Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 14 (1995), No. 4, 913–916

An Example for Different Associated Spaces in the Case of Pseudoanalytic Functions

U. Yüksel

Abstract. The theory of solving initial-value problems in associated spaces makes it possible to solve initial-value problems with more general initial functions. In many cases the associated space is not uniquely determined, i.e. there exist various so-called co-associated spaces in which initial-value problems can be solved. The present paper is aimed at giving an example to this fact.

Keywords: *Associated operators, initial-value problems, Cauchy-Kovalevska theorem*
AMS subject classification: 30 G 20, 35 A 10

The paper [3] deals with the initial-value problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= Lw := C_0(t,z)\frac{d_E w}{dz} + A_0(t,z)w + B_0(t,z)\overline{w} \\ w(0,z) &= w_0(z) \end{aligned} \tag{1}$$

in the space of functions which satisfy the equation

$$lw := \frac{\partial w}{\partial \bar{z}} - a_E(z)w - b_E(z)\overline{w} = 0 \tag{2}$$

where

$$\frac{d_E w}{dz} = \frac{\partial w}{\partial z} - A_E(z)w - B_E(z)\overline{w}$$

and a_E, b_E, A_E, B_E are the characteristic coefficients defined by Bers (see [1: p. 5]). The functions which satisfy the equation (2) are known as *E-pseudoanalytic* in the sense of Bers (cf. [1: p. 6]). Thus the initial-value problem (1) is investigated in the space of pseudoanalytic functions. In the first part of the paper [3] the authors proved that L is associated to l, i.e. $lw = 0$ implies $l(Lw) = 0$ if the following sufficient conditions hold:

$$A_{0,\bar{z}} - C_0(A_{E,\bar{z}} - a_{E,z}) = 0 \tag{3}$$

$$C_0 b_{E,z} + \overline{C}_0 b_{E,\bar{z}} + b_E \overline{C_0'} + 2ib_E \operatorname{Im}\big(C_0(\bar{a}_E - A_E) + A_0\big) = 0 \tag{4}$$

$$B_0 - B_E C_0 - b_E \overline{C}_0 = 0 \tag{5}$$

U. Yüksel: University of Ankara, Dep. Math., 06100 Tandogan Ankara, Turkey. – The author is most grateful to Prof. Dr. W. Tutschke for some helpful hints improving the final version of the paper.

ISSN 0232-2064 / $ 2.50 © Heldermann Verlag

while C_0 is holomorphic. If $0 \neq C_0$ is a real constant and $A_E = B_E \equiv 0$ (for that it is necessary and sufficient that F, G are antiholomorphic where $E = (F, G)$, see [1: p. 5]), then these conditions are reduced to

$$A_{0,\bar{z}} + C_0 a_{E,z} = 0 \tag{6}$$

$$C_0(b_{E,z} + b_{E,\bar{z}}) + 2ib_E(\operatorname{Im} A_0 - C_0 \operatorname{Im} a_E) = 0 \tag{7}$$

$$B_0 - b_E C_0 = 0.$$

They are a special case of the conditions (3) – (5). If b_E does not vanish identically, then A_0 must satisfy the inhomogeneous Cauchy-Riemann equation (6) and the linear algebraic side condition (7) at the same time. Provided $b_E \neq 0$ everywhere, $\operatorname{Im} A_0$ is uniquely determined by (7) which implies that (6) is a first order system for $\operatorname{Re} A_0$. The sufficient conditions for the solvability of this system can be written by the paper [3] as

$$\operatorname{Re}\left(\frac{1}{b_E}(b_{E,z} + b_{E,\bar{z}})\right) = 0 \tag{8}$$

$$\operatorname{Im}\left(\frac{b_{E,z}}{b_E}\right)_{z\bar{z}} = \operatorname{Im} a_{E,z\bar{z}} + \operatorname{Im} a_{E,zz}. \tag{9}$$

It can be easily shown that the characteristic coefficients a_E and b_E satisfy the relations

$$a_{E,z} = -b_E \bar{b}_E \tag{10}$$

$$b_{E,z} = -b_E \bar{a}_E \tag{11}$$

if $A_E = B_E \equiv 0$. By virtue of the equality

$$\operatorname{Im} a_{E,z\bar{z}} + \operatorname{Im} a_{E,zz} = (\operatorname{Im} a_{E,z})_z + (\operatorname{Im} a_{E,z})_{\bar{z}}$$

and the relation (10), the right-hand side of condition (9) vanishes identically. On the other hand by the aid of the relations (10) and (11) one gets

$$\operatorname{Im}\left(\frac{b_{E,z}}{b_E}\right)_{z\bar{z}} \equiv 0.$$

Therefore (9) is an identity. Moreover by the relation (11), condition (8) can be rewritten as

$$\operatorname{Re}\left(\frac{b_{E,\bar{z}}}{b_E}\right) = \operatorname{Re} a_E. \tag{12}$$

So we have proved the following

Lemma: *Let D be a simply connected and bounded domain, and $E = (F, G)$ be a smooth generating pair defined on D such that the generating functions F and G are antiholomorphic. Assume, further, that $0 \neq C_0$ is a real constant, $b_E \neq 0$ everywhere,*

$B_0 = b_E C_0$ and the condition (12) holds. Then there exists a coefficient A_0 which is uniquely determined up to an arbitrary real constant such that the operator

$$Lw = C_0 \frac{\partial w}{\partial z} + A_0 w + B_0 \overline{w}$$

is associated to the operator

$$lw = \frac{\partial w}{\partial \bar{z}} - a_E w - b_E \overline{w}.$$

Thus if the properties which are formulated in the lemma are fulfilled, then the initial-value problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= Lw := C_0 \frac{\partial w}{\partial z} + A_0 w + B_0 \overline{w} \\ w(0,z) &= w_0(z) \end{aligned}$$

can be solved in the space of pseudoanalytic functions by the abstract Cauchy-Kovalevska theory (see [3]).

Now we assume that C_0 is holomorphic and $b_E \equiv 0$ besides $A_E = B_E \equiv 0$. Then (4) holds identically, (5) implies $B_0 \equiv 0$, (10) shows that a_E is antiholomorphic and, consequently, (3) implies that A_0 is holomorphic. Hence the operator L,

$$Lw = C_0 \frac{\partial w}{\partial z} + A_0 w$$

is associated to the operator l,

$$lw = \frac{\partial w}{\partial \bar{z}} - a_E w$$

if C_0 and A_0 are holomorphic and a_E is antiholomorphic. Therefore the Cauchy-Kovalevska problem

$$\begin{aligned} \frac{\partial w}{\partial t} &= Lw := C_0 \frac{\partial w}{\partial z} + A_0 w \\ w(0,z) &= w_0(z) \end{aligned} \tag{13}$$

is solvable in the space of pseudoanalytic functions by the same theory. In the classical case we have $E = (1, i)$ and, consequently, all characteristic coefficients vanish identically. Then the initial-value problem (13) has a solution in the space of holomorphic functions in virtue of the classical Cauchy-Kovalevska theorem (in the linear case).

The above example shows that possibly there are various associated spaces in which initial-value problems are solvable. In such cases a permissible initial function has not necessarily to belong to a fixedly chosen associated space. On the contrary, more general initial functions are permissible, too, if only they can be decomposed into components belonging to one of the associated spaces (see [2]).

References

[1] Bers, L.: *Theory of Pseudoanalytic Functions.* New York: New York University 1953.

[2] Heersink, R. and W. Tutschke: *A decomposition theorem for solving initial-value problems in associated spaces.* Rend. Circolo Mat. Palermo 43 (1994), 419 - 434.

[3] Tutschke, W. and C. Withalm: *The Cauchy-Kovalevska theorem for pseudoanalytic functions in the sense of L. Bers.* Complex Variables 1 (1983), 389 - 393.

Received 14.03.1995