3.5}$$

where C denotes the $L^\infty(\Omega)$-norm of $\nabla\cdot\nu$, i.e. $C = |\nabla\cdot\nu|_\infty$. Using the Young inequality $ab \le \frac{\varepsilon^2}{2}a^2 + \frac{1}{2\varepsilon^2}b^2$ we obtain

$$\begin{aligned} \int_{\Gamma_1} u^{q+1}\,d\sigma(x) \le \frac{(q+1)\varepsilon^2}{2}\int_\Omega |\nabla u|^2dx + \frac{q+1}{2\varepsilon^2}\int_\Omega u^{2q}dx \\ + \frac{C\varepsilon^2}{2}\int_\Omega u^2dx + \frac{C}{2\varepsilon^2}\int_\Omega u^{2q}dx. \end{aligned}$$

Due to the Poincaré inequality one has for some constant K

$$\int_\Omega u^2dx \le K\int_\Omega |\nabla u|^2dx$$

so that we derive

$$\int_{\Gamma_1} u^{q+1} d\sigma(x) \leq \frac{\varepsilon^2}{2}\{(q+1)+CK\} \int_\Omega |\nabla u|^2 dx + \left\{\frac{q+1}{2\varepsilon^2} + \frac{C}{2\varepsilon^2}\right\} \int_\Omega u^{p+1} dx$$

since $2q = p+1$. Combining this with (3.3) and selecting ε such that $\frac{\varepsilon^2}{2}\{(q+1)+CK\} = 1$ we obtain for some constant C

$$a \int_\Omega u^{p+1} dx \leq C \int_\Omega u^{p+1} dx$$

hence a contradiction when a is large enough, $u \not\equiv 0$ ∎

In the case where $2q < p+1$, then, as in the one-dimensional case we can show that the problem (1.1) can fail to have a solution when the size of Ω is too small. More precisely let us show

Theorem 3.2. *Assume that $2q < p+1$ and $p \leq \frac{n+2}{n-2}$ when $n \geq 3$. Then, if the size of Ω is small enough the problem (1.1) cannot have a non-trivial solution.*

Proof. Consider for instance for $\varepsilon \in (0,1]$

$$\Omega_\varepsilon = (-1,1)^{n-1} \times (0,\varepsilon) \qquad \text{and} \qquad \Gamma_1 = (-1,1)^{n-1} \times \{0\}$$

and denote by $u = u_\varepsilon$ the solution to problem (1.1) corresponding to $\Omega = \Omega_\varepsilon$. Recall that by (3.2) one has

$$\int_\Omega |\nabla u|^2 dx + a \int_\Omega u^{p+1} dx = \int_{\Gamma_1} u^{q+1} d\sigma(x). \tag{3.6}$$

Next, remark that due to the Young Inequality

$$\begin{aligned}
\int_{\Gamma_1} u^{q+1}\, d\sigma(x) &= -\int_\Omega \frac{\partial}{\partial x_n} u^{q+1} dx \\
&= -(q+1) \int_\Omega u^q \frac{\partial u}{\partial x_n} dx \\
&\leq (q+1) \left\{ \frac{\delta^2}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2\delta^2} \int_\Omega u^{2q} dx \right\} \\
&\leq (q+1) \left\{ \frac{\delta^2}{2} \int_\Omega |\nabla u|^2 dx + \frac{1}{2\delta^2} \left(\int_\Omega u^{p+1} dx \right)^{2q/(p+1)} |\Omega|^{1-2q/(p+1)} \right\}.
\end{aligned}$$

Combining with (3.6) and selecting $(q+1)\frac{\delta^2}{2} = \frac{1}{2}$ we obtain for some constant C

$$\frac{1}{2} \int_\Omega |\nabla u|^2 dx + a \int_\Omega u^{p+1} dx \leq C \left(\int_\Omega u^{p+1} dx \right)^{2q/(p+1)} |\Omega|^{1-2q/(p+1)}.$$

Thus, if we denote by $|u|_{p+1}$ the usual $L^{p+1}(\Omega)$-norm we get for some constants

$$|u|_{p+1}^{p+1} \le C|\Omega| \tag{3.7}$$

$$\int_\Omega |\nabla u|^2 dx \le C|u|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)}. \tag{3.8}$$

Next, from the Sobolev embedding Theorem (see [2: p. 148]) we know that there exists a constant C such that

$$|v|_{p+1}^2 \le C \int_{\Omega_1} |\nabla v|^2 dx$$

for any $v \in H^1(\Omega_1)$ vanishing on $\partial\Omega_1 \setminus \Gamma_1$. So, extending $u = u_\varepsilon$ by 0 outside of $\Omega = \Omega_\varepsilon$ we derive

$$|u|_{p+1}^2 \le C \int_\Omega |\nabla u|^2 dx. \tag{3.9}$$

Combining (3.8) and (3.9) we obtain

$$|u|_{p+1}^2 \le C|u|_{p+1}^{2q} |\Omega|^{1-2q/(p+1)} \quad \text{and (if } u \ne 0) \quad 1 \le C|u|_{p+1}^{2q-2} |\Omega|^{1-2q/(p+1)}.$$

Hence by (3.7)

$$1 \le C|\Omega|^{2(q-1)/(p+1)} |\Omega|^{1-2q/(p+1)} = C|\Omega|^{1-2/(p+1)}$$

and a contradiction when $|\Omega| = |\Omega_\varepsilon|$ is small enough ∎

In fact, as we are going to see, what is important is the size of $|\Gamma_1|$ with respect to the one of $|\Omega|$. So, we would like to conclude this paper by an existence result referring the reader to forthcoming works for more on this topic. In what follows we will assume that

$$q < \frac{n}{n-2} \qquad \text{when } n \ge 3 \tag{3.10}$$

so that the trace operator is compact from $H^1(\Omega)$ into $L^{q+1}(\Gamma)$. We define

$$E(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx + \frac{a}{p+1} \int_\Omega |v|^{p+1} dx - \frac{1}{q+1} \int_{\Gamma_1} |v|^{q+1} d\sigma(x). \tag{3.11}$$

Then we have

Theorem 3.3. *Assume that* $2q < p+1$ *and that* (3.10) *holds. Set*

$$V_0 = \{v \in H^1(\Omega) | \, v = 0 \text{ on } \Gamma_0\} \qquad \text{and} \qquad V = V_0 \cap L^{p+1}(\Omega).$$

Then there exists $u \in V$ *such that* $E(u) \le E(v)$ *for all* $v \in V$.

Proof. First remark that arguing as in (3.4) and (3.5) one has for $v \in V$

$$\int_{\Gamma_1} |v|^{q+1} d\sigma(x) \le (q+1) \int_\Omega |v|^q |\nabla u| dx + C \int_\Omega |v|^{q+1} dx. \tag{3.12}$$

Then, since $q < \frac{p+1}{2} < p$, by Hölder's inequality

$$\begin{aligned}\int_\Omega |v|^{q+1} dx &\le \left(\int_\Omega |v|^{p+1} dx\right)^{(q+1)/(p+1)} |\Omega|^{1-(q+1)/(p+1)} \\ &= |\Omega|^{1-(q+1)/(p+1)} |v|_{p+1}^{q+1}.\end{aligned} \tag{3.13}$$

Moreover, using the Young inequality one has for some ε and some constant C_ε

$$\begin{aligned}(q+1)\int_\Omega |v|^q |\nabla u| dx &\le \varepsilon \int_\Omega |\nabla v|^2 dx + C_\varepsilon \int_\Omega |v|^{2q} dx \\ &\le \varepsilon \int_\Omega |\nabla v|^2 dx + C_\varepsilon \left(\int_\Omega |v|^{p+1} dx\right)^{2q/(p+1)} \\ &= \varepsilon \int_\Omega |\nabla v|^2 dx + C_\varepsilon |v|_{p+1}^{2q}.\end{aligned} \tag{3.14}$$

Thus, collecting (3.11) - (3.14) we obtain for some constants C_1 and C_2 depending eventually of ε

$$E(v) \ge \left(\frac{1}{2} - \varepsilon\right) \int_\Omega |\nabla v|^2 dx + \frac{a}{p+1} |v|_{p+1}^{p+1} - C_1 |v|_{p+1}^{q+1} - C_2 |v|_{p+1}^{2q}.$$

Selecting ε such that $\varepsilon < \frac{1}{2}$ and denoting

$$|v| = |\nabla v|_2 + |v|_{p+1} \tag{3.15}$$

it is clear since $q+1 < p+1$ and $2q < p+1$ that

$$\lim_{|v| \to +\infty} E(v) = +\infty. \tag{3.16}$$

Let us denote by $\{v_k\}$ a minimizing sequence of E on V, i.e. a sequence $\{v_k\}$ satisfying

$$\lim_{k \to +\infty} E(v_k) = \inf_{v \in V} E(v). \tag{3.17}$$

By (3.15) and (3.16) one has, for some constant C, $|\nabla v_k|_2 \le C$ and $|v_k|_{p+1} \le C$. So, one can extract a subsequence that for convenience we will still denote by v_k such that for some $u \in V$ one has

$$\begin{aligned} v_k &\rightharpoonup u && \text{in } V_0 \\ v_k &\rightharpoonup u && \text{in } L^{p+1}(\Omega) \\ v_k &\to u && \text{in } L^{q+1}(\Gamma) \end{aligned}$$

(recall (3.10)). Using now the lower semicontinuity of the maps $v \to |\nabla v|_2^2$ and $v \to |v|_{p+1}^{p+1}$ one deduces

$$\begin{aligned}\inf_{v \in V} E(v) &= \lim_{k \to +\infty} E(v_k) \\ &\ge \frac{1}{2} \liminf_k |\nabla v_k|_2^2 + \frac{a}{p+1} \liminf_k |v_k|_{p+1}^{p+1} - \frac{1}{q+1} \lim_k \int_{\Gamma_1} |v_k|^{q+1} d\sigma(x) \\ &\ge E(u).\end{aligned}$$

So, u is a minimizer of E and the result follows ∎

Remark 3.1. At this stage, nothing prevents the solution u to be equal to 0. As we will see this happens for instance under the assumptions of Theorem 3.2. Note also that the proof of Theorem 3.3 holds when $|\Gamma_0| = 0$.

Let us now turn to our existence result.

Theorem 3.4. *Assume that* $2q < p + 1$ *and that* (3.10) *holds. Set*

$$d(x) = \operatorname{dist}(x, \Gamma_0) = \inf_{y \in \Gamma_0} |x - y| \qquad \textit{and} \qquad D_1 = \{x \in \mathbb{R}^n \,|\, d(x) \le 1\},$$

where $|\cdot|$ *denotes either the Lebesgue measure, either the superficial measure on* Γ. *Then if*

$$\frac{1}{2}|D_1| + \frac{a}{p+1}|\Omega| - \frac{1}{q+1}|\Gamma_1 \setminus D_1| < 0, \tag{3.18}$$

there exists a non-trivial solution u *to problem* (1.1).

Proof. Consider the function $v = d \wedge 1$ where $\wedge$ denotes the minimum of two functions. It is clear that $v \in V$. Moreover, since d is a Lipschitz continuous function with a Lipschitz constant less than 1, $|\nabla d(x)| \le 1$ for a.e. $x \in \Omega$. So, we have

$$\begin{aligned} E(v) &= \frac{1}{2}\int_{D_1} |\nabla d|^2 dx + \frac{a}{p+1}\int_\Omega |v|^{p+1} dx - \frac{1}{q+1}\int_{\Gamma_1} |v|^{q+1} d\sigma(x) \\ &\le \frac{1}{2}|D_1| + \frac{a}{p+1}|\Omega| - \frac{1}{q+1}|\Gamma_1 \setminus D_1| \\ &< 0. \end{aligned}$$

Thus, the infimum (3.17) is negative and achieved for a non-zero function u. Noting that $|u| \in V$ and $E(u) = E(|u|)$, there is no loss of generality in assuming $u \ge 0$. But then, it is easy to see that u is solution to problem (1.1). This completes the proof of the theorem ∎

Remark 3.2. Note that it is very easy to find an open set Ω for which (3.18) holds. Assuming Ω included in some fixed domain it is enough to choose $|\Gamma_1 \setminus D_1|$ large enough.

Remark 3.3. In the case where $|\Gamma_0| = 0$ one remarks that since $q < p$,

$$E(\varepsilon) = \frac{a}{p+1}|\Omega|\varepsilon^{p+1} - \frac{1}{q+1}|\Gamma|\varepsilon^{q+1} < 0$$

for ε small enough. So, in this case problem (1.1) has always a solution (compare with [3]).

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Received 24.02.1995