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Contractive Intertwining Dilations of Quasi-Contractions

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Abstract. We prove that quasi-contractions in Kreĭn spaces always have contractive intertwining dilations. This result covers many of the known lifting of commutant theorems in both Hilbert and Kreĭn spaces. The approach is an adaptation of the angular operator method and uses the existence of invariant maximal non-negative subspaces for certain operators.

Keywords: *Contractive intertwining dilations, lifting of commutants, Kreĭn spaces, W-spaces, quasi-contractions, angular operators*

AMS subject classification: 47 A 20, 47 B 50, 47 A 15

1. Introduction

One possible approach of problems of interpolation of functions is by means of the so-called commutant lifting of contractions, or contractive intertwining dilations, in Hilbert spaces, initiated by D. Sarason in [19] and B. Sz.-Nagy and C. Foiaş in [20] (cf. [21]) where the problem of description of solutions was of most interest and culminated with the description of all solutions in terms of Schur parameters (see [3] and [5]).

An alternate direction was opened by the approach of J. A. Ball and J. W. Helton in [5] (see also the corrections in [6]) who put this into a different perspective by associating a certain indefinite inner product space in such a way that the set of contractive intertwining dilations corresponds to the set of maximal non-negative invariant subspaces of the shift operator. This approach was also applied to operators which are no longer contractions, but finite-rank perturbations of contractions (we call these operators quasi-contractions). For the existence of invariant maximal non-negative subspaces a generalized Beurling-Lax type theorem and an approximation pattern are used.

Motivated by similar problems in which the operators act in indefinite inner product spaces, variants of commutant lifting theorems were obtained for contractions by T. Constantinescu and the author in [8, 9] and by M. A. Dritschel in [11] (see also [12]) and for quasi-contractions by T. Constantinescu and the author in [9, 10]. These papers use the approach of four-block completion for contractions and quasi-contractions, respectively, and then the stepwise construction of intertwining dilations.

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The present paper arose from the author's attempt of using the approach of J. A. Ball and J. W. Helton when applied to quasi-contraction operators in Kreĭn spaces. The main obstructions appear because we no longer have a fundamental decomposition in which one can write explicitly the angular operators. In this paper we prove the existence of contractive intertwining dilations for quasi-contractions in Kreĭn spaces, in case the added spaces are positive definite (cf. Theorem 4.4). An impetus to write this note was given to us also by the recent paper of S. Treil and A. Volberg [23] who solve some generalized Nehari problems following the angular operator method combined with invariant maximal non-negative subspace theorems, via fixed point theorems.

Thus, following [5] and [23], we show that the correspondence between contractive intertwining dilations and invariant maximal non-negative subspaces in a certain fixed subspace holds in this case, too, and then we use a refinement of theorems of I. S. Iokhvidov [18] on invariant maximal non-negative subspaces, as in [23]. This approach avoids the use of Beurling-Lax type theorems as well as the approximation and some pseudo-regularity assumptions as in [5].

Investigations in this direction were recently carried over also by R. Arocena, T. Ya. Azizov, A. Dijksma and S. A. M. Marcantognini [2]. There are two major differences with respect to their results: first, in our proofs we do not use the known lifting of commutant theorems and hence we obtain these as particular cases from our Theorem 4.4, and second, we are able to prove the existence of intertwining dilations of quasi-contractions with no other additional technical assumption (compare with [2: Theorem 4.1]).

We thank T. Constantinescu for calling our attention and providing the reference [23], as well as for some useful discussions on the subject.

A few words about notation and background facts. For the elementary theory of indefinite inner product spaces and their linear operators that we use here, we refer to the monographs [1, 4, 7]. Also, we use the symbol $^\sharp$ to denote the involution with respect to the indefinite inner product spaces, and leave the * to its usual meaning. If $\mathcal{A}$ is a linear manifold in some Hilbert space, then cl$\mathcal{A}$ denotes its closure. If $\mathcal{L}$ is a *subspace* (that is, a closed linear manifold) of a Hilbert space $\mathcal{H}$, then $P_{\mathcal{L}}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$. The orthogonal direct sum $\oplus$ and the orthogonal subtraction $\ominus$ have their usual meaning in Hilbert spaces, and by no means in Kreĭn spaces. As a matter of fact, on our Kreĭn spaces we usually fix a Hilbert-space inner product and refer constantly to it. If $(\mathcal{K}, [\cdot,\cdot])$ is a Kreĭn space, then $[\perp]$ refers to the orthogonality with respect to the indefinite inner product. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are some Hilbert spaces, then $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the space of all bounded linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$.

Let $(\mathcal{X}, [\cdot,\cdot])$ be an indefinite inner product space. The *negative rank* $\kappa_-(\mathcal{X})$ is the maximal algebraic dimension over all negative subspaces $\mathcal{L}$ of $\mathcal{X}$, that is, $[x, x] < 0$ for all $x \in \mathcal{L} \setminus \{0\}$. If A is some Hermitian operator on $\mathcal{X}$, that is $[Ax, y] = [x, Ay]$ for all $x, y \in \mathcal{X}$, then the negative rank $\kappa_-(A)$ is, by definition, the negative rank of the indefinite inner product space $(\mathcal{X}, [\cdot,\cdot]_A)$, where $[x, y]_A = [Ax, y]$ for all $x, y \in \mathcal{X}$.

2. W-spaces, generalized angular operators, and invariant subspaces

We start this section with a universal property of Kreĭn spaces. First we introduce the following

Definition 2.1. An indefinite inner product space $(\mathcal{H}, [\cdot,\cdot])$ is called a *W-space* if there exists a positive definite inner product $\langle\cdot,\cdot\rangle$ with respect to which $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space and the inner product $[\cdot,\cdot]$ is jointly continuous.

As a consequence of the Open Mapping Principle the Hilbert space topology of a W-space is unique with the property that it makes the indefinite inner product $[\cdot,\cdot]$ jointly continuous (see, e.g., [7]). Thus, without any ambiguity, we call this topology the *strong topology* and all the continuity properties will be referred to it.

Proposition 2.2. *Let $(\mathcal{H}, [\cdot,\cdot])$ be a W-space. Then there exists a Kreĭn space $(\mathcal{K}, [\cdot,\cdot])$ which contains $(\mathcal{H}, [\cdot,\cdot])$ as a subspace.*

Proof. Let $\langle\cdot,\cdot\rangle$ be a positive definite inner product with respect to which $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is a Hilbert space and the inner product $[\cdot,\cdot]$ is jointly continuous. By the Riesz Representation Theorem, there exists an operator $G: \mathcal{H} \to \mathcal{H}$, called the *Gram operator*, such that

$$[x,y] = \langle Gx, y\rangle \qquad (x,y \in \mathcal{H}). \tag{2.1}$$

The operator G is bounded and selfadjoint with respect to the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. Modulo the multiplication of the positive inner product $\langle\cdot,\cdot\rangle$ with a positive constant we can assume that

$$|[x,y]| \leq \langle x,x\rangle^{\frac{1}{2}} \langle y,y\rangle^{\frac{1}{2}} \qquad (x,y \in \mathcal{H})$$

or, equivalently, that G is a contraction, i.e. $\langle Gx, x\rangle \leq \langle x,x\rangle$.

Consider now the defect operator $D_G = (I - G^2)^{\frac{1}{2}}$ and the defect space $\mathcal{D}_G = \mathrm{cl}\mathcal{R}(D_G)$ and let $\mathcal{K} = \mathcal{H} \oplus \mathcal{D}_G$ as Hilbert spaces. On the Hilbert space $\mathcal{K}$ we define the operator

$$J = \begin{bmatrix} G & D_G \\ D_G & -G|\mathcal{D}_G \end{bmatrix}. \tag{2.2}$$

It is a straightforward calculation to prove that J is a symmetry, that is $J^* = J = J^{-1}$. Thus, defining the indefinite inner product $[\cdot,\cdot]_{\mathcal{K}}$ by

$$[x,y]_{\mathcal{K}} = \langle Jx, y\rangle \qquad (x,y \in \mathcal{K})$$

this turns $\mathcal{K}$ into a Kreĭn space and J is a fundamental symmetry on it. $\mathcal{H}$ is naturally identified with the subspace $\mathcal{H} \oplus 0$ of $\mathcal{K}$ as Hilbert spaces. If $P_{\mathcal{H}}$ denotes the orthogonal projection of the Hilbert space $\mathcal{K}$ onto its subspace $\mathcal{H}$, then, taking into account (2.2), for all $x,y \in \mathcal{H}$ we have

$$[x,y]_{\mathcal{K}} = \langle Jx, y\rangle = \langle P_{\mathcal{H}} J P_{\mathcal{H}} x, y\rangle = \langle Gx, y\rangle = [x,y]$$

and the assertion is proved ∎

Let $\mathcal{H}$ be a W-space and let $G \in \mathcal{L}(\mathcal{H})$ be the Gram operator, that is G is selfadjoint with respect to the positive definite inner product $\langle\cdot,\cdot\rangle$ and (2.1) holds. Consider the Jordan decomposition $G = G_+ - G_-$ of G, and let $\mathcal{H}_+$ and $\mathcal{H}_-$ denote the spectral subspaces corresponding to the non-negative semiaxis $[0,+\infty)$ and the negative semiaxis $(-\infty,0)$, respectively. Clearly we have the decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.3}$$

and, if $x = x_+ + x_-$ and $y = y_+ + y_-$ are the corresponding representations of arbitrary vectors $x, y \in \mathcal{H}$, then

$$[x,y] = \langle G_+x_+, y_+\rangle - \langle G_-x_-, y_-\rangle.$$

In the following, we will use the notions of positivity, negativity, neutrality, etc. with respect to the indefinite inner product space $(\mathcal{H},[\cdot,\cdot])$ and fix the decomposition (2.3).

Let $\mathcal{M}$ be a non-negative subspace of $\mathcal{H}$, that is a closed linear manifold such that $[x,x] \geq 0$ for all $x \in \mathcal{M}$. With respect to the decomposition (2.3) this means

$$\langle G_+x_+, x_+\rangle \geq \langle G_-x_-, x_-\rangle \qquad (x = x_+ + x_- \in \mathcal{H}). \tag{2.4}$$

As in the case of Kreĭn spaces this enables us to introduce an angular operator. Let $P_\pm$ denote the projection of $\mathcal{H}_\pm$ with respect to the decomposition (2.3). Clearly $P_\pm$ are orthogonal projections in the Hilbert space $\mathcal{H}$, in particular their norms are less or equal to 1. Let us define an operator $K_\mathcal{M} : P_+\mathcal{M} \to \mathcal{H}_-$ by

$$K_\mathcal{M} : P_+x \mapsto P_-x \qquad (x \in \mathcal{M}). \tag{2.5}$$

By (2.4) and taking into account that G_- is injective on $\mathcal{H}_-$, this definition is correct and

$$\mathcal{M} = \{x + K_\mathcal{M}x \mid x \in P_+\mathcal{M}\}. \tag{2.6}$$

Since $\mathcal{M}$ is closed, this implies that the operator $K_\mathcal{M}$ is closed. The operator $K_\mathcal{M}$ is called the *generalized angular operator* of the non-negative subspace $\mathcal{M}$. Also note that in this general setting there is no reason to conclude that $P_+\mathcal{M}$ is a (closed) subspace. This anomaly is remedied if an extra condition is imposed.

Remark 2.3. The condition that in the Jordan decomposition of the Gram operator G the operator G_- has closed range is equivalent with the condition that the spectrum of G has a gap $(-\varepsilon,0)$. In particular, this shows that this condition is independent on which admissible positive definite inner product $\langle\cdot,\cdot\rangle$ we consider on the W-space $\mathcal{H}$ since, by changing it with another Gram operator, say B, we have $B = C^*GC$ for some boundedly invertible $C \in \mathcal{L}(\mathcal{H})$ (the inner products on the incoming Hilbert space $\mathcal{H}$ and the outgoing Hilbert space $\mathcal{H}$ are different) and this transformation preserves the topology of the spectrum.

A basic step in this approach is the possibility of handling generalized angular operators in W-spaces in a similar fashion as the angular operators in Kreĭn spaces. The next result comes from a paper of S. Treil [22] (cf. [23]). We give a detailed proof, for the reader's convenience (the article [22] was unaccesible to us) as well as for the reason that it will play a major role during the next section.

Proposition 2.4. *With the previous notation, assume that the operator G_- has closed range. Then:*

(1) *$\mathcal{M}$ is a non-negative subspace of $\mathcal{H}$ if and only if $P_+\mathcal{M}$ is closed, $K_\mathcal{M}$ is bounded and the inequality*

$$K_\mathcal{M}^* G_- K_\mathcal{M} \leq P_{P_+\mathcal{M}} G_+ | P_+\mathcal{M} \tag{2.7}$$

holds.

(2) *Let $\mathcal{M}$ and $\mathcal{N}$ be non-negative subspaces. Then $\mathcal{M} \subseteq \mathcal{N}$ if and only if $K_\mathcal{M} \subseteq K_\mathcal{N}$, that is $P_+\mathcal{M} \subseteq P_+\mathcal{N}$ and $K_\mathcal{M} x = K_\mathcal{N} x$ for all $x \in P_+\mathcal{M}$.*

(3) *For any non-negative subspace $\mathcal{M}$ there exists a maximal non-negative subspace $\widetilde{\mathcal{M}}$ such that $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$.*

(4) *A non-negative subspace $\mathcal{M}$ is maximal if and only if $P_+\mathcal{M} = \mathcal{H}_+$.*

Proof. (1). Let $\mathcal{M}$ be a non-negative subspace and consider $K_\mathcal{M}$ – its generalized angular operator as in (2.5). We first prove that the linear manifold $P_+\mathcal{M}$ is closed. To this end, let $(x_n)_{n\geq 1} \subset \mathcal{M}$ be a sequence of vectors such that $P_+x_n \to y$ as $n \to \infty$, for some vector $y \in \mathcal{H}_+$. Then

$$\|G_+^{\frac{1}{2}} P_+ x_n\|^2 = \langle G_+ P_+ x_n, P_+ x_n \rangle \geq \langle G_- P_- x_n, P_- x_n \rangle = \|G_-^{\frac{1}{2}} P_- x_n\|^2.$$

This implies that $(G_-^{\frac{1}{2}} P_- x_n)_{n\geq 1}$ is a Cauchy sequence and hence there exists a $z \in \mathcal{H}_-$ such that $G_-^{\frac{1}{2}} P_- x_n \to z$ as $n \to \infty$. Since G_- has closed range, it is invertible on $\mathcal{H}_-$ and the same is its square root $G_-^{\frac{1}{2}}$. Therefore, $P_- x_n \to G_-^{-\frac{1}{2}} z$ as $n \to \infty$ and hence

$$x_n = P_+ x_n + P_- x_n \;\to\; y + G_-^{-\frac{1}{2}} z = x \qquad \text{as } n \to \infty.$$

Since the projection P_+ is bounded, this implies that $P_+ x_n \to P_+ x = y$ as $n \to \infty$ and hence $y \in P_+\mathcal{M}$. Thus $P_+\mathcal{M}$ is closed.

Since the generalized angular operator $K_\mathcal{M}$ is closed and its domain $P_+\mathcal{M}$ is closed, by the Closed Graph Theorem we infer that the operator $K_\mathcal{M}$ is bounded. Further, the inequality (2.4) can be written as

$$\langle G_+ x_+, x_+ \rangle \geq \langle G_- K_\mathcal{M} x_+, K_\mathcal{M} x_+ \rangle \qquad (x_+ \in P_+\mathcal{M}) \tag{2.8}$$

and, since the operator $K_\mathcal{M}$ is bounded, this is equivalent to the inequality (2.7). This equivalence proves also the converse implication.

(2). If $\mathcal{M} \subseteq \mathcal{N}$ are non-negative subspaces, then $P_+\mathcal{M} \subseteq P_+\mathcal{N}$ and the inclusion $K_\mathcal{M} \subseteq K_\mathcal{N}$ comes directly from the definition of the generalized angular operator. Conversely, if $K_\mathcal{M} \subseteq K_\mathcal{N}$, then

$$\mathcal{M} = \{x + K_\mathcal{M} x \,|\, x \in P_+\mathcal{M}\} \subseteq \{x + K_\mathcal{N} \,|\, x \in P_+\mathcal{N}\} = \mathcal{N}.$$

(3) and (4). Let $\mathcal{M}$ be a non-negative subspace and $K_\mathcal{M}$ its angular operator. As in (1) $P_+\mathcal{M}$ is closed, $K_\mathcal{M}$ is bounded and the inequality (2.7) holds. We first remark that the inequality (2.7) is equivalent to

$$K_\mathcal{M} = G_-^{-\frac{1}{2}} C G_+^{\frac{1}{2}} P_{P_+\mathcal{M}} \tag{2.9}$$

for a uniquely determined contraction $C : \mathrm{cl}\mathcal{R}(G_+P_+|\mathcal{M}) \to \mathcal{H}_-$. We now remark that the inequality

$$\|CG_+^{\frac{1}{2}}P_{P_+\mathcal{M}}P_+h\| \le \|G_+^{\frac{1}{2}}P_+h\| \qquad (h \in \mathcal{M}) \tag{2.10}$$

holds. This inequality enables us to define the operator $\widetilde{C} : \mathrm{cl}\mathcal{R}(G_+) \to \mathcal{H}_-$ as follows:

$$\begin{cases} \widetilde{C}G_+^{\frac{1}{2}}P_+h = CG_+^{\frac{1}{2}}P_{P_+\mathcal{M}}h & \text{for } h \in \mathcal{M} \\ \widetilde{C}x = 0 & \text{for } x \in \mathcal{H}_+ \ominus (G_+^{\frac{1}{2}}P_+\mathcal{M}). \end{cases}$$

Again by (2.10), the operator $\widetilde{C}$ is contractive. Define the operator $\widetilde{K} : \mathcal{H}_+ \to \mathcal{H}_-$ by

$$\widetilde{K} = G_-^{-\frac{1}{2}}\widetilde{C}G_+^{\frac{1}{2}}.$$

Then the subspace $\widetilde{\mathcal{M}} = \{x + \widetilde{K}x \mid x \in \mathcal{H}_+\}$ is maximal non-negative. From the definition of the operator $\widetilde{K}$ it follows that $K_{\mathcal{M}} \subseteq \widetilde{K}$ and hence $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$ ∎

Corollary 2.5. *If the operator G_- has closed range, then the set*

$$\mathcal{X} = \left\{K_{\mathcal{M}} \,\middle|\, \mathcal{M} \text{ a maximal non-negative subspace}\right\}$$

is convex and compact with respect to the weak operator topology on $\mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$.

Proof. It follows from the proof of Proposition 2.4 (see (2.9)) that

$$\mathcal{X} = \left\{G_-^{-\frac{1}{2}}CG_+^{\frac{1}{2}} \,\middle|\, C : \mathrm{cl}\mathcal{R}(G_+) \to \mathcal{H}_-,\ \|C\| \le 1\right\}.$$

This shows that $\mathcal{X}$ is convex and, in view of the Alaoglu Theorem, it is also compact with respect to the weak operator topology on $\mathcal{L}(\mathcal{H}_+, \mathcal{H}_-)$ ∎

We conclude this section by recalling a theorem of existence of maximal invariant subspaces (cf. S. Treil and A. Volberg [23]) which is a generalization of a theorem of I. S. Iokhvidov [18]. Its proof is based on Proposition 2.4 and its Corollary 2.5 (cf. [22]) and a fixed point theorem of Ky Fan [13] and I. L. Glicksberg [17].

Theorem 2.6. *With the notation as before, assume that the operator G_- has closed range and let $V \in \mathcal{L}(\mathcal{H})$ be an operator subject to the following conditions:*

(i) *For any non-negative vector $x \in \mathcal{H}$ the vector Vx is also non-negative.*

(ii) *The operator $G_+^{1/2}P_{\mathcal{H}_+}VP_{\mathcal{H}_-}$ is compact.*

Then there exists a maximal non-negative subspace $\mathcal{M}$ in $\mathcal{H}$ which is invariant under the operator V.

3. Contractive dilations of quasi-contractions

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Kreĭn spaces and $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. We consider Kreĭn space extensions $\mathcal{G}_1 \supseteq \mathcal{H}_1$ and $\mathcal{G}_2 \supseteq \mathcal{H}_2$ such that $\mathcal{G}_1 \cap \mathcal{H}_1^{[\perp]}$ and $\mathcal{G}_2 \cap \mathcal{H}_2^{[\perp]}$ are positive definite, that is Hilbert spaces.

Definition 3.1. The set of *contractive dilations* of A, denoted by $\mathrm{CD}(A; \mathcal{G}_1, \mathcal{G}_2)$ consists of pairs $(A_\infty, \mathcal{E})$ subject to the following properties:

(1) $\mathcal{E}$ is a subspace of $\mathcal{G}_1$ of codimension at most $\kappa_-(I - A^\sharp A)$.

(2) $A_\infty \in \mathcal{B}(\mathcal{E}, \mathcal{G}_2)$ is a contraction, that is $[A_\infty x, A_\infty x] \leq [x, x]$ for all $x \in \mathcal{E}$.

(3) $P_{\mathcal{H}_2} A_\infty = A P_{\mathcal{H}_1} | \mathcal{E}$.

We now describe the basic construction, following closely the approach in [5]. Let

$$\mathcal{K} = \mathcal{G}_1 \oplus \mathcal{G}_2 \tag{3.1}$$

on which we consider the indefinite inner product $[\cdot, \cdot]$ defined by

$$[x_1 + x_2, y_1 + y_2] = [x_1, y_1] - [x_2, y_2] \qquad (x_1, y_1 \in \mathcal{G}_1,\ x_2, y_2 \in \mathcal{G}_2).$$

Then $(\mathcal{K}, [\cdot, \cdot])$ becomes a Kreĭn space. Fix fundamental symmetries J_1 and J_2 on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. On $\mathcal{K}$ we have the fixed fundamental symmetry J where, with respect to the decomposition

$$\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus (\mathcal{G}_1 \ominus \mathcal{H}_1) \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2),$$

the operator J has the representation

$$J = \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & -J_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}.$$

Let $\widetilde{A}$ denote the trivial extension of A onto the whole $\mathcal{G}_1$, that is $\widetilde{A} = A P_{\mathcal{H}_1}$. We consider the linear manifold $\mathcal{H}$ in $\mathcal{K}$ defined by

$$\mathcal{H} = \{x + \widetilde{A}x \mid x \in \mathcal{G}_1\} \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2). \tag{3.2}$$

Taking into account that $\mathcal{H}$ is the direct orthogonal sum of the graph of a bounded operator, hence a subspace, with another subspace, it follows that $\mathcal{H}$ itself is closed, that is, it is a subspace of $\mathcal{K}$. Endowing $\mathcal{H}$ with the indefinite inner product $[\cdot, \cdot]$, we thus obtain a W-space $(\mathcal{H}, [\cdot, \cdot])$ with the strong toplogy induced by the strong toplogy of $\mathcal{K}$. This implies that the Gram operator of $\mathcal{H}$ is $G = P_{\mathcal{H}} J | \mathcal{H}$.

In the following it will be useful to make use also of the decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus (\mathcal{G}_1 \ominus \mathcal{H}_1) \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2) \tag{3.3}$$

where $\mathcal{H}_0 = \{x + Ax \mid x \in \mathcal{H}_1\}$ is the graph of A. Letting $G_0 = P_{\mathcal{H}_0} J | \mathcal{H}_0$, with respect to the decomposition (3.3) we have

$$G = \begin{bmatrix} G_0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix}. \tag{3.4}$$

$\mathcal{H}_0$ is also a subspace of $\mathcal{K}$, and of $\mathcal{H}$ as well. Let us remark that $\kappa_-(\mathcal{H}_0) = \kappa_-(I - A^\sharp A)$. To see this, just note that for arbitrary $x \in \mathcal{H}_1$ we have

$$[x + Ax, x + Ax] = [x, x] - [Ax, Ax] = [(I - A^\sharp A)x, x].$$

Consider now the Jordan decomposition $G_0 = G_{0+} - G_{0-}$ of the Gram operator G_0 and let $\mathcal{H}_{0-} = \mathrm{cl}\mathcal{R}(G_{0-})$ and $\mathcal{H}_{0+} = \mathcal{H}_0 \ominus \mathcal{H}_{0-}$. Therefore

$$\operatorname{rank} G_{0-} = \dim \mathcal{H}_{0-} = \kappa_-(\mathcal{H}_0) = \kappa_-(I - A^\sharp A). \tag{3.5}$$

Further, letting

$$G_+ = G_{0+} \oplus I_{\mathcal{G}_1 \ominus \mathcal{H}_1} \qquad \text{and} \qquad G_- = G_{0-} \oplus I_{\mathcal{G}_2 \ominus \mathcal{H}_2}$$

from (3.4) it follows that $G = G_+ - G_-$ is the Jordan decomposition of G and $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is the corresponding spectral decomposition, where

$$\mathcal{H}_+ = \mathcal{H}_{0+} \oplus (\mathcal{G}_1 \ominus \mathcal{H}_1) \qquad \text{and} \qquad \mathcal{H}_- = \mathcal{H}_{0-} \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2). \tag{3.6}$$

Remark 3.2. It is interesting to note that we can compute explicitly the operator G_0, and hence the operator G. More precisely, taking into acount the form of the orthogonal projection onto the graph of an operator (see, e.g., [16]), with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_2$ we have

$$P_{\mathcal{H}_0} = \begin{bmatrix} (I + A^*A)^{-1} & (I + A^*A)^{-1}A^* \\ (I + AA^*)^{-1}A & -(I + AA^*)^{-1}AA^* \end{bmatrix},$$

and hence

$$G_0 = P_{\mathcal{H}_0} J P_{\mathcal{H}_0} = B \begin{bmatrix} J_1 - A^*J_2A & (J_1 + A^*J_2A)A^* \\ A(J_1 + A^*J_2A) & A(J_1 - A^*J_2A)A^* \end{bmatrix} B \tag{3.7}$$

where

$$B = \begin{bmatrix} (I + A^*A)^{-1} & 0 \\ 0 & (I + AA^*)^{-1} \end{bmatrix}.$$

But, it seems that this does not help us too much, unless we can calculate the Jordan decomposition of a selfadjoint operator as in the left side of (3.7).

The basic observation in [15] is that the $\mathcal{H}$-maximal non-negative subspaces correspond, via the angular operator method, to contractive dilations of A. We next adapt this idea to our setting, even though due to the indefiniteness of the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we cannot simply use the angular operator method from the Kreĭn space theory. On the other hand, we are able to perform this only in case A is a *quasi-contraction*, that is $\kappa_-(I - A^\sharp A) < \infty$. Note that in this case the operator G_- has closed range and the results in Lemma 2.4 apply.

Lemma 3.3. *Assume that* $\kappa_-(I - A^\sharp A) = \kappa < \infty$ *and let* $\mathcal{L}$ *be an* $\mathcal{H}$*-maximal non-negative subspace. Then*

$$A_\infty : \; \mathcal{E} = P_{\mathcal{G}_1}\mathcal{L} \ni P_{\mathcal{G}_1} f \mapsto P_{\mathcal{G}_2} f \qquad (f \in \mathcal{L}) \tag{3.8}$$

is correctly defined, $\operatorname{codim}_{\mathcal{G}_1}\mathcal{E} = \kappa$, *and the pair* $(A_\infty, \mathcal{E})$ *is in* $\mathrm{CD}(A; \mathcal{G}_1, \mathcal{G}_2)$.

Proof. Let $\mathcal{L}$ be an $\mathcal{H}$-maximal non-negative subspace. We first note that, since $\mathcal{L} \subseteq \mathcal{H}$ and (3.3), we have

$$AP_{\mathcal{H}_1} f = P_{\mathcal{H}_2} f \qquad (f \in \mathcal{L}). \tag{3.9}$$

We first prove that $P_{\mathcal{G}_1}|\mathcal{L}$ is injective. To this end, let $f \in \mathcal{L}$ be such that $P_{\mathcal{G}_1} f = 0$. Since $\mathcal{H}_1 \subseteq \mathcal{G}_1$ we have $P_{\mathcal{H}_1} f = P_{\mathcal{H}_1} P_{\mathcal{G}_1} f = 0$ and hence, by (3.9), it follows that $P_{\mathcal{H}_2} f = 0$ as well. Therefore, $f \in \mathcal{K} \ominus (\mathcal{G}_1 \oplus \mathcal{H}_2) = \mathcal{G}_2 \ominus \mathcal{H}_2$. This implies $f = 0$ since f is a non-negative vector and the subspace $\mathcal{G}_2 \ominus \mathcal{H}_2$ is negative.

As a consequence of the injectivity of the operator $P_{\mathcal{G}_1}|\mathcal{L}$, we get that the operator A_∞ as in (3.8) is correctly defined and

$$\mathcal{L} = \{f + A_\infty f \mid f \in \mathcal{E}\}. \tag{3.10}$$

For the moment all we can say is that A_∞ is a closed operator and $\mathcal{E}$ is a linear manifold in $\mathcal{G}_1$. Since $\mathcal{L}$ is non-negative we have

$$[A_\infty x, A_\infty x] \le [x, x] \qquad (x \in \mathcal{E}),$$

that is A_∞ is contractive. In addition, for arbitrary $x \in \mathcal{E}$ we have

$$P_{\mathcal{H}_2} A_\infty x = P_{\mathcal{H}_2}(x + A_\infty x) = AP_{\mathcal{H}_1}(x + A_\infty x) = AP_{\mathcal{H}_1} x$$

and hence $P_{\mathcal{H}_2} A_\infty = AP_{\mathcal{H}_1}|\mathcal{E}$ holds.

We now prove that the codimension of the linear manifold $\mathcal{E}$ in $\mathcal{G}_1$ is exacly κ. Since $\mathcal{L}$ is a $\mathcal{H}$-maximal non-negative subspace, by Proposition 2.4 there exists the generalized angular operator $K_{\mathcal{L}} : \mathcal{H}_+ \to \mathcal{H}_-$ such that

$$\mathcal{L} = \{x + K_{\mathcal{L}} x \mid x \in \mathcal{H}_+\}.$$

Taking into account (3.6) we get

$$P_{\mathcal{G}_1}\mathcal{L} + \mathcal{H}_{0-} \supseteq \mathcal{H}_1 \oplus (\mathcal{G}_1 \ominus \mathcal{H}_1) = \mathcal{G}_1. \tag{3.11}$$

We claim now that the operator $P_{\mathcal{G}_1}$ is injective also when restricted to the subspace $\mathcal{L} + \mathcal{H}_{0-}$. Indeed, let $l \in \mathcal{L}$ and $h \in \mathcal{H}_{0-}$ be such that $P_{\mathcal{G}_1}(l + h) = 0$. Since $h \in \mathcal{H}_1 \oplus \mathcal{H}_2$ this implies $P_{\mathcal{G}_1} h = P_{\mathcal{H}_1} h$ and hence $P_{\mathcal{G}_1} l = -P_{\mathcal{H}_1} h$, therefore $P_{\mathcal{G}_1 \ominus \mathcal{H}_1} l = 0$. Thus, $l \in \mathcal{H}_1 \oplus (\mathcal{G}_2 \ominus \mathcal{H}_2)$. Taking into account (3.3) it follows that $l = (x + Ax) + g_2$ for some $g_2 \in \mathcal{G}_2 \ominus \mathcal{H}_2$ and $x = -P_{\mathcal{H}_1} h$. But, by the construction of the space $\mathcal{H}_{0-}$ we have $h = x + Ax$ where $x = -P_{\mathcal{H}_1} h$, and hence $l = -h + g_2$. Now remark that the subspaces

$\mathcal{H}_{0-}$ and $\mathcal{G}_2 \ominus \mathcal{H}_2$ are negative and orthogonal with respect to the inner product $[\cdot,\cdot]$ and hence the vector $l = -h + g_2$ is either negative or null. But l is non-negative, as any other vector in $\mathcal{L}$, and hence $l = 0$ and $h = 0$. The claim is proved.

Since $\mathcal{L}$ is a non-negative subspace and $\mathcal{H}_{0-}$ a negative subspace it follows that the sum $\mathcal{L} + \mathcal{H}_{0-}$ is direct and, taking into account that $P_{\mathcal{G}_1}$ is injective on $\mathcal{L} \dot{+} \mathcal{H}_{0-}$, from (3.11) we get

$$\mathcal{E} \dot{+} P_{\mathcal{G}_1}\mathcal{H}_{0-} = \mathcal{G}_1$$

which proves that the codimension of $\mathcal{E}$ in $\mathcal{G}_1$ is exactly $\dim \mathcal{H}_{0-} = \kappa$.

We now prove that $\mathcal{E}$ is closed. First consider the subspace $\mathcal{H}'_+ = \ker(P_{\mathcal{H}_{0-}} K_{\mathcal{L}}) \subseteq \mathcal{H}_+$ and remark that $\operatorname{codim}_{\mathcal{H}_+}\mathcal{H}'_+ \leq \dim \mathcal{H}_{0-} = \kappa$. Define the subspace $\mathcal{L}'$ of $\mathcal{L}$ by

$$\mathcal{L}' = \{x + K_{\mathcal{L}}x \mid x \in \mathcal{H}'_+\}$$

and note that since $K_{\mathcal{L}}\mathcal{H}'_1 \subseteq \mathcal{G}_2 \ominus \mathcal{H}_2$ it follows $P_{\mathcal{G}_1}\mathcal{L}' = P_{\mathcal{G}_1}\mathcal{H}'_+$. Since $\mathcal{H}'_+$ is a subspace of $\mathcal{H}_+$ it follows that

$$\mathcal{H}'_+ = \{f + \widetilde{A}f \mid f \in P_{\mathcal{G}_1}\mathcal{H}'_+\}.$$

Since $\mathcal{H}'_+$ is closed and $\widetilde{A}$ is bounded it follows that $P_{\mathcal{G}_1}\mathcal{H}'_+ = P_{\mathcal{G}_1}\mathcal{L}'$ is closed. Taking into account that $\operatorname{codim}_{\mathcal{E}} P_{\mathcal{G}_1}\mathcal{L}' \leq \kappa < \infty$ it follows that the linear manifold $\mathcal{E}$ is closed, too.

Finally, since the operator A_∞ is closed and its domain $\mathcal{E}$ is a subspace, the Closed Graph Theorem implies that A_∞ is bounded ∎

Lemma 3.4. *Assume that* $\kappa_-(I - A^\sharp A) = \kappa < \infty$. *If* $(A_\infty; \mathcal{E})$ *is in* $\mathrm{CD}(A; \mathcal{G}_1, \mathcal{G}_2)$, *then*

$$\mathcal{L} = \{x + A_\infty x \mid x \in \mathcal{E}\} \tag{3.12}$$

is an $\mathcal{H}$*-maximal non-negative subspace and* $\operatorname{codim}_{\mathcal{G}_1}\mathcal{E} = \kappa$.

Moreover, this is a bijective correspondence between the class $\mathrm{CD}(A; \mathcal{G}_1, \mathcal{G}_2)$ *and the class of* $\mathcal{H}$*-maximal non-negative subspaces. This correspondence is inverse to that in Lemma* 3.3.

Proof. Since A_∞ is contractive we readily check that $\mathcal{L}$ is non-negative. To prove that $\mathcal{L}$ is a subspace of $\mathcal{H}$, let $f = x + A_\infty x$ for some vector $x \in \mathcal{E}$. Then

$$AP_{\mathcal{H}_1}f = AP_{\mathcal{H}_1}(x + A_\infty x) = AP_{\mathcal{H}_1}x = P_{\mathcal{H}_2}A_\infty x = P_{\mathcal{H}_2}(x + A_\infty x) = P_{\mathcal{H}_2}f.$$

In view of the definition of $\mathcal{H}$ this implies that $\mathcal{L} \subseteq \mathcal{H}$.

From Lemma 2.4 it follows that there exists an $\mathcal{H}$-maximal non-negative subspace $\widetilde{\mathcal{L}} \supseteq \mathcal{L}$. Then, by Lemma 3.3 we get that $P_{\mathcal{G}_1}\widetilde{\mathcal{L}}$ is a subspace of $\mathcal{G}_1$ of codimension κ. Since $P_{\mathcal{G}_1}\widetilde{\mathcal{L}} \supseteq P_{\mathcal{G}_1}\mathcal{L} = \mathcal{E}$ is a subspace of codimension in $\mathcal{G}_1$ at most κ it follows that $P_{\mathcal{G}_1}\widetilde{\mathcal{L}} = \mathcal{E}$, in particular $\mathcal{L} = \widetilde{\mathcal{L}}$ is an $\mathcal{H}$-maximal non-negative subspace and $\operatorname{codim}_{\mathcal{G}_1}\mathcal{E} = \kappa$ ∎

4. The main result

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Kreĭn spaces and consider two operators $T_i \in \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$. We assume that there exists pairs $(V_i; \mathcal{G}_i)$ $(i = 1, 2)$, subject to the following conditions:

(a$_i$) $\mathcal{G}_i$ is a Kreĭn space extension of $\mathcal{H}_i$ such that $\mathcal{G}_i \cap \mathcal{H}_i^{[\perp]}$ is positive definite, that is a Hilbert space.

(b$_i$) $V_i \in \mathcal{B}(\mathcal{G}_i)$ is a dilation of T_i, that is $P_{\mathcal{H}_i} V_i = T_i P_{\mathcal{H}_i}$.

(c$_1$) V_1 is expansive, that is $[V_1 x, V_1 x] \geq [x, x]$ for all $x \in \mathcal{G}_1$.

(c$_2$) V_2 is contractive, that is $[V_2 y, V_2 y] \leq [y, y]$ for all $y \in \mathcal{G}_2$.

As a consequence of assumption (b$_i$) it follows that $\mathcal{G}_i \cap \mathcal{H}_i^{[\perp]}$ is invariant under the operator V_i $(i = 1, 2)$.

Definition 4.1. Let $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator intertwining the operators T_1 and T_2, that is $AT_1 = T_2 A$. The set of *contractive intertwining dilations* of A, denoted by $\mathrm{CID}(A; T_1, T_2)$, consists of pairs $(A_\infty, \mathcal{E})$ subject to the following properties:

(1) $\mathcal{E}$ is a subspace of $\mathcal{G}_1$ of codimension at most $\kappa_-(I - A^\sharp A)$, invariant under V_1.

(2) $A_\infty \in \mathcal{B}(\mathcal{E}, \mathcal{G}_2)$ is a contraction, that is $[A_\infty x, A_\infty x] \leq [x, x]$ for all $x \in \mathcal{E}$.

(3) $P_{\mathcal{H}_2} A_\infty = A P_{\mathcal{H}_1} | \mathcal{E}$.

(4) $A_\infty V_1 | \mathcal{E} = V_2 A_\infty$.

We consider the W-space $\mathcal{H}$ as in (3.2). With respect to the decomposition (3.1) of the Kreĭn space $\mathcal{K}$ we define

$$V = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}. \tag{4.1}$$

Lemma 4.2. *The subspace $\mathcal{H}$ is invariant under V.*

Proof. Let h be an arbitrary vector in $\mathcal{H}$. By (3.2) we have $h = x + AP_{\mathcal{H}_1} x + y$ for some vectors $x \in \mathcal{G}_1$ and $y \in \mathcal{G}_2 \ominus \mathcal{H}_2$. Then, by the definition of V, we have

$$\begin{aligned} V(x + AP_{\mathcal{H}_1} x + y) &= V_1 x + V_2 A P_{\mathcal{H}_1} x + V_2 y \\ &= (P_{\mathcal{H}_1} V_1 x + P_{\mathcal{H}_2} V_2 A P_{\mathcal{H}_1} x) + P_{\mathcal{G}_1 \ominus \mathcal{H}_1} V_1 x \\ &\quad + (P_{\mathcal{G}_2 \ominus \mathcal{H}_2} V_2 A P_{\mathcal{H}_1} x + V_2 y). \end{aligned} \tag{4.2}$$

We take into account that V_i are dilations of T_i and that $AT_1 = T_2 A$, and get

$$P_{\mathcal{H}_2} V_2 A P_{\mathcal{H}_1} x = T_2 A P_{\mathcal{H}_1} x = A T_1 P_{\mathcal{H}_1} x = A P_{\mathcal{H}_1} V x,$$

therefore

$$P_{\mathcal{H}_1} V_1 x + P_{\mathcal{H}_2} V_2 A P_{\mathcal{H}_1} x = P_{\mathcal{H}_1} V_1 x + A P_{\mathcal{H}_1} V_1 x \in \mathcal{H}_0.$$

Since $\mathcal{G}_2 \ominus \mathcal{H}_2$ is invariant under V_2, we use (4.2) to conclude that $Vz \in \mathcal{H}$ ∎

By means of Lemma 4.2 and in view of the results in the previous section, we conclude that the invariant subspace representation of the set $\mathrm{CID}(A;T_1;T_2)$ holds in this setting, similarly as in [5].

Lemma 4.3. *Let $(A_\infty,\mathcal{E})$ be an element in $\mathrm{CID}(A;T_1;T_2)$. Then the subspace $\mathcal{L}$ defined as in (3.12) is $\mathcal{H}$-maximal non-negative and invariant under V. This correspondence is bijective between the set $\mathrm{CID}(A;T_1;T_2)$ and the set of all $\mathcal{H}$-maximal non-negative subspaces invariant under V.*

Proof. Most of the statements are already proved in Lemmas 3.3 and 3.4. Only the invariance property must be checked. Let $(A_\infty;\mathcal{E})$ be in $\mathrm{CID}(A;T_1,T_2)$. If f is an arbitrary vector in $\mathcal{L}$, then $f = x + A_\infty x$ for some $x \in \mathcal{E}$. Then

$$Vf = V(x + A_\infty x) = V_1 x + V_2 A_\infty x = V_1 x + A_\infty V_1 x \in \mathcal{L}.$$

Conversely, let $\mathcal{L}$ be an $\mathcal{H}$-maximal non-negative subspace invariant under V. If x is an arbitrary vector in $\mathcal{E}$, then $V(x + A_\infty x) \in \mathcal{L}$ and hence, $V_1 x = y$ and $V_2 A_\infty x = A_\infty y$ for some $y \in \mathcal{E} = P_{\mathcal{G}_1}\mathcal{L}$. Therefore, $V_1\mathcal{E} \subseteq \mathcal{E}$ and $V_2 A_\infty = A_\infty V_1|\mathcal{E}$ ∎

The main result of this paper is the existence of contractive intertwining dilations for quasi-contractions in Kreĭn spaces.

Theorem 4.4. *If A is a quasi-contraction, then the set $\mathrm{CID}(A;T_1,T_2)$ is non-void.*

Proof. We verify that the hypotheses of Theorem 2.6 are fulfilled. First, since V_1 is expansive and V_2 is contractive, for any vector $x = x_1 + x_2$ with $x_1 \in \mathcal{G}_1$ and $x_2 \in \mathcal{G}_2$ we have

$$[Vx, Vx] = [V_1x_1, V_1x_1] - [V_2x_2, V_2x_2] \geq [x_1, x_1] - [x_2, x_2] = [x, x],$$

that is V is expansive and hence it maps non-negative vectors into non-negative ones. Since $\dim \mathcal{H}_{0-} = \kappa_-(I - A^\sharp A) < \infty$ it follows that the operator G_- has closed range (cf. (3.5)). We now take into account (3.3) and get that

$$\begin{aligned} P_{\mathcal{H}_+} V P_{\mathcal{H}_-} &= (P_{\mathcal{H}_{0-}} + P_{\mathcal{G}_1 \ominus \mathcal{H}_1}) V (P_{\mathcal{H}_{0-}} + P_{\mathcal{G}_2 \ominus \mathcal{H}_2}) \\ &= (P_{\mathcal{H}_{0-}} + P_{\mathcal{G}_1 \ominus \mathcal{H}_1}) V P_{\mathcal{H}_{0-}} + P_{\mathcal{H}_{0+}} V P_{\mathcal{G}_2 \ominus \mathcal{H}_2} + P_{\mathcal{G}_1 \ominus \mathcal{H}_1} V P_{\mathcal{G}_2 \ominus \mathcal{H}_2}. \end{aligned}$$

From (4.1) it follows that $P_{\mathcal{G}_1 \ominus \mathcal{H}_1} V P_{\mathcal{G}_2 \ominus \mathcal{H}_2} = 0$ and, since $\mathcal{G}_2 \ominus \mathcal{H}_2$ is invariant under V_2 and $\mathcal{H}_{0+} \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$, we also have that $P_{\mathcal{H}_{0+}} V P_{\mathcal{G}_2 \ominus \mathcal{H}_2} = 0$. Therefore

$$P_{\mathcal{H}_+} V P_{\mathcal{H}_-} = (P_{\mathcal{H}_{0-}} + P_{\mathcal{G}_1 \ominus \mathcal{H}_1}) V P_{\mathcal{H}_{0-}}$$

and hence $\operatorname{rank} P_{\mathcal{H}_+} V P_{\mathcal{H}_-} \leq \dim \mathcal{H}_{0-} = \kappa < \infty$, in particular, the operator $P_{\mathcal{H}_+} V P_{\mathcal{H}_-}$ is compact.

The assumptions of Theorem 2.6 are verifed and hence there exists an $\mathcal{H}$-maximal non-negative subspace invariant under V. In view of Lemma 4.3 this implies that there exists a contractive intertwining dilation of A ∎

Theorem 4.4 is mostly useful in the case that T_i are contractions and $V_i \in \mathcal{B}(\mathcal{G}_i)$ are isometric dilations of T_i $(i = 1, 2)$. These isometric dilations always exist, even with the minimality property $\bigvee_{n\geq 0} V_i^n \mathcal{H}_i = \mathcal{G}_i$ in addition, and, in this case, they are unique modulo unitary equivalence (see, e.g., [9]).

We finally indicate how our Theorem 4.4 contains some other results on the existence of contractive intertwining dilations.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and A is a contraction, that is $\kappa_-(I - A^*A) = 0$, then this is the classical theorem of lifting of commutant of D. Sarason and B. Sz.-Nagy and C. Foiaş, as well as its other generalizations (see, e.g., [23]).

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces but A is a genuine quasi-contraction with $0 \neq \kappa_-(I - A^*A) < \infty$, then we obtain the result of Ball and Helton (cf. [5: Theorem 4.2], but also [3: Theorem 3.1]).

Assume now that both $\mathcal{H}_1$ and $\mathcal{H}_2$ are Kreĭn spaces and that A is a contraction with $\kappa_-(I - A^\sharp A) = 0$. If in addition $A^\sharp$ is a contraction, one obtains a particular case of Theorem 3.5 in [9] and otherwise one gets Theorem 3.2.1 in [12].

Finally, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are Kreĭn spaces and A is a genuine quasi-contraction, our Theorem 4.4 covers Theorem 4.1 in [2].

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