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# On a Fixed Point Theorem by Brosowski and Singh

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**Abstract.** We generalize a fixed point theorem of B. Brosowski and S. P. Singh to multi-valued non-expansive maps in Banach spaces with convex structure. Such maps have applications, for example, in game theory and in the mathematical modelling of some economical problems.

**Keywords:** *Fixed point theorems, multi-valued maps, non-expansive maps, contractive maps, best approximations, convex structures*

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Let $C$ be a non-empty set in a real normed space $X$. A *convex structure* on $C$ is, by definition, a continuous map $F : [0,1] \times C \times C \to C$ having the two properties

$$F(\lambda, x, F(\mu, y, z)) = F\left(\lambda + (1-\lambda)\mu, F\left(\frac{\lambda}{\lambda + (1-\lambda)\mu}, x, y\right), z\right) \tag{1}$$

$$F(\lambda, x, x) = x \tag{2}$$

for $\lambda, \mu \in [0,1]$ with $\lambda + (1-\lambda)\mu \neq 0$ and $x, y, z \in C$. From (1) and (2) it follows that

$$\left.\begin{aligned} F(1,x,y) &= x \\ F(0,x,y) &= y \end{aligned}\right\} \qquad (x, y \in C). \tag{3}$$

The general theory of convex structures has applications, for example, in game theory and mathematical economics [10], but also in some problems of colour vision and petroleum engineering [4, 7, 8]. A nice survey on basic properties and applications of convex structures may be found in [5].

Obviously, every convex set $C$ in a normed space $X$ gives rise to a canonical convex structure by putting $F(\lambda, x, y) = \lambda x + (1-\lambda)y$. Another example may be obtained as follows: Given a Hilbert space $H$, let $C$ be the set of all self-adjoint positive definite operators in $X = \mathcal{L}(H)$. For $\lambda \in [0,1]$ and $A, B \in C$ we put $F(\lambda, A, B) = A^\lambda B^{1-\lambda}$, where $A^\lambda$ denotes the usual fractional power (see, e.g., [3])

$$A^\lambda x = \frac{\sin \pi\lambda}{\pi} \int_0^\infty \tau^{\lambda-1}(A - \tau I)^{-1} x \, d\tau \qquad (0 < \lambda < 1).$$

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In what follows, we assume throughout that the convex structure $F$ is *regular* in the sense that

$$\big\|F(\lambda,x,p)-F(\lambda,y,q)\big\| \le \varphi(\lambda)\,\|x-y\|+\big(1-\psi(\lambda)\big)\,\|p-q\| \tag{4}$$

for $\lambda \in [0,1]$ and $x,y,p,q \in C$, where $\varphi,\psi : [0,1] \to [0,1]$ are continuous functions satisfying

$$\varphi(0)=\psi(0)=0, \qquad \varphi(1)=\psi(1)=1, \qquad 0<\varphi(\lambda),\psi(\lambda)<1 \text{ for } 0<\lambda<1.$$

Of course, the canonical convex structure on a convex set is always regular with $\varphi(\lambda) = \psi(\lambda) = \lambda$. Finally, we recall that a set $D \subseteq C$ is called *F-starshaped* if there exists a point $p \in D$ such that $F(\lambda,x,p) \in D$ for all $\lambda \in [0,1]$ and $x \in D$.

**Theorem.** *Given a real normed space $X$ and a non-empty closed set $C \subset X$ with regular convex structure $F : [0,1]\times C\times C \to C$, let $f : C \to 2^X$ be a multi-valued map with compact values, $f(\partial C) \subseteq C$ and some fixed point $x_0 \in C$, i.e. $x_0 \in f(x_0)$. Suppose the following:*

**(i)** *The set $D$ of best $C$-approximants to $x_0$, i.e.*

$$D=\Big\{x : x\in C \text{ such that } \|x_0-x\| = d(x_0,C)\Big\} \tag{5}$$

*is non-empty and $F$-starshaped.*

**(ii)** *The map $f$ is non-expansive on $D$ with respect to the Hausdorff metric $h$ on $X$,*

$$h\big(f(x),f(x_0)\big) \le \|x-x_0\| \qquad (x\in D), \tag{6}$$

*and $f(D)$ is compact.*

*Then $f$ has a fixed point closest to $x_0$.*

**Proof.** Observe first that all points $x \in D$ must belong to the boundary $\partial C$ of $C$. In fact, if $x \in D$ were an interior point of $C$, we would have $x_\lambda = F(\lambda,x_0,x) \in C$ and

$$\|x_\lambda - x_0\| = \big\|F(\lambda,x_0,x)-F(\lambda,x_0,x_0)\big\| \le \big(1-\psi(\lambda)\big)\|x-x_0\| < \|x-x_0\|$$

for $\lambda > 0$ sufficiently small, contradicting the fact that $x$ is a point of best approximation to $x_0$ in $C$.

The inclusion $D \subset \partial C$ implies that $f$ maps the set $D$ into itself: for $x \in D$ and $y \in f(x)$ we have $y \in f(\partial C) \subseteq C$ and

$$\|x_0-y\| \le h\big(f(x_0),f(x)\big) \le \|x_0-x\| = d(x_0,C),$$

by (6), hence $y \in D$.

Since $D$ is $F$-starshaped, we may choose $p \in D$ such that $F(\lambda,x,p) \in D$ for all $\lambda \in [0,1]$ and $x \in D$. Now let $(\lambda_n)_n$ be a sequence in $(0,1)$ converging to 1, as $n \to \infty$, and define $f_n : D \to 2^X$ by

$$f_n(x) = \bigcup_{y\in f(x)} F(\lambda_n,y,p) \qquad (x\in D). \tag{7}$$

Since $D$ is $F$-starshaped, $f_n(x)$ is actually a subset of $D$ for any $x \in D$. We claim that $f_n$ is a multi-valued contraction on $D$. In fact, fix $x_i \in D$ and choose $z_i \in f_n(x_i)$, hence $z_i = F(\lambda_n, y_i, p)$ for some $y_i \in f(x_i)$ $(i = 1, 2)$. We get then the estimate

$$\begin{aligned} \|z_1 - z_2\| &= \|F(\lambda_n, y_1, p) - F(\lambda_n, y_2, p)\| \\ &\leq \varphi(\lambda_n)\|y_1 - y_2\| \leq \varphi(\lambda_n) h(f(x_1), f(x_2)) \leq \varphi(\lambda_n)\|x_1 - x_2\|, \end{aligned}$$

since $F$ is regular and $f$ is non-expansive on $D$. We conclude that the map $f_n$ is a multi-valued contraction, and thus, by Nadler's fixed point theorem [6], there exists $x_n \in D$ such that $x_n \in f(x_n)$.

Now we employ the hypothesis that $f(D)$ is compact and choose a subsequence $(x_{n_k})_k$ such that $x_{n_k} \to x_*$ as $k \to \infty$. The relation $x_{n_k} \in f_{n_k}(x_{n_k})$ means that $x_{n_k} = F(\lambda_{n_k}, y_{n_k}, p)$ for some $y_{n_k} \in f(x_{n_k})$, hence $y_{n_k} \to y_*$ for some $y_* \in f(x_*)$. By the continuity of $F$, this implies that

$$x_{n_k} = F(\lambda_{n_k}, y_{n_k}, p) \longrightarrow F(1, y_*, p) = y_* \qquad (k \to \infty). \tag{8}$$

From (8) we conclude that

$$d(x_*, f(x_*)) \leq \|x_* - x_{n_k}\| + \|x_{n_k} - y_*\| + d(y_*, f(x_*)) \to 0 \qquad (k \to \infty)$$

which shows that $x_*$ is a fixed point of $f$ in $D$, since $D$ is closed ∎

Our theorem may be considered as a generalization of a fixed point theorem of S. P. Singh [9] (where $f$ is single-valued and $C$ is star-shaped in the usual sense) which in turn extends a fixed point theorem of B. Brosowski [1] (where $f$ is linear and $C$ is convex). Moreover, some results of our recent paper [2] may be regarded as special variants of the above theorem for single-valued non-expansive maps $f$.

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