

An Integro-Differential Parabolic Variational Inequality Connected with the Problem of the American Option Pricing

L. Mastroeni and M. Matzeu

Abstract. An existence and regularity result for a linear integro-differential inequality of parabolic type, connected with the problem of the American option pricing, is stated. The proof is based on the use of some estimates of Lewy-Stampacchia type for parabolic variational inequalities and a fixed point argument.

Keywords: *Variational inequalities, integro-differential parabolic operators, Lewy-Stampacchia type estimates, Tychonov fixed point theorem, American option pricing*

AMS subject classification: 49 J 40, 45 K 05, 90 A 09, 60 H 99

1. Introduction

The problem of finding explicit pricing formulas for European call and put options on stocks which do not pay dividends was solved by Black and Scholes in [3]. As for the American options, the problem can be formulated in the framework of optimal stopping time theory. Relying on the connection between optimal stopping and variational inequalities (see [1] and [2]), Jaillet, Lamberton and Lapeyre stated in [9] that the price of the American option is the unique solution of a variational inequality of parabolic type. In their paper the price process is supposed to be a diffusion. In [10] Merton derived a model allowing for jumps in the pricing problem and proposed some tractable formulas for the price of European options. In this case the infinitesimal generator related to the process is given by a linear integro-differential operator of parabolic type. In the one-dimensional case and referring to the elliptic part with constant coefficients, in [11] Zhang stated that the American option price in the mentioned Merton's jump diffusion model is the unique solution of a suitable linear integro-differential variational inequality. Moreover, some interesting numerical implementations are developed.

The aim of the present paper is to state an existence and regularity result for a variational inequality of the same kind as that considered in [11]. However, in our

L. Mastroeni: Univ. di Roma "Tor Vergata", Dip. di Studi Econ.-Fin. e Metodi Quantitativi,
Via di Tor Vergata, s.n.c. IT - 00133 Roma

M. Matzeu: Univ. di Roma "Tor Vergata", Dip. di Mat., Viale della Ric. Sci., IT - 00133
Roma

case, the dimension of the space is arbitrary and the elliptic part of the differential operator has variable coefficients. As in the other mentioned papers, the constraint in the variational inequality is represented by a single obstacle and the framework of the space variable is the whole space \mathbb{R}^N (for other results about equations or weak solutions of variational inequalities see [2]). In the case that \mathbb{R}^N is replaced by an open bounded subset and referring to two obstacles variational inequalities of the same type considered in the present paper, many interesting results are stated in [7] and [8]. Actually, in those papers, the elliptic part of the integro-differential operator can be even quasi-linear and the integral term is more general than the one we consider here (see, e.g., (1.3), (1.4), (1.8) of [8]). (Indeed the case considered in the present paper could be extended to more general ones, in particular we could assume that the measure ν appearing in the integral operator also depends on the time variable in a suitable way.) However, the techniques we use here are quite different from the ones used in [7] and [8], where the proofs are based on Green representations for the solutions of the equations (see [6]) and on penalization methods.

Indeed, in the particular case we are interested in, a simple argument can be carried on. More precisely, starting from some results about parabolic variational inequalities stated in [4] and [5], we develop a fixed point argument.

First, thanks to the Lewy-Stampacchia inequalities, we are able to find a suitable ball in some Sobolev space Y which is stable under the map $S : v \rightarrow Sv$, where Sv is the “unique” solution of the parabolic variational inequality corresponding to the (fixed) value of the integral operator at v . Then, the solution is found as a fixed point of S , taking into account that S is shown to be weakly lower semicontinuous with respect to a suitable “graph-norm” with respect to the time derivative operator. The suitable regularity result for the solution u which allows to interpret u in the mentioned economical framework is derived by the use of the Lewy-Stampacchia inequalities and by some standard regularity results for parabolic equations (see [1]).

2. The existence and uniqueness result

Let us consider the evolution variational inequality

$$\begin{aligned}
 \text{(VI)} \quad & u \in X_\mu, \quad \frac{\partial u}{\partial t} \in X'_\mu, \quad u \geq \psi, \quad u(T, x) = \psi(T, x) \text{ for a.e. } x \in \mathbb{R}^N \\
 & \left\langle -\frac{\partial u}{\partial t} + Au, v - u \right\rangle_\mu \geq \int_0^T \int_{\mathbb{R}^N} Bu(t, x)(u(t, x) - v(t, x))e^{-\mu|x|} dx dt \\
 & \text{for all } \psi \leq v \in X_\mu.
 \end{aligned}$$

Here the following notations are used:

- For a fixed $\mu > 0$, X_μ is the Hilbert space $L^2(0, T; H^1_\mu(\mathbb{R}^N))$ ($T > 0, N \in \mathbb{N}$) and $H^1_\mu(\mathbb{R}^N)$ is the Sobolev space of all functions $v \in L^2(\mathbb{R}^N, e^{-\mu|x|} dx)$ (i.e. $|v(x)|^2 e^{-\mu|x|}$ is integrable on \mathbb{R}^N) whose first weak derivatives $\frac{\partial v}{\partial x_i}$ belong to $L^2(\mathbb{R}^N, e^{-\mu|x|} dx)$ ($i =$

$1, \dots, N$). The space X_μ is equipped with the inner product

$$(v, w)_\mu = \int_0^T \left(\int_{\mathbb{R}^N} v(t, x)w(t, x)e^{-\mu|x|} dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \frac{\partial v(t, x)}{\partial x_i} \frac{\partial w(t, x)}{\partial x_i} e^{-\mu|x|} dx \right) dt$$

and relative norm $\|v\|_\mu = (v, v)_\mu^{1/2}$.

- X'_μ is the dual space of X_μ .
- $\langle \cdot, \cdot \rangle_\mu$ is the pairing between X_μ and X'_μ .
- ψ is a fixed element in X_μ such that the mapping $t \rightarrow \psi(t, x)$ is continuous in $[0, T]$ for almost every $x \in \mathbb{R}^N$.
- A is the differential operator given by

$$A = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij}(t, x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^N a_i(t, x) \frac{\partial}{\partial x_i} + a_0(t, x)$$

where a_{ij}, a_i, a_0 belong to $L^\infty([0, T] \times \mathbb{R}^N)$. Then A is linear and continuous from X_μ into X'_μ and one has

$$\begin{aligned} (Av, w)_\mu &= \sum_{i,j=1}^N \int_0^T \int_{\mathbb{R}^N} a_{ij}(t, x) \frac{\partial v(t, x)}{\partial x_i} \frac{\partial w(t, x)}{\partial x_j} e^{-\mu|x|} dx dt \\ &+ \sum_{i=1}^N \int_0^T \int_{\mathbb{R}^N} \left(a_i(t, x) - \mu \sum_{j=1}^N a_{ij}(t, x) \frac{x_j}{|x|} \right) \frac{\partial v(t, x)}{\partial x_i} w(t, x) e^{-\mu|x|} dx dt \\ &+ \int_0^T \int_{\mathbb{R}^N} a_0(t, x) v(t, x) w(t, x) e^{-\mu|x|} dx dt \quad \text{for all } v, w \in X_\mu. \end{aligned}$$

- B is the integral operator defined as

$$Bv(t, x) = \int_{\mathbb{R}^N} (v(t, x + z) - v(t, x)) d\nu(z)$$

where ν is a positive Radon measure on \mathbb{R}^N such that $\int_{\mathbb{R}^N} e^{\mu|x|} d\nu(x) < +\infty$. Note that B is linear and continuous from X_μ into itself and from $L^2([0, T]; L^2_\mu)$ into itself with $L^2_\mu = L^2(\mathbb{R}^N; e^{-\mu|x|} dx)$. Let us put

$$\|B\| = \sup_{\|v\|_{L^2([0, T]; L^2_\mu)} \leq 1} \left\{ \int_0^T \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} (v(t, x + z) - v(t, x)) d\nu(z) \right)^2 e^{-\mu|x|} dx dt \right\}^{1/2}.$$

By using some results about parabolic variational inequalities (see [4] and [5]) and a fixed point method, we are able to prove the following

Theorem 1. Let a_{ij}, a_i, a_0 be chosen in such a way that, for some $c(A) > 0$,

$$\langle Av, v \rangle_\mu \geq c(A) \|v\|_\mu^2 \quad \text{for all } v \in X_\mu \tag{1}$$

and let ψ satisfy the condition

$$\left(-\frac{\partial \psi}{\partial t} + A\psi \right) \in X_\mu^* \tag{2}$$

where X_μ^* is the order dual space of X_μ (i.e. $F \in X_\mu^*$ if and only if $F = F^+ - F^-$, where F^+ and F^- are positive elements of X'_μ). Let moreover the coefficients a_{ij}, a_i, a_0 and the measure ν be chosen in such a way that putting

$$\|A\| = \|A\|_{\mathcal{L}(X_\mu, X'_\mu)},$$

the relation

$$\|B\| < \min \left\{ \frac{1}{3} \left(\frac{c(A)}{c(A) + \|A\|} \right), c(A) \right\} \tag{3}$$

is verified. Then problem (VI) admits a unique solution u .

Proof. The uniqueness of the solution u can be easily proved by standard arguments (see, e.g., [1: Proof of Theorem 2.2]) due to the coerciveness of A and to the fact that one has

$$\langle Av, v \rangle_\mu + \int_0^T \int_{\mathbb{R}^N} Bv(t, x)v(t, x)e^{-\mu|x|} dx dt \geq \tilde{c} \|v\|_\mu^2$$

with $\tilde{c} > 0$ (precisely one takes \tilde{c} as $\tilde{c} = c(A) - \|B\|$ and \tilde{c} is positive as a consequence of (3)).

Let us now prove the existence. First of all, let us fix an arbitrary element v in the space

$$Y_\mu = \left\{ v \in X_\mu : \frac{\partial v}{\partial t} \in X'_\mu \right\}$$

and consider the parabolic variational inequality ¹⁾

$$\left. \begin{aligned} w \in Y_\mu : \left\langle -\frac{\partial w}{\partial t} + Aw, z - w \right\rangle_\mu &\geq \int_0^T \int_{\mathbb{R}^N} Bv(w - z)e^{-\mu|x|} dx dt \\ \text{for all } z \in X_\mu, z &\geq \psi \\ w &\geq \psi \\ w(T, x) = \psi(T, x) &\text{ for a.e. } x \in \mathbb{R}^N. \end{aligned} \right\} \tag{4}$$

¹⁾ From now on the dependence on the variable (t, x) is understood.

Conditions (1) and (2) enable to state (see [4] for a similar case dealing with open bounded subsets of \mathbb{R}^N , and [5] for a very general case) that, for any $v \in Y_\mu$, there exists a unique solution $w = Sv$ of (4) which further satisfies the so called "Lewy-Stampacchia inequalities"

$$-Bv \leq -\frac{\partial w}{\partial t} + Aw \leq -Bv + \left(-\frac{\partial \psi}{\partial t} + A\psi + Bv\right)^+ \tag{5}$$

Here the inequalities and the "positive part" are to be intended in the sense of the order dual space X_μ^* of X_μ .

In order to prove the statement about the existence in Theorem 1, it is sufficient, by definition of S , to state that there exists a fixed point of the map S from Y_μ into itself. Actually it is possible to show that S has the following properties:

(S₁) S is *weakly continuous* from Y_μ into Y_μ , where Y_μ is equipped with the graph norm with respect to the operator $\frac{\partial}{\partial t}$, that is $\|v\|_{Y_\mu} = \|v\|_\mu + \|\frac{\partial v}{\partial t}\|_{X'_\mu}$.

(S₂) There exists a *weakly compact convex* set $D \subset Y_\mu$ such that $S(D) \subset D$.

At this point the existence result will follow from the application of the *Tychonov fixed-point* theorem.

Therefore let us verify property (S₁). Let $\{v_n\} \rightarrow v$ in Y_μ and let $w_n = Sv_n$ for all $n \in \mathbb{N}$. If one considers the solution \tilde{w} of the problem

$$\left. \begin{aligned} -\frac{\partial \tilde{w}}{\partial t} + A\tilde{w} &= 0 \quad (\tilde{w} \in X_\mu) \\ \tilde{w}(T, x) &= \psi(T, x) \end{aligned} \right\}$$

and put $\bar{w}_n = w_n - \tilde{w}$, then it is easy to verify that \bar{w}_n solves the variational inequality

$$\left. \begin{aligned} \bar{w}_n \in Y_\mu : \left\langle -\frac{\partial \bar{w}_n}{\partial t} + A\bar{w}_n, z - \bar{w}_n \right\rangle_\mu &\geq \int_0^T \int_{\mathbb{R}^N} Bv_n(\bar{w}_n - z)e^{-\mu|x|} dx dt \\ \text{for all } z \in X_\mu, z &\geq \bar{\psi} = \psi - \tilde{w} \\ \bar{w}_n &\geq \bar{\psi} = \psi - \tilde{w} \\ \bar{w}_n(T, x) &= 0 \text{ for a.e. } x \in \mathbb{R}^N. \end{aligned} \right\} \tag{6}$$

We claim that $\{\bar{w}_n\}$ is bounded in Y_μ . Indeed, by the Lewy-Stampacchia inequalities

$$-Bv_n \leq -\frac{\partial \bar{w}_n}{\partial t} + A\bar{w}_n \leq -Bv_n + \left(-\frac{\partial \bar{\psi}}{\partial t} + A\bar{\psi} + Bv_n\right)^+ \tag{7}$$

and the positivity of the operator $-\frac{\partial}{\partial t}$ on the closed subspace Y_μ^0 of Y_μ given by

$$Y_\mu^0 = \left\{ v \in Y_\mu : v(T, x) = 0 \text{ for a.e. } x \in \mathbb{R}^N \right\},$$

one easily deduces the relation

$$\langle A\bar{w}_n, \bar{w}_n \rangle_\mu \leq \left(3\|Bv_n\|_\mu + \left\| -\frac{\partial \bar{\psi}}{\partial t} + A\bar{\psi} \right\|_{X'_\mu} \right) \|\bar{w}_n\|_\mu.$$

So the coerciveness of A and the boundedness of $\{Bv_n\}$ in X_μ imply

$$\|\bar{w}_n\|_\mu \leq \text{const.} \tag{8}$$

On the other hand, still (7), the boundedness of A and (8) yield

$$\left\| \frac{\partial \bar{w}_n}{\partial t} \right\|_{X'_\mu} \leq \left\| -\frac{\partial \bar{w}_n}{\partial t} + A\bar{w}_n \right\|_{X'_\mu} + \text{const} \|\bar{w}_n\|_\mu \leq \text{const.} \tag{9}$$

Thus (8) and (9) give the boundedness of $\{\bar{w}_n\}$ in Y_μ^0 .

Therefore at least a subsequence of $\{\bar{w}_n\}$, still named $\{\bar{w}_n\}$, weakly converges to some \bar{w} in Y_μ^0 . At this point, still taking into account the positivity of $-\frac{\partial}{\partial t}$ on Y_μ^0 , one observes that, for any $z \in X_\mu$, the functional

$$F_z(v) = \left\langle -\frac{\partial v}{\partial t}, v - z \right\rangle_\mu$$

is convex. Moreover, F_z is weakly lower semicontinuous on Y_μ^0 since it is continuous in the Y_μ -norm. Then one gets

$$\left\langle -\frac{\partial}{\partial t} \bar{w}, \bar{w} - z \right\rangle_\mu \leq \liminf_{n \rightarrow \infty} \left\langle -\frac{\partial \bar{w}_n}{\partial t}, \bar{w}_n - z \right\rangle_\mu. \tag{10}$$

The same argument applied to the functional G_z defined, for any $z \in X_\mu$, as

$$G_z(v) = \langle Av, v - z \rangle_\mu$$

(G_z too is weakly lower semicontinuous and convex, thanks to the coerciveness of A) yields

$$\langle A\bar{w}, \bar{w} - z \rangle_\mu \leq \liminf_{n \rightarrow \infty} \langle A\bar{w}_n, \bar{w}_n - z \rangle_\mu. \tag{11}$$

Thus (6), (10), (11) and the weak convergence of Bv_n to Bv in X_μ give

$$\begin{aligned} & \left\langle -\frac{\partial \bar{w}}{\partial t} + A\bar{w}, \bar{w} - z \right\rangle_\mu \\ & \leq \int_0^T \int_{\mathbb{R}^N} Bv z e^{-\mu|x|} dx dt - \limsup_{n \rightarrow \infty} \int_0^T \int_{\mathbb{R}^N} Bv_n \bar{w}_n e^{-\mu|x|} dx dt. \end{aligned} \tag{12}$$

Now, let us take any number $\gamma > \mu$. Due to the compact embedding of H_μ^1 into $L^2_\gamma = L^2(\mathbb{R}^N, e^{-\gamma|x|} dx)$ (see [11] for a proof in the one-dimensional case, which can be

easily extended to the N -dimensional case), one has that $\{Bv_n\}$ strongly converges, up to subsequences, to Bv in L^2_γ . Therefore, since $\{\bar{w}_n e^{\eta|z|}\}$ is weakly convergent to $\{\bar{w} e^{\eta|z|}\}$ in $L^2(0, T; H^1_\gamma)$ with $\eta = \gamma - \mu$, one easily deduces that

$$\begin{aligned}
 & \limsup \int_0^T \int_{\mathbb{R}^N} Bv_n \bar{w}_n e^{-\mu|z|} dx dt \\
 &= \limsup \int_0^T \int_{\mathbb{R}^N} Bv_n (\bar{w}_n e^{\eta|z|}) \cdot e^{-\gamma|z|} dx dt \\
 &= \lim \int_0^T \int_{\mathbb{R}^N} Bv_n (\bar{w}_n e^{\eta|z|}) e^{-\gamma|z|} dx dt \tag{13} \\
 &= \int_0^T \int_{\mathbb{R}^N} Bv \bar{w} e^{\eta|z|} e^{-\gamma|z|} dx dt \\
 &= \int_0^T \int_{\mathbb{R}^N} Bv \bar{w} e^{-\mu|z|} dx dt
 \end{aligned}$$

so that (12) and (13) imply that \bar{w} solves the variational inequality

$$\left. \begin{aligned}
 & \bar{w} \in Y_\mu^0 : \left\langle -\frac{\partial \bar{w}}{\partial t} + A\bar{w}, z - \bar{w} \right\rangle_\mu \geq \int_0^T \int_{\mathbb{R}^N} Bv(\bar{w} - z) e^{-\mu|z|} dx dt \\
 & \text{for all } z \in X_\mu, z \geq \bar{\psi} \\
 & \bar{w} \geq \bar{\psi}.
 \end{aligned} \right\} \tag{14}$$

Now, recalling the definition of $\bar{w}_n = w_n - w$, one easily checks that $w = \bar{w} + \tilde{w} = \lim \bar{w}_n + \tilde{w} = \lim w_n$ is the solution of the variational inequality

$$\left. \begin{aligned}
 & w \in Y_\mu : \left\langle -\frac{\partial w}{\partial t} + Aw, z - w \right\rangle_\mu \geq \int_0^T \int_{\mathbb{R}^N} Bv(w - z) e^{-\mu|z|} dx dt \\
 & \text{for all } z \in X_\mu, z \geq \psi \\
 & w \geq \psi \\
 & w(T, x) = \psi(T, x) \text{ for a.e. } x \in \mathbb{R}^N
 \end{aligned} \right\}$$

that is $w = Sv$, and S is weakly continuous on Y_μ .

Now let us verify property (S_2) . At this purpose let us put, for any $r > 0$,

$$D_r = \{v \in Y_\mu : \|v\|_{Y_\mu} \leq r\}$$

and let us show that D_r verifies

$$S(D_r) \subset D_r \tag{15}$$

for any $r \geq \bar{r} > 0$ suitable chosen so that, for $r \geq \bar{r}$, as D_r is a weakly compact convex subset of Y_μ , property (S_1) will be satisfied with $D = D_r$. Indeed, let $v \in D_r$, $w = Sv$ and $\tilde{w} = w - \psi$, so that \tilde{w} belongs to Y_μ^0 and $\langle -\frac{\partial \tilde{w}}{\partial t}, \tilde{w} \rangle_\mu > 0$. By the Lewy-Stampacchia inequalities, one deduces

$$\begin{aligned} -Bv + \left(-\frac{\partial \psi}{\partial t} + A\psi \right) &\leq -\frac{\partial \tilde{w}}{\partial t} + A\tilde{w} \\ &\leq -Bv + \left(-\frac{\partial \psi}{\partial t} + A\psi + Bv \right)^+ + \left(-\frac{\partial \psi}{\partial t} + A\psi \right). \end{aligned}$$

Hence, the usual calculations based on the positivity of $-\frac{\partial}{\partial t}$ on Y_μ^0 and the coerciveness of A easily yield

$$c(A) \|\tilde{w}\|_{X_\mu} \leq \left\| -\frac{\partial}{\partial t} \tilde{w} + A\tilde{w} \right\|_{X'_\mu} \leq 3\|Bv\|_{L^2([0,T];L^2_\mu)} + 3 \left\| -\frac{\partial \psi}{\partial t} + A\psi \right\|_{X'_\mu} \tag{16}$$

Moreover, taking the second inequality in (16) into account

$$\left\| \frac{\partial}{\partial t} \tilde{w} \right\|_{X'_\mu} \leq 3\|Bv\|_{L^2([0,T];L^2_\mu)} + 3 \left\| -\frac{\partial \psi}{\partial t} + A\psi \right\|_{X'_\mu} + \|A\| \|\tilde{w}\|_\mu \tag{17}$$

Finally, by (16) and (17) and the definition of $\tilde{w} = w - \psi$, one easily deduces

$$\|Sv\|_{Y_\mu} = \|w\|_{Y_\mu} \leq R_1(A, B) \|v\|_{Y_\mu} + R_2(A, B, \psi)$$

where

$$R_1(A, B) = 3\|B\| \left(1 + \frac{\|A\| + 1}{c(A)} \right)$$

and $R_2(A, B, \psi)$ is a suitable positive number depending on A, B and ψ . At this point, taking into account the elementary inequality

$$sx + R_2(A, B, \psi) \leq x \quad \text{for } 0 \leq s < 1 \text{ and } x = x(s) > 0 \text{ sufficiently large}$$

one easily deduces that relation (3) (which implies $R_1(A, B) < 1$) and the choice of a sufficiently large $r > 0$ (precisely it is sufficient to take r in such a way that $R_1(A, B)r + R_2(A, B, \psi) \leq r$) guarantee the inclusion (15). So property (S_2) is verified with $D = D_r$ ■

3. The regularity results

The aim of this section is to give some regularity results for problem (VI). They allow to give an economical interpretation of the solution u at least in the case $N \leq 5$. This interpretation was already proposed in [11] for the one-dimensional case under some weaker assumptions (the operator A was not supposed to be coercive and no constraint was made on B in dependence of A) as well as under some stronger assumptions (the coefficients of A were supposed to be constant).

A first "regularity" (in some sense) result can be obtained as a consequence of the method itself followed in order to construct the solution u to problem (VI). It is expressed by the following

Theorem 2. *The solution u of problem (VI) verifies the inequalities*

$$0 \leq -\frac{\partial u}{\partial t} + Au + Bu \leq \left(-\frac{\partial \psi}{\partial t} + A\psi + Bu\right)^+ \tag{18}$$

in the sense of the order dual space.

Proof. Using the notations introduced in the proof of Theorem 1, as immediate consequence of the fact that $u = Su$ and that, for any $v \in Y_\mu$,

$$-Bv \leq -\frac{\partial(Sv)}{\partial t} + A(Sv) \leq -Bv + \left(-\frac{\partial \psi}{\partial t} + A\psi + Bv\right)^+$$

the statement follows ■

The other regularity results can be obtained as corollaries of Theorem 1, using the fact that Bu belongs to X_μ and applying some general regularity results for parabolic equations (see [1]).

At this purpose, let us define, for $m \in \mathbb{N} \cup \{0\}$ and $q \in [1, +\infty)$, the space $W_\mu^{m,q}(\mathbb{R}^N)$ as the Sobolev space of all functions $v \in L^q(\mathbb{R}^N, e^{-\mu|x|} dx)$ whose weak derivatives up to the order m belong to $L^q(\mathbb{R}^N, e^{-\mu|x|} dx)$, equipped with the norm ²⁾

$$\|v\|_{W_\mu^{m,q}(\mathbb{R}^N)} = \sum_{\substack{j=1,\dots,N \\ |\alpha| \leq m}} \left(\int_{\mathbb{R}^N} \left| \frac{\partial^\alpha v}{\partial x_j^\alpha} \right|^q e^{-\mu|x|} dx \right)^{1/q}$$

(note that $W_\mu^{1,2}(\mathbb{R}^N) = H_\mu^1(\mathbb{R}^N)$ and $W_\mu^{0,q}(\mathbb{R}^N) = L_\mu^q(\mathbb{R}^N)$).

An immediate consequence of Theorem 2 is given by the following

Corollary 1. *Let u be the solution of problem (VI) and let the element*

$$g = \left(-\frac{\partial \psi}{\partial t} + A\psi + Bu\right)^+ - Bu \tag{19}$$

²⁾ α denotes a multi-index $(\alpha_1, \dots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$ and $|\alpha| = \alpha_1 + \dots + \alpha_N$.

belong to some space Z of the type $L^p(0, T; W_{\mu}^{m,q}(\mathbb{R}^N))$, for some choice of $p, q \in [1, +\infty]$ and $m \in \mathbb{N} \cup \{0\}$. Then, for a suitable $f \in Z$, u solves the problem

$$\begin{aligned} -u' + Au &= f \\ u(T, x) &= \psi(T, x). \end{aligned} \tag{20}$$

Proof. It follows from (18) and from the fact that the space Z is a lattice, thus any element in the "order interval" $[0, g]$ belongs to Z ■

In order to give the economical interpretation of the solution u of problem (VI), we need to use the following regularity result which is a consequence of Corollary 1.

Theorem 3. *Let the coefficients a_{ij}, a_i, a_0 defining the operator A satisfy the conditions*

$$a_{ij}, a_i, a_0 \in C^1([0, T] \times \mathbb{R}^N) \tag{21}$$

$$a_{ij}(x, t) = a_{ji}(x, t) \text{ for all } i, j \in \{1, \dots, N\} \tag{22}$$

$$a_0(x, t) \geq \beta \text{ for some } \beta > 0 \tag{23}$$

$$\left| \frac{\partial a_{ij}}{\partial t} \right| \leq \text{const} \tag{24}$$

$$\left| \frac{\partial a_{ij}}{\partial x_k} \right| \leq \text{const for all } k \in \{1, \dots, N\} \tag{25}$$

and let ψ verify

$$\left. \begin{aligned} \psi &\in L^p(0, T; W_{\mu}^{2,p}(\mathbb{R}^N)) \\ \frac{\partial \psi}{\partial t} &\in L^p(0, T; L_{\mu}^p(\mathbb{R}^N)) \\ A\psi &\in L^p(0, T; L_{\mu}^p(\mathbb{R}^N)) \\ -\frac{\partial \psi}{\partial t} + A\psi &\in L^2(0, T; H_{\mu}^1(\mathbb{R}^N)) \end{aligned} \right\} \tag{26}$$

with any $p \in [2, +\infty)$ if $N \leq 2$ and $p = 2^*$ if $N > 2$ ³⁾ Then the solution u of problem (VI) verifies the conditions

$$u \in L^p(0, T; W_{\mu}^{2,p}(\mathbb{R}^N)) \tag{27}$$

$$\frac{\partial u}{\partial t} \in L^p(0, T; L_{\mu}^p(\mathbb{R}^N)) \tag{28}$$

for any $p \in [2, +\infty)$ if $N \leq 2$ and $p = 2^*$ if $N > 2$.

Proof. First of all, the fact that $-\frac{\partial \psi}{\partial t} + A\psi$ belongs to $L^2(0, T; H_{\mu}^1(\mathbb{R}^N))$ implies that u is the solution of a problem of the type

$$\left. \begin{aligned} -\frac{\partial u}{\partial t} + Au &= f \in L^2(0, T; H_{\mu}^1(\mathbb{R}^N)) \\ u(T, x) &= \psi(T, x) \in H_{\mu}^1(\mathbb{R}^N) \end{aligned} \right\} \tag{29}$$

³⁾ For $N > 2$, 2^* is defined as $2^* = \frac{2N}{N-2}$.

thanks to Corollary 1. Then a general result about parabolic inequalities (see [1: Theorem 6.11]) assures that u belongs to $L^\infty(0, T; H_\mu^1(\mathbb{R}^N))$ with $\frac{\partial u}{\partial t} \in L^2(0, T; H_\mu^1(\mathbb{R}^N))$. Therefore Bu itself belongs to $L^\infty(0, T; H_\mu^1(\mathbb{R}^N))$, thus, in particular, it belongs to $L^p(0, T; L_\mu^p(\mathbb{R}^N))$, due to the well-known result about the embeddings of Sobolev spaces into L^p spaces. Therefore f itself in (29) belongs to the space

$$L^2(0, T; H_\mu^1(\mathbb{R}^n)) \cap L^p(0, T; L_\mu^p(\mathbb{R}^N)).$$

At this point, conditions (21) - (25) yield (27) and (28), as a consequence of another general result about parabolic equations (see [1: Theorem 6.12⁴⁾]) ■

4. An economical interpretation

Let us illustrate now an economical interpretation of the solution of problem (VI) in the one-dimensional case and in case that the coefficients of the operator A are constant.

Let us consider an American put option over a stock whose price is described by a stochastic process $(S_t)_{t \geq 0}$ given by the solution of the Cauchy problem

$$\left. \begin{aligned} S_0 &= y \\ \frac{dS_t}{S_t} &= \mu dt + \sigma dB_t + d \left(\sum_{j=1}^{N_t} U_j \right) \end{aligned} \right\}$$

where y is the so called "spot price" at the time $t = 0$, $(B_t)_{t \geq 0}$ is a standard \mathbb{R} -valued Brownian motion, $(U_j)_{j \geq 1}$ is a sequence of identically distributed random variables in $(-1, +\infty)$, μ and σ are constants with $\sigma > 0$, and U_j represent the jumps of the process, which are connected with a Poisson process $(N_t)_{t \geq 0}$. The processes $(B_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$ and $(U_j)_{j \geq 1}$ are independent.

Let us suppose that the interest rate r is a strictly positive constant and that $\mu = r - \lambda EU_1$ (λ is the "intensity" of the Poisson process $(N_t)_{t \geq 0}$). Let us suppose that the American option, of expiring date T , allows a profit $f(S_t) = (K - S_t)^+$ if it is exercised at time t where K is the so called "exercise price". One defines

$$P(t, S_t) = \sup_{r \in \mathcal{T}_t, T} \mathbf{E}(e^{-r(r-t)} f(S_r^{t,y}))$$

as the value of the option at time t , where \mathcal{T}_t, T is the set of the stopping times in $[t, T]$ and $(S_s^{t,y})_{s \geq t}$ is the process defined by

$$S_s^{t,y} = ye^{(\mu - \sigma^2/2)(s-t) + \sigma(B_s - B_t)} \prod_{j=N_t+1}^{N_s} (1 + U_j).$$

⁴⁾ Actually in that result the term $\bar{u}(x) = u(x, T)$ is supposed to be equal to zero, but it is easy to check that the same result holds if there exists some $\tilde{u} \in L^p(0, T; W_\mu^{2,p}(\mathbb{R}^N))$ with $(-\frac{\partial \tilde{u}}{\partial t} + A\tilde{u}) \in L^2(0, T; H_\mu^1(\mathbb{R}^N)) \cap L^p(0, T; L_\mu^p(\mathbb{R}^N))$ such that $\tilde{u}(T, x) = \bar{u}(x)$. In our case one takes $\tilde{u} = \psi$.

Let us consider the change of variable $X_t = \log S_t$ and define

$$u^*(t, x) = \sup_{\tau \in \mathcal{T}_t, \tau \leq E} (e^{-r(\tau-t)} \psi(X_\tau^t, x))$$

where $\psi(x) = (k - e^x)^+$. It is easy to check that $P(t, x) = u^*(t, \log x)$. That is, in order to evaluate the price $P(t, x)$, it is sufficient to compute u^* . In [11] the author shows also that u^* coincides with the unique solution of the variational inequality of type (VI) related to the choices $N = 1$ and constant coefficients for the operator A . The results contained in the present paper enable us to give a suitable generalization of this economical interpretation to the case $N \leq 5$ and to variable coefficients of A . Indeed a main argument in the proof proposed by [11] is the fact that the exponent p appearing in (26) must satisfy $p > \frac{N}{2}$. This relation is obviously satisfied for $N = 1$ (the Zhang case where $p = 2$ by definition of solution of problem (VI)) and $N = 2$ (in this case one can consider any $p \geq 2$, thus $p > \frac{2}{2} = 1$). Otherwise, if $N \geq 3$ one has to take $p = 2^*$ in (26) and condition $p > \frac{N}{2}$ is equivalent to that of $N \leq 5$. A detailed exposition of these results will be given in a forthcoming paper.

References

- [1] Bensoussan, A. and J. L. Lions: *Applications des inéquations variationnelles en contrôle stochastique*. Paris: Dunod 1978.
- [2] Bensoussan, A. and J. L. Lions: *Contrôle impulsionnel et inéquations quasi variationnelles*. Paris: Dunod 1982.
- [3] Black, F. and M. Scholes: *The pricing of options and corporate liabilities*. J. Pol. Econ. 81 (1973), 635 - 654.
- [4] Charrier, P. and G. M. Troianello: *On strong solutions of parabolic unilateral problems with obstacle dependent on time*. J. Math. Anal. Appl. 65 (1978), 110 - 125.
- [5] Donati, F. and M. Matzeu: *On the strong solutions of some nonlinear evolution problems in ordered Banach spaces*. Boll. Un. Mat. Ital. 16-B (1979)5, 54 - 73.
- [6] Garroni, M. G. and J. L. Menaldi: *Green Functions for Second Order Parabolic Integro-Differential Problems* (Pitman Research Notes in Math. Series). London: Longman 1992.
- [7] Garroni, M. G., Solonnikov, V. A. and M. A. Vivaldi: *Quasi-linear integro-differential parabolic problems with non homogeneous conditions*. Houston J. Math. 18 (1992), 481 - 532.
- [8] Garroni, M. G. and M. A. Vivaldi: *Quasi-linear parabolic integro-differential problems with non linear oblique boundary conditions*. Nonlin. Anal. 16 (1991), 1089 - 1116.
- [9] Jaillet, P., Lamberton, D. and B. Lapeyre: *Variational inequalities and the pricing of american options*. Acta Appl. Math. 21 (1990), 263 - 289.
- [10] Merton, R. C.: *Option pricing when underlying stock returns are discontinuous*. J. Fin. Econ. 3 (1976), 125 - 144.
- [11] Zhang, X.: *Options américaines et modèles des diffusion avec sauts*. Comp. Rend. Acad. Sci. Paris (Séries I) 317 (1993), 857 - 862.

Received 15.02.1995