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Vector-Valued Integration in BK-Spaces

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Abstract. Questions of convergence in BK-spaces, i.e. Banach spaces of complex-valued sequences $x = (x_k)_{k\in\mathbb{Z}}$ with continuity of all functionals $x \mapsto x_k$ $(k \in \mathbb{Z})$ will be studied by methods of Fourier analysis. An elegant treatment is possible if the Cesàro sections of a BK-space element x can be represented by vector-valued Riemann integrals. This was done by Goes [2] following the example of Katznelson [5: pp. 10 - 12]. The purpose of this paper is to make precise the conditions in [2] concerning Riemann integration and to demonstrate relations between BK-spaces which are generated by a given BK-space.

Keywords: *BK-spaces, Riemann integration, Cesàro-sectional (weak) convergence and boundedness*

AMS subject classification: Primary 46 B 45, 46 A 45, 28 B 05, secondary 40 G 05, 42 A 24

1. Introduction

This paper is motivated by a letter of Boettcher to Goes (*Beispiel eines translationsinvarianten BK-Raumes, der nicht die Eigenschaft σB hat*) from May 31, 1990, in which the space $E = \widehat{L^2(\mathbf{T})} \oplus \widehat{M^d(\mathbf{T})}$ (cf. Example 3.7) is considered as example of a translation-invariant BK-space which fails to have the so-called property σB. Choosing some element $x = \hat{\delta}_0$ Boettcher proves that the sequence $(\sigma_n x)_{n\in\mathbb{N}_0}$ with $\sigma_n x \in \widehat{L^2(\mathbf{T})} \subseteq E$ for all $n \in \mathbb{N}_0$ is not bounded.

This shows that in general Proposition 4.1/(i) in Goes [2] is not valid. Actually the BK-valued Riemann integral used in the proof of Proposition 4.1/(i) may not exist. It is evident that the function xe (cf. Definition 2.7) is not Riemann integrable because $\sigma_n x \notin \widehat{M^d(\mathbf{T})}$ for all $n \in \mathbb{N}_0$. Thus Riemann integrability of xe is sufficient for x to have property σB (hence [2: Proposition 4.1/(i)] is valid in this case (cf. Theorem 3.5)). However it is not necessary, as $u \in E_3$ in Example 3.7 shows.

From this point of view the advantages and the limitations in the representation of $\sigma_n x$ as vector-valued integral shall be demonstrated. Especially some properties of Riemann and Bochner integration in translation-invariant BK-spaces are considered. Beyond this significant relations between certain BK-subspaces of the linear space Ω of all complex-valued sequences are presented.

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2. Notations, definitions and preliminary remarks

2.1 Convergence and boundedness in BK-spaces. Let $\mathbb{R}$ and $\mathbb{C}$ be the set of real and complex numbers, respectively, let $\mathbb{N}$ and $\mathbb{N}_0$ be the set of positive and non-negative integers, respectively, and $\mathbb{Z}$ the set of integers. Furthermore let Ω be the linear space of complex-valued sequences on $\mathbb{Z}$, i.e.

$$\Omega = \left\{(x_k)_{k\in\mathbb{Z}}\middle| x_k \in \mathbb{C} \text{ for all } k \in \mathbb{Z}\right\}.$$

For $k \in \mathbb{Z}$ let δ^k the Kronecker symbol and define $\sigma_n : \Omega \to \Omega \quad (n \in \mathbb{N}_0)$ by

$$\sigma_n : x \longmapsto \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) x_k \delta^k \qquad (x \in \Omega).$$

The function $\sigma_n x$ is called *n-th Cesàro section of order one* of x.

Definition 2.1 (BK-space). $(E; \|\cdot\|_E)$ with $E \subseteq \Omega$ is called *BK-space* if

1. E endowed with the norm $\|\cdot\|_E$ is a Banach space
2. $Pr_k : E \to \mathbb{C}$ with $x \mapsto x_k$ is continuous for all $k \in \mathbb{Z}$.

Definition 2.2 (Convergence and boundedness). Let E be a BK-space, $x \in \Omega$ and $\sigma_n x \in E$ for all $n \in \mathbb{N}_0$. Then x has the

1. property σK of *Cesàro-sectional convergence* if $(\sigma_n x)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in E, i.e. $\lim_{n\to\infty} \sigma_n x = x \in E$;

2. property σB of *Cesàro-sectional boundedness* if $\sup_{n\in\mathbb{N}_0} \|\sigma_n x\|_E < \infty$;

3. property $S\sigma K$ of *weak Cesàro-sectional convergence* if $x \in E$ and $\lim_{n\to\infty} \phi(\sigma_n x) = \phi(x)$ for all $\phi \in E'$.

Remark 2.3. Let P be one of the properties $\sigma K, \sigma B$ or $S\sigma K$. Then the space $E_P = \{x \in \Omega | \, x \text{ has the property } P\}$ endowed with the norm $\|x\|_{E_P} = \sup_{n\in\mathbb{N}_0} \|\sigma_n x\|_E$ is a BK-space with $\|x\|_E \leq \sup_{n\in\mathbb{N}_0} \|\sigma_n x\|_E$ for all $x \in E_{S\sigma K}$, and thus for all $x \in E_{\sigma K}$ (cf. Yosida [6: Theorem 2/p. 120]) as well known.

Definition 2.4. A closed subspace G of E has *sectional density* (*Abschnittsdichte AD*) if the set $\Phi = \{x \in \Omega | \, \{k \in \mathbb{Z} | \, x_k \neq 0\} \text{ is finite}\}$ is dense in G, i.e. if $G = \overline{\Phi \cap G}$. In particular let E_{AD} be defined by $E_{AD} = \overline{\Phi \cap E}$.

According to Zeller [7: Sätze 2.2, 3.3 and 3.4] we have the following lemma.

Lemma 2.5. *Let $(E; \|\cdot\|_E)$ be a BK-space. Then the following assertions are true.*

1. *E has property σK if and only if $E = E_{AD}$ and $E \subseteq E_{\sigma B}$.*
2. *E has property $S\sigma K$ if and only if E has property σK.*

Remarks 2.6. Obviously we have the relation $E_{\sigma K} \subseteq E_{S\sigma K} \subseteq E_{AD}$, and if $E_{AD} \subseteq E_{\sigma B}$, then even $E_{\sigma K} = E_{S\sigma K} = E_{AD}$. If $E_{S\sigma K}$ is closed with respect to $\|\cdot\|_E$, then $E_{S\sigma K} = E_{\sigma K}$.

Definition 2.7 (Invariance and continuity of translation). Let $\mathbf{T} = \mathbb{R}/2\pi\mathbb{Z}$, where $\mathbb{R}$ is the additive group of real numbers. Then

$$e(t) = (e^{ikt})_{k\in\mathbf{Z}} \qquad \text{for all } \ t \in \mathbf{T}.$$

A BK-space E is called

1. *translation invariant* if $xe(t) = (x_k e^{ikt})_{k\in\mathbf{Z}} \in E$ and $\|xe(t)\|_E = \|x\|_E$ for all $x \in E$ and $t \in \mathbf{T}$

and it is called

2. *homogeneous BK-space* if in addition the translation is continuous, i.e. the convergence $\lim_{t\to t_0} \|xe(t) - xe(t_0)\|_E = 0$ for all $x \in E$ and $t_0 \in \mathbf{T}$ is true.

A translation invariant BK-space E has *weakly continuous translation* if

$$\lim_{t\to t_0} \phi\big(xe(t) - xe(t_0)\big) = 0 \qquad \text{for all } \ x \in E, \phi \in E' \text{ and } t_0 \in \mathbf{T}.$$

Remark 2.8. The continuity of xe for $x \in E$ in a particular point $t_0 \in \mathbf{T}$ implies trivially the continuity of xe on $\mathbf{T}$.

2.2 Vector-valued integration. In this subsection we refer to Gordon [3], a survey article, where essential criteria of Riemann integration are stated.

The Riemann integral and some of its properties. Let $[a, b]$ be a real finite interval and X a Banach space. Furthermore let a *partition* $\widetilde{\mathcal{P}}$ of $[a, b]$ be given with

$$\widetilde{\mathcal{P}} = \Big\{t_i \Big| 0 \le i \le N;\ a = t_0 < t_1 < \ldots < t_N = b\Big\}$$

and

$$|\widetilde{\mathcal{P}}| = \max\big\{t_i - t_{i-1} : 1 \le i \le N\big\}$$

its *norm*. If $\widetilde{\mathcal{P}}_1 \subseteq \widetilde{\mathcal{P}}_2$, then $\widetilde{\mathcal{P}}_2$ is called a *refinement* of $\widetilde{\mathcal{P}}_1$.

If we choose $s_i \in [t_{i-1}, t_i]$ for all $1 \le i \le N$, we obtain from $\widetilde{\mathcal{P}}$ a *tagged partition* $\mathcal{P}$ of $[a, b]$, i.e.

$$\mathcal{P} = \Big\{\big(s_i; [t_{i-1}; t_i]\big)\Big| 1 \le i \le N;\ a = t_0 < t_1 < \ldots < t_N = b;\ s_i \in [t_{i-1}; t_i]\Big\}.$$

For $f : [a, b] \to X$ we call

$$f(\mathcal{P}) = \sum_{i=1}^{N} f(s_i)(t_i - t_{i-1})$$

a *Riemann sum* of f.

Definition 2.9 (Riemann integral). The function $f : [a, b] \to X$ is called *Riemann integrable* (*R-integrable*) if

$$\exists z \in X : \forall \varepsilon > 0 : \exists \delta > 0 : \forall \mathcal{P} \text{ tagged with } |\tilde{\mathcal{P}}| < \delta : \quad \|f(\mathcal{P}) - z\|_X < \varepsilon$$

and $z = (R) - \int_a^b f(t)\,dt$ is called the *Riemann integral* of f.

Evidently an R-integrable function f must be bounded (cf. [3: p. 924]). The proofs of the following two theorems are obvious and omitted.

Theorem 2.10 (Cauchy criteria). *Let a function $f : [a,b] \to X$ be given. Then the following assertions are pairwise equivalent.*

1. *f is R-integrable on $[a,b]$.*

2. $\forall \varepsilon > 0 : \exists \delta > 0 : \forall \mathcal{P}_1, \mathcal{P}_2$ *tagged with* $|\widetilde{\mathcal{P}}_1|, |\widetilde{\mathcal{P}}_2| < \delta : \; \|f(\mathcal{P}_1) - f(\mathcal{P}_2)\|_X < \varepsilon$.

3. $\forall \varepsilon > 0 : \exists \widetilde{\mathcal{P}}_\varepsilon : \forall \widetilde{\mathcal{P}}_1, \widetilde{\mathcal{P}}_2$ *refinements of* $\widetilde{\mathcal{P}}_\varepsilon; \mathcal{P}_1, \mathcal{P}_2$ *tagged* : $\|f(\mathcal{P}_1) - f(\mathcal{P}_2)\|_X < \varepsilon$.

Theorem 2.11. *Let the function $f : [a,b] \to X$ be R-integrable on $[a,b]$. Then we have:*

1. *f is R-integrable on every subinterval of $[a,b]$.*

2. *If $\|f(t)\|_X \le M$ on $[a,b]$, then $\left\| \int_a^b f(t)\,dt \right\|_X \le M(b-a)$.*

3. *If $h : [a,b] \to X$ is continuous, then h is R-integrable.*

4. *If Y is a Banach space and $T : X \to Y$ a continuous linear operator, then*

$$\int_a^b T(f(t))\,dt = T\left(\int_a^b f(t)\,dt\right).$$

5. *If $g : [a;b] \to X$ is R-integrable on $[a,b]$, then $f+g$ is R-integrable on $[a,b]$ and*

$$\int_a^b (f+g)(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt.$$

Theorem 2.12. *Let $f : [a,b] \to X$ be an R-integrable and $g : [a,b] \to \mathbb{C}$ a continuous function. Then the product function $gf : [a,b] \to X$ is R-integrable.*

Proof. Let $\|f(t)\|_X \le M$ on $[a,b]$. We consider a sequence of step functions $(g_n)_{n\in\mathbb{N}}$ with $g_n : [a,b] \to \mathbb{C}$ converging uniformly on $[a,b]$ to g. Each function g_n can be represented as

$$g_n = \sum_{k=1}^{m_n} \alpha_k^{(n)} \chi_{[\tau_{k-1},\tau_k)} + \alpha_{m_n}^{(n)} \chi_{\{b\}}$$

where $a = \tau_0 < \tau_1 < \ldots < \tau_{m_n} = b$ and $\alpha_k^{(n)} \in \mathbb{C}$ for all $1 \le k \le m_n$. Obviously by the assertions 1 and 5 of Theorem 2.11 $g_n f$ is R-integrable for all $n \in \mathbb{N}$, and we get

$$\int_a^b g_n f(t)\,dt = \sum_{k=1}^{m_n} \alpha_k^{(n)} \int_{\tau_{k-1}}^{\tau_k} f(t)\,dt.$$

We choose $n \in \mathbb{N}$ such that

$$\sup_{t\in[a,b]} |g_n(t) - g(t)| < \frac{\varepsilon}{3M(b-a)}.$$

Let $\widetilde{\mathcal{P}}_\varepsilon$ be a partition such that for two arbitrarily chosen refinements $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ of $\widetilde{\mathcal{P}}_\varepsilon$ with $\mathcal{P}_1$ and $\mathcal{P}_2$ tagged

$$\|g_n f(\mathcal{P}_1) - g_n f(\mathcal{P}_2)\|_X < \frac{\varepsilon}{3}.$$

(cf. Theorem 2.10/3.). With

$$\|gf(\mathcal{P}_1) - gf(\mathcal{P}_2)\|_X \le \|gf(\mathcal{P}_1) - g_n f(\mathcal{P}_1)\|_X + \|g_n f(\mathcal{P}_1) - g_n f(\mathcal{P}_2)\|_X + \|g_n f(\mathcal{P}_2) - gf(\mathcal{P}_2)\|_X$$

we obtain

$$\|gf(\mathcal{P}_1) - g_n f(\mathcal{P}_1)\|_X < \frac{\varepsilon}{3M(b-a)} M(b-a) = \frac{\varepsilon}{3}$$

and analogously

$$\|gf(\mathcal{P}_2) - g_n f(\mathcal{P}_2)\|_X < \frac{\varepsilon}{3}.$$

Thus $\|gf(\mathcal{P}_1) - gf(\mathcal{P}_2)\|_X < \varepsilon$ for all tagged partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ with refinements $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ of $\widetilde{\mathcal{P}}_\varepsilon$, i.e. $\int_a^b gf(t)\,dt$ exists ∎

The Bochner integral and some of its properties. Beside the Riemann integral we will have a look at the Bochner integral. Later we shall see consequences of these two possibilities of vector-valued integration for BK-spaces.

Definition 2.13 (Bochner integral). Let $(\Sigma; \mathcal{A}; \mu)$ be a measure space. A function $f : \Sigma \to X$ is called *simple* if there exist $E_1, \ldots, E_n \in \mathcal{A}$ and $x_1, \ldots, x_n \in X$ such that $f = \sum_{i=1}^n x_i \chi_{E_i}$. For a simple function f the *Bochner integral* is defined by

$$(B)\text{-}\int_\Sigma f\,d\mu = \sum_{i=1}^n x_i \mu(E_i).$$

A function $f : \Sigma \to X$ is called *μ-measurable* if f is μ-almost everywhere the limit of a sequence $(f_n)_{n\in\mathbf{N}}$ of simple functions, i.e. if

$$\lim_{n\to\infty} \|f_n - f\|_X = 0 \qquad a.e.$$

A μ-measurable function $f : \Sigma \to X$ is called *Bochner integrable* if there exists a sequence $(f_n)_{n\in\mathbf{N}}$ of simple functions such that for the sequence of Lebesgue integrals $\left(\int_\Sigma \|f_n - f\|_X d\mu\right)_{n\in\mathbf{N}}$ we have

$$\lim_{n\to\infty} \int_\Sigma \|f_n - f\|_X d\mu = 0.$$

Then the *Bochner integral* is defined by

$$(B)\text{-}\int_\Sigma f d\mu = \lim_{n\to\infty} (B)\text{-} \int_\Sigma f_n d\mu.$$

In the following we consider only Lebesgue-measure spaces $(\Sigma; \mathcal{L}; \lambda)$. The next theorem is due to Diestel and Uhl [1: Theorem 9/p. 49].

Theorem 2.14. *Let the function f be Bochner integrable on $[a,b]$ with respect to the Lebesgue measure λ. Then*

$$\lim_{h\to 0} \frac{1}{h} \int_s^{s+h} \|f(t) - f(s)\|_X d\lambda(t) = 0$$

for almost all $s \in [a,b]$.

Now we obtain

Lemma 2.15. *Let $h > 0$ and $f : [a;b] \to X$ be a Bochner integrable function. Then*

$$\lim_{h\to 0} \frac{1}{h}\lambda\Big(\Big\{t \in [s;s+h] \Big| \|f(t) - f(s)\|_X < \varepsilon\Big\}\Big) = 1$$

for almost all $s \in [a,b]$ and all $\varepsilon > 0$.

Proof. For $\varepsilon > 0$ set

$$a_\varepsilon(s) = \lambda\Big(\Big\{t \in [s;s+h] \Big| \|f(t) - f(s)\|_X < \varepsilon\Big\}\Big)$$

$$b_\varepsilon(s) = \lambda\Big(\Big\{t \in [s;s+h] \Big| \|f(t) - f(s)\|_X \geq \varepsilon\Big\}\Big).$$

Then $a_\varepsilon(s) + b_\varepsilon(s) = h$. Hence by Theorem 2.14

$$\frac{1}{h}(\varepsilon b_\varepsilon(s)) \leq \frac{1}{h}\int_s^{s+h} \|f(t) - f(s)\|_X d\lambda(t) \longrightarrow 0 \quad \text{as } h \to 0$$

for almost all $s \in [a,b]$. Thus $\lim_{h\to 0} \frac{1}{h} a_\varepsilon(s) = 1$ ∎

3. Properties of BK-valued integrals

In the following we consider BK-subspaces of Ω which are induced by a translation-invariant BK-space E and certain properties.

Definition 3.1. For a translation invariant BK-space E we define

$$E_c = \Big\{x \in E \Big| xe \text{ is continuous on } \mathbf{T}\Big\}$$

$$E_\lambda = \Big\{x \in E \Big| xe \text{ is } \lambda\text{-measurable on } \mathbf{T}\Big\}$$

$$E_{Ri} = \Big\{x \in E \Big| (R)\text{-}\frac{1}{2\pi}\int_0^{2\pi} xe(t)\,dt \text{ exists}\Big\}.$$

Remarks 3.2. E_c and E_{Ri} are with respect to $\|\cdot\|_E$ translation-invariant BK-spaces. Obviously $E_c \subseteq E_{Ri}$ because every continuous function is R-integrable. Furthermore E_c is a homogeneous BK-space, and xe is Bochner integrable for all $x \in E_\lambda$ (cf. [1: Theorem 2/p. 45]).

It will be our aim to demonstrate relations between the spaces E_P for P equal one of the properties $\sigma K, \sigma B, S\sigma K, AD$, c, Ri or λ and to compare these relations with results in [2].

Theorem 3.3. *The function xe is continuous if and only if xe is λ-measurable, i.e. $E_c = E_\lambda$.*

Proof. Evidently the continuity of the function xe implies its measurability. Lemma 2.15 and the translation invariance of E imply

$$\lim_{h\to 0} \frac{1}{h}\lambda\Big(\Big\{\tau \in [t;t+h] \Big| \|xe(\tau) - xe(t)\|_E < \varepsilon\Big\}\Big) = 1$$

for all $t \in \mathbf{T}$ and all $\varepsilon > 0$. Supposing that xe is not continuous we get

$$\exists \delta > 0 : \exists (t_n)_{n\in\mathbb{N}} : t_n \downarrow t_0 : \forall t_n : \quad \left\| xe(t_n) - xe(t_0) \right\|_E \geq \delta \tag{1}$$

for $t_0 \in (0, 2\pi)$. Let $\varepsilon = \frac{\delta}{16}$ and

$$A_k(h) = \left(\left\{ \tau \in [t_k, t_k + h] \middle| \left\| xe(\tau) - xe(t_k) \right\|_E < \frac{\delta}{16} \right\} \right)$$

for all $k \in \mathbb{N}_0$. Choose h such that $\lambda(A_0(h)) \geq \frac{9}{10}h$. Then by the translation invariance we also have $\lambda(A_n(h)) \geq \frac{9}{10}h$ for all $n \in \mathbb{N}$. Let now be $n_0 \in \mathbb{N}$ such that $|t_n - t_0| < \frac{1}{10}h$ for all $n \geq n_0$. Then $A_n(h) \cap A_0(h) \neq \emptyset$ for $n \geq n_0$ because otherwise

$$\frac{11}{10}h > \lambda([t_0, t_n + h]) \geq \lambda(A_n(h) \cup A_0(h)) = \lambda(A_n(h)) + \lambda(A_0(h)) \geq \frac{18}{10}h.$$

With $\tau_n \in A_n(h) \cap A_0(h)$ we obtain

$$\left\| xe(t_n) - xe(t_0) \right\|_E \leq \left\| xe(t_n) - xe(\tau_n) \right\|_E + \left\| xe(\tau_n) - xe(t_0) \right\|_E < \frac{\delta}{8}$$

for all $n \geq n_0$. This is a contradiction to (1). Thus xe is continuous in t_0 und therefore on $\mathbf{T}$ ∎

Theorem 3.4. *Let $(K_n)_{n\in\mathbb{N}_0}$ with*

$$K_n(t) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ikt}$$

be the Fejér kernel. Then

$$\sigma_n x = \frac{1}{2\pi} \int_0^{2\pi} K_n(t) xe(-t)\, dt$$

for all $n \in \mathbb{N}_0$ and all $x \in E_{Ri}$.

Proof. Since $\frac{1}{2\pi}\int_0^{2\pi} xe(-t)\, dt$ exists for all $x \in E_{Ri}$, we obtain by Theorem 2.12 for all $n \in \mathbb{N}_0$ the existence of the integrals $\frac{1}{2\pi}\int_0^{2\pi} K_n(t)xe(-t)dt \in E$. Using assertion 4 of Theorem 2.11 we have for all $k \in \mathbb{Z}$

$$\begin{aligned} Pr_k\left(\frac{1}{2\pi}\int_0^{2\pi} K_n(t)xe(-t)\, dt\right) &= \frac{1}{2\pi}\int_0^{2\pi} Pr_k(K_n(t)xe(-t))\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_n(t) x_k e^{-ikt}\, dt \\ &= \begin{cases} \left(1 - \frac{|k|}{n+1}\right) x_k & \text{for } |k| \leq n \\ 0 & \text{for } |k| > n. \end{cases} \end{aligned}$$

Thus $\frac{1}{2\pi}\int_0^{2\pi} K_n(t)xe(-t)\, dt = \sigma_n x$ ∎

According to [2] we obtain the following Theorems 3.5 and 3.6.

Theorem 3.5. *Let E be a translation invariant BK-space. Then*

1. $E_{Ri} \subseteq E_{\sigma B}$

2. $E_{S\sigma K} = E_{AD} = E_c = E_{\sigma K}$.

Proof. We prove the two assertions in the following way.

1. Let $x \in E_{Ri}$. Then for all $n \in \mathbb{N}_0$

$$\|\sigma_n x\|_E = \left\| \frac{1}{2\pi} \int_0^{2\pi} K_n(t) xe(-t)\, dt \right\|_E \leq \frac{1}{2\pi} \int_0^{2\pi} K_n(t) \|x\|_E \, dt = \|x\|_E.$$

Thus $\sup_{n\in\mathbb{N}_0} \|\sigma_n x\|_E \leq \|x\|_E < \infty$, i.e. $E_{Ri} \subseteq E_{\sigma B}$.

2. We prove $E_{S\sigma K} \subseteq E_{AD} \subseteq E_c \subseteq E_{\sigma K} \subseteq E_{S\sigma K}$. Let $x \in E_{S\sigma K}$. Then by [6: Theorem 2/p. 120]

$$\forall \varepsilon > 0 : \exists \alpha_i^{(n)} \geq 0 \ (0 \leq i \leq n) : \quad \sum_{i=0}^{n} \alpha_i^{(n)} = 1 \ \text{ and } \ \left\| \sum_{i=0}^{n} \alpha_i^{(n)} \sigma_i x - x \right\|_E < \varepsilon,$$

i.e. $x \in E_{AD}$. Hence $E_{S\sigma K} \subseteq E_{AD}$. From $x \in E_{AD}$ one obtains that for all $\varepsilon > 0$ there exists an $z_x \in \Phi \cap E$ with $\|x - z_x\|_E < \frac{\varepsilon}{3}$. Obviously $z_x \in E_c$ and therefore

$$\forall t_0 \in \mathbf{T} : \exists U_\delta(t_0) : \forall t \in U_\delta(t_0) : \quad \|z_x e(t) - z_x e(t_0)\|_E < \frac{\varepsilon}{3}.$$

From this we get

$$\begin{aligned} \|xe(t) - xe(t_0)\|_E &\leq \|xe(t) - z_x e(t)\|_E \\ &\quad + \|z_x e(t) - z_x e(t_0)\|_E + \|z_x e(t_0) - xe(t_0)\|_E \\ &< \varepsilon, \end{aligned}$$

for all $t \in U_\delta(t_0)$, i.e. $x \in E_c$. Trivially $x \in E_{Ri}$ and $\|\sigma_n x - x\|_E \to 0$ (cf. [2: p. 246]), i.e. $x \in E_{\sigma K}$. Consequently $|\phi(\sigma_n x) - \phi(x)| \to 0$ for all $\phi \in E'$, i.e. $x \in E_{S\sigma K}$ ∎

Theorem 3.6. *Let E_{wc} be the space of those elements $x \in E$ for which xe has weakly continuous translation. Then $E_{wc} = E_{\sigma K} = E_c$. Thus in E weakly continuous translation is equivalent to continuous translation.*

Proof. We have $E_{\sigma K} = E_c$ and $E_c \subseteq E_{wc}$. We only have to prove $E_{wc} \subseteq E_c$. Let $x \in E_{wc}$. Then xe is weakly continuous and therefore weakly measurable. The range of xe, i.e. $\{xe(t) | t \in \mathbf{T}\}$, is a subset of the closure of

$$\left\{ \sum_{k=0}^{n} \beta_k xe(t_k) \,\middle|\, n \in \mathbb{N}_0 \ \text{ and } \ \beta_k \in \mathbb{Q}, t_k \in \mathbb{Q} \cap \mathbf{T} \ \text{ for all } \ 0 \leq k \leq n \right\}$$

(cf. [6: Theorem 2/p. 120]), where $\mathbb{Q}$ denotes the set of all rational numbers. Therefore $\{xe(t) | t \in \mathbf{T}\}$ is a separable set. Thus the conditions of the Pettis theorem [6: p. 131] are fulfilled, and we obtain $x \in E_\lambda = E_c$ (cf. Theorem 3.3), i.e. $E_{wc} = E_c = E_{\sigma K}$ ∎

Boettcher and Goes noticed that Theorem 3.6 can also be interpreted as an application of the theorem in [6: p. 233]. Furthermore we have to remark that in the proof of Proposition 4.3 in [2] there is not paid attention to the fact that the additional condition $E \subseteq E_{Ri}$ is used.

The following example illustrates inclusion relations between the considered BK-spaces.

Example 3.7. First we define the following spaces: Let $L^p(\mathbf{T})$ $(1 \leq p < \infty)$ be the space of all complex-valued Lebesgue measurable functions on $\mathbf{T}$ with $\int_0^{2\pi} |f|^p d\lambda < \infty$ and $\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^p d\lambda\right)^{1/p}$ (cf. Katznelson [5: p. 14]) and $L^\infty(\mathbf{T})$ the subspace of $L^1(\mathbf{T})$ of all essentially bounded functions endowed with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in\mathbf{T}} |f(t)|$. Furthermore let $M^d(\mathbf{T})$ be the space of all purely discontinuous (Borel) measures on $\mathbf{T}$ (cf. Hewitt and Ross [4: Definition 19.13/p. 269] and [5: p. 37]).

Now we consider the associated translation invariant BK-spaces $\widehat{L^p(\mathbf{T})}, \widehat{L^\infty(\mathbf{T})}$ and $\widehat{M^d(\mathbf{T})}$ of sequences of Fourier(-Stieltjes) coefficients, $l^\infty = \{u \in \Omega \mid \sup_{k\in\mathbb{Z}} |u_k| < \infty\}$ and $c_0 = \{v \in l^\infty \mid \lim_{|k|\to\infty} |v_k| = 0\}$. Then the translation invariant BK-space $(E; \|\cdot\|_E)$ shall be constructed by

$$E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$$

and endowed with the norm $\|\cdot\|_E = \sum_{k=1}^4 \|\cdot\|_{E_k}$. Let

$$E_1 = \left\{x \in \widehat{L^\infty(\mathbf{T})} \,\middle|\, x_{2k} = 0 \text{ for all } k \in \mathbb{Z}\setminus\{0\}\right\}.$$

Then

$$\forall x \in E_1 : \exists f \in L^\infty(\mathbf{T}) : \forall k \in \mathbb{Z} : \quad x_k = \hat{f}(k).$$

Let be $\|x\|_{E_1} = \|f\|_\infty$. Then E_1 endowed with this norm is a BK-space. With

$$E_2 = \left\{y \in \widehat{M^d(\mathbf{T})} \,\middle|\, y_{2k} = 0 \text{ for all } k \in \mathbb{Z}\setminus\{0\}\right\}$$

one obtains

$$\forall y \in E_2 : \exists \mu \in M^d(\mathbf{T}) : \forall k \in \mathbb{Z} : \quad y_k = \hat{\mu}(k).$$

Correspondingly let be $\|y\|_{E_2} = \operatorname{var}(\mu)$ (total variation of μ). According to [4: Theorem 19.20/p. 273] we have $M^d(\mathbf{T}) \cap L^1(\mathbf{T}) = \{0\}$, and with $L^\infty(\mathbf{T}) \subseteq L^1(\mathbf{T})$ we have $M^d(\mathbf{T}) \cap L^\infty(\mathbf{T}) = \{0\}$ respectively $E_1 \cap E_2 = \{0\}$. Furthermore let E_3 and E_4 be defined by

$$E_3 = \left\{u \in l^\infty \,\middle|\, u_{2k+1} = u_{4k} = 0 \text{ for all } k \in \mathbb{Z}\right\}$$

and

$$E_4 = \left\{v \in c_0 \,\middle|\, v_0 = v_{2k+1} = v_{4k+2} = 0 \text{ for all } k \in \mathbb{Z}\right\},$$

and let E_3 and E_4 be endowed with the usual norm $\|w\|_{E_3} = \|w\|_\infty = \sup_{k\in\mathbb{Z}} |w_k|$ and $\|w\|_{E_4} = \|w\|_\infty = \sup_{k\in\mathbb{Z}} |w_k|$, respectively.

As direct sum of BK-spaces E is evidently a BK-space. First we prove that in general E_c is a proper subset of E_{Ri}. Choose $f \in L^\infty(\mathbf{T})$ with

$$f(t) = \begin{cases} 0 & \text{for } t \in [0,\pi) \\ 1 & \text{for } t \in [\pi, 2\pi). \end{cases}$$

Then $x \in E_1$ with

$$x_k = \hat{f}(k) = \begin{cases} 0 & \text{for } 0 \neq k \text{ even} \\ \frac{1}{2} & \text{for } k = 0 \\ \frac{i}{\pi k} & \text{for } k \text{ odd} \end{cases}$$

and $xe(\tau) = \hat{f}_\tau$ (where $f_\tau(t) = f(t+\tau)$ for all $t \in \mathbf{T}$) such that

$$\left\| xe(\tau_1) - xe(\tau_2) \right\|_E = \left\| f_{\tau_1} - f_{\tau_2} \right\|_\infty = 1$$

for all τ_1 and τ_2 with $\tau_1 \neq \tau_2$. Thus $x \notin E_c$. Now we prove the existence of $\frac{1}{2\pi}\int_0^{2\pi} xe(t)\,dt$. For $\varepsilon > 0$ let

$$\widetilde{\mathcal{P}}_\varepsilon = \left\{ t_j^{(\varepsilon)} \middle| \, 0 = t_0^{(\varepsilon)} < t_1^{(\varepsilon)} < \ldots < t_{N_\varepsilon}^{(\varepsilon)} = 2\pi \right\}$$

be a partition with $|\widetilde{\mathcal{P}}_\varepsilon| < \frac{\varepsilon\pi}{4}$, and let $\widetilde{\mathcal{P}}_1$ and $\widetilde{\mathcal{P}}_2$ be two refinements of $\widetilde{\mathcal{P}}_\varepsilon$. Then $\widetilde{\mathcal{P}}_k = \bigcup_{j=1}^{N_\varepsilon} \widetilde{\mathcal{P}}_{kj}$ $(k \in \{1,2\})$ with

$$\widetilde{\mathcal{P}}_{kj} = \left\{ t_i^{(kj)} \middle| \, t_{j-1}^{(\varepsilon)} = t_0^{(kj)} < t_1^{(kj)} < \ldots < t_{N_{kj}}^{(kj)} = t_j^{(\varepsilon)} \right\} \qquad (1 \le j \le N_\varepsilon).$$

For corresponding tagged partitions $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_\varepsilon$ we get

$$\begin{aligned}
&\left\| \frac{1}{2\pi} xe(\mathcal{P}_k) - \frac{1}{2\pi} xe(\mathcal{P}_\varepsilon) \right\|_{E_1} \\
&= \frac{1}{2\pi} \left\| \sum_{j=1}^{N_\varepsilon} \sum_{i=1}^{N_{kj}} \left(xe(s_i^{(kj)}) - xe(s_j^{(\varepsilon)}) \right) \left(t_i^{(kj)} - t_{i-1}^{(kj)} \right) \right\|_{E_1} \\
&= \frac{1}{2\pi} \left\| \sum_{j=1}^{N_\varepsilon} \sum_{i=1}^{N_{kj}} \left(f_{s_i^{(kj)}} - f_{s_j^{(\varepsilon)}} \right) \left(t_i^{(kj)} - t_{i-1}^{(kj)} \right) \right\|_\infty \\
&\le \frac{1}{2\pi} \left\| \sum_{j=1}^{N_\varepsilon} \left(\chi_{[\pi - t_j^{(\varepsilon)}, \pi - t_{j-1}^{(\varepsilon)}]} + \chi_{[2\pi - t_j^{(\varepsilon)}, 2\pi - t_{j-1}^{(\varepsilon)}]} \right) \left(t_j^{(\varepsilon)} - t_{j-1}^{(\varepsilon)} \right) \right\|_\infty \\
&\le \frac{1}{2\pi} 4 |\widetilde{\mathcal{P}}_\varepsilon| \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Thus we have

$$\begin{aligned}&\left\|\frac{1}{2\pi}xe(\mathcal{P}_1)-\frac{1}{2\pi}xe(\mathcal{P}_2)\right\|_{E_1}\\&\qquad\le\left\|\frac{1}{2\pi}xe(\mathcal{P}_1)-\frac{1}{2\pi}xe(\mathcal{P}_\varepsilon)\right\|_{E_1}+\left\|\frac{1}{2\pi}xe(\mathcal{P}_2)-\frac{1}{2\pi}xe(\mathcal{P}_\varepsilon)\right\|_{E_1}\\&\qquad<\varepsilon.\end{aligned}$$

Therefore $\frac{1}{2\pi}xe$ is R-integrable by assertion 3 in Theorem 2.10 (cf. also [3: Example 12/p. 930]).

Furthermore there exist elements of E which do not have σB. These elements cannot be R-integrable. Let $\mu \in M^d(\mathbf{T})$ be such that μ is the difference of the two Dirac measures δ_0 and δ_π, i.e. $\mu = \delta_0 - \delta_\pi$. Then $y \in E_2$ with

$$y_k = \hat{\mu}(k) = \begin{cases} 0 & \text{for } k \text{ even} \\ \dfrac{1}{\pi} & \text{for } k \text{ odd.} \end{cases}$$

We have $\sigma_n y \in E_1$ for all $n \in \mathbb{N}_0$, but $(\|\sigma_n y\|_{E_1})_{n\in\mathbb{N}_0}$ is not bounded. (This is a modification of Boettcher's example.)

We know that $E_{Ri} \subseteq E_{\sigma B}$. In general this inclusion is proper. Let $u \in E_3$ be such that

$$u_k = \begin{cases} 1 & \text{for } k = 4p+2 \quad (p \in \mathbb{Z}) \\ 0 & \text{otherwise.} \end{cases}$$

Evidently $\|u\|_E = \|u\|_{E_3} = 1$ and $\sup_{n\in\mathbb{N}_0} \|\sigma_n u\|_E = 1$, i.e. $u \in E_{\sigma B}$. The function ue is not R-integrable, otherwise $u_0 = 0$ would imply

$$\frac{1}{2\pi}\int_0^{2\pi} ue(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} ue(-t)\,dt = \sigma_0 u = 0 \; (\in E)$$

or written as a limit

$$\lim_{k\to\infty}\left\|\frac{1}{2\pi}\sum_{j=1}^{k} ue\left(\frac{2\pi j}{k}\right)\frac{2\pi}{k}\right\|_E = 0.$$

But for any $k = 4p+2$ one obtains

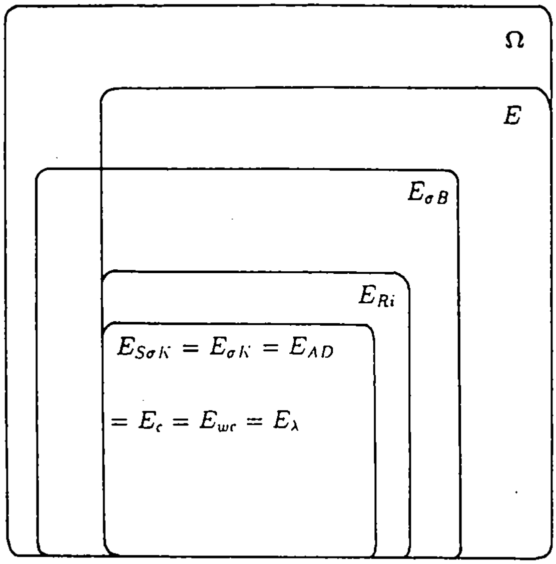
$$\left\|\frac{1}{2\pi}\sum_{j=1}^{k} ue\left(\frac{2\pi j}{k}\right)\frac{2\pi}{k}\right\|_E \ge \left|\frac{1}{2\pi}\sum_{j=1}^{k} u_k e^{ik\frac{2\pi j}{k}}\frac{2\pi}{k}\right| = 1.$$

The function ue cannot be R-integrable, i.e. E_{Ri} is a proper subset of $E_{\sigma B}$. Finally it is well known that there exist elements in $E_{\sigma B}$ which are not in E. Let $w = (w_k)_{k\in\mathbf{Z}}$ be such that

$$w_k = \begin{cases} 1 & \text{for } k = 4p \quad (p \neq 0) \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $w \notin E$, $\sigma_n w \in E_4$ for all $n \in \mathbb{N}_0$ and $\sup_{n \in \mathbb{N}_0} \|\sigma_n w\|_E < \infty$.

The following chart shows the relations between the spaces, which are generated by a translation-invariant BK-space E.



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