A New Algebra
of Periodic Generalized Functions

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Abstract. Let n denote a strictly positive integer. We construct a complex differential algebra $\mathcal{G}_n$ of so-called 2π-periodic generalized functions. We show that the space $\mathcal{D}_{2\pi}^{'(n)}$ of 2π-periodic distributions on $\mathbb{R}^n$ can be canonically embedded into $\mathcal{G}_n$. Next we lay the foundation for calculation in $\mathcal{G}_n$. This algebra $\mathcal{G}_n$ enables one to solve, in a canonical way, differential problems with strong singular periodic data which have no solution in $\mathcal{D}_{2\pi}^{'(n)}$.

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1. Introduction

In the last two decades, the attempts to overcome the impossibility result of L. Schwartz concerning the multiplication of distributions (see [10]) have progressively lead to the elaboration of the theory of generalized functions. The main contributors are H. A. Biagioni [1], J. F. Colombeau [3], Yu. V. Egorov [6], M. Obergguggenberger [7] and E. Rosinger [9]. See [7, 8] for a comprehensive account of the topic.

One major motivation of this theory is that some differential problems such as nonlinear problems or even linear problems which have no solution in the space of distributions can admit solutions in algebras of generalized functions.

The existence of differential algebras with a canonical embedding of distributions as a linear subspace and having optimal properties was evidenced by J. F. Colombeau (see [4]). These algebras of generalized functions are named *Colombeau algebras*.

Nevertheless, apart from the author's works in [11, 12] where the embeddings of distributions in algebras of generalized functions are not canonical, nothing has been done in case involving periodicity.

This paper is devoted to the construction of an algebra, whose elements are called *periodic generalized functions*, with a canonical linear embedding of 2π-periodic distributions, and where periodic of mathematical objects attached to it, such as Fourier series, play a basic role. More precisely, let n denotes a strictly positive integer and let $\mathcal{A}^{(n)}$ be a subset of the smooth 2π-periodic functions on $\mathbb{R}^n$. Then our algebra, denoted

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by $\mathcal{G}_n$, is a factor of an algebra of functions defined on $\mathcal{A}^{(n)}$ with values in the algebra of smooth 2π-periodic functions on $\mathbb{R}^n$; in this way we follow Colombeau's idea.

2. Notations

We set $\mathcal{I} = [-\pi, +\pi]$, $\mathbb{R}_+^* = (0, +\infty)$ and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$, where $\mathbb{N}$ is the set of non-negative integers. For each $n \in \mathbb{N}^*$, let $\mathcal{F}_{2\pi}^{(n)}$ denote the algebra of 2π-periodic complex-valued measurable functions on $\mathbb{R}^n$ and $\mathcal{E}_{2\pi}^{(n)}$ the subalgebra of C^∞-functions in $\mathcal{F}_{2\pi}^{(n)}$ with the topology of uniform convergence of functions and their derivarives. If $u \in \mathcal{E}_{2\pi}^{(n)}$ and $\alpha = (\alpha_1, ldots, \alpha_n) \in \mathbb{N}^n$, we set

$$\partial^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

For $q \geq 1$ we set

$$L_{2\pi}^{q(n)} = \left\{f \in \mathcal{F}_{2\pi}^{(n)} \,\Big|\, \|f\|_q = \left(\frac{1}{(2\pi)^n}\int_{\mathcal{I}^n} |f(x)|^q dx\right)^{1/q} < +\infty\right\}.$$

$$L_{2\pi}^{\infty(n)} = \left\{f \in \mathcal{F}_{2\pi}^{(n)} \,\Big|\, \|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathcal{I}^n} |f(x)| < +\infty\right\}.$$

We denote by $\mathcal{D}_{2\pi}^{'(n)}$ the space of 2π-periodic distributions on $\mathbb{R}^n$, that is the topological dual of $\mathcal{E}_{2\pi}^{(n)}$ which is considered as a subspace of $\mathcal{D}'^{(n)}$. An element T of $\mathcal{D}_{2\pi}^{'(n)}$ which is a function acts on f in $\mathcal{E}_{2\pi}^{(n)}$ by $\langle T, f\rangle = \frac{1}{(2\pi)^n}\int_{\mathcal{I}^n} T(x)f(x)\,dx$. If $p \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$ and $T \in \mathcal{D}_{2\pi}^{'(n)}$, then the Fourier coefficient of index p of T is denoted by $\widehat{T}(p)$ or $\widehat{T}_p$, and e^{ipx} is denoted by $e_p(x)$, where $px = p_1 x_1 + \ldots + p_n x_n$. The periodic Dirac measure, that is $\sum e_p$, will be denoted by δ_n, or by δ if no confusion can arise. We have $(\widehat{T}_p)_p \in \mathcal{S}'(\mathbb{Z}^n)$ and $(\widehat{u}_p)_p \in \mathcal{S}(\mathbb{Z}^n)$ for $u \in \mathcal{E}_{2\pi}^{(n)}$ where $\mathcal{S}'(\mathbb{Z}^n)$ and $\mathcal{S}(\mathbb{Z}^n)$ are the complex spaces of *slowly increasing* and *rapidly decreasing* sequences, respectively.

3. The definition of the subset $\mathcal{A}^{(n)}$

We now define some special subsets of $\mathcal{E}_{2\pi}^{(1)}$ and $\mathcal{E}_{2\pi}^{(n)}$ which are needed for the construction of our algebra. Let $m \in \mathbb{N}$, $I_m = \{p \in \mathbb{Z}\,|\,|p| \leq m\}$, $n \in \mathbb{N}^*$ and $\chi \in \mathcal{E}_{2\pi}^{(1)}$. We define the following subsets:

$$\mathcal{A}_m^{(1)} = \left\{\varphi \in \mathcal{E}_{2\pi}^{(1)} : \widehat{\varphi}(p) = 1 \text{ if } p \in I_m\right\}$$

$$\mathcal{A}_m^{(n)} = \left\{\varphi \in \mathcal{E}_{2\pi}^{(n)} : \varphi = \psi^{\otimes n} \text{ for some } \psi \in \mathcal{E}_{2\pi}^{(n)}\right\}$$

$$\mathcal{A}^{(n)} = \bigcup_{m \in \mathbb{N}} \mathcal{A}_m^{(n)}$$

$$\mathcal{A}_m^{(1)}(\chi) = \left\{\varphi \in \mathcal{E}_{2\pi}^{(1)} : \widehat{\varphi}(p) = 1 \text{ if } p \in I_m, \ |\widehat{\varphi}(p)| \leq |\widehat{\chi}(p)| \text{ if } p \notin I_m\right\}$$

$$\mathcal{A}_m^{(n)}(\chi) = \left\{\varphi \in \mathcal{E}_{2\pi}^{(n)} : \varphi = \psi^{\otimes n} \text{ for some } \psi \in \mathcal{A}_m^{(1)}\right\}.$$

We note that

$$\mathcal{A}_m^{(n)} = \bigcup_{\chi \in \mathcal{E}_{2\pi}^{(1)}} \mathcal{A}_m^{(n)}(\chi).$$

Let $\psi \in \mathcal{E}_{2\pi}^{(1)}$ and $\varphi = \psi^{\otimes n}$. We have $\widehat{\varphi}(p) = \widehat{\psi}(p_1) \cdots \widehat{\psi}(p_n) = \widehat{\psi}^{\otimes n}(p)$ so that if $\varphi \in \mathcal{A}^{(n)}$, then $\widehat{\varphi}(p) = 1$ for $p \in I_m$, and if $\varphi \in \mathcal{A}_m^{(n)}(\chi)$, then $\widehat{\varphi}(p) = 1$ for $p \in I_m$ and $|\widehat{\varphi}(p)| \leq |\widehat{\chi}^{\otimes n}(p)|$ for $p \notin I_m$.

We define $\rho_m = \sum_{p \in I_m} e_p$. Clearly we have that $\rho_m \in \mathcal{A}_m^{(1)}$, so $\rho_m^{\otimes n} \in \mathcal{A}_m^{(n)}$. We note that

$$\rho_m(t) = \frac{\sin(m + \frac{1}{2})t}{\sin \frac{t}{2}} \qquad \text{and} \qquad \lim_{m \to +\infty} \rho_m^{\otimes n} = \delta \quad \text{in } \mathcal{D}_{2\pi}^{'(n)}.$$

4. The construction of the algebra $\mathcal{G}_n$

Let $\mathcal{X}_n$ denote the algebra of $\mathcal{E}_{2\pi}^{(n)}$-valued maps on $\mathcal{A}^{(n)}$. We define the sets

$$\mathcal{X}^{(n)} = \left\{ u \in \mathcal{X}_n \left| \begin{array}{l} \text{for all } \chi \in \mathcal{E}_{2\pi}^{(1)} \text{ and } \alpha \in \mathbb{N}^n \text{ there exist } r \in \mathbb{R} \\ N \in \mathbb{N} \text{ and } c > 0 \text{ such that, for all } m \geq N, \\ \varphi \in \mathcal{A}_m^{(n)}(\chi) \Longrightarrow \|\partial^\alpha(u(\varphi))\|_\infty \leq c(m+1)^r \end{array} \right. \right\}$$

$$\mathcal{N}^{(n)} = \left\{ u \in \mathcal{X}_n \left| \begin{array}{l} \text{for all } \chi \in \mathcal{E}_{2\pi}^{(1)}, \alpha \in \mathbb{N}^n \text{ and } q > 0 \text{ there exist} \\ N \in \mathbb{N} \text{ and } c > 0 \text{ such that, for all } m \geq N, \\ \varphi \in \mathcal{A}_m^{(n)}(\chi) \Longrightarrow \|\partial^\alpha(u(\varphi))\|_\infty \leq c(m+1)^{-q} \end{array} \right. \right\}.$$

Proposition 1. *$\mathcal{X}^{(n)}$ is a subalgebra of $\mathcal{X}_n$ and $\mathcal{N}^{(n)}$ is a subalgebra of $\mathcal{X}^{(n)}$.*

Proof. Obviously $\mathcal{X}^{(n)}$ and $\mathcal{N}^{(n)}$ are algebras and $\mathcal{N}^{(n)} \subset \mathcal{X}^{(n)}$. By using the Leibniz formula it is easy to show that $\mathcal{N}^{(n)}$ is an ideal of $\mathcal{X}^{(n)}$ ∎

The following lemma will be useful in the proof of many propositions of the first part of this paper.

Lemma 2. *Let $u \in \mathcal{X}_n$ fulfil the following property:*

For all $\chi \in \mathcal{E}_{2\pi}^{'(1)}$ there exists $(b_p)_p \in S(\mathbb{Z}^n)$ such that for all $\alpha \in \mathbb{N}^n$ there exist $(a_p)_p \in S'(\mathbb{Z}^n)$ and $N \in \mathbb{N}$ that, for all $m \geq N$, $\varphi \in A_m^{(n)}(\chi)$ implies $\|\partial^\alpha(u(\varphi))\|_\infty \leq \sum_{p \notin I_m^n} a_p b_p$.

Then u belongs to $\mathcal{N}^{(n)}$.

Proof. Let $u \in \mathcal{X}_n$ satisfy the above property, $\chi \in \mathcal{E}_{2\pi}^{(1)}, \alpha \in \mathbb{N}^n$ and $q > 0$. There exists $N \in \mathbb{N}$ such that if $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ and $m \geq N$ one has, with the previous notations,

$$\|\partial^\alpha(u(\varphi))\|_\infty \leq \sum_{p \notin I_m^n} a_p b_p.$$

On the other hand, since $(1 + |p|)^{-q} < (1 + m)^{-q}$ when p does not belong to I_m^n, we set

$$\|\partial^\alpha(u(\varphi))\|_\infty \leq \big(\sum (1 + |p|)^q a_p b_p\big)(1 + m)^{-q}$$

for any φ in $\mathcal{A}_m^{(n)}(\chi)$ with $m \geq N$ ∎

Now we give a characterization of the subalgebras $\mathcal{X}^{(n)}$ and $\mathcal{N}^{(n)}$ by the Fourier coefficients $\widehat{u(\varphi)}(p)$ of $u(\varphi)$ for $\varphi \in \mathcal{A}^{(n)}$, $u \in \mathcal{X}_n$ and $p \in \mathbb{Z}^n$.

Proposition 3. *Let $u \in \mathcal{X}_n$. Then we have the following assertions:*

(a) *$u \in \mathcal{X}^{(n)}$ if and only if for all $\chi \in \mathcal{E}_{2\pi}^{(1)}$ and $s \in \mathbb{N}$ there exist $r \in \mathbb{R}, N \in \mathbb{N}$ and $c > 0$ such that, for all $m \geq N$, $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ and $p \in \mathbb{Z}^n$,*

$$|\widehat{u(\varphi)}(p)| \leq c(m + 1)^r(1 + \|p\|^2)^{-s}.$$

(b) *$u \in \mathcal{N}^{(n)}$ if and only if for all $\chi \in \mathcal{E}_{2\pi}^{(1)}, s \in \mathbb{N}$ and $q > 0$ there exist $N \in \mathbb{N}$ and $c > 0$ such that, for all $m \geq N$, $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ and $p \in \mathbb{Z}^n$,*

$$|\widehat{u(\varphi)}(p)| \leq c(m + 1)^{-q}(1 + \|p\|^2)^{-s}.$$

Proof. Since assertion (b) is similar to assertion (a), we shall only prove the last one. For this let $u \in \mathcal{X}^{(n)}, \chi \in \mathcal{E}_{2\pi}^{(1)}$ and $s \in \mathbb{N}$. From the definition of $\mathcal{X}^{(n)}$ we derive that for each $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2s$ there exists $r_\alpha \in \mathbb{R}, N_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that, for any $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ with $m \geq N_\alpha$, one has

$$\|\partial^\alpha(u(\varphi))\|_\infty \leq c_\alpha(m + 1)^{r_\alpha}.$$

On the other hand, for any $p \in \mathbb{Z}^n$ we have

$$\widehat{\partial^\alpha(u(\varphi))}(p) = (ip)^\alpha \widehat{u(\varphi)}(p) \qquad \text{and} \qquad |\widehat{\partial^\alpha(u(\varphi))}(p)| \leq \|\partial^\alpha(u(\varphi))\|_\infty$$

so that

$$\Big(\sup_{|\alpha| \leq 2s} |p^\alpha|\Big)|\widehat{u(\varphi)}(p)| \leq \sum_{\alpha \leq 2s} \|\partial^\alpha(u(\varphi))\|_\infty.$$

Moreover, it is well-known that there exists a strictly positive constant γ depending only on n and s such that, for any $x \in \mathbb{R}^n$, one has

$$\sum_{|\alpha| \leq 2s} |x^\alpha| \geq \gamma(1 + \|x\|^2)^s.$$

By taking

$$r = \sum_{|\alpha| \leq 2s} r_\alpha, \qquad N = \sum_{|\alpha| \leq 2s} N_\alpha, \qquad c = \gamma^{-1} \sum_{|\alpha| \leq 2s} c_\alpha$$

the assertion now follows ∎

We are now in position to give the definition of the algebra $\mathcal{G}_n$.

Definition 1. The *algebra $\mathcal{G}_n$ of periodic generalized functions* is the factor algebra $\mathcal{X}^{(n)}/\mathcal{N}^{(n)}$.

If $u \in \mathcal{X}_n$ and $x \in \mathbb{R}^n$ we set $u = (u_\varphi)_{\varphi \in \mathcal{A}^{(n)}}$ or $u = (u_\varphi)_\varphi$, $[u(\varphi)](x) = u(\varphi, x)$, and for any $\alpha \in \mathbb{N}^n$ we define $\partial^\alpha u$ as $(\partial^\alpha u_\varphi)_\varphi$. Clearly it holds $\partial^\alpha(\mathcal{X}^{(n)}) \subset \mathcal{X}^{(n)}$. So if $U \in \mathcal{G}_n$ and u is a representative of U, writing $\mathrm{cl}(u)$ to denote the class of u, we can consider $\mathrm{cl}(\partial^\alpha u)$. Because $\partial^\alpha(\mathcal{N}^{(n)}) \subset \mathcal{N}^{(n)}$, $\mathrm{cl}(\partial^\alpha u)$ is independent of the representative chosen. Then if we define $\partial^\alpha U$ as $\mathrm{cl}(\partial^\alpha u)$, $\mathcal{G}_n$ becomes a differential algebra which is obviously assocative and commutative, with $\mathrm{cl}((1_\varphi)_\varphi)$ as unit element where $1_\varphi = 1$ for any φ in $\mathcal{A}^{(n)}$.

5. Some basic properties of $\mathcal{G}_n$

We recall that the convolution of two elements f and g in $L_{2\pi}^{1(n)}$ is defined by

$$(f * g)(x) = \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} f(x - y)g(y)\, dy = \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} f(y)g(x - y)\, dy.$$

Whenever $T \in \mathcal{D}_{2\pi}^{'(n)}$ and $\varphi \in \mathcal{E}_{2\pi}^{(n)}$, the convolution $T * \varphi = \sum \widehat{T}(p)\widehat{\varphi}(p)e_p$ belongs to $\mathcal{E}_{2\pi}^{(n)}$. On the other hand we have $\langle T, \varphi \rangle = \sum \widehat{T}(p)\widehat{\varphi}(-p)$.

The definition of a canonical linear embedding of $\mathcal{D}_{2\pi}^{'(n)}$ into $\mathcal{G}_n$ is based on the following two propositions.

Proposition 4. *If $T \in \mathcal{D}_{2\pi}^{'(n)}$, then $(T * \varphi)_\varphi \in \mathcal{X}^{(n)}$.*

Proof. Given $T \in \mathcal{D}_{2\pi}^{'(n)}$, $\chi \in \mathcal{E}_{2\pi}^{(1)}$ and $\alpha \in \mathbb{N}^n$, there exist $s, \lambda \in \mathbb{R}$ such that $|\widehat{T}(p)| \leq \lambda(1 + \|p\|)^s$ for all $p \in \mathbb{Z}^n$. We can write

$$\partial^\alpha(T * \varphi) = \sum_{p \in I_m^n} (ip)^\alpha \widehat{T}(p)\widehat{\varphi}(p)e_p + \sum_{p \notin I_m^n} (ip)^\alpha \widehat{T}(p)\widehat{\varphi}(p)e_p.$$

We have $|(ip)^\alpha| \leq m^{n|\alpha|}$ whenever $p \in I_m^n$ and $\mathrm{card}\, I_m^n = (2m + 1)^n$. Then for $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ it follows

$$\|\partial^\alpha(T * \varphi)\|_\infty \leq \lambda\Big((2m + 1)^n m^{n|\alpha|}(1 + m\sqrt{n})^s + \sum |p^\alpha|(1 + \|p\|)^s |\widehat{\chi}^\otimes(p)|\Big)$$

and from this last inequality we obtain

$$\|\partial^\alpha(T * \varphi)\|_\infty \leq \lambda\Big(2^n n^{s/2} + \sum |p^\alpha|(1 + \|p\|)^s |\widehat{\chi}^\otimes(p)|\Big)(m + 1)^{n(|\alpha|+1)+s}$$

which means that $(T * \varphi)_\varphi \in \mathcal{X}^{(n)}$ ∎

Proposition 5. *If* $\mathbf{i} : \mathcal{D}_{2\pi}^{'(n)} \to \mathcal{X}^{(n)}$ *is defined by* $\mathbf{i}(T) = (T * \varphi)_\varphi$, *then* $\mathbf{i}$ *is a linear one-to-one map satisfying* $\mathbf{i}^{-1}(\mathcal{N}^{(n)}) = \{0\}$.

Proof. Obviously $\mathbf{i}$ is linear. Let $T \in \mathcal{D}_{2\pi}^{'(n)}$ such that $\mathbf{i}(T) = 0$. Then $T * \varphi = 0$ for all $\varphi \in \mathcal{A}^n$, hence if m is any element of $\mathbb{N}$ we have $T * \rho_m^{\otimes n} = 0$. Since $\lim_{m \to +\infty} \rho_m^\otimes = \delta$ in $\mathcal{D}_{2\pi}^{'(n)}$ and $T * \delta = T$, it follows that $T = 0$.

Let us now prove the second claim. Clearly we have $\mathbf{i}(0) = 0 \in \mathcal{N}^{(n)}$. Suppose that $T \in \mathcal{D}_{2\pi}^{'(n)}$ and $\mathbf{i}(T) \in \mathcal{N}^{(n)}$. It follows from the definition of $\mathcal{N}^{(n)}$ that there exists $(N, c) \in \mathbb{N} \times \mathbb{R}_+^*$ such that $\|(T * \rho_m^{\otimes n})\|_\infty \leq c(m+1)^{-1}$ for $m \geq N$. Therefore $\lim_{m \to +\infty} (T * \rho_m^{\otimes n}) = 0$ in $\mathcal{D}_{2\pi}^{'(n)}$ from which one concludes that $T = 0$ ∎

Corollary 6. *The linear map* $\mathbf{j} : \mathcal{D}_{2\pi}^{'(n)} \to \mathcal{G}_n$ *defined by* $\mathbf{j}(T) = \mathrm{cl}(\mathbf{i}(T))$ *is one-to-one and the relation* $\mathbf{j}(\partial^\alpha T) = \partial^\alpha(\mathbf{j}(T))$ *holds true for any* $\alpha \in \mathbb{N}^n$.

Proof. This is an obvious consequence of Proposition 5 and the well-known relation $\partial^\alpha(T * \varphi) = (\partial^\alpha T) * \varphi$ ∎

For $f \in \mathcal{E}_{2\pi}^{(n)}$, let $(f)_\varphi$ denote the sequence taking the constant value f. Clearly $(f)_\varphi \in \mathcal{X}^{(n)}$ hence we can consider $\mathrm{cl}((f)_\varphi)$ in $\mathcal{G}_n$. We have the following

Proposition 7. *If* $f \in \mathcal{E}_{2\pi}^{(n)}$, *then* $(f * \varphi - f)_\varphi \in \mathcal{N}^{(n)}$.

Proof. Given $\chi \in \mathcal{E}_{2\pi}^{(1)}$ and $\varphi \in \mathcal{A}_m^{(n)}(\chi)$, we have

$$f * \varphi - f = \sum_{p \notin I_m^n} (\widehat{\varphi}(p) - 1)\widehat{f}(p)e_p.$$

Hence for any $\alpha \in \mathbb{N}^n$ it holds

$$\|\partial^\alpha(f * \varphi - f)\|_\infty \leq \sum_{p \notin I_m^n} |p^\alpha|(|\widehat{\chi}^{\otimes n}(p)| + 1)|\widehat{f}(p)|.$$

Now by applying Lemma 2 to $u = (f * \varphi - f)_\varphi$ with $a_p = |p^\alpha|(|\widehat{\chi}^{\otimes n}(p)| + 1)$ and $b_p = |\widehat{f}(p)|$, we conclude that $(f * \varphi - f)_\varphi \in \mathcal{N}^{(n)}$ ∎

Corollary 8. *The map* $\sigma : \mathcal{E}_{2\pi}^{(n)} \to \mathcal{G}_n$ *defined by* $\sigma(f) = \mathrm{cl}((f)_\varphi)$ *is a one-to-one morphism of differential algebras and* $\mathbf{j}_{|\mathcal{E}_{2\pi}^{(n)}} = \sigma$.

Proof. Obviously σ is a one-to-one morphism of differential algebras and the second claim follows straigtforwardly from Proposition 7 ∎

6. The relation of association

We can identify an element $T \in \mathcal{D}_{2\pi}^{'(n)}$ with $\mathbf{j}(T)$ and say that a generalized function is a *distribution* if it belongs to $\mathbf{j}(\mathcal{D}_{2\pi}^{'(n)})$. But there is another natural way to compare distributions and generalized functions, namely the association process.

Definition 2. An element $U \in \mathcal{D}_{2\pi}^{'(n)}$ is said to *admit* $T \in \mathcal{D}_{2\pi}^{'(n)}$ as an *associated distribution* if U has a representative u such that

$$\lim_{m \to +\infty} \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} [u(\rho_m^{\otimes n})](x) f(x) \, dx = \langle T, f \rangle$$

for all $f \in \mathcal{E}_{2\pi}^{(n)}$. This relation called *association* will be denoted by $U \approx T$.

We claim that this definition is independent of the chosen representative u. For, if $U = 0$, that is $u \in \mathcal{N}^{(n)}$, there exist $N \in \mathbb{N}$ and $c > 0$ such that $\|u(\rho_m^{\otimes n})\|_\infty \le c(m+1)^{-1}$ for $m \ge N$, from which it follows that $U \approx 0$.

It is easy to see that every distribution is associated with itself, that is, $\mathbf{j}(T) \approx T$ for $T \in \mathcal{D}_{2\pi}^{'(n)}$.

Definition 3. Two elements U and V in $\mathcal{G}_n$ are said to be *associated* or *weakly equal* if $U - V$ is associated with 0. In this case we write $U \approx V$.

Obviously association is an equivalence relation which is compatible with addition and differentiation but not with multiplication as shown by the following example.

Example 1. Let $d \in \mathcal{X}^{(1)}$ be defined by $d(\varphi) = \varphi$. Then d and d^2 are representatives of $\mathbf{j}(\delta)$ and $\mathbf{j}(\delta)^2$, respectively. We have $\lim_{m \to +\infty} \sin x \cdot \rho_m = 0$ in $\mathcal{D}_{2\pi}^{'(1)}$, that is $\mathbf{j}(\sin x)\mathbf{j}(\delta) \approx 0$, but $\mathbf{j}(\sin x)^2 \mathbf{j}(\delta)^2 \approx \mathbf{j}(\cos^2 \frac{x}{2})$. So $\mathbf{j}(\sin x)^2 \mathbf{j}(\delta)^2 \not\approx 0$.

Proposition 9. *We have the following assertions:*

(a) *If f and g are two elements of $L_{2\pi}^{2(n)}$, then $\mathbf{j}(f)\mathbf{j}(g) \approx fg$.*

(b) *If f belongs to $\mathcal{E}_{2\pi}^{(n)}$ and T to $\mathcal{D}_{2\pi}^{'(n)}$, then $\mathbf{j}(f)\mathbf{j}(T) \approx fT$.*

Proof. Let $\psi \in \mathcal{E}_{2\pi}^{(n)}$. If we set $d\mu = \frac{1}{(2\pi)^n} dx$, then we can write

$$\int_{\mathcal{I}^n} \left[(f * \rho_m^{\otimes n})(g * \rho_m^{\otimes n}) - fg \right] \psi \, d\mu$$

$$= \int_{\mathcal{I}^n} (f * \rho_m^{\otimes n})(g * \rho_m^{\otimes n} - g)\psi \, d\mu + \int_{\mathcal{I}^n} (f * \rho_m^{\otimes n} - f)g\psi \, d\mu$$

whence we obtain, by applying the Cauchy-Schwarz inequality,

$$\left| \int_{\mathcal{I}^n} \left[(f * \rho_m^{\otimes n})(g * \rho_m^{\otimes n}) - fg \right] \psi \, d\mu \right|$$

$$\le \left(\|f * \rho_m^{\otimes n}\|_2 \|g * \rho_m^{\otimes n} - g\|_2 + \|f * \rho_m^{\otimes n} - f\|_2 \|g\|_2 \right) \|\psi\|_\infty.$$

Now by using the Parseval equality we have the following relations:

$$\|f * \rho_m^{\otimes n}\|_2 \le \|f\|_2,$$

$$\|g * \rho_m^{\otimes n} - g\|_2^2 = \sum_{p \notin \mathcal{I}_m^n} |\widehat{g}(p)|^2$$

$$\|f * \rho_m^{\otimes n} - f\|_2^2 = \sum_{p \notin \mathcal{I}_m^n} |\widehat{f}(p)|^2.$$

Whence from the convergence of the series $\sum |\widehat{f}(p)|^2$ and $\sum |\widehat{g}(p)|^2$ we derive that the last two sums converge to 0 whenever $m \to +\infty$, proving assertion (a). Since $\mathbf{j}(f) = \sigma(f)$, assertion (b) becomes obvious ∎

7. Restriction to subspaces

Let k and n be integers such that $1 \le k < n$ and $U \in \mathcal{G}_n$. We denote by u a representative of U and by $\mathcal{F}$ the subspace $\{x \in \mathbb{R}^n : x_{k+1}, \ldots, x_n = 0\}$ of $\mathbb{R}^n$ that we shall identify with $\mathbb{R}^k$. We define the *restriction* of u to $\mathcal{F}$ as the element of $\mathcal{X}^{(k)}$, denoted by $u\big|_{\mathcal{F}}$ such that for any $\chi \in \mathcal{A}^{(n)}$

$$(u_{|\mathcal{F}})(\chi^{\otimes k}) = \left[u(\chi^{\otimes n})\right]_{|\mathcal{F}}.$$

If $v \in \mathcal{X}^{(n)}$ and $u - v \in \mathcal{N}^{(n)}$, then $u_{|\mathcal{F}} - v_{|\mathcal{F}} = (u - v)_{|\mathcal{F}} \in \mathcal{N}^{(k)}$. So we have the following definition.

Definition 4. If $U \in \mathcal{G}_n$ and u is a representative of U, then the restriction of U to the subspace $\mathcal{F}$ denoted by $U_{|\mathcal{F}}$ is the element of $\mathcal{G}_k$ defined by $U_{|\mathcal{F}} = \mathrm{cl}\,(u_{|\mathcal{F}})$.

If no confusion can arise, for any strictly positive integer l we shall denote the embedding of $\mathcal{D}_{2\pi}^{'(l)}$ into $\mathcal{G}_l$ by the letter $\mathbf{j}$. The following statement is a result of coherence.

Theorem 10. *We have the following assertions.*

(a) *If f is an element of $\mathcal{E}_{2\pi}^{(n)}$, then $\mathbf{j}(f)_{|\mathcal{F}} = \mathbf{j}(f_{|\mathcal{F}})$.*

(b) *If $T \in \mathcal{D}_{2\pi}^{'(n)}$ is a continuous function with continuous partial derivatives of order lesser or equal l with respect to the last $n - k$ variables where $l > \frac{n-k}{2}$, then $\mathbf{j}(T)_{|\mathcal{F}} \approx \mathbf{j}(T_{|\mathcal{F}})$.*

(c) *If $T \in \mathcal{D}_{2\pi}^{'(n)}$ is continuous with $\sum |\widehat{T}(p)| < +\infty$, then $\mathbf{j}(T)_{|\mathcal{F}} \approx \mathbf{j}(T_{|\mathcal{F}})$.*

Proof. We identify $\mathbb{R}^n$ with $\mathbb{R}^k \times \mathbb{R}^{n-k}$ and denote an element x of $\mathbb{R}^n$ by (x', x''); we shall use the same convention for $\mathbb{Z}^n$. The partial derivatives of T with respect to x'' will be denoted by ∂_2^β for $\beta \in \mathbb{N}^{n-k}$. Since assertion (a) follows directly from Proposition 7, we shall only prove assertions (b) and (c).

If $f \in \mathcal{E}_{2\pi}^{(k)}$ and $\xi \in \mathcal{I}^k$, we set $f_\xi = f(\cdot - \xi)$ and $T_\xi = T(-\xi, \cdot)$. We have

$$\langle (T * \rho_m^{\otimes n})|_{\mathcal{F}}, f \rangle = \frac{1}{(2\pi)^k} \int_{\mathcal{I}^k} \left[\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} T(\xi - y', -y'') \rho_m^{\otimes n}(y) \, dy \right] f(\xi) \, d\xi.$$

By the Fubini theorem and the substitutions $\xi = y' - \eta$, $z' = y'$ and $z'' = -y''$ we derive that

$$\langle (T * \rho_m^{\otimes n})|_{\mathcal{F}}, f \rangle = \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} \left[\frac{1}{(2\pi)^k} \int_{\mathcal{I}^k} T_\xi(y'') f_\xi(y') \, d\xi \right] \rho_m^{\otimes n}(y) \, dy.$$

For ξ fixed , let $a_p(\xi)$ denotes the Fourier coefficient of index p of $T_\xi \otimes f_\xi$. Since $\widehat{f_\xi}(p') = e^{-ip'\xi} \widehat{f}(p')$, we have $a_p(\xi) = e^{-ip'\xi} \widehat{f}(p') \widehat{T_\xi}(p'')$. Now we want to prove that the sum $\sum |a_p(\xi)|$ is finite in order to derive that $T_\xi \otimes f_\xi = \sum a_p(\xi) e_p$ by using the continuity of $T_\xi \otimes f_\xi$. Since $(\widehat{f}(p'))_{p'}$ belongs to $\mathcal{S}(\mathbb{Z}^k)$ we only need to prove that $\sum |\widehat{T_\xi}(p'')|$ is finite for $\sum |a_p(\xi)| = (\sum |\widehat{f}(p')|)(\sum |\widehat{T_\xi}(p'')|)$.

Let $\beta \in \mathbb{Z}^{n-k}$ with $|\beta| \leq l$. Since $\partial_2^\beta T_\xi$ is a continuous function, it follows that $\partial_2^\beta T_\xi$ belongs to $L^2(\mathcal{I}^{n-k})$, and since $\widehat{\partial_2^\beta T_\xi}(p'') = (ip'')^\beta \widehat{T_\xi}(p'')$, by the Parseval equality we derive that

$$\sum \left[\sum_{|\beta| \leq l} (p'')^{2\beta} \right] |\widehat{T_\xi}(p'')|^2 = \sum_{|\beta| \leq l} \|\partial_2^\beta T_\xi\|_2^2$$

where $\|\cdot\|_2$ is the L^2-norm in $L^2(\mathcal{I}^{n-k})$. On the other hand there exists $c > 0$ such that $\sum_{|\beta| \leq l} (p'')^{2\beta} \geq c(1 + \|p''\|^2)^l$, thus

$$\sum (1 + \|p''\|^2)^l |\widehat{T_\xi}(p'')|^2 \leq c^{-1} \sum_{|\beta| \leq l} \|\partial_2^\beta T_\xi\|_2^2 < +\infty.$$

Now using the Cauchy-Schwarz inequality we obtain

$$\sum |\widehat{T_\xi}(p'')| \leq \left[\sum (1 + \|p''\|^2)^l |\widehat{T_\xi}(p'')|^2 \right]^{1/2} \left[\sum (1 + \|p''\|^2)^{-l} \right]^{1/2}.$$

When $l > \frac{n-k}{2}$, then the sum $\sum (1 + \|p''\|^2)^{-l}$ is finite, and so is $\sum |\widehat{T_\xi}(p'')|$, proving the expected result.

From the equality $T_\xi \otimes f_\xi = \sum a_p(\xi) e_p$, by the Fubini theorem and the uniform convergence of this series in $\mathbb{R}^n$ we get

$$\langle (T * \rho_m^{\otimes n})|_{\mathcal{F}}, f \rangle = \frac{1}{(2\pi)^k} \int_{\mathcal{I}^k} \left(\sum_{p \in I_m^n} a_p(\xi) \right) d\xi.$$

Since $\partial_2^\beta T$ is continuous for $|\beta| \leq l$, so is $\xi \rightarrow \|\partial_2^\beta T_\xi\|_2^2$. Thus this last function is bounded on $\mathcal{I}^k$. Then we can derive from the above inequalities that $\sum_{p \in I_m^n} a_p(\xi)$ is uniformly bounded with respect to m on the set $\mathcal{I}^k$. Whence we derive from the Lebesgue bounded convergence theorem that

$$\lim_{m \to +\infty} \int_{\mathcal{I}^k} \left(\sum_{p \in I_m^n} a_p(\xi) \right) d\xi = \int_{\mathcal{I}^k} \left(\sum a_p(\xi) \right) d\xi = \int_{\mathcal{I}^k} T(-\xi, 0) f(-\xi) \, d\xi.$$

Finally we have

$$\lim_{m \to +\infty} \left\langle (T * \rho_m^{\otimes n})_{|\mathcal{F}}, f \right\rangle = \frac{1}{(2\pi)^k} \int_{\mathcal{I}^k} T(\xi, 0) f(\xi) \, d\xi$$

which implies that $\mathbf{j}(T)_{|\mathcal{F}} \approx \mathbf{j}(T_{|\mathcal{F}})$, and assertion (b) is proved.

Now we shall prove assertion (c). It follows from the fact that $\sum |\widehat{T}(p)|$ is finite and from the continuity of T that $T = \sum \widehat{T}(p) e_p$ holds, whence we obtain

$$T * \rho_m^{\otimes n} = \sum_{p \in I_m^n} \widehat{T}(p) e_p \qquad \text{and} \qquad (T * \rho_m^{\otimes n})_{|\mathcal{F}} = \sum_{p \in I_m^n} \widehat{T}(p) e_{p'}.$$

Now we can use again the Lebesgue bounded convergence theorem to obtain the weak equality of $\mathbf{j}(T)_{|\mathcal{F}}$ and $\mathbf{j}(T_{|\mathcal{F}})$ ∎

8. Periodic generalized complex numbers and point values

Let $\mathcal{C}$ denote the algebra of complex-valued maps λ defined on $\mathcal{A}^{(1)}$ and satisfying the property:

For all $\chi \in \mathcal{E}_{2\pi}^{(1)}$ there exist $r \in \mathbb{R}$, $N \in \mathbb{N}$ and $c > 0$ such that, for all $m \geq N$ and $\psi \in \mathcal{A}_m^{(1)}(\chi)$, $|\lambda(\psi)| \leq c(m+1)^r$.

Further, let $\mathcal{J}$ denote the subset of $\mathcal{C}$ consisting of elements λ satisfying the property:

For all $\chi \in \mathcal{E}_{2\pi}^{(1)}$ and $q \in \mathbb{R}$ there exist $N \in \mathbb{N}$ and $c > 0$ such that, for all $m \geq N$ and $\psi \in \mathcal{A}_m^{(1)}(\chi)$, $|\lambda(\psi)| \leq c(m+1)^q$.

We have the following

Proposition 11. *If $T \in \mathcal{D}_{2\pi}^{'(n)}$, we set $\langle T, u \rangle = \left(T, u(\psi^{\otimes n}) \right)_{\psi}$ for $u \in \mathcal{X}_n$ and $\psi \in \mathcal{A}^{(1)}$. Then $T(\mathcal{X}^{(n)}) \subset \mathcal{C}$ and $T(\mathcal{N}^{(n)}) \subset \mathcal{J}$.*

Proof. These two inclusions can be proved in the same way, thus we just prove the first one only. Let $u \in \mathcal{X}^{(n)}$. If $\varphi \in \mathcal{A}^{(n)}$, we have $\langle T, u(\varphi) \rangle = \sum \widehat{T}(p) \widehat{u(\varphi)}(-p)$. Let $\chi \in \mathcal{E}_{2\pi}^{(1)}$. We choose $s \in \mathbb{N}$ fulfilling $\sum |\widehat{T}(p)|(1 + \|p\|^2)^{-s} < +\infty$. From the definition of $\mathcal{X}^{(n)}$ there exist $r \in \mathbb{R}$, $N \in \mathbb{N}$ and $c > 0$ such that for any $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ with $m \geq N$ and any $p \in \mathbb{Z}^n$ one has

$$|\widehat{u(\varphi)}(p)| \leq c(m+1)^r (1 + \|p\|^2)^{-s}.$$

Then we have

$$|\langle T, u(\varphi) \rangle| \leq \left[c \sum |\widehat{T}(p)|(1 + \|p\|^2)^{-s} \right] (m+1)^r$$

for any $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ proving that $\langle T, u \rangle \in \mathcal{C}$ ∎

Obviously $\mathcal{C}$ is an algebra and $\mathcal{J}$ is an ideal of $\mathcal{C}$, therefore we give the

Definition 5. The *algebra of periodic complex numbers* is defined as factor algebra $\overline{C} = C/\mathcal{J}$.

Subsequently, speaking about elements of the algebra $\overline{C}$, we shall omit to mention periodic.

To each complex number z we can assign the class in $\overline{C}$ of the constant map λ such that $\lambda(\psi) = z$ for all $\psi \in \mathcal{A}^{(1)}$. Thus we define a morphism between the algebras $\mathbb{C}$ and $\overline{C}$ which is clearly a canonical embedding. We denote it by $\overline{\mathbf{i}}$.

For any $n \in \mathbb{N}$ there also exists a canonical embedding of $\overline{C}$ in $\mathcal{G}_n$ as follows. If Λ denotes a generalised complex number with λ as representative, then we assign to it the element of $\mathcal{G}_n$ which is the class of $[\psi^{\otimes n} \longmapsto \lambda(\psi)]$. Obviously this class does not depend on the chosen representative λ. Then we define a one-to-one morphism which is denoted by $\overline{\mathbf{j}}$.

Definition 6. An element U of $\mathcal{G}_n$ will be called *constant* if there exists Λ in $\overline{C}$ such that $U = \overline{\mathbf{j}}(\Lambda)$.

Let $x \in \mathbb{R}^n$ and $U \in \mathcal{G}_n$. If u and v are two representatives of U, it is easy to see that $[\psi \longmapsto (u - v)(\psi^{\otimes n})(x)]$ is an element of $\mathcal{J}$. Thus we give the following

Definition 7. Let $x \in \mathbb{R}^n$ and $U \in \mathcal{G}_n$. Then the *point value* of U at x is $\mathrm{cl}\,[\psi \longmapsto u(\psi^{\otimes n})(x)]$.

Clearly if U is constant, then U takes constant point values but the converse is false as shown by the following example.

Example 2. For $m \in \mathbb{N}^*$ let φ_m denote the function in $\mathcal{E}_{2\pi}^{(n)}$ defined by $\varphi_m(x) = \sin x_1 \cdot \exp(-m \sin^2 x_1)$. Let $u \in \mathcal{X}_n$ be such that $u(\rho_m^{\otimes n}) = \varphi_m$ and $u(\varphi) = 0$ otherwise. Clearly $u \in \mathcal{X}^{(n)}$. Then we can set $U = \mathrm{cl}(u)$. Let $x \in \mathbb{R}^n$. If $\sin x_1 = 0$, then $\varphi_m(x) = 0$, otherwise $|\varphi_m(x)| \le \exp(-m \sin^2 x_1)$ with $\sin^2 x_1 > 0$. Then we conclude that $U(x) = 0$ for any $x \in \mathbb{R}^n$. On the other hand we have

$$\frac{\partial}{\partial x_1} \varphi_m(x) = \cos x_1 (1 - 2m \sin^2 x_1) \exp(-m \sin^2 x_1).$$

Hence we derive that

$$\frac{\partial}{\partial x_1} \varphi_m(0, x_2, \ldots, x_n) = 1 \qquad \text{for any } m \in \mathbb{N}^*$$

whence $u \notin \mathcal{N}^{(n)}$, that is $U \ne 0$.

Remark. We can also take φ_m such that $\varphi_m(x) = \cos x_1 \cdot [1 - E_m(\sin x_1)]$ where E_m is an elementary factor of Weierstrass, that is the function defined on $\mathbb{C}$ by $E_m(z) = (1 - z) \exp(z + \frac{z^2}{2!} + ldots + \frac{z^m}{m!})$.

The notion of association in $\overline{C}$ is provided by the following

Definition 8. Two elements Λ and Λ' in $\overline{C}$ are said o be *associated* if they admit representatives λ and λ' such that $\lim_{m \to +\infty}[\lambda(\rho_m) - \lambda'(\rho_m)] = 0$ in $\mathbb{C}$; we write $\Lambda \approx \Lambda'$. One says that Λ is *associated* with $z \in \mathbb{C}$ if $\Lambda \approx \overline{\mathbf{i}}(z)$, that is $\lim_{m \to +\infty} \lambda(\rho_m) = z$.

Obviously the association in $\overline{C}$ is an equivalence relation. Now we shall compute the point values of $[\mathbf{j}(\delta)]$.

Example 3. For any φ in $\mathcal{A}^{(n)}$ we set $o(\varphi) = \sup\{m \in \mathbb{N}|\, \varphi \in \mathcal{A}_m^{(n)}\}$. Let φ be defined by $d_1(\varphi) = \rho_{o(\varphi)}^{\otimes n}$. Obviously $d_1 \in \mathcal{X}^{(n)}$, and if $\varphi = \psi^{\otimes n}$ with $\psi \in \mathcal{A}^{(1)}$, then $o(\varphi) = o(\psi)$. We recall that d is defined by $d(\varphi) = \varphi$ (see Example 1).

If $\varphi \in \mathcal{A}_m^{(n)}$, then $d(\varphi) - d_1(\varphi) = \sum_{p \notin I_m^n} \widehat{\varphi}(p) e_p$. Thus for $\chi \in \mathcal{E}_{2\pi}^{(1)}$, $\alpha \in \mathbb{N}^n$ and $\varphi \in \mathcal{A}_m^{(n)}(\chi)$ we have that

$$\left\| \partial^\alpha \big(d(\varphi) - d_1(\varphi)\big) \right\|_\infty \leq \sum_{p \notin I_m^n} |p^\alpha|\,|\widehat{\chi}^{\otimes n} p)|.$$

Now from Lemma 2 it follows that $d_1 - d_2$ belongs to $\mathcal{N}^{(n)}$, hence $\mathbf{j}(\delta) = \mathrm{cl}(d_1)$. Consequently if $x \in \mathbb{R}^n$, we have

$$[\mathbf{j}(\delta)](x) = \mathrm{cl}\left(d_1(\psi^{\otimes n}, x) \right)_\psi = \mathrm{cl}\left[\prod_{k=1}^n \frac{\sin\left(o(\psi) + \frac{1}{2}\right) x_k}{\sin \frac{x_k}{2}} \right]_\psi$$

Since $o(\rho_m) = m$,

$$d_1(\rho_m^{\otimes n}, x) = \prod_{k=1}^n \frac{\left[\sin\left(m + \frac{1}{2}\right) x_k \right]}{\sin \frac{x_k}{2}}.$$

In particular, $d_1(\rho_m^{\otimes n}, 0) = (2m+1)^n$ whence $\lim_{m \to +\infty} d_1(\rho_m^{\otimes n}, 0) = +\infty$. Subsequently we shall denote this last equality by $[\mathbf{j}(\delta)](0) \approx +\infty$.

Now we introduce some rudiments of the integration theory for periodic generalized functions on compact subsets of $\mathbb{R}^n$. This will be sufficient for our purposes, in particular, to define the Fourier coefficients of periodic generalized funtions (which involve only integration on $\mathcal{I}^n$).

9. Integration on compact subsets of $\mathbb{R}^n$

Let K denote a compact subset of $\mathbb{R}^n$ and u an element of $\mathcal{X}^{(n)}$. Then for any $\psi \in A_m^{(1)}$, $u(\psi^{\otimes n})$ is integrable on K and

$$\psi \mapsto \int_K [u(\psi^{\otimes n})](x)\, dx)$$

is clearly an element of $\mathcal{C}$. On the other hand if $v \in \mathcal{X}^{(n)}$ is such that $u - v \in \mathcal{N}^{(n)}$, then

$$\psi \mapsto \int_K [(u - v)(\psi^{\otimes n})](x)\, dx$$

belongs to $\mathcal{J}$. Therefore the following definition makes sense.

Definition 9. Given $U \in \mathcal{G}_n$, the *integral* of U on the compact set K of $\mathbb{R}^n$ is the element of $\overline{\mathcal{C}}$ denoted by $\int_K U(x)\, dx$ and defined by

$$\int_K U(x)\, dx = cl\left(\psi \mapsto \int_K [u(\psi^{\otimes n})](x)\, dx \right)$$

where u is an arbitrary representative of U and $\psi \in \mathcal{A}^{(1)}$.

Note that the expression $U(x)$ under the integral is just a notation and must not be confused with the point value of U at x. It is obvious that this kind of integration has the property of linearity as the classical one. Thus we are going to give some further results of coherence.

Proposition 12. *If U and V denote two elements of $\mathcal{G}_n$, then for any $\alpha \in \mathbb{N}^n$ we have*

$$\int_{\mathcal{I}^n} \partial^\alpha U(x) \, V(x) \, dx = (-1)^{|\alpha|} \int_{\mathcal{I}^n} U(x) \, \partial^\alpha V(x) \, dx.$$

Proof. This result is a direct consequence of the fact that on the one hand $u(\varphi)$ and $v(\varphi)$ are 2π-periodic for any $\varphi \in \mathcal{A}^{(n)}$, u and v being representatives of U and V, respectively, and on the other hand, integrating by parts, this equality holds with U and V in $\mathcal{E}_{2\pi}^{(n)}$ ∎

Proposition 13. *Let K denote a compact subset of $\mathbb{R}^n$ and f an element of $\mathcal{F}_{2\pi}^{(n)}$. Then we have the following assertions:*

(a) *If $f \in L_n^{2(n)}$, then $\int_K [\mathbf{j}(f)](x) \, dx \approx \int_K f(x) \, dx$.*

(b) *If $f \in \mathcal{E}_{2\pi}^{(n)}$, then $\int_K [\mathbf{j}(f)](x) \, dx = \bar{\mathbf{i}} \left(\int_K f(x) \, dx \right)$.*

Proof. Let $f \in L^2(\mathcal{I}^n)$. Then by the Cauchy-Schwarz inequality we have

$$\left| \int_K (f * \rho_m^{\otimes n} - f)(x) \, dx \right| \leq \sqrt{\operatorname{mes} K} \left(\int_K |f * \rho_m^{\otimes n} - f|^2(x) \, dx \right)^{1/2}.$$

On the other hand, there exist $k \in \mathbb{N}^*$ and k translations $\tau_1, \ldots, \tau_k$ such that $K \subset \cup_{i=1}^k \tau_i(\mathcal{I}^n)$. Thus we obtain

$$\int_K |f * \rho_m^{\otimes n} - f|^2(x) \, dx \leq \sum_{i=1}^k \int_{\tau_i(\mathcal{I}^n)} |f * \rho_m^{\otimes n} - f|^2(x) \, dx$$

and since the integrals of the right-hand side are equal to $\int_{\mathcal{I}^n} |f * \rho_m^{\otimes n} - f|^2(x) \, dx$ it follows that

$$\left| \int_K (f * \rho_m^{\otimes n} - f)(x) \, dx \right| \leq \sqrt{k \operatorname{mes} K} \left(\sum_{p \notin \mathcal{I}_m^n} |\widehat{f}(p)|^2 \right)^{1/2}.$$

By assumption $f \in L^2(\mathcal{I}^n)$, hence $\lim_{m \to +\infty} \sum_{p \notin \mathcal{I}_m^n} |\widehat{f}(p)|^2 = 0$ whence we deduce that assertion (a) holds. Assertion (b) results directly from Proposition 7 ∎

Proposition 14. *We have the following assertions.*

(a) *If S and T are two elements of $\mathcal{D}_{2\pi}^{'(n)}$, then*

$$\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} [\mathbf{j}(S)](x) \, [\mathbf{j}(T)](x) \, dx = \operatorname{cl} \left(\langle S * \check{T}, \psi^{\otimes n} \rangle \right)_{\psi \in \mathcal{A}^{(1)}}.$$

(b) *If T is an element of $\mathcal{D}_{2\pi}^{'(n)}$ and f belongs to $\mathcal{E}_{2\pi}^{(n)}$, then*

$$\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} [\mathbf{j}(T)](x)\, [\mathbf{j}(f)](x)\, dx = \bar{\mathbf{i}}(\langle T, f\rangle).$$

In particular

$$\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} [\mathbf{j}(T)](x)\, [\mathbf{j}(e_{-p})](x)\, dx = \bar{\mathbf{i}}(\widehat{T}(p)).$$

Proof. Let $\chi \in \mathcal{E}_{2\pi}^{(n)}$ and $\psi \in \mathcal{A}_m^{(1)}(\chi)$. By setting $\varphi = \psi^{\otimes n}$ we have

$$\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} (S*\varphi)(x)\,(T*\varphi)(x)\, dx = \sum \widehat{S}_p \widehat{\varphi}_p \widehat{T}_{-p} \widehat{\varphi}_{-p}$$

$$= \sum_{p \in I_m^n} \widehat{S}_p \widehat{T}_{-p} \widehat{\varphi}_{-p} + \sum_{p \notin I_m^n} \widehat{S}_p \widehat{T}_{-p} \widehat{\varphi}_{-p} \widehat{\varphi}_p.$$

Since $\langle S*\check{T}, \varphi\rangle = \sum \widehat{S}_p \widehat{T}_{-p} \widehat{\varphi}_{-p}$ we derive that

$$\left| \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} (S*\varphi)(x)\,(T*\varphi)(x)\, dx - \langle S*\check{T}, \varphi\rangle \right| \le \sum_{p \notin I_m^n} (|\widehat{\chi}_{-p}^{\otimes n}| + 1)\, |\widehat{S}_p \widehat{T}_{-p}|\, |\widehat{\chi}_{-p}^{\otimes n}|.$$

Now we obtain assertion (a) by applying Lemma 2.

In the same way, for $f \in \mathcal{E}_{2\pi}^{(n)}$ we obtain

$$\left| \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} (T*\varphi)(x)\, dx - \langle T, f\rangle \right| \le \sum_{p \notin I_m^n} (|\widehat{\chi}_{-p}^{\otimes n}| + 1)\, |\widehat{T}(p)|\, |\widehat{f}(-p)|$$

whence we conclude that assertion (b) holds ∎

10. Convolution

Convolution in $\mathcal{E}_{2\pi}^{(n)}$ can be extended straightforwardly to $\mathcal{G}_n$. This extension is compatible with the embedding $\mathbf{j}$ and the association process as it will be proved in Proposition 16.

Definition 10. Let u and v denote two elements of $\mathcal{X}_n$. Then the *convolution* of u and v, denoted by $u*v$, is defined by $(u*v)(\varphi) = u(\varphi)*v(\varphi)$ for $\varphi \in \mathcal{A}^{(n)}$.

Proposition 15. *Equipped with convolution, $\mathcal{X}_n$ is an associative and commutative algebra. $\mathcal{X}^{(n)}$ is a subalgebra of $\mathcal{X}_n$ and $\mathcal{N}^{(n)}$ an ideal of $\mathcal{X}^{(n)}$.*

Proof. The first assertion is obvious and the second one follows from the fact that $\|\partial^\alpha(f*g)\|_\infty \le \|\partial^\alpha f\|_\infty \|g\|_\infty$ for f, g in $\mathcal{E}_{2\pi}^{(n)}$ and $\alpha \in \mathbb{N}^n$ ∎

Now let u, v, w, ϕ denote four elements of $\mathcal{X}^{(n)}$ such that $u - w$ and $v - \phi$ are both in $\mathcal{N}^{(n)}$. By writing

$$u * v - w * \phi = (u - w) * (v - \phi) + w * (v - \phi) + \phi * (u - w)$$

it becomes obvious from Proposition 15 that $u * v - w * \phi \in \mathcal{N}^{(n)}$. We give the following

Definition 11. Let U and V denote two elements of $\mathcal{G}_n$. The *convolution* of U and V is the element of $\mathcal{G}_n$ denoted by $U * V$ and defined by

$$U * V = \text{cl}\,[(u * v)(\varphi)]_\varphi$$

where u and v are arbitrary representatives of U and V, respectively.

Obviously, equipped with convolution, $\mathcal{G}_n$ is an associative and commutative algebra. We have the following

Proposition 16. *We have the following assertions.*

(a) *If S and T are elements of $\mathcal{D}_{2\pi}^{'(n)}$, then $\mathbf{j}(S) * \mathbf{j}(T) = \mathbf{j}(S * T)$. In particular $\mathbf{j}(\delta) * \mathbf{j}(T) = \mathbf{j}(T)$.*

(b) *Let U, V in $\mathcal{G}_n$ and S, T in $\mathcal{D}_{2\pi}^{'(n)}$. If $U \approx S$ and $V \approx T$, then $U * V \approx S * T$.*

Proof. Let us prove assertion (a). Since S and T are distributions we must compute

$$\partial^\alpha \left[(S * \varphi) * (T * \varphi) - (S * T) * \varphi \right] = \partial^\alpha \left[(S * T) * (\varphi * \varphi - \varphi) \right]$$

where $\alpha \in \mathbb{N}^n$ and $\varphi \in A_m^{(n)}(\chi)$ with $\chi \in \mathcal{E}_{2\pi}^{(1)}$. We have successively

$$\partial^\alpha \left[(S * T) * (\varphi * \varphi - \varphi) \right] = \sum (ip)^\alpha \widehat{S}(p) \widehat{T}(p)(\widehat{\varphi}(p)^2 - \widehat{\varphi}(p)) e_p$$

$$\left\| \partial^\alpha \left[(S * T) * (\varphi * \varphi - \varphi) \right] \right\|_\infty \leq \sum_{p \notin I_m^n} |p^\alpha \widehat{S}(p) \widehat{T}(p)|(|\widehat{\chi}^{\otimes n}(p)| + 1)|\widehat{\chi}^{\otimes n}(p)|$$

and by aplying Lemma 2 we obtain assertion (a).

To prove assertion (b) we use the following lemma (cf. [2: p. 54 - 56]).

Lemma 17. *If $(T_m)_{m \in \mathbb{N}}$ is a convergent sequence of periodic distributions on $\mathbb{R}^n$, then there exist two constants M and μ such that*

$$|\widehat{T_m}(p)| \leq M(1 + \|p\|)^\mu$$

for all $m \in \mathbb{N}$ and $p \in \mathbb{Z}^n$.

Suppose that U and V satisfy the conditions of assertion (b) in Proposition 16. Letting u and v denote their representatives we set $u_m = u(\rho_m^{\otimes n})$ and $v_m = v(\rho_m^{\otimes n})$. By assumption, we have $\sum_{m \to +\infty} u_m = S$ and $\sum_{m \to +\infty} v_m = T$ in $\mathcal{D}_{2\pi}^{'(n)}$. It follows that, for any p in $\mathbb{Z}^n$, $\sum_{m \to +\infty} \widehat{u_m}(p) = \widehat{S}(p)$ and $\sum_{m \to +\infty} \widehat{v_m}(p) = \widehat{T}(p)$. If f denotes an element of $\mathcal{E}_{2\pi}^{(n)}$, then we have

$$\langle u_m * v_m, f \rangle = \sum \widehat{u_m}(p) \widehat{v_m}(p) \widehat{f}(-p).$$

Therefore using Lemma 17, we obtain

$$\sum_{m \to +\infty} \langle u_m * v_m, f \rangle = \sum \widehat{S}(p) \widehat{T}(p) \widehat{f}(-p)$$

by the Lebesgue bounded convergence theorem. Then $\sum_{m \to +\infty} \langle u_m * v_m, f \rangle = \langle S * T, f \rangle$ which signifies that $U * V \approx S * T$ ∎

11. Fourier coefficients

In this section we define Fourier coefficients for generalized functions. This notion is compatible with the embedding **j** and the convolution operation. We use it to characterize a generalized function having an associate distribution.

Definition 12. Let $U \in \mathcal{G}_n$ and $p \in \mathbb{Z}^n$. The *Fourier coefficient of index* p *of* U is the generalized complex number $\widehat{U}(p) = \frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} U(x)[\mathbf{j}(e_{-p})](x) \, dx$.

This definition means that if u is any representative of U, then we have

$$\widehat{U}(p) = \mathrm{cl} \left[\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} [u(\psi^{\otimes n})](x) e^{-ipx} dx \right]_\psi .$$

Proposition 18. *We have the following assertions.*

(a) *If* $T \in \mathcal{D}_{2\pi}^{'(n)}$ *and* $p \in \mathbb{Z}^n$, *then* $\widehat{\mathbf{j}(T)}(p) = \bar{\mathbf{i}}(\widehat{T}(p))$.

(b) *If* U *and* V *belong to* $\mathcal{G}_n$, *then, for all* $p \in \mathbb{Z}^n$, $\widehat{(U * V)}(p) = \widehat{U}(p) \, \widehat{V}(p)$.

Proof. We have

$$\widehat{\mathbf{j}(T)}(p) = \mathrm{cl} \left[\frac{1}{(2\pi)^n} \int_{\mathcal{I}^n} (T * \psi^{\otimes n})(x) e^{-ipx} dx \right]_\psi$$

which means that $\widehat{\mathbf{j}(T)}(p) = \mathrm{cl}\,[\widehat{T}(p)\, \widehat{\psi}^{\otimes n}(p)]_\psi$. When $\psi \in \mathcal{A}_m^{(1)}$, the equality $\widehat{\psi}^{\otimes n}(p) = 1$ holds for $m \geq \sup_{1 \leq k \leq n} |p_k|$, whence we conclude that $\widehat{\mathbf{j}(T)}(p) = \bar{\mathbf{i}}(\widehat{T}(p))$ proving assertion (a). Assertion (b) results from the definition and the relation $\widehat{(u * v)}(p) = \widehat{u}(p)\widehat{v}(p)$ for u and v in $\mathcal{X}_n$ ∎

In the following statement we use the notation $u(\rho_m^{\otimes n}) = u_m$ for $u \in \mathcal{X}_n$.

Theorem 19. *If* U *and* V *belong to* $\mathcal{G}_n$, *then* U *and* V *are weakly equal if and only if the following conditions are satisfied:*

(i) U *and* V *admit representatives* u *and* v *for which there exist two constants* M *and* μ *such that* $|\widehat{u_m}(p) - \widehat{v_m}(p)| \leq M(1 + \|p\|)^\mu$ *for any* $p \in \mathbb{Z}^n$ *and any* $m \in \mathbb{N}$.

(ii) *For all* p *in* $\mathbb{Z}^n$, $\widehat{U}(p) \approx \widehat{V}(p)$.

Proof. We may assume that $V = 0$. If u is a representative of U such that $\lim_{m \to +\infty} u_m = 0$ in $\mathcal{D}_{2\pi}^{'(n)}$, then for all p in $\mathbb{Z}^n$ we have $\lim_{m \to +\infty} \widehat{u_m}(p) = 0$, that is condition (ii). Condition (i) follows obviously from Lemma 17.

Conversely, if conditions (i) and (ii) hold for a representative u of U, we conclude as in the proof of part (b) of Proposition 16 that $U \approx 0$ ∎

12. The principle of nonlinear operations in $\mathcal{G}_n$ and $\overline{C}$

For $k \in \mathbb{N}$, let F denote a complex-valued function defined on $\mathbb{R}^n \times \mathbb{C}^k$ which is periodic with respect to the first variables. We suppose that, when identifying $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{C}^k$ with $\mathbb{R}^{2k}$, F is of class C^∞ on $\mathbb{R}^n \times \mathbb{R}^{2k}$ and satisfies the following condition:

For all $(\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^k \times \mathbb{N}^k$, there exist $M > 0$ and $\mu > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\left| \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \xi} \right)^\beta \left(\frac{\partial}{\partial \eta} \right)^\gamma F(x, \xi + i\eta) \right| \le M(1 + \|\xi + i\eta\|)^\mu$$

where $(\xi, \eta) \in \mathbb{R}^{2k}$ and $\xi + i\eta = (\xi_1 + i\eta_1, \ldots, \xi_k + i\eta_k)$.

A function F which satisfies the above inequality with $(\alpha, \beta, \gamma) = (0, 0, 0)$ is said to be *slowly increasing at infinity uniformly with respect to* x. Thus this condition means that F and all its derivatives are slowly increasing at infinity uniformly with respect to x.

If $u_1, \ldots, u_k$ are k elements in $\mathcal{X}_n$ we set $F(u_1, \ldots, u_k) = \left[F\left(u_1(\varphi), \ldots, u_k(\varphi) \right) \right]_\varphi$, which is obviously an element of $\mathcal{X}_n$. We have the following

Proposition 20. *If $u_1, \ldots, u_k$ are k elements in $\mathcal{X}^{(n)}$, then $F(u_1, \ldots, u_k) \in \mathcal{X}^{(n)}$. Moreover if $v_1, \ldots, v_k$ are k elements in $\mathcal{X}^{(n)}$ such that $u_i - v_i \in \mathcal{N}^{(n)}$ for $i = 1, \ldots, k$, then $F(u_1, \ldots, u_k) - F(v_1, \ldots, v_k) \in \mathcal{N}^{(n)}$.*

Proof. It is a straightforward adaptation of Colombeau's proof for the non-periodic case (see [3: p. 21 - 24]) ∎

Consequently if $(U_1, \ldots, U_k) \in (\mathcal{G}_n)^k$, we can set $F(U_1, \ldots, U_k) = \mathrm{cl}\,[F(u_1, \ldots, u_k)]$ where $u_1, \ldots, u_k$ are representatives of $U_1, \ldots, U_k$, respectively. In the same way, if $(\Lambda_1, \ldots, \Lambda_k) \in \overline{C}^k$, we set $F(\Lambda_1, \ldots, \Lambda_k) = \mathrm{cl}\,[F(\lambda_1, \ldots, \lambda_k)]$ where $\lambda_1, \ldots, \lambda_k$ are representatives of $\Lambda_1 \ldots, \Lambda_k$, respectively.

These operations enable one to solve nonlinear differential problems with strong non-linearities (see [11, 12]). We can show that the results on the Goursat problem obtained in [11] are valid in the new algebra $\mathcal{G}_2$.

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