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On some Subclasses of Nevanlinna Functions

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Abstract. For any function $Q = Q(\ell)$ belonging to the class $\mathbf{N}$ of Nevanlinna functions, the function $Q_\tau = Q_\tau(\ell)$ defined by $Q_\tau(\ell) = \frac{Q(\ell)-\tau(\mathrm{Im}\,Q(\mu))^2}{\tau Q(\ell)+1}$ belongs to $\mathbf{N}$ for all values of $\tau \in \mathbb{R} \cup \{\infty\}$. The class $\mathbf{N}$ possesses subclasses $\mathbf{N}_0 \subset \mathbf{N}_1$, each defined by some additional asymptotic conditions. If a function Q belongs to such a subclass, then for all but one value of $\tau \in \mathbb{R} \cup \{\infty\}$ the function Q_τ belongs to the same subclass and the corresponding exceptional function can be characterized (cf. [4]). In this note we introduce two subclasses $\mathbf{N}_{-2} \subset \mathbf{N}_{-1}$ of $\mathbf{N}_0$ which can be described in terms of the moments of the spectral measures in the associated integral representations. We characterize the corresponding exceptional function in a purely function-theoretic way by suitably estimating certain quadratic terms. The behaviour of the exceptional function connects the subclasses $\mathbf{N}_{-2}$ and $\mathbf{N}_{-1}$ to the classes $\mathbf{N}_0$ and $\mathbf{N}_1$, respectively, as studied in [4]. In operator-theoretic terms these notions have a translation in terms of Q-functions of selfadjoint extensions of a symmetric operator with defect numbers $(1,1)$. In this sense the exceptional function has an interpretation in terms of a generalized Friedrichs extension of the symmetric operator.

Keywords: *Symmetric operators, selfadjoint extensions, Friedrichs extension, Q-functions, Nevanlinna functions*

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0. Introduction

A scalar function $Q = Q(\ell)$ is said to be a *Nevanlinna function* if it is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ and satisfies $\overline{Q(\ell)} = Q(\bar{\ell})$ and $\frac{\mathrm{Im}\,Q(\ell)}{\mathrm{Im}\,\ell} \geq 0$ for all $\ell \in \mathbb{C} \setminus \mathbb{R}$. The set of all Nevanlinna functions is denoted by $\mathbf{N}$. The subclass $\mathbf{N}_1$ is the set of functions Q which belong to $\mathbf{N}$ and for which

$$\int_1^\infty \frac{\mathrm{Im}\,Q(iy)}{y}\, dy < \infty.$$

Similarly, the subclass $\mathbf{N}_0$ is the set of functions Q which belong to $\mathbf{N}$ and for which

$$\sup_{y>0} y\,\mathrm{Im}\,Q(iy) < \infty.$$

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The inclusion $\mathbf{N}_0 \subset \mathbf{N}_1$ is clear. Bilinear transforms of functions in $\mathbf{N}$, $\mathbf{N}_1$ and $\mathbf{N}_0$ were studied in [4]. For each function Q in $\mathbf{N}$, which does not reduce to a real constant, the bilinear transform $Q_\tau = Q_\tau(\ell)$ is defined by

$$Q_\tau(\ell) = \frac{Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2}{\tau Q(\ell) + 1} \qquad (\tau \in \mathbb{R} \cup \{\infty\}) \tag{0.1}$$

where $\mu \in \mathbb{C} \setminus \mathbb{R}$ is a fixed number. For $\tau = \infty$ we mean that

$$Q_\infty(\ell) = -\frac{(\operatorname{Im} Q(\mu))^2}{Q(\ell)}, \tag{0.2}$$

i.e. Q_∞ can be seen as a limiting case of (0.1). It follows that for each $\tau \in \mathbb{R} \cup \{\infty\}$ the function Q_τ belongs to $\mathbf{N}$. Moreover, if Q belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, then for all but one $\tau \in \mathbb{R} \cup \{\infty\}$ the corresponding function Q_τ belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, respectively. The exceptional value of τ is given by $\frac{1}{\tau} + \gamma = 0$, where $\gamma = \lim_{y\to\infty} Q(iy)$ (cf. [4]). Note that this limit is a real number as $Q \in \mathbf{N}_1$ (cf. [6]). In fact, if Q belongs to $\mathbf{N}_0$ and $\frac{1}{\tau} + \gamma = 0$, then the corresponding exceptional function $H = Q_\tau$ has $\lim_{y\to\infty} \frac{\operatorname{Im} Q_\tau(iy)}{y}$ positive and the function

$$H(\ell) - \left(\lim_{y\to\infty} \frac{\operatorname{Im} H(iy)}{y}\right) \ell, \tag{0.3}$$

belongs to $\mathbf{N}$. Conversely, each Nevanlinna function H for which $\lim_{y\to\infty} \frac{\operatorname{Im} H(iy)}{y}$ is positive is the exceptional function of the bilinear transform of a function from $\mathbf{N}_0$.

In this note we introduce two subclasses of $\mathbf{N}_0$. The subclass $\mathbf{N}_{-1}$ is the set of functions Q in $\mathbf{N}_0$ for which

$$\int_1^\infty \left(\sup_{y>0} y \operatorname{Im} Q(iy) - y \operatorname{Im} Q(iy)\right) dy < \infty.$$

Similarly, the subclass $\mathbf{N}_{-2}$ is the set of functions Q in $\mathbf{N}_0$ for which

$$\sup_{y>0} y^2 \left(\sup_{y>0} y \operatorname{Im} Q(iy) - y \operatorname{Im} Q(iy)\right) < \infty.$$

Observe in these definitions that the function $\sup_{y>0} y \operatorname{Im} Q(iy) - y \operatorname{Im} Q(iy)$ is non-negative and non-increasing for $y > 0$. The inclusion $\mathbf{N}_{-2} \subset \mathbf{N}_{-1}$ is clear. Necessary and sufficient for a function to belong to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$ is that the moments

$$\int_{\mathbb{R}} t^k \, d\sigma(t)$$

in terms of the corresponding spectral measure (see (1.1)) are absolutely convergent integrals for $k = 0, 1$ and $k = 0, 1, 2$, respectively. We show that if Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, then for all but the exceptional value of $\tau \in \mathbb{R} \cup \{\infty\}$, the corresponding function Q_τ belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively. Hence, all these Nevanlinna functions behave in a similar way by having finite moments of the same orders. A connection to the classes $\mathbf{N}_0$ and $\mathbf{N}_1$ as considered in [4] is obtained by looking at the behaviour of the bilinear transform (0.1) corresponding to the exceptional value of τ. When $\frac{1}{\tau}+\gamma=0$, then the corresponding exceptional function $H=Q_\tau$ has the property that the function in (0.3) belongs to $\mathbf{N}_1$ or to $\mathbf{N}_0$, respectively. Moreover, a converse is also valid: each Nevanlinna function H for which $\lim_{y\to\infty}\frac{\operatorname{Im} H(iy)}{y}$ is positive, and for which the function in (0.3) belongs to $\mathbf{N}_1$ or to $\mathbf{N}_0$, is the exceptional function of the bilinear transform of a function from $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively.

In this note we provide function-theoretic proofs of these facts. They are based on the integral representations for functions in each of the classes $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$ (see Proposition 1.2). In each of the arguments suitable estimates for certain quadratic terms are needed.

The above results have operator-theoretic interpretations. Let S be a closed symmetric operator in a Hilbert space with defect numbers $(1,1)$. The selfadjoint extensions (in the given Hilbert space) of such an operator are in one-to-one correspondence with $\tau\in\mathbb{R}\cup\{\infty\}$. The Q-function of a selfadjoint extension belongs to $\mathbf{N}$ and then (0.1) provides a parametrization for the Q-functions of the other selfadjoint extensions. Moreover, each function in $\mathbf{N}$ determines a closed symmetric operator with defect numbers $(1,1)$ and a selfadjoint extension. If S has a selfadjoint extension whose Q-function Q belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, then for each $\tau\in\mathbb{R}\cup\{\infty\}$ with $\frac{1}{\tau}+\gamma\neq 0$ the corresponding selfadjoint extension of S has a Q-function Q_τ given by (0.1) and belonging to $\mathbf{N}_1$ or $\mathbf{N}_0$, respectively. The exceptional selfadjoint extension corresponding to the case $\frac{1}{\tau}+\gamma=0$ can be characterized in a similar way as the Friedrichs extension for semibounded operators (cf. [2 - 4]). In fact, if Q belongs to $\mathbf{N}_0$, the operator S is not densely defined and the exceptional extension is the only selfadjoint extension which is not an operator. If H is the Q-function of this exceptional extension, then the function in (0.3) is the Q-function of the (orthogonal) operator part of the exceptional selfadjoint extension and a natural restriction of the graph of S to the corresponding closed subspace. Therefore, the first and second order moments associated to the Q-functions of the selfadjoint operator extensions of S are finite if and only if the function in (0.3) belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, respectively.

For further operator-theoretic considerations of the facts in this note and in [4], we refer to [5].

1. Integral representations

We present integral representations for functions belonging to the classes $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$. As these classes are subsets of $\mathbf{N}_0$, this means that we give necessary and sufficient conditions on the spectral measure in the integral representation of functions belonging to $\mathbf{N}_0$, to characterize the classes $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$, respectively.

We briefly collect the integral representations of functions in the classes $\mathbf{N}$, $\mathbf{N}_1$ and $\mathbf{N}_0$ (cf. [4, 6]). The class $\mathbf{N}$ coincides with the class of functions Q with integral representation

$$Q(\ell) = \alpha + \beta\ell + \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) d\sigma(t) \tag{1.1}$$

where $\alpha \in \mathbb{R}$ and $\beta \geq 0$, and where the function σ is non-decreasing on $\mathbb{R}$ and satisfies

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+1} < \infty. \tag{1.2}$$

The class $\mathbf{N}_1$ coincides with the class of functions Q with integral representation

$$Q(\ell) = \gamma + \int_{\mathbb{R}} \frac{d\sigma(t)}{t-\ell} \tag{1.3}$$

where $\gamma = \lim_{y\to\infty} Q(iy)$ belongs to $\mathbb{R}$ and

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{|t|+1} < \infty. \tag{1.4}$$

The class $\mathbf{N}_0$ coincides with the class of functions with integral representation (1.3), where

$$\int_{\mathbb{R}} d\sigma(t) < \infty. \tag{1.5}$$

Note that for functions Q in $\mathbf{N}$ it follows from (1.1) that

$$\begin{aligned} \mathrm{Re}Q(iy) &= \alpha + \int_{\mathbb{R}} \left(\frac{t}{t^2+y^2} - \frac{t}{t^2+1} \right) d\sigma(t) \\ \mathrm{Im}Q(iy) &= \beta y + \int_{\mathbb{R}} \frac{y}{t^2+y^2} d\sigma(t). \end{aligned} \tag{1.6}$$

This implies that

$$\lim_{y\to\infty} \frac{\mathrm{Re}Q(iy)}{y} = 0, \qquad \lim_{y\to\infty} \frac{\mathrm{Im}Q(iy)}{y} = \beta, \qquad \lim_{y\to\infty} \frac{|Q(iy)|}{y} = \beta. \tag{1.7}$$

For functions Q in $\mathbf{N}_1$ it follows from (1.3) that

$$\begin{aligned} \mathrm{Re}Q(iy) - \gamma &= \int_{\mathbb{R}} \frac{t}{t^2+y^2}\, d\sigma(t) \\ \mathrm{Im}Q(iy) &= \int_{\mathbb{R}} \frac{y}{t^2+y^2}\, d\sigma(t). \end{aligned} \tag{1.8}$$

In particular, for functions Q in $\mathbf{N}_0$, this leads to

$$\sup_{y>0} y\mathrm{Im}Q(iy) = \lim_{y\to\infty} y\mathrm{Im}Q(iy) = \lim_{y\to\infty} y|Q(iy)-\gamma| = \int_{\mathbb{R}} d\sigma(t), \tag{1.9}$$

which is positive, if Q does not reduce to a real constant. Moreover, in this case, it follows from (1.8) and (1.9) that

$$\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy) = \int_{\mathbb{R}} \frac{t^2}{t^2+y^2}\, d\sigma(t). \tag{1.10}$$

The following lemma will be used to prove integral representations for functions in $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$. Each quantity in it is non-negative and may be equal to ∞.

Lemma 1.1. *Let Q be a function in $\mathbf{N}_0$, i.e. a function with integral representation* (1.3) *such that* (1.5) *is satisfied. Then*

$$\int_1^\infty \left(\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy)\right) dy = \int_{\mathbb{R}} |t| \left(\frac{\pi}{2} - \arctan\frac{1}{|t|}\right) d\sigma(t) \qquad (\leq \infty) \tag{1.11}$$

and

$$\sup_{y>0} y^2 \left(\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy)\right) = \int_{\mathbb{R}} t^2 d\sigma(t) \qquad (\leq \infty). \tag{1.12}$$

Proof. The statement in (1.11) follows from (1.10), the identity

$$\int_1^\infty \frac{dy}{t^2+y^2} = \frac{1}{|t|}\left(\frac{\pi}{2} - \arctan\frac{1}{|t|}\right) \qquad (t \neq 0) \tag{1.13}$$

and an application of the Fubini theorem. Multiplying (1.10) by y^2 and applying the monotone convergence theorem we obtain the identity (1.12). This completes the proof ∎

Proposition 1.2. *The classes $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$ coincide with the classes of functions Q with integral representation* (1.3), *where $\gamma \in \mathbb{R}$ and σ is a non-decreasing function on $\mathbb{R}$, which satisfies*

$$\int_{\mathbb{R}} (|t|+1)\, d\sigma(t) < \infty \tag{1.14}$$

and

$$\int_{\mathbb{R}} (t^2+1)\, d\sigma(t) < \infty, \tag{1.15}$$

respectively.

Proof. Suppose that the function Q has the integral representation (1.3), where $\gamma \in \mathbb{R}$ and σ is a non-decreasing function on $\mathbb{R}$, which satisfies (1.14) or (1.15), respectively. As

$$\int_{\mathbb{R}} d\sigma(t) \leq \int_{\mathbb{R}} (|t|+1)\, d\sigma(t) \qquad \text{or} \qquad \int_{\mathbb{R}} d\sigma(t) \leq \int_{\mathbb{R}} (t^2+1)\, d\sigma(t),$$

it follows that Q belongs to $\mathbf{N}_0$. The identities (1.11) and (1.12) give that Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively.

Conversely, let Q belong to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively. By definition Q belongs to $\mathbf{N}_0$. Hence Q has the integral representation (1.3) with integrability condition (1.5). However, since we require the left-hand side of (1.11) and (1.12), respectively, to be finite, it follows that σ satisfies the integrability condition (1.14) or (1.15), respectively. This completes the proof ∎

For a different characterization of the subclasses $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$ in terms of operators we refer to [5].

2. Bilinear transforms

It was shown in [4] that for any function Q in $\mathbf{N}_1$ or $\mathbf{N}_0$ the bilinear transform Q_τ, given by (0.1), belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, respectively, for all but the exceptional value of $\tau \in \mathbb{R} \cup \{\infty\}$. In this section we prove corresponding facts when Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$. From now on we assume tacitly that Q does not reduce to a real constant.

As in [4] we use the following consequence of the formula (0.1) for the bilinear transform:

$$\mathrm{Im} Q_\tau(\ell) = \frac{\mathrm{Im} Q(\ell)}{|1+\tau Q(\ell)|^2}\left(1+\left(\tau \mathrm{Im} Q(\mu)\right)^2\right) \qquad \left(\tau \in \mathbb{R}\cup\{\infty\}\right). \tag{2.1}$$

The main result in this section is based on the next lemma. (For $\tau = \infty$ we use (0.2) and the formulas below simplify accordingly.)

Lemma 2.1. *Let the function Q belong to $\mathbf{N}_0$. Then for all $\tau \in \mathbb{R} \cup \{\infty\}$ with $\frac{1}{\tau} + \gamma \neq 0$ we have*

$$\begin{aligned} &\frac{|1+\tau\gamma|^2}{1+(\tau \mathrm{Im} Q(\mu))^2}\left(\sup_{y>0} y \mathrm{Im} Q_\tau(iy) - y \mathrm{Im} Q_\tau(iy)\right) \\ &\qquad = \left(\sup_{y>0} y \mathrm{Im} Q(iy) - y \mathrm{Im} Q(iy)\right) + \frac{y \mathrm{Im} Q(iy)}{|1+\tau Q(iy)|^2} R(y) \end{aligned} \tag{2.2}$$

where the function R is defined by

$$R(y) = \left(\mathrm{Re} Q(iy) - \gamma\right)\left(2\tau + \tau^2\left(\mathrm{Re} Q(iy) + \gamma\right)\right) + \tau^2 (\mathrm{Im} Q(iy))^2. \tag{2.3}$$

Proof. Note that if $\frac{1}{\tau} + \gamma \neq 0$, it follows from (2.1) that $\sup_{y>0} y \mathrm{Im} Q_\tau(iy)$ is finite and that the left-hand side of (2.2) is equal to

$$\sup_{y>0} y \mathrm{Im} Q(iy) - y \mathrm{Im} Q(iy) + \frac{y \mathrm{Im} Q(iy)}{|1+\tau Q(iy)|^2}\left(|1+\tau Q(iy)|^2 - |1+\tau\gamma|^2\right).$$

The lemma follows by writing out the terms in the difference $|1+\tau Q(iy)|^2 - |1+\tau\gamma|^2$. This completes the proof ∎

Proposition 2.2. *If the function Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, then for all $\tau \in \mathbb{R} \cup \{\infty\}$ with $\frac{1}{\tau} + \gamma \neq 0$ the function Q_τ in (0.1) belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively.*

Proof. Since the function Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, it belongs to $\mathbf{N}_0$ and we may apply Lemma 2.1. The factor of R in (2.2) is bounded, due to (1.9) and the fact that $\lim_{y\to\infty} Q(iy) = \gamma$. Hence, the proposition is proved once we show that the function R in (2.3) satisfies

$$\int_1^\infty |R(y)|\, dy < \infty \qquad \text{or} \qquad \sup_{y>0} y^2 |R(y)| < \infty, \tag{2.4}$$

respectively. As the factor $2\tau + \tau^2(\mathrm{Re}Q(iy) + \gamma)$ has a limit, it suffices to show that conditions similar to (2.4) are satisfied by the functions

$$\mathrm{Re}Q(iy) - \gamma \qquad \text{or} \qquad (\mathrm{Im}Q(iy))^2, \tag{2.5}$$

respectively. According to Proposition 1.2 the function Q has the integral representation (1.3) and in particular (1.8) holds. Moreover, either (1.14) or (1.15) is satisfied, respectively.

First assume that the function Q belongs to $\mathbf{N}_{-1}$. According to (2.5) it suffices to show that

$$\int_1^\infty |\mathrm{Re}Q(iy) - \gamma|\, dy < \infty \qquad \text{and} \qquad \int_1^\infty (\mathrm{Im}Q(iy))^2 dy < \infty. \tag{2.6}$$

From (1.8) and (1.13) we obtain

$$\int_1^\infty |\mathrm{Re}Q(iy) - \gamma|\, dy \le \int_{\mathbb{R}} \left(\frac{\pi}{2} - \arctan\frac{1}{|t|}\right) d\sigma(t) < \infty, \tag{2.7}$$

as (1.14) implies (1.5). It follows from (1.8) and the Cauchy-Schwarz inequality that

$$\begin{aligned}(\mathrm{Im}Q(iy))^2 &\le \left(\int_{\mathbb{R}} \frac{y^2}{(t^2+y^2)^2}\, d\sigma(t)\right)\left(\int_{\mathbb{R}} d\sigma(t)\right)\\ &\le \left(\int_{\mathbb{R}} \frac{1}{t^2+y^2}\, d\sigma(t)\right)\left(\int_{\mathbb{R}} d\sigma(t)\right).\end{aligned}$$

This yields in a similar way

$$\int_1^\infty (\mathrm{Im}Q(iy))^2 dy \le \left(\int_{\mathbb{R}} \frac{1}{|t|}\left(\frac{\pi}{2} - \arctan\frac{1}{|t|}\right) d\sigma(t)\right)\left(\int_{\mathbb{R}} d\sigma(t)\right) < \infty$$

as (1.14) guarantees (1.4) and (1.5). We conclude therefore that Q_τ belongs to $\mathbf{N}_{-1}$.

Next assume that the function Q belongs to $\mathbf{N}_{-2}$. It suffices to show that

$$\sup_{y>0} y^2|\mathrm{Re}Q(iy) - \gamma| < \infty \qquad \text{and} \qquad \sup_{y>0} y^2(\mathrm{Im}Q(iy))^2 < \infty$$

(cf. (2.5)). From (1.8) we see that

$$\sup_{y>0} y^2|\mathrm{Re}Q(iy) - \gamma| \le \sup_{y>0} \int_{\mathbb{R}} \frac{|t|y^2}{t^2+y^2}\, d\sigma(t) = \int_{\mathbb{R}} |t|\, d\sigma(t) < \infty \tag{2.8}$$

as (1.15) implies (1.14). Moreover

$$\sup_{y>0} y^2(\mathrm{Im}Q(iy))^2 = \left(\int_{\mathbb{R}} d\sigma(t)\right)^2 < \infty$$

as (1.15) guarantees (1.5), so that (1.9) can be used. We conclude that Q_τ belongs to $\mathbf{N}_{-2}$. This completes the proof ∎

3. Exceptional functions

In this section we consider the bilinear transform Q_τ in (0.1) of functions Q belonging to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, when τ is the exceptional value given by $\frac{1}{\tau}+\gamma=0$.

We begin with the following lemma (compare [4]).

Lemma 3.1. *Let Q be a function in $\mathbf{N}_0$, i.e. a function with the integral representation (1.3) such that (1.5) holds. If $\frac{1}{\tau}+\gamma=0$, then the function $H=Q_\tau$ satisfies*

$$\lim_{y\to\infty}\frac{\mathrm{Im}H(iy)}{y}=\frac{\gamma^2+(\mathrm{Im}Q(\mu))^2}{\int_{\mathbb{R}}d\sigma(t)} \tag{3.1}$$

so that this limit is positive. Moreover, the function in (0.3) belongs to $\mathbf{N}$. Its imaginary part for $\ell=iy$ is given by

$$\left(\frac{\mathrm{Im}Q(iy)\int_{\mathbb{R}}d\sigma(t)-y|Q(iy)-\gamma|^2}{|Q(iy)-\gamma|^2\int_{\mathbb{R}}d\sigma(t)}\right)\left(\gamma^2+(\mathrm{Im}Q(\mu))^2\right). \tag{3.2}$$

Proof. It follows from (2.1) with $\frac{1}{\tau}+\gamma=0$ and $H=Q_\tau$ that

$$\mathrm{Im}H(iy)=\frac{\mathrm{Im}Q(iy)}{|Q(iy)-\gamma|^2}\left(\gamma^2+(\mathrm{Im}Q(\mu))^2\right). \tag{3.3}$$

The desired limit follows from (1.9). The function in (0.3) belongs to $\mathbf{N}$ as the coefficient of the linear term in the integral representation of $H(\ell)$ is given by (3.1) (see (1.1) and (1.7)). Finally the expression (3.2) is a consequence of (3.3) as the imaginary part of the function in (0.3) for $\ell=iy$ is given by

$$\left(\frac{\mathrm{Im}Q(iy)}{|Q(iy)-\gamma|^2}-\frac{y}{\int_{\mathbb{R}}d\sigma(t)}\right)\left(\gamma^2+(\mathrm{Im}Q(\mu))^2\right).$$

This completes the proof ∎

Proposition 3.2. *Let the function Q belong to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$. For the exceptional function $H=Q_\tau$ with $\frac{1}{\tau}+\gamma=0$ the function defined in (0.3) belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, respectively.*

Proof. The function Q has the integral representation (1.3) such that (1.14) or (1.15) holds, respectively. We introduce the function

$$S(y)=y^2\left(\mathrm{Im}Q(iy)\int_{\mathbb{R}}d\sigma(t)-y|Q(iy)-\gamma|^2\right).$$

According to Lemma 3.1 this function is non-negative. Due to (3.2) and (1.9) it suffices to prove that

$$\int_1^\infty\frac{S(y)}{y}\,dy<\infty \qquad \text{or} \qquad \sup_{y>0}yS(y)<\infty$$

when Q belongs to $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively. Again using (1.9) we see that

$$\frac{S(y)}{y} = (y\mathrm{Im}Q(iy))\left(\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy)\right) - y^2\left(\mathrm{Re}Q(iy) - \gamma\right)^2. \tag{3.4}$$

Assume that Q belongs to $\mathbf{N}_{-1}$. We show that the function $\frac{S(y)}{y}$ is integrable. The first term in the right-hand side of (3.4) is integrable as the factor $y\mathrm{Im}Q(iy)$ has an upper bound $\int_{\mathbb{R}} \sigma(t)$ because (1.14) implies (1.5). The second term in the right-hand side of (3.4) is also integrable: according to (2.8) the function $y^2|\mathrm{Re}Q(iy) - \gamma|$ has an upper bound $\int_{\mathbb{R}} |t|\, d\sigma(t)$ as (1.14) is valid and according to (2.7) the function $\mathrm{Re}Q(iy) - \gamma$ is integrable as (1.14) guarantees (1.5). We conclude that the function $\frac{S(y)}{y}$ is integrable.

Next assume that Q belongs to $\mathbf{N}_{-2}$. We show that $yS(y)$ is bounded above. It follows from (3.4) that

$$yS(y) = (y\mathrm{Im}Q(iy))\, y^2\left(\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy)\right) - y^4\left(\mathrm{Re}Q(iy) - \gamma\right)^2. \tag{3.5}$$

According to (1.9) the first term in the right-hand side of (3.5) is bounded above since (1.15) implies (1.5). According to (2.8) the absolute value of the second term in the right-hand side of (3.5) is bounded above since (1.15) implies (1.14). We conclude that $\sup_{y>0} yS(y) < \infty$. This completes the proof ∎

As explained in the introduction the exceptional function corresponds in the terminology of operators to the Q-function of the generalized Friedrichs extension of a symmetric operator with defect numbers $(1,1)$. The main result in this section describes the behaviour of this special extension.

4. Characterization of exceptional functions

In this section we show that Proposition 3.2 has an converse. We characterize all possible exceptional functions relative to functions in the subclasses $\mathbf{N}_{-1}$ and $\mathbf{N}_{-2}$, respectively.

We begin with a lemma which can be found in [4], but is repeated here to make the paper selfcontained.

Lemma 4.1. *Let H be a Nevanlinna function for which* $\lim_{y\to\infty} \frac{\mathrm{Im}H(iy)}{y}$ *is positive, so that the function in* (0.3) *belongs to* $\mathbf{N}$. *Then there exists a function Q in* $\mathbf{N}_0$ *such that H is the exceptional function in the bilinear transform* (0.1) *of Q.*

Proof. Let H have the integral representation (1.1) with condition (1.2) and $\beta = \lim_{y\to\infty} \frac{\mathrm{Im}H(iy)}{y}$. Define Q by

$$Q(\ell) = -\frac{|H(\mu)|^4}{(\mathrm{Im}H(\mu))^2 H(\ell)}. \tag{4.1}$$

Then

$$\mathrm{Im}Q(\ell) = \frac{|H(\mu)|^4}{(\mathrm{Im}H(\mu))^2}\,\frac{\mathrm{Im}H(\ell)}{|H(\ell)|^2}. \tag{4.2}$$

This implies that Q is a Nevanlinna function. Moreover, it follows from (1.7) that

$$\sup_{y>0} y\mathrm{Im}Q(iy) = \sup_{y>0} \frac{|H(\mu)|^4}{(\mathrm{Im}H(\mu))^2} \frac{\mathrm{Im}H(iy)}{y} \frac{y^2}{|H(iy)|^2} = \frac{|H(\mu)|^4}{\beta(\mathrm{Im}H(\mu))^2}.$$

This shows that $Q \in \mathbf{N}_0$. Clearly $\lim_{y\to\infty} Q(iy) = 0$. Thus $\tau = \infty$ is the exceptional value for the bilinear transform of Q. Moreover $H = Q_\infty$ is the corresponding exceptional function, as is easily checked by means of (0.2). This completes the proof ∎

Proposition 4.2. *Let H be a Nevanlinna function for which $\lim_{y\to\infty} \frac{\mathrm{Im}H(iy)}{y}$ is positive. If the function in* (0.3) *belongs to $\mathbf{N}_1$ or $\mathbf{N}_0$, then there exists a function Q in $\mathbf{N}_{-1}$ or $\mathbf{N}_{-2}$, respectively, such that H is the exceptional function in the bilinear transform* (0.1) *of Q.*

Proof. Assume that the function H has the integral representation (1.1) with (1.2) and that $\beta = \lim_{y\to\infty} \frac{\mathrm{Im}H(iy)}{y}$. Let the function Q be defined as in (4.1) and introduce the function T by

$$T(y) = \frac{(\mathrm{Im}H(\mu))^2}{|H(\mu)|^4} \left(\sup_{y>0} y\mathrm{Im}Q(iy) - y\mathrm{Im}Q(iy) \right). \tag{4.3}$$

By Lemma 4.1 it suffices to show that $\int_1^\infty T(y)\,dy < \infty$ and $\sup_{y>0} y^2 T(y) < \infty$, respectively. It follows from (4.3), (4.2) and (1.6) that

$$\begin{aligned} T(y) &= \frac{1}{\beta} - \beta\frac{y^2}{|H(iy)|^2} - \left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2}\right) \frac{y^2}{|H(iy)|^2} \\ &= \frac{y^2}{|H(iy)|^2} \left[\frac{1}{\beta}\left(\frac{|H(iy)|^2}{y^2} - \beta^2 \right) - \int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2} \right]. \end{aligned} \tag{4.4}$$

Again, by using (1.6) we obtain

$$\left(\frac{\mathrm{Im}H(iy)}{y}\right)^2 - \beta^2 = \left(2\beta + \int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2}\right)\left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2}\right)$$

and therefore we see that

$$T(y) = \frac{y^2}{|H(iy)|^2} \left[\frac{1}{\beta}\left(\frac{\mathrm{Re}H(iy)}{y}\right)^2 + \left(1 + \frac{1}{\beta}\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2}\right)\left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2}\right)\right]. \tag{4.5}$$

Since we assume that H belongs to $\mathbf{N}$ and that $\beta > 0$, it follows from (1.7) and (1.2) that the terms

$$\frac{y}{|H(iy)|} \quad \text{and} \quad 1 + \frac{1}{\beta}\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+y^2} \tag{4.6}$$

in (4.5) have finite limits as $y \to \infty$, and hence are bounded on the interval $[1, \infty)$.

Assume that the function in (0.3) belongs to $\mathbf{N}_1$. The term

$$\mathrm{Re}H(iy) \tag{4.7}$$

in (4.5) coincides with the real part of the function in (0.3). Therefore it has a finite limit as $y \to \infty$ and, in particular, it is bounded on the interval $[1, \infty)$. Observe that $\int_1^\infty \frac{1}{y^2} dy = 1$ and

$$\int_1^\infty \left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2 + y^2} \right) dy = \int_{\mathbb{R}} \frac{1}{|t|} \left(\frac{\pi}{2} - \arctan \frac{1}{|t|} \right) d\sigma(t) < \infty$$

since σ satisfies (1.4). Hence $\int_1^\infty T(y)\, dy < \infty$ and it follows that Q belongs to $\mathbf{N}_{-1}$.

Next assume that the function in (0.3) belongs to $\mathbf{N}_0$. Then the terms (4.6) and (4.7) are still bounded. Furthermore, we observe that

$$y^2 \int_{\mathbb{R}} \frac{d\sigma(t)}{t^2 + y^2}$$

has a finite limit since now σ satisfies (1.5). Hence $\sup_{y>0} y^2 T(y) < \infty$ and it follows that Q belongs to $\mathbf{N}_{-2}$. This completes the proof ∎

In the framework of operator theory Propositions 3.2 and 4.2 show the interaction between all selfadjoint operator extensions of a non-densely defined symmetric operator with defect numbers $(1,1)$ on the one hand, and the behaviour of the operator part of the generalized Friedrichs extension on the other hand.

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