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An Interpolation Problem with Symmetry and Related Questions

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Abstract. An interpolation problem in the class of Carathéodory functions subject to a supplementary symmetry is solved; the problem originates with the theory of periodically correlated stochastic processes. The methods used are from state space theory.

Keywords: *Carathéodory–Fejér interpolation, state space methods*

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1. Introduction and motivations

In the present paper we study an interpolation problem with symmetry in the class of Carathéodory functions. The motivation originates with the theory of periodically correlated stochastic processes and is now explicited. Let us first recall that a discrete time complex-valued random process $(X_n)_{n\in\mathbb{Z}}$ is centered if $E(X_n) = 0$ for each n where E denotes mathematical expectation. It is said to be wide sense stationary if $E(X_{n+m}X_m^*)$ depends only on n. This last function is denoted by $(R(n))_{n\in\mathbb{Z}}$, and is called the autocovariance function of X. The function R is positive semidefinite (we will also use the term non-negative), in the sense that $\sum_{i,j=1}^{M} a_i a_j^* R(n_i - n_j) \geq 0$ for every choice of $M \in \mathbb{N}$, of $n_1, \dots n_M \in \mathbb{Z}$ and of $a_1, \dots, a_M \in \mathbb{C}$. This implies that $R(n)$ is characterized by a unique non-negative measure μ on the interval $[0, 2\pi]$ such that, for each n,

$$R(n) = \int_0^{2\pi} e^{int} d\mu(t).$$

Let us now consider a finite sequence $R(0), \dots, R(N)$ of complex numbers. It is well known that these numbers coincide with the first values of an autocovariance function if and only if the $(N+1) \times (N+1)$ Toeplitz matrix with ij entry $R(i-j)$ is non-negative (it is understood that $R(-n) = R(n)^*$ for $n = 0, \dots, N$). Moreover, the set of all autocovariance functions whose first coefficients coincide with $R(0), \dots, R(N)$ can be

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parameterized : this is the trigonometric moment problem, which is equivalent to the Carathéodory–Fejér interpolation problem. For more information on these problems we refer to [3], [8: p. 132] and [9: Chapter 6].

In 1961, Gladyšev [10] introduced the concept of a periodically correlated random processes. A centered stochastic process $(X_n)_{n\in\mathbb{Z}}$ is said to be *periodically correlated* if there exists a $T \in \mathbb{N}$ such that

$$E(X_{n+m+T}X^*_{m+T}) = E(X_{n+m}X^*_m)$$

for every choice of n and m in $\mathbb{Z}$. Therefore, X is periodically correlated if and only if the function $R(m,n) = E(X_{m+n}X^*_m)$ is periodic with respect to its first variable. In this case, it can be written as

$$R(m,n) = \sum_{k=0}^{T-1} R_k(n) \exp \frac{2i\pi km}{T},$$

and the function of two variables $R(m,n)$ is characterized by the T functions of one variable $R_0, \dots R_{T-1}$, sometimes called the *cyclic autocovariance functions* of X in the engineering literature. Gladyšev showed that the cyclic autocovariance functions are characterized as follows.

Theorem 1.1. *Let $R_0, \dots, R_{T-1}$ be T functions defined on $\mathbb{Z}$. They represent the cyclic autocovariance functions of a periodically correlated stochastic process X if and only if the $T \times T$ matrix-valued sequence $(S_n)_{n\in\mathbb{Z}}$ defined by*

$$(S_n)_{j,k} = R_{k-j}(n) \exp \frac{2ij\pi n}{T}$$

is positive semidefinite (it is understood that, for every $k \in \{0, \dots, T-1\}$ and every $q \in \mathbb{Z}$, $R_{k+qT}(n) = R_k(n)$).

The positivity of the sequence $(S_n)_{n\in\mathbb{Z}}$ is equivalent to the existence of (complex-valued) measures μ_k on the interval $[0, 2\pi]$ for which

$$R_k(n) = \int_0^{2\pi} e^{int} d\mu_k(t)$$

and for which the matrix-valued measure μ defined by

$$d\mu_{j,k}(t) = d\mu_{k-j}\left(t - \frac{2j\pi}{T}\right)$$

is non-negative (we also set $\mu_{k+qT} = \mu_k$ for $k = 0, \dots, T-1$ and $q \in \mathbb{Z}$).

However, the following problem has not yet been considered :

Let $R_k(n)$ $(k = 0, \ldots, T-1;\ n = 0, \ldots, N)$ be given finite sequences. Find necessary and sufficient conditions which guarantee the existence of a periodically correlated process X whose first $N+1$ values of the cyclic autocovariance functions coincide with $\{R_k(n) : k = 0, \ldots, T-1;\ n = 0, \ldots N\}$, and describe the set of all such cyclic autocovariance functions.

A stochastic process $(X_n)_{n\in\mathbb{Z}}$ is periodically correlated with period T if and only if the T-valued process $Y_k = (X_{kT+k_0}, \ldots, X_{kT+k_0+T-1})^\tau$ (where τ denotes transpose, and where k_0 is fixed, $0 \le k_0 \le T-1$) is stationary. Therefore, a tempting approach to solve the above problem is to investigate whether it is equivalent to a solvable matrix-valued tangential trigonometric moment problem. Unfortunately, this is not the case since the data of the problem may lead to an unsolvable interpolation problem. For example, even in the simplest case $T = 2$, the reader may check that if $N = 2L-1$, then the data $E(X_m X_n^*)$ for $m = 0, \ldots, T-1$ and $n = 0, \ldots, N$ are equivalent to the knowledge of the full matrices $E(Y_{k+l}Y_k^*)$ for $l = 0, L-1$ and of $(1,0)E(Y_{k+L}Y_k^*)\binom{0}{1}$. To the best of our knowledge, the associated tangential matrix-valued trigonometric moment problem cannot be solved using the existing methods. Hence, we propose to use a quite different approach.

In this paper, we restrict ourselves to the case $T = 2$. The general case can be treated similarly, but involves rather complicated notations. Using Theorem 1.1, the problem above can be shown to be equivalent to the existence of infinite extensions $(R_0(n))_{n>N}$ and $(R_1(n))_{n>N}$ of $(R_0(n))_{n=0,\ldots,N}$ and $(R_1(n))_{n=0,\ldots,N}$ for which the 2×2 matrix-valued sequence $(S(n))_{n\in\mathbb{Z}}$ defined by

$$S(n) = \begin{pmatrix} R_0(n) & R_1(n) \\ (-1)^n R_1(n) & (-1)^n R_0(n) \end{pmatrix}$$

is non-negative. Let J be the 2×2 signature matrix $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $S(n)$ satisfies $S(n) = (-1)^n J S(n) J$, this turns out to be equivalent to the following structured trigonometric moment problem.

Problem 1.2. *Find the set of all (2×2)-valued non-negative measures μ defined on $[0, 2\pi]$ for which*

$$S(n) = \int_0^{2\pi} e^{int}\, d\mu(t) \quad \textit{for } n = 0, N$$

$$d\mu(t) = J\, d\mu(t-\pi) J$$

Recall that a $\mathbb{C}^{p\times p}$-valued function Φ analytic in the unit disk $\mathbb{D}$ is called a Carathéodory function if $\operatorname{Re}\Phi(z) \ge 0$ for $z \in \mathbb{D}$. Problem 1.2 is equivalent to the following structured Carathéodory-Fejér problem.

Problem 1.3. *Find the set of all $\mathbb{C}^{2\times 2}$ Carathéodory functions $\Phi(z)$ for which*

$$\Phi(z) = S(0) + 2\sum_{n=1}^{N} S(n) z^n + z^N o(z)$$

$$\Phi(z) = J\Phi(-z)J$$

where J is the signature matrix introduced above.

The connection between these two problems follows directly from the Riesz-Herglotz theorem (see [6: Chapter 1/Theorem 4.5] and [8: p. 70]) which states that a $\mathbb{C}^{p\times p}$-valued function Φ is a Carathéodory function with $\Phi(0) \geq 0$ if and only if it can be written (in a unique way) as

$$\Phi(z) = \int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t) = \mu_0 + 2\sum_{n\geq 1} \mu_n z^n$$

where μ is a non-negative measure, and where $\mu_n = \int_0^{2\pi} e^{-int}\, d\mu(t)$.

Motivated by the above discussion, we study more general interpolation problems in the class of $\mathbb{C}^{p\times p}$-valued Carathéodory functions $\Phi(z)$ for which $\Phi(0)$ is non-negative and which satisfy the symmetry $\Phi(z) = J\Phi(-z)J$ where J is now an arbitrary signature matrix (i.e. $J = J^* = J^{-1}$). These problems and the outline of the paper are given in the next section.

2. The main results

In this section, we introduce the interpolation problems to be studied and outline the strategy used to solve them. For M a meromorphic $\mathbb{C}^{p\times p}$-valued function and J some preassigned signature matrix we set

$$\mathring{M}(z) = JM(-z)J. \tag{2.1}$$

If Θ is $\mathbb{C}^{2p\times 2p}$-valued, then $\mathring{\Theta}$ is defined by $(\mathring{\Theta}_{ij})$, where $\Theta = (\Theta_{ij})$ is the decomposition of Θ into four $\mathbb{C}^{p\times p}$-valued blocks.

Problem 2.1 (Tangential Nevanlinna-Pick problem). *Given n points $\omega_1, \ldots, \omega_n \in \mathbb{D}\setminus\{0\}$, given n pairs of vectors $(\eta_1, \xi_1), \ldots, (\eta_n, \xi_n)$ of $\mathbb{C}^p$, and given a non-negative matrix $\Phi_0 \in \mathbb{C}^{p\times p}$, find all Carathéodory functions Φ such that, for $i = 1, \ldots, n$,*

$$\Phi(\omega_i)^*\xi_i = \eta_i \tag{2.2}$$

$$\Phi(0) = \Phi_0 \tag{2.3}$$

and which satisfy the symmetry constraint

$$\Phi(z) = \mathring{\Phi}(z). \tag{2.4}$$

Note that since Φ_0 is non-negative, every solution $\Phi(z)$ of problem 2.1 is such that $\Phi(0) \geq 0$. If Φ is a solution of this interpolation problem, then the symmetry constraint (2.4) forces the supplementary interpolation conditions

$$\Phi(-\omega_i)^* J\xi_i = J\eta_i \tag{2.5}$$

for $i = 1, \ldots, n$.

In order to describe the set of all Carathéodory functions $\Phi(z)$ which satisfy $\Phi(0) = \Phi_0$ and conditions (2.2) and (2.5) (but which do not necessarily satisfy the symmetry condition (2.4)) we first need some notation. We suppose that the points $\omega_1, \ldots, \omega_n$ have been indexed in such a way that, for $i, j = 1, \ldots, s$, $\omega_i + \omega_j \neq 0$, and that $\omega_{s+1} = -\omega_{s+2}, \ldots, \omega_{n-1} = -\omega_n$ (obviously, $n - s$ is even). Let us introduce matrices A and C as follows: We let $A(\omega) = \operatorname{diag}(\omega, -\omega)$ and A is the block diagonal matrix whose first block entry is the $p \times p$ zero matrix, and whose next n diagonal blocks are $A(\omega_i)$ $(i = 1, \ldots, n)$. The matrix C is a $(2p \times (p + 2n))$-matrix whose first $2p \times p$ block is $\begin{pmatrix} -\Phi_0 \\ I_p \end{pmatrix}$, whose next s blocks are $(2p \times 2)$-matrices given by $\begin{pmatrix} -\eta_i & -J\eta_i \\ \xi_i & J\xi_i \end{pmatrix}$ and whose $\frac{n-s}{2}$ last blocks are $(2p \times 4)$-matrices given by

$$\begin{pmatrix} -J\eta_{s+2i} & -J\eta_{s+2i-1} & -\eta_{s+2i} & -\eta_{s+2i-1} \\ J\xi_{s+2i} & J\xi_{s+2i-1} & \xi_{s+2i} & \xi_{s+2i-1} \end{pmatrix},$$

for $i = 1, \ldots, \frac{n-s}{2}$.

Let $\mathbf{J}$ be the signature matrix $\mathbf{J} = \begin{pmatrix} 0 & -I_p \\ -I_p & 0 \end{pmatrix}$. As is well known (see [3, 8] and [9: Chapter 6]), Problem 2.1 (without the constraint (2.4)) has a solution if and only if the Stein equation

$$\mathbb{P} - A^*\mathbb{P}A = C^*\mathbf{J}C \tag{2.6}$$

has a non-negative solution. If $\mathbb{P} > 0$, the set of solutions $\Phi(z)$ of Problem 2.1 is given by

$$\Phi(z) = \Big(\Theta_{11}(z)\alpha(z) + \Theta_{12}(z)\beta(z)\Big)\Big(\Theta_{21}(z)\alpha(z) + \Theta_{22}(z)\beta(z)\Big)^{-1} \tag{2.7}$$

where Θ is a $\mathbb{C}^{2p \times 2p}$-valued rational function built from the interpolation data (and is given by formula (3.8)), $\mathbf{J}$-unitary on the unit circle and uniquely defined up to a constant $\mathbf{J}$-unitary matrix on the right. Here (α, β) runs through the set of Carathéodory pairs. We recall that a pair (α, β) of $\mathbb{C}^{p \times p}$-valued functions meromorphic in $\mathbb{D}$ is called a *Carathéodory pair* if the matrix $\begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix}$ has rank p for all z in its domain of analyticity, at the possible exception of the set of zeros of a non-identically vanishing function meromorphic in the open unit disk, and if moreover $\beta(z)^*\alpha(z) + \alpha(z)^*\beta(z) \geq 0$ for all z in the domain of analyticity of the pair.

This concept is intimately connected with that of a Carathéodory function: If $\beta(z)$ is of full rank in the unit disk, then $\alpha(z)\beta^{-1}(z)$ is a $\mathbb{C}^{p \times p}$-valued Carathéodory function, or equivalently, $(I_p, \alpha(z)\beta^{-1}(z))$ is a Carathéodory pair. Two Carathéodory pairs (α, β) and (γ, δ) are said to be equivalent if there is a $\mathbb{C}^{p \times p}$-valued function G meromorphic in $\mathbb{D}$ with non-identically vanishing determinant such that $(\alpha, \beta) = (\gamma, \delta)G$. When we say that two Carathéodory pairs are equal, we always mean that they belong to the same equivalence class.

Problem 2.2 (Tangential Carathéodory-Fejér problem). *Given $\xi, \eta_1, \ldots, \eta_n \in \mathbb{C}^p$ and given a non-negative matrix Φ_0 of $\mathbb{C}^{p \times p}$, find all Carathéodory functions $\Phi(z)$ such that $\Phi(0) = \Phi_0$ and*

$$\Phi(0)^{(i)*}\xi = 2i!\eta_i \qquad (i = 1, \ldots, n) \tag{2.8}$$

and which satisfy the symmetry constraint (2.4).

From condition (2.4) it follows that $\Phi^{(i)}(0) = (-1)^i J\Phi^{(i)}(0)J$. Therefore, a solution $\Phi(z)$ of Problem 2.2 also satisfies the interpolation conditions

$$\Phi(0)^{(i)*} J\xi = 2(-1)^i i! J\eta_i \qquad (i = 1, \dots, n). \tag{2.9}$$

If $J\xi = \xi$, these constraints force that $(-1)^i J\eta_i = \eta_i$ $(i = 1, \dots, n)$ (and similarly for $J\xi = -\xi$), in which cases equations (2.9) are equivalent to the original tangential constraints. Here, we focus on the case where $J\xi \neq \pm\xi$. As above, let us describe the set of solutions of this new interpolation problem (without the constraint (2.4)). Since the space $\mathcal{M}$ spanned by ξ and $J\xi$ is J-invariant, so is its orthogonal complement $\mathcal{M}^\perp$ in $\mathbb{C}^p$ (endowed with the usual Euclidean metric). Hence, we can find a basis $\{e_1, \dots, e_{p-2}\}$ of $\mathcal{M}^\perp$ which consists of eigenvectors of J :

$$Je_i = (-1)^{\varepsilon_i} e_i \qquad (\varepsilon_i = \pm 1,\, i = 1, p-2).$$

We rewrite the interpolation Problem 2.2 as follows :

$$\begin{aligned}
\Phi(0)^* e_i &= \Phi_0 e_i && (i = 1, \dots, p-2)\\
\Phi(0)^{i*} \xi &= 2i!\eta_i && (i = 0, \dots, n)\\
\Phi(0)^{i*} J\xi &= 2(-1)^i i! J\eta_i && (i = 0, \dots, n)
\end{aligned}$$

As above, the existence of solutions is subordinated to the non-negativity of the solution of the Stein equation (2.6) where now the matrices A and C are given as follows: A is a block diagonal matrix with three blocks. The first block is a $(p-2) \times (p-2)$ zero matrix and the next two blocks are Jordan cells of size $n+1$ with zero diagonal. C is a block row matrix $C = (C_1, C_2, C_3)$ with

$$\begin{aligned}
C_1 &= \begin{pmatrix} -\Phi_0^* e_1 & \cdots & -\Phi_0^* e_{p-2} \\ e_1 & \cdots & e_{p-2} \end{pmatrix}\\
C_2 &= \begin{pmatrix} -\Phi_0^* \xi & -2\eta_1 & \cdots & -2\eta_n \\ \xi & 0 & \cdots & 0 \end{pmatrix}\\
C_3 &= \begin{pmatrix} -\Phi_0^* J\xi & 2J\eta_1 & \cdots & 2(-1)^{n+1} J\eta_n \\ J\xi & 0 & \cdots & 0 \end{pmatrix}.
\end{aligned}$$

The new interpolation Problem 2.2 (without the symmetry constraint (2.4)) is solvable if and only if the solution $\mathbb{P}$ of the Stein equation (2.6) is non-negative. If $\mathbb{P}$ is positive definite, the set of all solutions $\Phi(z)$ of Problem 2.2 is given as before by formula (2.7), in which the function $\Theta(z)$ is a J-unitary matrix on the unit circle given by formula (3.8) (with the present choice of A and C).

Let us now indicate how we solve the two interpolation Problems 2.1 and 2.2 with the symmetry constraint (2.4). We will prove that

$$\mathring{\Theta}(z) = \Theta(z)U \tag{2.10}$$

for some J-unitary constant matrix U. Therefore, a solution $\Phi(z)$ of Problem 2.2 given by (2.7) for some Carathéodory pair $(\alpha(z), \beta(z))$ satisfies $\Phi(z) = \mathring{\Phi}(z)$ if and only if

$$\begin{pmatrix} \mathring{\alpha}(z) \\ \mathring{\beta}(z) \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha(z) \\ \beta(z) \end{pmatrix} \tag{2.11}$$

where we recall that (2.11) is an equality between equivalence classes of Carathéodory pairs. Therefore, the structured interpolation problems have solutions if and only if there exist Carathéodory pairs such that (2.11) holds; such pairs parameterize the set of all structured solutions.

That Carathéodory pairs which meet (2.11) always exist follows from the next lemma.

Lemma 2.3. *Let W be a constant* **J***-unitary matrix. There exist constant Carathéodory pairs (α, β) such that*

$$W \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \mathring{\alpha} \\ \mathring{\beta} \end{pmatrix}. \tag{2.12}$$

Proof. The matrix $\mathbf{W} = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right) W$ is still **J**-unitary, and (2.12) can be rewritten as

$$\mathbf{W} W \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha J \\ \beta J \end{pmatrix}. \tag{2.13}$$

Let us endow the space $\mathbb{C}^{2p}$ with the indefinite inner product $[x, y] = y^* \mathbf{J} x$. Then, $\mathbb{C}^{2p}$ is a Pontryagin space of index p. The matrix $\mathbf{W}$ is **J**-unitary with respect to this metric. Thus (see, e.g., [5: Theorem 7.1]), $\mathbf{W}$ admits a p-dimensional invariant positive subspace. Let $\{x_1, \ldots, x_p\}$ be a basis of such a subspace, and set

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (x_1 x_2 \cdots x_p).$$

Then (α, β) is a Carathéodory pair. Furthermore, from the invariance property, there exists an invertible matrix $G \in \mathbb{C}^{p \times p}$ such that

$$W(x_1\, x_2\, \cdots\, x_p) = (x_1\, x_2\, \cdots\, x_p) G$$

from which it follows that, as Carathéodory pairs, (2.13) holds, which ends the proof ∎

In the next section, we characterize the matrix-valued rational functions Θ which are unitary on the unit circle, and which satisfy condition (2.10) for some **J**-unitary constant matrix U. Finally, in Section 4 we check that the resolvent matrices $\Theta(z)$ associated to Problems 2.1 and 2.2 satisfy equation (2.10).

3. Some remarks on rational matrix-valued functions

We recall that $\mathbf{J} = \begin{pmatrix} 0 & -I_p \\ -I_p & 0 \end{pmatrix}$, and we put $\hat{J} = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$. Note that for any choice of the signature matrix $J \in \mathbb{C}^{p\times p}$ we have $\hat{J}\mathbf{J}\hat{J} = \mathbf{J}$. In this section, we characterize matrix-valued rational functions Θ which are $\mathbf{J}$-unitary on the unit circle $\mathbb{T}$ and satisfy $\mathring{\Theta}(z) = \Theta(z)U$ where $\mathring{\Theta}$ is defined by $(\mathring{\Theta}_{ij})$ with $\Theta = (\Theta_{ij})$ the decomposition of Θ into four $\mathbb{C}^{p\times p}$-valued blocks, and where U is a $\mathbf{J}$-unitary constant.

We begin with two well-known preliminary results, the proofs of which are given for completeness. We recall (see [4, 7]) that a pair $(C, A) \in \mathbb{C}^{p\times n} \times \mathbb{C}^{n\times n}$ is called observable if $\cap_{k=0}^{\infty} \operatorname{Ker} CA^k = \{0\}$.

Lemma 3.1. *Let $(C, A) \in \mathbb{C}^{2p\times n} \times \mathbb{C}^{n\times n}$ be an observable pair of matrices. Then there is at most one matrix $T \in \mathbb{C}^{n\times n}$ such that*

$$AT = -TA \tag{3.1}$$

$$-\hat{J}CT = C. \tag{3.2}$$

If such a matrix T exists and is invertible, it satisfies $T^2 = I_n$.

Proof. Let T_1 and T_2 be two matrices satisfying (3.1) and (3.2). Then the matrix $\Delta = T_1 - T_2$ satisfies $A\Delta = -\Delta A$ and $C\Delta = 0$. Therefore, for $n \in \mathbb{N}_0$, $CA^n\Delta = (-1)^n C\Delta A^n = 0$. Since (C, A) is an observable pair, it follows that $\Delta = 0$.

Suppose now that T satisfies (3.1) and (3.2) and is invertible. Then T^{-1} is readily seen to satisfy the same equations and by uniqueness, we have $T = T^{-1}$ ∎

The following lemma is valid for an arbitrary signature matrix. The specific form of $\mathbf{J}$ plays no role in the proof.

Lemma 3.2. *Let $(C, A) \in \mathbb{C}^{2p\times n} \times \mathbb{C}^{n\times n}$ be an observable pair of matrices and suppose that the Stein equation $\mathbb{P} - A^*\mathbb{P}A = C^*\mathbf{J}C$ has a solution $\mathbb{P}$ which is invertible and Hermitian. Let $\omega \in \mathbb{T}$ be such that $(\omega I_n - A)$ is invertible. Then the function*

$$\Theta(z, \omega) = \mathbf{J} - (1 - z\omega^*)C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n - \omega A)^{-*}C^* \tag{3.3}$$

is $\mathbf{J}$-unitary on the unit circle. For ω and ν of modulus one and in the resolvent set of A, [illegible]

$$\Theta(z, \nu) = \Theta(z, \omega)\Theta(\omega, \nu)$$

is fulfilled.

Proof. Since $\mathbb{P}$ is Hermitian, it is readily checked that

$$\frac{\mathbf{J} - \Theta(z, \omega)\mathbf{J}\Theta(\lambda, \omega)^*}{1 - z\lambda^*} = C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n - \lambda A)^{-*}C^*$$

(see [1: Theorem 3.4] and [9]). In particular,

$$\Theta(z, \omega)\mathbf{J}\Theta(\lambda, \omega)^* = \Theta(z, \nu)\mathbf{J}\Theta(\lambda, \nu)^*$$

and the functions $\Theta(z, \omega)$ and $\Theta(z, \nu)$ differ by a multiplicative $\mathbf{J}$-unitary constant U from the right: $\Theta(z, \nu) = \Theta(z, \omega)U$. Setting $z = \omega$ in the previous equation, one gets $U = \Theta(\omega, \nu)$ and hence the required result ∎

Combining these two lemmas, we have the following

Proposition 3.3. *Let $(C, A) \in \mathbb{C}^{2p\times n} \times \mathbb{C}^{n\times n}$ be an observable pair of matrices and suppose that the Stein equation $\mathbb{P} - A^*\mathbb{P}A = C^*\mathbf{J}C$ has a unique solution $\mathbb{P}$, which furthermore is invertible (and therefore Hermitian). Suppose that there is an invertible matrix T such that* (3.1) *and* (3.2) *hold, and let ω be of modulus one and in the resolvent set of A. Let $\Theta(z,\omega)$ be given by* (3.3). *Then,*

$$\mathring{\Theta}(z,\omega) = \Theta(z,-\omega) = \Theta(z,\omega)\Theta(\omega,-\omega).$$

Proof. Using (3.1) and (3.2), we note that $T^*\mathbb{P}T$ is also a solution of the Stein equation (2.6). By the presumed uniqueness of the solution, we have $\mathbb{P} = T^*\mathbb{P}T$. Furthermore, still using (3.1) and (3.2), we can write

$$\begin{aligned}
\mathring{\Theta}(z,\omega) &= \hat{J}\Theta(-z,\omega)\hat{J} \\
&= \hat{J}\mathbf{J}\hat{J} - (1 + z\omega^*)\hat{J}C(I_n + zA)^{-1}\mathbb{P}^{-1}(I_n - \omega A)^{-*}C^*\hat{J} \\
&= \mathbf{J} + (1 + z\omega^*)CT^{-1}(I_n + zA)^{-1}\mathbb{P}^{-1}(I_n - \omega A)^{-*}C^*\hat{J} \\
&= \mathbf{J} + (1 + z\omega^*)C(I_n - zA)^{-1}T^{-1}\mathbb{P}^{-1}(I_n - \omega A)^{-*}C^*\hat{J} \\
&= \mathbf{J} + (1 + z\omega^*)C(I_n - zA)^{-1}\mathbb{P}^{-1}T^*(I_n - \omega A)^{-*}C^*\hat{J} \\
&= \mathbf{J} + (1 + z\omega^*)C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n + \omega A)^{-*}T^*C^*\hat{J} \\
&= \mathbf{J} - (1 + z\omega^*)C(I_n - zA)^{-1}\mathbb{P}^{-1}(I_n + \omega A)^{-*}C^* \\
&= \Theta(z,-\omega)
\end{aligned}$$

which ends the proof of the proposition ∎

We recall that the Stein equation (2.7) has a unique solution if and only if the spectrum of A has no pairs of points symmetric with respect to the unit circle (see, e.g., [7: Appendix E]). Proposition 3.3 is what will be used in Section 4. We complete the analysis of this section by the following statement.

Theorem 3.4. *Let Θ be a matrix-valued rational function analytic at the origin and let $\mathbf{J}$ be unitary on the unit circle. Let $\Theta(z) = D + Cz(I_n - zA)^{-1}B$ be a minimal realization of Θ, and suppose that the Stein equation $\mathbb{P} - A^*\mathbb{P}A = C^*\mathbf{J}C$ has a unique solution. Then the following assertions are equivalent:*

(1) *There exists a $\mathbf{J}$-unitary constant U such that* (2.10) *holds.*

(2) *There exists an invertible matrix T such that* (3.1) *and* (3.2) *hold.*

Proof. Let us suppose that $\mathring{\Theta}(z) = \Theta(z)U$ for some J-unitary constant matrix U. Then, since

$$\mathring{\Theta}(z) = \hat{J}D\hat{J} - z\hat{J}C(I_n + zA)^{-1}B\hat{J}$$

is a minimal realization of $\mathring{\Theta}(z)$, there exists a (unique) similarity matrix T such that

$$\begin{pmatrix} A & BU \\ C & DU \end{pmatrix}\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} -A & B\hat{J} \\ -\hat{J}C & \hat{J}D\hat{J} \end{pmatrix}$$

and, in particular, $AT = -TA$ and $CT = -\hat{J}C$. To prove the converse we note that, using (3.1) and (3.2),

$$\mathring{\Theta}(z) = \hat{J}D\hat{J} + zC(I_n - zA)^{-1}T^{-1}B\hat{J}.$$

Thus, Θ and $\mathring{\Theta}$ are two matrix-valued functions which are **J**-unitary on the unit circle and have left pole structure (see [11]) defined by the same pair (C, A). Since the Stein equation (2.6) has a unique solution, Θ and $\mathring{\Theta}$ differ by a **J**-unitary constant on the right (see [1: Theorem 3.2]) ∎

4. Solution of the problems with condition $\Phi = \mathring{\Phi}$

In this section we show that (2.10) holds for Problems 2.1 and 2.2. We first suppose that the solution of the associated Stein equation (2.6) is strictly positive (the non-negative case will be discussed at the end of the section). First, we recall the expression of the associated functions $\Theta(z)$ introduced in Section 2. Let $\omega \in \mathbb{T}$ be such that $(\omega I_n - A)$ is invertible. Then, in both cases, a particular resolvent matrix is the function $z \to \Theta(z, \omega)$ given by (3.3). In view of Proposition 3.3, it suffices then to show that equations (3.1) and (3.2) have an invertible solution T to establish that Problems 2.1 and 2.2 are solvable.

In the case of the Nevanlinna-Pick problem, T is a block-diagonal matrix with blocks given as follows: The first block is equal to $-J$. The next s blocks are all equal to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The last $\frac{n-s}{2}$ blocks are equal to $\left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$.

In the Carathéodory-Fejér case, T is given by

$$T = -\begin{pmatrix} T_0 & 0 & 0 \\ 0 & 0 & T_1 \\ 0 & T_1 & 0 \end{pmatrix}$$

with $T_0 \in \mathbb{C}^{(p-2)\times(p-2)}$ and $T_1 \in \mathbb{C}^{n\times n}$ given by

$$T_0 = \operatorname{diag}\left((-1)^{\varepsilon_1}, \dots, (-1)^{\varepsilon_{p-2}}\right) \qquad \text{and} \qquad T_1 = \operatorname{diag}\left(1, -1, 1, \dots\right)$$

respectively. With these choices of T, equations (3.1) and (3.2) are readily seen to be met in both instances.

Let us finally briefly consider the case where $\mathbb{P}$ is only non-negative. It is possible to establish the existence of solutions of Problems 2.1 and 2.2 by approximation methods (see, e.g., [2: p. 17]). Thus, in the Carathéodory-Fejér case, we replace Φ_0 by $\Phi_{0,\varepsilon} = \Phi_0 + \varepsilon I_p$ and consider Problem 2.2 associated with the same ξ and η_i and this new $\Phi_{0,\varepsilon}$.

For each ε, it is easy to show that the matrix $\mathbb{P}(\varepsilon)$ associated to the modified Stein equation is positive definite. Therefore, the corresponding interpolation problem has a solution $\Phi(z,\varepsilon)$. The set of matrices $\Phi(0,\varepsilon)$ is bounded for $\varepsilon \leq 1$. Therefore, their associated non-negative measures $\mu(\varepsilon)$ for $\varepsilon \leq 1$ are also bounded, and by the Helly selection theorem (see [6: Chapter 1/Theorem 4.3] and [8: p. 68]), the family $\{\Phi(z,\varepsilon)\}_\varepsilon$ has a convergent subsequence which is a solution of Problem 2.2.

In the Nevalinna-Pick problem, we replace Φ_0 by $\Phi_{0,\varepsilon} = \Phi_0 + \varepsilon I_p$ and the vectors $(\eta_k)_{k=1,\ldots,n}$ by the vectors $(\eta_{k,\ldots,\varepsilon})_{k=1,n}$ defined by $\eta_{k,\varepsilon} = \eta_k + \varepsilon\xi_k$. It is easily seen that for ε small enough, the modified Stein equation has a positive definite solution $\mathbb{P}(\varepsilon)$. Therefore, the modified problem is solvable for ε small enough, and a solution of the initial problem can be constructed by using the Helly selection theorem.

References

[1] Alpay, D. and I. Gohberg: *Unitary rational matrix functions.* Oper. Theory: Adv. Appl. 33 (1988), 175 - 222.

[2] Alpay, D. and Ph. Loubaton: *The tangential trigonometric moment problem on an interval and related topics.* Oper. Theory: Adv. Appl. 64 (1993), 1 - 21.

[3] Ball, J., Gohberg, I. and L. Rodman: *Interpolation of Rational Matrix Functions.* Oper. Theory: Adv. Appl. 45 (1990), 1 - 605.

[4] Bart, H., Gohberg, I. and M. Kaashoek: *Minimal factorization of matrix and operator functions.* Oper. Theory: Adv. Appl. 1 (1979), 1 - 227.

[5] Bognár, J.: *Indefinite Inner Product Spaces.* Berlin: Springer - Verlag 1974.

[6] Brodskii, M. S.: *Triangular and Jordan Representations of Linear Operators* (Transl. Math. Monographs: Vol. 32). Providence (R. I.): Amer. Math. Soc. 1971.

[7] Chen, C.: *Introduction to Linear System Theory.* New York: Holt, Rinehart and Winston Inc. 1970.

[8] Dubovoi, V., Fritzsche B. and B. Kirstein: *Matricial Version of the Classical Schur Problem* (Teubner-Texte zur Mathematik: Vol. 129). Stuttgart- Leipzig: Teubner - Verlag 1992.

[9] Dym, H.: *J-Contractive Matrix Functions, Reproducing Kernel Spaces and Interpolation* (Reg. Conf. Ser.: Vol 71). Providence (R. I.): Amer. Math. Soc. 1989.

[10] Gladysěv, E.: *Periodically correlated random sequences.* Soviet Math. 12 (1961), 382 - 385.

[11] Gohberg, I, Kaashoek, M. A., Lerer, L. and L. Rodman: *Minimal divisors of rational matrix functions with prescribed zero and pole structures.* Oper. Theory: Adv. Appl. 12 (1984), 241 - 275.

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