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Nonregular Pseudo-Differential Operators

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Abstract. We study the boundedness properties of pseudo-differential operators $a(x, D)$ and their adjoints $a(x, D)^*$ with symbols in a certain vector-valued Besov space on Besov spaces $B^s_{p,q}$ and Triebel spaces $F^s_{p,q}$ $(0 < p, q \leq \infty)$. Applications are given to multiplication properties of Besov and Triebel spaces. We show that our results are best possible for both pseudo-differential estimates and multiplication. Denoting by $(\cdot,\cdot)$ the duality between Besov and between Triebel spaces we derive general conditions under which $(a(x, D)f, g) = (f, a(x, D)^*g)$ holds. This requires a precise definition of $a(x, D)f$ and $a(x, D)^*f$ for $f \in F^s_{p,q}$ and $f \in B^s_{p,q}$.

Keywords: *Pseudo-differential operators, function spaces, nonregular symbols, paramultiplication, pointwise multiplication*

AMS subject classification: 47 G 30, 46 E 35, 35 S 50

0. Introduction

One approach to nonlinear partial differential equations is based on the study of linear differential equations with limited regularity. This leads naturally to applications of pseudo-differential operators with nonregular symbols to nonlinear differential equations. In the monograph by M. Taylor [17] one can find many applications of the calculus of nonregular pseudo-differential operators to nonlinear differential equations. And in this book there are also applications of adjoint pseudo-differential operators. In [7, 9, 10, 12, 22] the reader can find a deeper study of the calculus and of some of the applications. Our operators contain as a limiting case the paradifferential operators in the form introduced by Y. Meyer [12]. As a rule, in order to obtain optimal results one needs the whole scale of symbols studied here and in [11]. For such results see, for instance, [9, 10].

In the paper [11] we studied among others the symbol class $SB^m_\delta(r, \mu; N, \lambda)$ which is defined by means of vector-valued Besov spaces $B^r_{\mu,\infty}(B^N_{\lambda,\infty})$. Among other things we proved the boundedness and compactness of the corresponding pseudo-differential operators and their adjoints on Triebel spaces $F^s_{p,q}$ for the values of the parameters $0 < p, q, \mu \leq \infty$. Here we prove the boundedness of these operators and their adjoints on Besov spaces $B^s_{p,q}$. By approximation [11: Lemma 1] this immediately implies compactness. And more generally, we introduce symbols related to $B^r_{\mu,\nu}(B^N_{\lambda,\infty})$. This allows us to prove sharp estimates. We are even able to prove some unexpected estimates for these operators on Triebel spaces. This is done in Section 3. In Section 2 we give a complete

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construction of $a(x,D)f$ and $a(x,D)^*f$ for $f \in F^s_{p,q}$ and $f \in B^s_{p,q}$. Usually one defines a pseudo-differential operator on $\mathcal{S}(\mathbb{R}^n)$ and then extends by continuity. But this works only if $0 < p,q < \infty$. The construction given here works in the full range $0 < p,q \le \infty$, and is consistent with the usual one. For another approach using elementary symbols see [22]. It should be remarked that we need much less regularity in the ξ-variable than [22]. In Section 4 we study in detail under which conditions $(a(x,D)f,g) = (f,a(x,D)^*g)$ holds. It turns out that this is the case, when the boundedness conditions of Section 3 hold. This involves a deeper study of the duality of Besov and Triebel spaces, and of the approximation of distributions in $F^s_{p,q}$ and $B^s_{p,q}$ by entire analytic functions of exponential type. And in the final Section 5 we apply our estimates, and we characterize the multiplication of Besov spaces $B^r_{\mu,\nu} \cdot B^s_{p,q} \subseteq B^s_{p,q}$ $(0 < p,q,\mu,\nu \le \infty)$, and also the mixed multiplication $B^r_{\mu,\nu} \cdot F^s_{p,q} \subseteq F^s_{p,q}$. This section ends with a discussion of the sharpness of our pseudo-differential estimates. And we obtain a complete characterization of the conditions, when boundedness of our pseudo-differential operators holds.

1. Preliminaries on function spaces

Denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions, and by $\mathcal{S}'(\mathbb{R}^n)$ its dual, the space of tempered distributions. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the Fourier transform and its inverse, respectively. Let $\varphi_k \in \mathcal{S}(\mathbb{R}^n)$ $(k \in \mathbb{N}_0)$ be real-valued such that

$$\begin{aligned} &\operatorname{supp}\varphi_0 \subseteq \{\xi : |\xi| \le 2\} \\ &\operatorname{supp}\varphi_k \subseteq \{\xi : 2^{k-1} \le |\xi| \le 2^{k+1}\} \qquad (k \in \mathbb{N}) \end{aligned} \tag{1}$$

and for any multi-index α there exists a constant $C_\alpha > 0$ such that

$$|D^\alpha \varphi_k(\xi)| \le C_\alpha 2^{-k|\alpha|} \qquad (k \in \mathbb{N}_0) \tag{2}$$

$$\sum_{k=0}^\infty \varphi_k(\xi) = 1 \qquad (\xi \in \mathbb{R}^n). \tag{3}$$

Such system exists (see Triebel [20: Remark 2.3.1.1]). One may even choose φ_k in such a way that $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, if $k \ge 1$, for some function $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Let $0 < p,q \le \infty$ and $s \in \mathbb{R}$. Let $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq \mathcal{S}(\mathbb{R}^n)$ be a system of functions satisfying (1) - (3). Define the Besov space $B^s_{p,q}$ and Triebel space $F^s_{p,q}$ as the spaces of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}} = \big\|\{2^{ks}\mathcal{F}^{-1}(\varphi_k \mathcal{F}f)\}\big\|_{l^q(L^p)} < \infty \tag{4}$$

$$\|f\|_{F^s_{p,q}} = \big\|\{2^{ks}\mathcal{F}^{-1}(\varphi_k \mathcal{F}f)\}\big\|_{L^p(l^q)} < \infty \quad \text{if } p < \infty. \tag{5}$$

For their basic properties see Triebel [19 - 21]. In particular these spaces are independent of the chosen system $\{\varphi_k\}_{k\in\mathbb{N}_0}$ and they are quasi-Banach spaces. The Triebel spaces $F^s_{\infty,q}$ $(0 < q \le \infty)$ are defined in Frazier and Jawerth [2]. Define $F^s_{\infty,q}$ $(0 < q \le \infty)$ to be the space of all tempered distributions such that

$$\|f\|_{F^s_{\infty,q}} = \sup_{l,j}\Big(2^{jn}\int_{Q_{l,j}} \sum_{k=j}^\infty 2^{ksq}|\mathcal{F}^{-1}(\varphi_k\mathcal{F}f)|^q\,dx\Big)^{1/q} < \infty \tag{6}$$

where the supremum is extended over all dyadic cubes

$$Q_{l,j} = \left\{x : 2^{-j}l_i \le x_i < 2^{-j}(l_i+1) \quad (i = 1,\dots,n)\right\}$$

where $l = (l_1,\dots,l_n) \in \mathbb{Z}^n$ and $j \in \mathbb{N}$. It holds that

$$B^s_{p,\min\{p,q\}} \hookrightarrow F^s_{p,q} \hookrightarrow B^s_{p,\max\{p,q\}} \tag{7}$$

for $0 < p,q \le \infty$ and $s \in \mathbb{R}$.

The following three lemmata are one of the main tools when we are estimating nonregular pseudo-differential operators.

Lemma 1. *Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions and $0 < c_2 < c_1$ such that*

$$\operatorname{supp}\mathcal{F}f_0 \subseteq \{\xi : |\xi| \le c_1\}$$

$$\operatorname{supp}\mathcal{F}f_k \subseteq \{\xi : c_2 2^k \le |\xi| \le c_1 2^k\} \qquad (k \in \mathbb{N}).$$

Let $s \in \mathbb{R}$ and $0 < p,q \le \infty$. Then it holds that, for some constant $C > 0$,

(a) $$\left\| \sum_{k=0}^{\infty} f_k \right\|_{B^s_{p,q}} \le C \left\|\{2^{ks}f_k\}\right\|_{l^q(L^p)}$$

(b) $$\left\| \sum_{k=0}^{\infty} f_k \right\|_{F^s_{p,q}} \le C \left\|\{2^{ks}f_k\}\right\|_{L^p(l^q)} \quad \text{if } p < \infty.$$

More precisely, if the right side of the inequality in (a) *or* (b) *is finite, then $\left\{\sum_{k=0}^N f_k\right\}_N$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $\sum_{k=0}^\infty f_k$ satisfying this inequality.*

Lemma 2. *Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions and $c > 0$ a constant such that*

$$\operatorname{supp}\mathcal{F}f_k \subseteq \left\{\xi : |\xi| \le c2^k\right\} \qquad (k \in \mathbb{N}_0).$$

Let $s \in \mathbb{R}$ and $0 < p,q \le \infty$.

(a) *If $s > n\left(\max\left\{1,\frac{1}{p}\right\} - 1\right)$, then it holds that*

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{B^s_{p,q}} \le C \left\|\{2^{ks}f_k\}\right\|_{l^q(L^p)}.$$

(b) *If $s > n\left(\max\left\{1,\frac{1}{p},\frac{1}{q}\right\} - 1\right)$, then it holds that*

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{F^s_{p,q}} \le C \left\|\{2^{ks}f_k\}\right\|_{L^p(l^q)} \quad \text{if } p < \infty.$$

More precisely, if the right side of the inequality in (a) *or* (b) *is finite, then $\left\{\sum_{k=0}^N f_k\right\}_N$ converges in $\mathcal{S}'(\mathbb{R}^n)$ to a distribution $\sum_{k=0}^\infty f_k$ satisfying this inequality.*

Both these lemmata are wellknown (see, for example, [5 - 8, 12, 20] or [22]). The counterpart for $F^s_{\infty,q}$ $(0 < q \le \infty)$ is [11: Lemma 13], it holds that

$$\left\| \sum_{k=0}^{\infty} f_k \right\|_{F^s_{\infty,q}} \le C \sup_{l,j} \left(2^{jn} \int_{Q_{l,j}} \sum_{k=j}^{\infty} 2^{ksq}|f_k|^q \, dx \right)^{1/q}. \tag{8}$$

In case $s = n \cdot \left(\max\left\{1,\frac{1}{p}\right\} - 1\right)$ there is the following version.

Lemma 3. *Let $\{f_k\}_{k\in\mathbb{N}_0}$ be a sequence of functions and $c>0$ a constant such that*

$$\operatorname{supp}\mathcal{F}f_k \subseteq \{\xi : |\xi| \le c2^k\} \qquad (k\in\mathbb{N}_0).$$

Let $s = n\cdot\left(\max\left\{1,\frac{1}{p}\right\}-1\right)$ and $0<p\le\infty$.

(a) *If $\gamma=\min\{1,p\}$, then it holds that, for some constant $C>0$,*

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{B^s_{p,\infty}} \le C\left\|\{2^{ks}f_k\}\right\|_{l^\gamma(L^p)}.$$

(b) *If $0<p<1$, then it holds that, for some constant $C>0$,*

$$\left\|\sum_{k=0}^{\infty} f_k\right\|_{B^s_{p,\infty}} \le C\left\|\{2^{ks}f_k\}\right\|_{L^p(l^\infty)}.$$

More precisely, if the right side of the inequality in (a) *or* (b) *is finite, then $\left\{\sum_{k=0}^{N} f_k\right\}_N$ converges in $S'(\mathbb{R}^n)$ to a distribution $\sum_{k=0}^{\infty} f_k$ satisfying this inequality.*

Proof. *Step* 1. Let $g=\sum_{k=0}^{\infty} f_k$ (we assume the convergence for a moment) and let $g_k=\mathcal{F}^{-1}(\varphi_k\mathcal{F}g)$. Then for some natural number $\kappa=\kappa(c)$ we have

$$g_k = \sum_{j=k-\kappa}^{\infty} \mathcal{F}^{-1}\varphi_k * f_j$$

which implies

$$\begin{aligned}
\|g\|_{B^s_{p,\infty}} &\le C\sup_k 2^{ks}\|g_k\|_{L^p}\\
&\le C\sup_k 2^{ks}\left(\sum_{j=k-\kappa}^{\infty}\|\mathcal{F}^{-1}\varphi_k * f_j\|^\gamma_{L^p}\right)^{1/\gamma}\\
&\le C\sup_k 2^{ks}\left(\sum_{j=k-\kappa}^{\infty}2^{jn(1-\gamma)}\|\mathcal{F}^{-1}\varphi_k\|^\gamma_{L^\gamma}\|f_j\|^\gamma_{L^p}\right)^{1/\gamma}\\
&\le C\left(\sum_{k=0}^{\infty}2^{ks\gamma}\|f_k\|^\gamma_{L^p}\right)^{1/\gamma}
\end{aligned}$$

where we have used the convolution inequality [20: Remark 1.5.3.2].

Step 2. Let $0<p<1$. Using the vector-valued version of the inequality of Plancherel-Polya-Nikol'skij (see [13: Proposition 2.4.1/(b)] and [11: Lemma 18]), we get

$$\begin{aligned}
|g_k| &= \left|\int\sum_{j=k-\kappa}^{\infty}\mathcal{F}^{-1}\varphi_k(x-y)f_j(y)\,dy\right|\\
&\le C\left(\int\left|\sup_{j\ge k-\kappa}2^{jn\left(\frac{1}{p}-1\right)}\left|\mathcal{F}^{-1}\varphi_k(x-y)f_j(y)\right|\right|^p dy\right)^{1/p}
\end{aligned}$$

and hence by the Fubini-Tonelli theorem

$$\|g_k\|_{L^p} \le C\left(\iint \left|\sup_{j\ge k-\kappa} 2^{jn\left(\frac{1}{p}-1\right)}\left|\mathcal{F}^{-1}\varphi_k(x-y)f_j(y)\right|\right|^p dxdy\right)^{1/p}$$
$$\le C2^{-kn\left(\frac{1}{p}-1\right)}\left\|\sup_{j\ge k-\kappa} 2^{jn\left(\frac{1}{p}-1\right)}|f_j|\right\|_{L^p}$$

which yields (b).

Step 3. It remains to prove that $\left\{\sum_{k=0}^N f_k\right\}_N$ converges in $\mathcal{S}'(\mathbb{R}^n)$. In case (a) it follows from the inequality just proven that

$$\left\|\sum_{k=M}^N f_k\right\|_{B^s_{p,\infty}} \le C\left(\sum_{k=M}^N 2^{ks\gamma}\|f_k\|^\gamma_{L^p}\right)^{1/\gamma} \longrightarrow 0 \quad \text{if } M,N\to\infty$$

and hence $\left\{\sum_{k=0}^N f_k\right\}_N$ is a Cauchy sequence in $B^s_{p,\infty}$ and thus convergent. In case (b) the vector-valued inequality of Plancherel-Polya-Nikol'skij yields

$$\left\|\sum_{k=0}^\infty |f_k|\right\|_{L^1} \le C\left\|\sup_k 2^{ks}|f_k|\right\|_{L^p}$$

and thus $\left\{\sum_{k=0}^N f_k\right\}_N$ converges in L^1 ∎

We conclude this section with a remark concerning the topology where the convergence takes place. Let

$$\begin{aligned} B_{F^s_{p,q}}(R) &= \left\{f\in F^s_{p,q} : \|f\|_{F^s_{p,q}} \le R\right\} \\ B_{B^s_{p,q}}(R) &= \left\{f\in B^s_{p,q} : \|f\|_{B^s_{p,q}} \le R\right\}. \end{aligned} \tag{9}$$

We provide both sets with the relativ topology of $\mathcal{S}'(\mathbb{R}^n)$. Let $f^{(j)} \to f$ in $\mathcal{S}'(\mathbb{R}^n)$. Then it holds that

$$\|f\|_{F^s_{p,q}} \le \liminf_{j\to\infty}\|f^{(j)}\|_{F^s_{p,q}} \qquad \text{and} \qquad \|f\|_{B^s_{p,q}} \le \liminf_{j\to\infty}\|f^{(j)}\|_{B^s_{p,q}} \tag{10}$$

for any $s\in\mathbb{R}$, $0<p,q\le\infty$ (see [1, 5]). Hence $B_{F^s_{p,q}}(R)$ and $B_{B^s_{p,q}}(R)$ are closed in $F^s_{p,q}$ and $B^s_{p,q}$, respectively. Now it is wellknown that bounded subsets of $\mathcal{S}'(\mathbb{R}^n)$ are relatively compact and metrizable (see [18]). Thus $B_{F^s_{p,q}}(R)$ and $B_{B^s_{p,q}}(R)$ with the relativ topology of $\mathcal{S}'(\mathbb{R}^n)$ are compact metric spaces. This is a generalization of the weak* compactness of the closed unit sphere in case $1\le p,q\le\infty$ to arbitrary $0<p,q\le\infty$.

In Section 4 we present an improvement of the convergence in Lemmata 1 - 3, which is better than convergence in $\mathcal{S}'(\mathbb{R}^n)$.

2. Non-regular pseudo-differential operators

For abbreviation we often write $f_k = \mathcal{F}^{-1}(\varphi_k \mathcal{F} f)$ and for a function $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ let us write

$$a_j(x,\xi) = \mathcal{F}^{-1}_{\eta\to x}\big(\varphi_j(\eta)\mathcal{F}_{x\to\eta}a(\cdot,\xi)\big).$$

Define the vector-valued Besov space $B^r_{\mu,\nu}(B^N_{\lambda,\infty})$ as the space of all $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\|a\|_{B^r_{\mu,\nu}(B^N_{\lambda,\infty})} = \Big\| \Big\{ 2^{jr} \|a_j(x,\cdot)\|_{B^N_{\lambda,\infty}} \Big\} \Big\|_{l^\nu(L^\mu(dx))} < \infty.$$

For their properties see Schmeißer and Triebel [13]. We need these spaces for the parameter values $0 < \mu \le \infty$, $1 \le \lambda \le \infty$, $r \ge \frac{n}{\mu}$ and $N > \frac{n}{\lambda}$. If in addition $r > \frac{n}{\mu}$, there is a description of these spaces by means of differences. We refer to [13: Theorems 2.3.4.1 and 2.3.4.2]. In the language of [13] $B^r_{\mu,\nu}(B^N_{\lambda,\infty})$ are the spaces $SB^{\overline{r}}_{\overline{p},\overline{q}}$ with $\overline{r} = (N, r)$, $\overline{p} = (\lambda, \mu)$ and $\overline{q} = (\infty, \nu)$.

Let $m, r \in \mathbb{R}$, $0 \le \delta \le 1$, $0 < \mu, \nu \le \infty$, $r \ge \frac{n}{\mu}$, $r > 0$, $1 \le \lambda \le \infty$ and $N > \frac{n}{\lambda}$. For a symbol a we write $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ if

$$\begin{aligned} &\sup_k 2^{-km} \Big\| \|a(x, 2^k\cdot)\varphi_k(2^k\cdot)\|_{B^N_{\lambda,\infty}} \Big\|_{L^\infty(dx)} < \infty \\ &\sup_k 2^{-k(m+\delta r)} \|a(x,2^k\cdot)\varphi_k(2^k\cdot)\|_{B^r_{\mu,\nu}(B^N_{\lambda,\infty})} < \infty. \end{aligned} \tag{11}$$

These two norms make $SB^m_\delta(r,\mu,\nu;N,\lambda)$ into a quasi-Banach space (into a Banach space, if $\mu,\nu \ge 1$). It follows from [13: Theorem 2.4.1] that under the present parameters values and if $r > \frac{n}{\mu}$ each symbol $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ is a continuous function.

The definition of these symbol classes is mainly motivated by the pointwise estimate in Proposition 5 (and also by Proposition 4). When using the Littlewood-Paley decomposition of Besov and Triebel spaces as a starting point, it almost immediately allows good estimates for our pseudo-differential operators. Choosing $\mu = \nu = N = \lambda = \infty$, one sees that our symbols include the classical Hörmander classes $S^m_{1,\delta}$. Let us compare our classes with those of Yamazaki [22]. He considers only the case $\delta = 0$, $1 \le \mu, \nu \le \infty$ and $N = \lambda = \infty$. In [22] estimates are obtained by decomposing a symbol into reduced (or elementary) symbols. This can be done with our symbols equally well. By comparing the symbol classes on the reduced symbol level one sees that our class $SB^m_0(r,\mu,\nu;\infty,1)$ equals the class $S'(B^{M,r}_{\mu,\nu})^m$ in [22] (here $M = (1,\dots,1)$). The class $SF^m_0(r,\mu,\nu;\infty,1)$ studied in [11] is even more general than the class $S'(F^{M,r}_{\mu,\nu})^m$ in [22]. On the other hand, in [22] general anisotropic spaces are allowed. However, our discussion extends to parabolic Besov and Triebel spaces (compare [8]). Thus our approach is more general than the one in [22], and it gives sharper results.

For a symbol $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ and a function $f \in \mathcal{S}(\mathbb{R}^n)$ define

$$a(x,D)f(x) = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}a(x,\xi)\mathcal{F}f(\xi)\,d\xi \tag{12}$$

to be the associated pseudo-differential operator. Let

$$a_{j,k}(x,\xi) = \mathcal{F}^{-1}_{\eta\to x}\big(\varphi_j(\eta)\mathcal{F}_{x\to\eta}a(\cdot,\xi)\big)\varphi_k(\xi).$$

We decompose the symbol into three parts

$$a(x,\xi) = a^{(1)}(x,\xi) + a^{(2)}(x,\xi) + a^{(3)}(x,\xi)$$

where

$$a^{(1)}(x,\xi) = \sum_{k=4}^{\infty}\sum_{j=0}^{k-4} a_{j,k}(x,\xi) \tag{13}$$

$$a^{(2)}(x,\xi) = \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} a_{j,k}(x,\xi) \tag{14}$$

$$a^{(3)}(x,\xi) = \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty} a_{j,k}(x,\xi). \tag{15}$$

The adjoint pseudo-differential operator is defined to be, if $f \in \mathcal{S}(\mathbb{R}^n)$,

$$a(x,D)^* f(x) = \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\overline{a(y,\xi)}f(y)\,dyd\xi \tag{16}$$

in the sense of oscillatory integrals. From

$$\mathcal{F}\big(a_{j,k}(x,D)^* f_l\big)(\xi) = \int e^{-iy\cdot\xi}\overline{a_{j,k}(y,\xi)}f_l(y)\,dy = \int \mathcal{F}_x\overline{a_{j,k}(\xi-\eta,\xi)}\mathcal{F}f_l(\eta)\,d\eta$$

one sees that there is only a contribution if $|\xi| \sim 2^k$ and

$$l\begin{cases} \le \max\{j,k\}+4 & \text{if } |j-k|\le 3\\ \sim \max\{j,k\} & \text{if } |j-k|\ge 4.\end{cases}$$

It follows that

$$a^{(1)}(x,D)^* f = \sum_{k=4}^{\infty}\sum_{j=0}^{k-4}\sum_{l=k-3}^{k+3} a_{j,k}(x,D)^* f_l \tag{17}$$

$$a^{(2)}(x,D)^* f = \sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3}\sum_{l=0}^{k+6} a_{j,k}(x,D)^* f_l \tag{18}$$

$$a^{(3)}(x,D)^* f = \sum_{k=0}^{\infty}\sum_{j=k+4}^{\infty}\sum_{l=j-3}^{j+3} a_{j,k}(x,D)^* f_l. \tag{19}$$

Moreover

$$\operatorname{supp}\mathcal{F}\big(a_{j,k}(x,D)^* f_l\big) \subseteq \{\xi : |\xi| \sim 2^k\}. \tag{20}$$

In case $0 < p, q < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{p,q}$ and $F^s_{p,q}$. In this case one can define $a(x, D)f$ and $a(x, D)^* f$ on $B^s_{p,q}$ and $F^s_{p,q}$ through extension by continuity if an estimate on $\mathcal{S}(\mathbb{R}^n)$ is known. If $p = \infty$ or $q = \infty$, this is no longer possible.

The following approach works in the whole range $0 < p, q \leq \infty$. It is probably wellknown to the experts but still unpublished. In [22] there is a different approach using elementary symbols. Starting from (12) and (16) the extension is done in two steps. In the first step we begin by defining $a_{j,k}(x, D)f_k$ and $a_{j,k}(x, D)^* f_l$ if f is in $B^s_{p,q}$ or in $F^s_{p,q}$. This is done in the following two propositions. This is the other main tool in the estimation of pseudo-differential operators.

Proposition 4. *Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a bounded and measurable symbol such that*

$$\operatorname{supp} a(x, \cdot) \subseteq \{\xi : |\xi| \leq c2^k\}.$$

Suppose $0 < p_1, p_2, \mu \leq \infty$, $1 \leq \lambda \leq \infty$ and $\frac{1}{p_1} = \frac{1}{\mu} + \frac{1}{p_2}$.

(a) *If $1 \leq p_1 \leq \infty$, or if $0 < p_1 < 1$ and*

$$\mathcal{F}f \subseteq \{\xi : |\xi| \leq c2^k\},$$

and if $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p_2}\right\}$, then for some constant $C > 0$

$$\|a(x, D)f\|_{L^{p_1}} \leq C \Big\| \|a(\cdot, 2^k \cdot)\|_{B^N_{\lambda,\infty}} \Big\|_{L^\mu} \|f\|_{L^{p_2}}.$$

(b) *If $1 \leq p_1 \leq \infty$, or if $0 < p_1 < 1$ and*

$$\operatorname{supp} \mathcal{F}_x a(\cdot, \cdot) \subseteq \{\eta : |\eta| \leq c2^j\} \times \{\xi : |\xi| \leq c2^k\}$$
$$\operatorname{supp} \mathcal{F}f \subseteq \{\xi : |\xi| \leq c2^l\},$$

and if $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, 1 - \frac{1}{p_1}, \frac{1}{p_1} - \frac{1}{2}\right\}$, then for some constant $C > 0$

$$\|a(x, D)^* f\|_{L^{p_1}} \leq C\, 2^{(\max\{j,k,l\}-k)\, n \cdot \left(\max\left\{1, \frac{1}{p_1}\right\} - 1\right)} \Big\| \|a(\cdot, 2^k \cdot)\|_{B^N_{\lambda,\infty}} \Big\|_{L^\mu} \|f\|_{L^{p_2}}.$$

For the proof of the proposition we need the Hardy-Littlewood maximal operator M_τ defined for $0 < \tau < \infty$ by

$$M_\tau f(x) = \sup_{r>0} \left(\frac{1}{|B(x, r)|} \int_{B(x,r)} |f(y)|^\tau \, dy \right)^{1/\tau}.$$

Recall that M_τ is bounded on L^p if $\tau < p \leq \infty$ and bounded on $L^p(l^q)$ if $\tau < p < \infty$ and $\tau < q \leq \infty$.

Proof of Proposition 4. *Step* 1. Let

$$K(x, x - y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} a(x, \xi)\, d\xi$$

be the kernel of $a(x,D)$. Then if $1 \le \tau \le 2$, it follows from the Hölder and the Hausdorff-Young inequalities that

$$\begin{aligned} |a(x,D)f(x)| &= \left| \int K(x,x-y)f(y)\,dy \right| \\ &\le C \sum_{\nu=-\infty}^{+\infty} 2^{\nu\frac{n}{\tau}} \left(\int \left| K(x,x-y)\varphi_\nu(x-y) \right|^{\tau'} dy \right)^{1/\tau'} M_\tau f(x) \\ &\le C \sum_{\nu=-\infty}^{+\infty} 2^{\nu\frac{n}{\tau}} \left\| \mathcal{F}_\xi^{-1}(\varphi_\nu \mathcal{F}_\xi a(x,\cdot)) \right\|_{L^\tau} M_\tau f(x) \\ &\le C \| a(x,2^k\cdot)\|_{\dot{B}^{n/\tau}_{\tau,1}} M_\tau f(x) \end{aligned}$$

where the dotted space $\dot{B}^{n/\tau}_{\tau,1}$ is the homogeneous Besov space. Then the boundedness of M_τ yields assertion (a) in case $1 < p_2 \le \infty$.

Step 2. Let $0 < \tau < 1$. Then by the the Plancherel-Polya-Nikol'skij inequality (see [20: Remark 1.3.2.1])

$$\begin{aligned} |a(x,D)f(x)| &= \left| \int K(x,x-y)f(y)\,dy \right| \\ &\le C 2^{kn\left(\frac{1}{\tau}-1\right)} \left(\int |K(x,x-y)f(y)|^\tau dy \right)^{1/\tau} \\ &\le C 2^{kn\left(\frac{1}{\tau}-1\right)} \|a(x,\cdot)\|_{\dot{B}^{n/\tau}_{1,\tau}} M_\tau f(x) \\ &\le C \|a(x,2^k\cdot)\|_{B^{n/\tau}_{1,\tau}} M_\tau f(x) \end{aligned}$$

from which assertion (a) follows. Note that in case $p_1 = p_2 = 1$ we have the additional restriction $\mathcal{F}f \subseteq \{\xi : |\xi| \le c2^k\}$.

Step 3. Let

$$K^*(y,x-y) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot\xi} \overline{a(y,\xi)}\, d\xi$$

be the kernel of $a(x,D)^*$. By considering the kernels it is clear that assertion (b) follows from (a) by duality if $1 \le p_1 < \infty$. Let $0 < p_1 < 1$. Since

$$\int e^{-iy\cdot\eta} K^*(y,x-y)f(y)\,dy = \frac{1}{(2\pi)^n} \iint e^{ix\cdot\xi} \mathcal{F}_x \overline{a(\xi+\eta-\zeta,\xi)} \mathcal{F}f(\zeta)\, d\zeta\, d\xi$$

the Plancherel-Polya-Nikol'skij inequality yields

$$\left| \int K^*(y,x-y)f(y)\,dy \right| \le C 2^{\rho n\left(\frac{1}{p_1}-1\right)} \left(\int |K^*(y,x-y)f(y)|^{p_1} dy \right)^{1/p_1} \tag{21}$$

where $\rho = \max\{j,k,l\}$. But then by the Hölder and the Bernstein inequalities

$$\begin{aligned}\left\|a(x,D)^* f\right\|_{L^{p_1}} &\le C 2^{\rho n\left(\frac{1}{p_1}-1\right)} \left(\int \left(\int |K^*(y,x-y)|^{p_1} dx\right)^{\mu/p_1} dy\right)^{1/\mu} \|f\|_{L^{p_2}} \\ &\le C 2^{\rho n\left(\frac{1}{p_1}-1\right)} \left\| \, \|a(\cdot,\cdot)\|_{\dot{B}^{n\left(\frac{1}{p_1}-\frac{1}{2}\right)}_{2,p_1}} \right\|_{L^\mu} \|f\|_{L^{p_2}} \\ &\le C 2^{(\rho-k)n\left(\frac{1}{p_1}-1\right)} \left\| \, \|a(\cdot,2^k\cdot)\|_{\dot{B}^{n\left(\frac{1}{p_1}-\frac{1}{2}\right)}_{2,p_1}} \right\|_{L^\mu} \|f\|_{L^{p_2}}.\end{aligned}$$

Step 4. Let $p_1 = p_2 = \infty$. We want to prove the pointwise estimate

$$\left|a(x,D)^* f(x)\right| \le C \left\| \, \|a(\cdot,2^k\cdot)\|_{B^N_{1,\infty}} \right\|_{L^\infty} M_1 f(x) \tag{22}$$

if $N > n$, from which the case $p_1 = p_2 = \infty$ of assertion (b) follows. By duality we obtain the case $p_1 = p_2 = 1$ of assertion (a), too. Now we have

$$\begin{aligned}\left|a(x,D)^* f(x)\right| &= \left|\int K^*(y,x-y) f(y)\, dy\right| \\ &\le C\left(\sum_{\nu=-\infty}^{+\infty} \sup_y \left|K^*(y,x-y)\varphi_\nu(x-y)\right|\right) M_1 f(x) \\ &\le C\left(\sum_{\nu=-\infty}^{+\infty} 2^{(\nu+k)n} \sup_y \left\|\mathcal{F}_\xi^{-1}(\varphi_{\nu+k}\mathcal{F}_\xi a(y,2^k\cdot))\right\|_{L^1}\right) M_1 f(x).\end{aligned}$$

If $N > n$, then

$$\sum_{\nu\ge -k} 2^{(\nu+k)n} \sup_y \left\|\mathcal{F}_\xi^{-1}(\varphi_{\nu+k}\mathcal{F}_\xi a(y,2^k\cdot))\right\|_{L^1} \le C \left\| \, \|a(\cdot,2^k\cdot)\|_{B^N_{1,\infty}} \right\|_{L^\infty}$$

and, since $\left\|\mathcal{F}^{-1}\varphi_{\nu+k}\right\|_{L^1} \le C$,

$$\sum_{\nu< -k} 2^{(\nu+k)n} \sup_y \left\|\mathcal{F}_\xi^{-1}(\varphi_{\nu+k}\mathcal{F}_\xi a(y,2^k\cdot))\right\|_{L^1} \le C \left\| \, \|a(\cdot,2^k\cdot)\|_{L^1} \right\|_{L^\infty}.$$

Now (22) follows ∎

Let us single out the maximal inequalities.

Proposition 5. *Let* $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ *be a symbol.*

(a) *Let*

$$\operatorname{supp} a(x,\cdot),\ \operatorname{supp} \mathcal{F}f \subseteq \{\xi : |\xi| \le c2^k\}.$$

If $0 < \tau \le 2$ *and* $N > \frac{n}{\tau}$, *then there exists a constant* $C > 0$ *such that*

$$\left|a(x,D)f(x)\right| \le C \|a(x,2^k\cdot)\|_{B^N_{\max\{1,\tau\},\infty}} M_\tau f(x).$$

(b) *Let*

$$\operatorname{supp} \mathcal{F}_x a(\cdot,\cdot) \subseteq \{\eta : |\eta| \le c2^j\} \times \{\xi : |\xi| \le c2^k\}$$

$$\operatorname{supp} \mathcal{F} f \subseteq \{\xi : |\xi| \le c2^l\}.$$

If $0 < \tau \le 1$ *and* $N > \frac{n}{\tau}$, *then there exists a constant* $C > 0$ *such that*

$$\left|a(x,D)^* f(x)\right| \le C 2^{(\max\{j,k,l\}-k)n\left(\frac{1}{\tau}-1\right)} \left\| \|a(\cdot, 2^k \cdot)\|_{B^N_{1,\infty}} \right\|_{L^\infty} M_\tau f(x).$$

Proof. It remains to prove assertion (b) for the case $0 < \tau < 1$. But the proof is the same as for the case $\tau = 1$ by taking (21) with $p_1 = \tau$ as a starting point ∎

Part (a) of Proposition 5 is from [5]. For general parabolic metrics of product type the proposition is found in [8]. In the present context the proof is much more readable. Proposition 4 is the main tool in the case of Besov spaces, and Proposition 5 is the main tool in the case of Triebel spaces. Note that there is a difference in the regularity needed for the ξ-variable.

Now we can define $a_{j,k}(x,D)f_k$ and $a_{j,k}(x,D)^* f_l$ if $f \in B^s_{p,q}$ or $f \in F^s_{p,q}$. For the definition of $a(x,D)f$ and $a(x,D)^* f$ for f in $B^s_{p,q}$ or in $F^s_{p,q}$ we then use Lemmata 1 - 3. For example, it holds that

$$\operatorname{supp} \mathcal{F}\left(\sum_{j=0}^{k-4} a_{j,k}(x,D) f_k \right) \subseteq \{\xi : |\xi| \sim 2^k\}.$$

Then Lemma 1 yields

$$\left\| \sum_{k=4}^{\infty} \sum_{j=0}^{k-4} a_{j,k}(x,D) f_k \right\|_{B^s_{p,q}} \le C \left\| \left\{ 2^{ks} \sum_{j=0}^{k-4} a_{j,k}(x,D) f_k \right\} \right\|_{l^q(L^p)}.$$

If we can show that for say $f \in B^{s+m}_{p,q}$ the right side of this inequality is finite we conclude from Lemma 1 that there exist $g_1 \in B^s_{p,q}$ such that

$$\sum_{k=4}^{N} \sum_{j=0}^{k-4} a_{j,k}(x,D) f_k \longrightarrow g_1 \quad \text{if } N \to \infty$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$. We define $a^{(1)}(x,D)f = g_1$. Similarly

$$\operatorname{supp} \mathcal{F}\left(\sum_{j=k-3}^{k+3} a_{j,k}(x,D) f_k \right) \subseteq \{\xi : |\xi| \le c2^k\}.$$

Here we can apply Lemma 2 or 3, and for suitable f we can conclude that

$$\sum_{k=0}^{N} \sum_{j=k-3}^{k+3} a_{j,k}(x,D) f_k \longrightarrow g_2 \quad \text{if } N \to \infty$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$, and we define $a^{(2)}(x,D)f = g_2$. Finally

$$\operatorname{supp}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{j,k}(x,D)f_k\right) \subseteq \{\xi : |\xi| \sim 2^j\}.$$

Here we can again apply Lemma 1 and for suitable f conclude that

$$\sum_{k=0}^{\infty}\sum_{j=k+4}^{N} a_{j,k}(x,D)f_k = \sum_{j=4}^{N}\sum_{k=0}^{j-4} a_{j,k}(x,D)f_k \longrightarrow g_3 \quad \text{if } N \to \infty$$

with convergence in $\mathcal{S}'(\mathbb{R}^n)$, and we define $a^{(3)}(x,D)f = g_3$. Thus $a(x,D)f$ is well defined for suitable f provided we can show that the right side of some inequality in one of the Lemmata 1 - 3 is finite. Since by (20)

$$\operatorname{supp}\mathcal{F}\left(\sum_j\sum_l a_{j,k}(x,D)^* f_l\right) \subseteq \{\xi : |\xi| \sim 2^k\}$$

the same reasoning applies to $a(x,D)^*f$.

It remains to prove that this definition is consistent with (12) and (16) when $f \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 6. *Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a symbol such that*

$$\sup_k 2^{-km}\left\| \|a(\cdot,2^k\cdot)\varphi_k(2^k\cdot)\|_{B^N_{\lambda,\infty}}\right\|_{L^\infty} < \infty$$

for some $N > n \cdot \max\left\{\frac{1}{2},\frac{1}{\lambda}\right\}$ with $1 \le \lambda \le \infty$ and let $f \in \mathcal{S}(\mathbb{R}^n)$. Then the definition given for $a(x,D)f$ above is consistent with (12), *and the definition given for $a(x,D)^*f$ above is consistent with* (16).

Proof. *Step* 1. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\psi(\xi) = 1$ in a neighbourhood of the origin, and let $\psi_N(\xi) = \psi(2^{-N}\xi)$. For each $\xi \in \mathbb{R}^n$ there exists a set E_ξ of measure zero – the complement of the Lebesgue set – such that if $x \notin E_\xi$, then

$$\mathcal{F}_x^{-1}\left(\psi_N\mathcal{F}_x a(\cdot,\xi)\right) \longrightarrow a(x,\xi) \quad \text{as } N \to \infty.$$

Let $\{\xi_l\}_l$ be dense in $\mathbb{R}^n$ and $E = \cup_{l=0}^\infty E_{\xi_l}$. This is a set of measure zero and we claim that for $(x,\xi) \notin E \times \mathbb{R}^n$

$$\mathcal{F}_x^{-1}\left(\psi_N\mathcal{F}_x a(\cdot,\xi)\right) \longrightarrow a(x,\xi) \quad \text{as } N \to \infty. \tag{23}$$

By hypothesis there exist a constant $C > 0$ and $0 < \tau \le 1$ such that if $|\eta| \le 1$, then

$$\left|a(x,\xi+\eta) - a(x,\xi)\right| \le C(1+|\xi|)^m|\eta|^\tau.$$

For $(x,\xi) \notin E \times \mathbb{R}^n$ choose a subsequence $\{\xi_{l_j}\}_j$ such that $\xi_{l_j} \to \xi$ as $j \to \infty$. Then we obtain

$$\begin{aligned}&\left|\mathcal{F}_x^{-1}(\psi_N \mathcal{F}_x a(\cdot,\xi)) - a(x,\xi)\right| \\ &\quad \le \left|\int \mathcal{F}_x^{-1}\psi_N(x-y)\big(a(y,\xi) - a(y,\xi_{l_j})\big)\,dy\right| \\ &\qquad + \left|\mathcal{F}_x^{-1}(\psi_N \mathcal{F}_x a(\cdot,\xi_{l_j})) - a(x,\xi_{l_j})\right| + \left|a(x,\xi_{l_j}) - a(x,\xi)\right|.\end{aligned}$$

Now choose j so large that the first and the third summand become small independent of N. Fix j and choose N so large that the second summand becomes small. Hence (23) follows.

Step 2. From (23) and the dominated convergence theorem we obtain the assertion for $a(x,D)f$. Then the same argument applies to $a(x,D)^*f$ if $m < -n$. In case $m \ge -n$ choose $l \in \mathbb{N}$ such that $2l > m+n$. Let $a_N^{(i)}(x,D)^*f$ $(i = 1,2,3)$ in obvious notation.Then

$$\begin{aligned}&\frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\overline{a_N^{(i)}(y,\xi)}(1+|\xi|^2)^{-l} f(y)\,dyd\xi \\ &\quad\longrightarrow \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\overline{a^{(i)}(y,\xi)}(1+|\xi|^2)^{-l} f(y)\,dyd\xi \quad \text{if } N \to \infty\end{aligned}$$

pointwise everywhere in $\mathbb{R}^n$, and hence by applying $(1-\Delta)^l$ to both sides we obtain

$$a_N^{(i)}(x,D)^* f(x) \longrightarrow \frac{1}{(2\pi)^n}\iint e^{i(x-y)\cdot\xi}\overline{a^{(i)}(y,\xi)} f(y)\,dyd\xi$$

with convergence in $S'(\mathbb{R}^n)$, where the double integral is understood as an oscillatory integral ∎

Thus we have completed the definition of $a(x,D)f$ and $a(x,D)^*f$ modulo the proof of the finitness of the right side of some inequalities in the Lemmata 1 - 3. This will be done in the next section.

3. Pseudo-differential estimates on $B^s_{p,q}$ and $F^s_{p,q}$

In this section we state and proof the main results of this paper. We begin with the action of $SB^m_\delta(r,\mu,\nu;N,\lambda)$ on $B^s_{p,q}$.

Theorem 7. *Let $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0 < \mu,\nu \le \infty$, $r > 0$, $(1-\delta)r \ge \frac{n}{\mu}$ and $1 \le \lambda \le \infty$. Let $0 < p,q \le \infty$ and $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p}\right\}$.*

(a) *If*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r < s < r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D) : B^{s+m}_{p,q} \to B^s_{p,q}$$

is bounded.

(b) *If* $(1-\delta)r > \frac{n}{\mu}$, $\nu \le q \le \infty$ *and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r < s = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D) : B^{s+m}_{p,q} \to B^s_{p,q}$$

is bounded.

(c) *If* $(1-\delta)r > \frac{n}{\mu}$, $0 < q \le \min\{1,p\}$ *and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r = s < r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D) : B^{s+m}_{p,q} \to B^s_{p,q}$$

is bounded.

Theorem 8. *Let* $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ *be such that* $m \in \mathbb{R}$, $0 < \mu, \nu \le \infty$, $r > 0$, $(1-\delta)r \ge \frac{n}{\mu}$ *and* $1 \le \lambda \le \infty$. *Let* $0 < p, q \le \infty$ *and* $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, 1 - \frac{1}{p}, \frac{1}{p} - \frac{1}{2}\right\}$.

(a) *If*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r < s < (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D)^* : B^s_{p,q} \to B^{s-m}_{p,q}$$

is bounded.

(b) *If* $(1-\delta)r > \frac{n}{\mu}$, $\frac{1}{\nu} + \frac{1}{q} \ge \max\left\{1, \frac{1}{p}\right\}$ *and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r = s < (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D)^* : B^s_{p,q} \to B^{s-m}_{p,q}$$

is bounded.

(c) *If* $(1-\delta)r > \frac{n}{\mu}$ *and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r < s = (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D)^* : B^s_{p,\infty} \to B^{s-m}_{p,\infty}$$

is bounded.

We remark that in both theorems the restrictions on q are sharp. At the end of Section 5 we will present counterexamples.

We continue with the action of $SB^m_\delta(r,\mu,\nu;N,\lambda)$ on $F^s_{p,q}$.

Theorem 9. *Let $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0 < \mu,\nu \le \infty$, $(1-\delta)r > \frac{n}{\mu}$ and $1 \le \lambda \le \infty$. Let $0 < p, q \le \infty$.*

(a) *If $0 < p \le 1$, $0 < p < 1$ in case $\mu = \infty$, $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p}, \frac{1}{q}\right\}$ and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r = s < r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D) : F^{s+m}_{p,q} \to F^s_{p,q}$$

is bounded.

(b) *If $\mu < p \le \infty$, $\nu \le p$, $N > n \cdot \max\left\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p}, \frac{1}{q}\right\}$ and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r < s = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D) : F^{s+m}_{p,q} \to F^s_{p,q}$$

is bounded.

(c) *If $\frac{1}{\mu} + \frac{1}{p} > 1$, $\frac{1}{\nu} + \frac{1}{p} \ge 1$, $N > n \cdot \max\left\{1, \frac{1}{p}, \frac{1}{q}\right\}$ and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r = s < (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D)^* : F^s_{p,q} \to F^{s-m}_{p,q}$$

is bounded.

(d) *If $0 < \mu < \infty$, $p = \infty$, $N > n \cdot \max\left\{1, \frac{1}{p}, \frac{1}{q}\right\}$ and*

$$n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r < s = (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\},$$

then the operator

$$a(x,D)^* : F^s_{\infty,q} \to F^{s-m}_{\infty,q}$$

is bounded.

In Theorem 9 we may always take $0 < q \le \infty$. This is surprising, since in the Besov space case the restrictions on the parameter q are sharp. At the end of Section 5 we discuss the conditions, under which boundedness holds. In particular, if $0 < \mu < \infty$, and in assertion (a) $1 < p \le \infty$ or in assertion (d) $0 < p < \infty$, then no boundedness result can hold whatever $0 < q \le \infty$ is. In the cases of assertions (b) and (c) not treated here there are conditions on q even when the operator is a multiplication operator. Also $\nu \le \mu$ becomes necessary.

For the proof of Theorems 7 - 9 we need two more lemmata.

Lemma 10. *If* $0 < \mu < \mu_1 \le \infty$ *and* $0 < \nu \le \infty$, *then*

$$SB_\delta^m(r,\mu,\nu;N,\lambda) \subseteq SB_{\delta_1}^m(r_1,\mu_1,\nu;N,\lambda)$$

where $r - \frac{n}{\mu} = r_1 - \frac{n}{\mu_1}$ *and* $(1-\delta)r - \frac{n}{\mu} = (1-\delta_1)r_1 - \frac{n}{\mu_1}$. *In particular it holds* $\delta r = \delta_1 r_1$.

This lemma is a consequence of [13: Theorem 2.4.1]. The next lemma is found in [5] or [22].

Lemma 11. *Let* $s \in \mathbb{R}$ *and* $0 < q, r \le \infty$.

(a) *If* $s < 0$, *then for some constant* $C > 0$

$$\left\| \left\{ 2^{js} \left(\sum_{k=0}^{j} |a_k|^r \right)^{1/r} \right\} \right\|_{l^q} \le C \| \{2^{js} a_j\} \|_{l^q}$$

holds.

(b) *If* $s > 0$, *then for some constant* $C > 0$

$$\left\| \left\{ 2^{js} \left(\sum_{k=j}^{\infty} |a_k|^r \right)^{1/r} \right\} \right\|_{l^q} \le C \| \{2^{js} a_j\} \|_{l^q}$$

holds.

In case $r = \infty$ the lemma holds with obvious modification.

Proof of Theorem 7. Step 1. It holds for every $s \in \mathbb{R}$

$$\|a^{(1)}(x,D)f\|_{B^s_{p,q}} \le C\|f\|_{B^{s+m}_{p,q}}.$$

In fact, by Proposition 4/(a)

$$\left\| \sum_{j=0}^{k-4} a_{j,k}(x,D)f_k \right\|_{L^p} \le C \left\| \left\| \sum_{j=0}^{k-4} a_{j,k}(x,2^k\cdot) \right\|_{B^N_{\lambda,\infty}} \right\|_{L^\infty} \|f_k\|_{L^p} \le C 2^{km} \|f_k\|_{L^p}$$

and the assertion follows from Lemma 1/(a).

Step 2. Let $\frac{1}{p_1} = \frac{1}{\mu} + \frac{1}{p}$ and $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - (1-\delta)r < s$. Then using Lemma 2/(a) and again Proposition 4/(a) it follows that

$$\begin{aligned}
&\|a^{(2)}(x,D)f\|_{B^{s+(1-\delta)r-\frac{n}{\mu}}_{p,q}} \\
&\le C \left\|a^{(2)}(x,D)f\right\|_{B^{s+(1-\delta)r}_{p_1,q}} \\
&\le C \left(\sum_{k=0}^{\infty} \sum_{j=k-3}^{k+3} 2^{k(s+(1-\delta)r)q} \|a_{j,k}(x,D)f_k\|^q_{L^{p_1}} \right)^{1/q} \\
&\le C \left(\sum_{k=0}^{\infty} \sum_{j=k-3}^{k+3} 2^{k(s+(1-\delta)r)q} \left\| \|a_{j,k}(\cdot,2^k\cdot)\|_{B^N_{\lambda,\infty}} \right\|^q_{L^\mu} \|f_k\|^q_{L^p} \right)^{1/q} \\
&\le C\|f\|_{B^{s+m}_{p,q}}.
\end{aligned}$$

Step 3. Let $\mu_1 = \max\{\mu, p\}$. Then by Lemma 10 $a \in SB^m_{\delta_1}(r_1, \mu_1, \nu; N, \lambda)$ where $r - \frac{n}{\mu} = r_1 - \frac{n}{\mu_1}$ and $(1-\delta)r - \frac{n}{\mu} = (1-\delta_1)r_1 - \frac{n}{\mu_1}$. If $s < r_1$, it follows as above with $\frac{1}{p} = \frac{1}{\mu_1} + \frac{1}{p_2}$ and $\rho = \min\{1, p\}$ that

$$\begin{aligned}\|a^{(3)}(x,D)f\|_{B^s_{p,q}} &\le C\left(\sum_{j=4}^{\infty} 2^{jsq}\left\|\sum_{k=0}^{j-4} a_{j,k}(x,D)f_k\right\|^q_{L^p}\right)^{1/q}\\ &\le C\left(\sum_{j=4}^{\infty} 2^{jsq}\left(\sum_{k=0}^{j-4}\|a_{j,k}(x,D)f_k\|^\rho_{L^p}\right)^{q/\rho}\right)^{1/q}\\ &\le C\|f\|_{B^{s+m+\frac{n}{\mu}-(1-\delta)r}_{p,q}}\end{aligned}$$

where we have used Lemma 11/(a). Now Steps 1 - 3 yield assertion (a).

Step 4. For assertion (b) we have only to improve the estimate for $a^{(3)}(x,D)$. Let $s = r_1$ and $\rho = \min\{1, p, q\}$. Then if $\nu \le q \le \infty$, then

$$\begin{aligned}\|a^{(3)}(x,D)f\|_{B^s_{p,q}} &\le C\left(\sum_{k=0}^{\infty}\left\|\sum_{j=k+4}^{\infty} a_{j,k}(x,D)f_k\right\|^\rho_{B^s_{p,q}}\right)^{1/\rho}\\ &\le C\left(\sum_{k=0}^{\infty}\left(\sum_{j=k+4}^{\infty} 2^{jsq}\|a_{j,k}(x,D)f_k\|^q_{L^p}\right)^{\rho/q}\right)^{1/\rho}\\ &\le C\left(\sum_{k=0}^{\infty}\left(\sum_{j=k+4}^{\infty} 2^{jsq}\left\|\,\|a_{j,k}(\cdot,2^k\cdot)\|_{B^N_{\lambda,\infty}}\right\|^q_{L^{\mu_1}}\|f_k\|^q_{L^{p_2}}\right)^{\rho/q}\right)^{1/\rho}\\ &\le C\|f\|_{B^{s+m+\frac{n}{\mu}-(1-\delta)r}_{p,\rho}}\end{aligned}$$

since $s = r_1$ and $(1-\delta)r - \frac{n}{\mu} = (1-\delta_1)r_1 - \frac{n}{\mu_1}$.

Step 5. For assertion (c) we have to improve the estimate for $a^{(2)}(x,D)$. If

$$\frac{1}{\mu} + \frac{1}{p}\begin{cases}\le 1, & \text{let } \frac{1}{p_1} = \frac{1}{\mu} + \frac{1}{p} \text{ and } \mu_1 = \mu\\ > 1, & \text{let } p_1 = \min\{1,p\} \text{ and } \mu_1 = \begin{cases}\mu & \text{if } p \le \mu\\ \max\{1,\mu\} & \text{if } p > \mu\end{cases}\end{cases}$$

and let $\rho = \min\{1, p_1\}$. Then using Lemma 3/(a) we obtain

$$\begin{aligned}\|a^{(2)}(x,D)f\|_{B^{s+(1-\delta)r-\frac{n}{\mu}}_{p,\infty}} &\le C\|a^{(2)}(x,D)f\|_{B^{n(\frac{1}{\rho}-1)}_{p_1,\infty}}\\ &\le C\left(\sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} 2^{kn(1-\rho)}\|a_{j,k}(x,D)f_k\|^\rho_{L^{p_1}}\right)^{1/\rho}\\ &\le C\|f\|_{B^{m+n(\frac{1}{\rho}-1)-(1-\delta_1)r_1}_{p_2,\rho}}\end{aligned}$$

where $\frac{1}{p_1} = \frac{1}{\mu_1} + \frac{1}{p_2}$. If $\frac{1}{\mu} + \frac{1}{p} \leq 1$, then $p_2 = p, \rho = 1$ and $(1-\delta_1)r_1 = (1-\delta)r = -s$. If $\frac{1}{\mu} + \frac{1}{p} > 1$, consider first the case $p \geq 1$. Then $\rho = p_1 = 1$ and $1 = \frac{1}{\mu_1} + \frac{1}{p_2}$. If $0 < p < 1$, then $\rho = p_1 = p$ and $n\left(\frac{1}{p} - \frac{1}{p_2}\right) = \frac{n}{\mu_1}$. Then in any case $p_2 \geq p$ and

$$n\left(\frac{1}{\rho} - 1\right) - (1-\delta_1)r_1 + n\left(\frac{1}{p} - \frac{1}{p_2}\right) = n\left(\frac{1}{\mu_1} + \frac{1}{p} - 1\right) - (1-\delta_1)r_1 = s.$$

Thus we obtain since $\rho = \min\{1,p\} \geq q$

$$\|a^{(2)}(x,D)f\|_{B^{s+(1-\delta)r-\frac{n}{\mu}}_{p,\infty}} \leq C\|f\|_{B^{s+m}_{p,q}}$$

and the proof is finished ∎

Proof of Theorem 8. Step 1. By Proposition 4/(b), (17) and Lemma 1/(a)

$$\left\|\sum_{j=0}^{k-4} a_{j,k}(x,D)^* f_k\right\|_{L^p} \leq C\left\|\left\|\sum_{j=0}^{k-4} a_{j,k}(\cdot,2^k\cdot)\right\|_{B^N_{\lambda,\infty}}\right\|_{L^\infty} \|f_k\|_{L^p} \leq C2^{km}\|f_k\|_{L^p}.$$

Hence by Lemma 1/(a), if $s \in \mathbb{R}$, then $a^{(1)}(x,D)^* : B^s_{p,q} \to B^{s-m}_{p,q}$.

Step 2. Let $\rho = \min\{1,p\}$, $\mu_1 = \max\{\mu,p\}$ and $s < (1-\delta_1)r_1 = (1-\delta)r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$. Then with $\frac{1}{p} = \frac{1}{\mu_1} + \frac{1}{p_2}$

$$\begin{aligned}
\|a^{(2)}(x,D)^* f\|_{B^{s-m}_{p,q}} &\leq C\sum_{j=-3}^{3}\left(\sum_{k=0}^{\infty} 2^{k(s-m)q}\left(\sum_{l=0}^{k+6}\|a_{k+j,k}(x,D)^* f_l\|^\rho_{L^p}\right)^{q/\rho}\right)^{1/q} \\
&\leq C\left(\sum_{k=0}^{\infty} 2^{k\left(s-(1-\delta_1)r_1\right)q}\left(\sum_{l=0}^{k+6}\|f_l\|^\rho_{L^{p_2}}\right)^{q/\rho}\right)^{1/q} \\
&\leq C\|f\|_{B^{s-(1-\delta_1)r_1}_{p_2,q}} \\
&\leq C\|f\|_{B^{s+\frac{n}{\mu}-(1-\delta)r}_{p,q}}.
\end{aligned}$$

Again we have used Lemma 11/(a).

Step 3. In case $s = (1-\delta_1)r_1$ one obtains analoguously

$$\|a^{(2)}(x,D)^* f\|_{B^{s-m}_{p,\infty}} \leq C\|f\|_{B^{s+\frac{n}{\mu}-(1-\delta)r}_{p,\min\{1,p\}}}.$$

Step 4. If

$$\frac{1}{\mu} + \frac{1}{p}\begin{cases} \leq 1, & \text{let } \mu_1 = \mu \text{ and } \frac{1}{p_1} = \frac{1}{\mu} + \frac{1}{p} \\ > 1, & \text{let } p_1 = \min\{1,p\} \text{ and } \mu_1 = \begin{cases} \mu & \text{if } p \leq \mu \\ \max\{1,\mu\} & \text{if } p > \mu \end{cases}\end{cases}$$

and let $\rho = \min\{1, p_1\}$. Then $p \le p_2$ and

$$n \cdot \left(\frac{1}{p_1} - \frac{1}{p}\right) + r_1 - n \cdot \left(\frac{1}{\mu_1} + \frac{1}{\rho} - 1\right) = r - n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right)$$

holds. Suppose $s > n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r$, and let $\frac{1}{p_1} = \frac{1}{\mu_1} + \frac{1}{p_2}$ and $s_1 = s + n \cdot \left(\frac{1}{p_1} - \frac{1}{p}\right)$. Then using Lemma 11/(b) we obtain similar to the above reasoning

$$\begin{aligned}
&\left\|a^{(3)}(x,D)^* f\right\|_{B_{p,q}^{s-m+(1-\delta)r-\frac{n}{\mu}}} \\
&\le C\left\|a^{(3)}(x,D)^* f\right\|_{B_{p_1,q}^{s_1-m+(1-\delta)r-\frac{n}{\mu}}} \\
&\le C \sum_{l=-3}^{3} \left(\sum_{k=0}^{\infty} 2^{k\left(s_1 - m + (1-\delta_1) r_1 - \frac{n}{\mu_1}\right)q} \left\| \sum_{j=k+4}^{\infty} a_{j,k}(x,D)^* f_{j+l} \right\|_{L^{p_1}}^{q}\right)^{1/q} \\
&\le C \left(\sum_{k=0}^{\infty} 2^{k\left(s_1 + r_1 - n\left(\frac{1}{\mu_1} + \frac{1}{\rho} - 1\right)\right)q} \left(\sum_{j=k+4}^{\infty} 2^{j\left(n\left(\frac{1}{\rho} - 1\right) - r_1\right)\rho} \|f_j\|_{L^{p_2}}^{\rho}\right)^{q/\rho}\right)^{1/q} \\
&\le C\|f\|_{B_{p_2,q}^{s_1 - \frac{n}{\mu_1}}} \\
&\le C\|f\|_{B_{p,q}^{s}}
\end{aligned}$$

since $s_1 - \frac{n}{\mu_1} + n \cdot \left(\frac{1}{p} - \frac{1}{p_2}\right) = s$. Now Steps 1, 2 and 4 yield assertion (a) of the theorem, and Step 3 yields assertion (b).

Step 5. In case $s = n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r$, if $\rho = \min\{1, p\}$ and $\frac{1}{\nu} + \frac{1}{q} \ge \max\left\{1, \frac{1}{p}\right\}$ one obtains by using Lemma 11/(b)

$$\begin{aligned}
&\left\|a^{(3)}(x,D)^* f\right\|_{B_{p,\infty}^{s-m+(1-\delta)r-\frac{n}{\mu}}} \\
&\le C \sup_k 2^{k\left(s_1 + r_1 - n \cdot \left(\frac{1}{\mu_1} + \frac{1}{\rho} - 1\right)\right)} \left(\sum_{j=k+4}^{\infty} \left\| \|a_{j,k}(\cdot, 2^k \cdot)\|_{B_{\lambda,\infty}^{N}} \right\|_{L^{\mu_1}}^{\rho} \|f_j\|_{L^{p_2}}^{\rho}\right)^{1/\rho} \\
&\le C\|f\|_{B_{p_2,q}^{n\left(\frac{1}{\rho}-1\right)-r_1}} \\
&\le C\|f\|_{B_{p,q}^{s}},
\end{aligned}$$

since

$$n\left(\frac{1}{\rho} - 1\right) - r_1 + n\left(\frac{1}{p} - \frac{1}{p_2}\right) = n\left(\frac{1}{\rho} - 1\right) - r + \frac{n}{\mu} + n\left(\frac{1}{p} - \frac{1}{p_1}\right) = s.$$

This proves assertion (b) ∎

In [11] we used a different argument to handle the estimation of $a^{(3)}(x,D)^*$. The case $1 < p \le \infty$ was derived by duality from the estimate for $a^{(3)}(x,D)$. It turns out that in [11] in most cases the duality argument can be avoided. In any case the B-space case treated here is easier than the F-space case treated in [11], where we have proved sharp estimates. We use them in the following proof of Theorem 9 without further reference. In fact, we show only those estimates which have to be improved.

Proof of Theorem 9. Step 1. If $\mu = \infty$ and $0 < p < 1$, then by Lemma 3/(b)

$$\begin{aligned}\|a^{(2)}(x,D)f\|_{B^{n(\frac{1}{p}-1)}_{p,\infty}} &\le C\sum_{l=-3}^{3}\Big\|\sup_k 2^{kn\left(\frac{1}{p}-1\right)}|a_{k+l,k}(x,D)f_k|\Big\|_{L^p}\\ &\le C\|f\|_{F^{m+n(\frac{1}{p}-1)-(1-\delta)r}_{p,\infty}}.\end{aligned}$$

If $0 < \mu < \infty$ and $0 < p \le 1$, let $\mu_1 = \max\{\mu, p\}$ and $\frac{1}{p} = \frac{1}{\mu_1} + \frac{1}{p_2}$. Then Lemma 3/(a) yields

$$\begin{aligned}\|a^{(2)}(x,D)f\|_{B^{n(\frac{1}{p}-1)}_{p,\infty}} &\le C\sum_{l=-3}^{3}\left(\sum_{k=0}^{\infty}2^{kn(1-p)}\|a_{k+l,k}(x,D)f_k\|^p_{L^p}\right)^{1/p}\\ &\le C\|f\|_{B^{m+n(\frac{1}{p}-1)-(1-\delta_1)r_1}_{p_2,p}}\\ &\le C\|f\|_{F^{m+n(\frac{1}{\mu}+\frac{1}{p}-1)-(1-\delta)r}_{p,\infty}}\end{aligned}$$

where we have used the embedding

$$F^{s+n(\frac{1}{p}-\frac{1}{q})}_{p,\infty} \hookrightarrow B^s_{q,p} \tag{24}$$

for $0 < p < q \le \infty$ (see [3]). Thus assertion (a) is proved.

Step 2. For assertion (b) we use the embedding

$$B^{s+n(\frac{1}{\mu}-\frac{1}{p})}_{\mu,p} \hookrightarrow F^s_{p,q} \tag{25}$$

for $0 < \mu < p \le \infty$ and $0 < q \le \infty$ (see [1, 6] and for the case $p = \infty$ see [11: Lemma 16]). The proof of this embedding extends to

$$SB^m_\delta(r,\mu,p;N,\lambda) \hookrightarrow SF^m_{\delta_1}(r_1,p,q;N,\lambda)$$

where $r - \frac{n}{\mu} = r_1 - \frac{n}{p}$ and $(1-\delta)r - \frac{n}{\mu} = (1-\delta_1)r_1 - \frac{n}{p}$. The symbol classes SF are defined in [11]. Assertion (b) follows now from [11: Theorem 14].

Step 3. For the case $\mu = \infty$ and $0 < p < 1$ of assertion (c) see [11: Theorem 14]. Hence suppose that $0 < \mu < \infty$, $\frac{1}{\mu} + \frac{1}{p} > 1$ and $\frac{1}{\nu} + \frac{1}{p} \ge 1$. Since trivially $\frac{1}{\nu} + \frac{1}{p} \ge \max\left\{1, \frac{1}{p}\right\}$, Step 5 of the proof of Theorem 8 yields

$$\|a^{(3)}(x,D)^*f\|_{B^{s-m+(1-\delta)r-\frac{n}{\mu}}_{p,\infty}} \le C\|f\|_{B^{n(\frac{1}{p}-1)-r_1}_{p_2,p}} \le C\|f\|_{F^s_{p,q}}$$

where we have used (24). Observe that $\frac{1}{\mu} + \frac{1}{p} > 1$ implies $p < p_2$. Now assertion (c) follows.

Step 4. Using (25), assertion (d) follows from

$$\begin{aligned}\left\|a^{(2)}(x,D)^* f\right\|_{F^{s-m}_{\infty,q}} &\le C\left\|a^{(2)}(x,D)^* f\right\|_{B^{(1-\delta)r-m}_{\mu,\infty}} \\ &\le C\sum_{l=-3}^{3}\sup_k 2^{k((1-\delta)r-m)}\left\|a_{k+l,k}(x,D)^*\left(\sum_{j=0}^{k+6} f_j\right)\right\|_{L^\mu} \\ &\le C\left\|f\right\|_{L^\infty}\end{aligned}$$

and the proof is finished ∎

4. Adjoints and duality

Let

$$(f,g) = \int f(x)\overline{g}(x)\,dx \tag{26}$$

be the L^2-scalar product. More generally, denote by $\langle\cdot,\cdot\rangle$ the $(\mathcal{S}',\mathcal{S})$-duality bracket, and let $(f,g) = \langle f,\overline{g}\rangle$. The question arises whether it holds

$$\big(a(x,D)f,g\big) = \big(f,a(x,D)^* g\big), \tag{27}$$

and as we will see, the answer is yes whenever the boundedness results of Section 3 hold. However, we begin with the following

Lemma 12. *Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a measurable symbol such that*

$$|a(x,\xi)| \le C(1+|\xi|)^m$$

for some constant $C > 0$ and $m \in \mathbb{R}$. Then for every $f, g \in \mathcal{S}(\mathbb{R}^n)$

$$\big(a(x,D)f,g\big) = \big(f,a(x,D)^* g\big).$$

Proof. Suppose first $m < -n$. Then by Fubini's theorem

$$\begin{aligned}\big(a(x,D)f,g\big) &= \frac{1}{(2\pi)^n}\iiint e^{i(x-y)\cdot\xi}a(x,\xi)f(y)\overline{g(x)}\,dy\,d\xi\,dx \\ &= \frac{1}{(2\pi)^n}\iiint f(y)\overline{e^{i(y-x)\cdot\xi}\overline{a(x,\xi)}g(x)}\,dx\,d\xi\,dy \\ &= \big(f,a(x,D)^* g\big).\end{aligned}$$

If $m \ge -n$, choose $l \in \mathbb{N}$ such that $2l > m+n$ and let $a^l(x,\xi) = a(x,\xi)(1+|\xi|^2)^{-l}$. Then by the definition of an oscillatory integral $a(x,D)^* g = (1-\Delta)^l a^l(x,D)^* g$ and hence

$$\big(a(x,D)f,g\big) = \big((1-\Delta)^l f,a^l(x,D)^* g\big) = \big(f,a(x,D)^* g\big)$$

and the proof is finished ∎

Denote by $\mathring{B}^s_{p,q}$ and $\mathring{F}^s_{p,q}$ the closure of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}$ and $F^s_{p,q}$, respectively. Recall that $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{p,q}$ and $F^s_{p,q}$ if and only if $0 < p,q < \infty$. Let $p' = \frac{p}{p-1}$ if $1 < p \le \infty$ and $p' = \infty$ if $0 < p \le 1$, and let $s' = s - n \cdot \left(\max\left\{1, \frac{1}{p}\right\} - 1\right)$. It holds

$$(\mathring{B}^s_{p,q})' = B^{-s'}_{p',q'} \tag{28}$$

(see [19]), and there are similar results for the Triebel spaces (see [6, 19, 20]). But we need a generalization of (28) and its analogue for Triebel spaces. Let X be a Banach space and M be any topological vector space such that there exists a continuous injection $M \hookrightarrow X'$, and such that M separates the points of X. Then (X, M) form a dual pairing and we can speak of the weak topologies $\sigma(X, M)$ on X and $\sigma(M, X)$ on M (see [18]).

Proposition 13. *Let $p' = \frac{p}{p-1}$ if $1 < p \le \infty$ and $p' = \infty$ if $0 < p \le 1$, and let $s' = s - n \cdot \left(\max\left\{1, \frac{1}{p}\right\} - 1\right)$.*

(a) *Let $q' = \frac{q}{q-1}$ if $1 < q \le \infty$ and $q' = \infty$ if $0 < q \le 1$. Then it holds*

$$B^s_{p,q} \hookrightarrow (B^{-s'}_{p',q'})'.$$

(b) *Let $q' = \frac{q}{q-1}$ if $1 < q \le \infty$ and $1 \le p \le \infty$, and let $q' = \infty$ if $0 < q \le 1$ or $0 < p < 1$. Then it holds*

$$F^s_{p,q} \hookrightarrow (F^{-s'}_{p',q'})'.$$

Proof. *Step* 1. Let $p_1 = \min\{1, p\}$, $f \in B^s_{p,q}$ and $g \in B^{-s'}_{p',q'}$. Then for some constant $C > 0$

$$\begin{aligned} |(f,g)| &\le \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} \int |f_j g_k|\,dx \\ &\le C \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} 2^{jn\left(\frac{1}{p_1}-1\right)} \|f_j g_k\|_{L^{p_1}} \\ &\le C \|f\|_{B^s_{p,q}} \|g\|_{B^{-s'}_{p',q'}}. \end{aligned}$$

This yields assertion (a). Note that in case $0 < p < 1$ we have used the inequality of Plancherel-Polya-Nikol'skij [20: Remark 1.3.2.1].

Step 2. Let $0 < p < 1$, $f \in F^s_{p,q}$ and $g \in F^{-s'}_{\infty,\infty} = B^{-s'}_{\infty,\infty}$. Then by using the vector-valued inequality of Plancherel-Polya-Nikol'skij [13: Proposition 2.4.1/(b)] we obtain

$$\begin{aligned} |(f,g)| &\le \int \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} |f_j g_k|\,dx \\ &\le C \left\| \sup_j \sup_{|k-j|\le 3} 2^{jn\left(\frac{1}{p}-1\right)} |f_j g_k| \right\|_{L^p} \\ &\le C \|f\|_{F^s_{p,q}} \|g\|_{F^{-s'}_{\infty,\infty}}. \end{aligned}$$

This yields the case $0 < p < 1$ of assertion (b). The case $1 < p < \infty$ can be treated analoguously.

Step 3. To treat the cases $p = 1$ and $p = \infty$ we need to describe the duality between $F^{-s}_{1,q'}$ and $F^{s}_{\infty;q}$ for $0 < q \le \infty$. If $Q = Q_{l,j}$, define (compare [2])

$$\sup\nolimits_Q(f) = 2^{j\frac{n}{2}} \sup_{x \in Q_{l,j}} |f_j(x)|.$$

Then

$$\begin{aligned} |(f,g)| &\le \int \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} |f_j g_k|\, dx \\ &\le \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} \sum_{l} \int_{Q_{l,j}} |f_j g_k|\, dx \\ &\le C \sum_{j=0}^{\infty} \sum_{l} \sum_{k=j-3}^{j+3} \sum_{|l-m| \le 8n} \sup\nolimits_{Q_{l,j}}(f) \sup\nolimits_{Q_{m,k}}(g) \end{aligned}$$

and hence, arguing as in the proof of [2: Theorem 5.9] we obtain by using [2: Lemma 2.5 and (5.6)]

$$|(f,g)| \le \int \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+3} |f_j g_k|\, dx \le C \|f\|_{F^{s}_{\infty,q}} \|g\|_{F^{-s}_{1,q'}}$$

and the proof is finished ∎

With that proposition we are able to improve in the following Lemmata 14 - 16 the convergence in Lemmata 1 - 3.

Lemma 14. *Let* $p' = \frac{p}{p-1}$ *if* $1 < p \le \infty$ *and* $p' = \infty$ *if* $0 < p \le 1$, *and let* $s' = s - n \cdot \left(\max\left\{1, \frac{1}{p}\right\} - 1\right)$. *Let* $\{f_k\}_{k \in \mathbb{N}_0}$ *be a sequence of functions and* $0 < c_2 < c_1$ *constants such that*

$$\begin{aligned} &\operatorname{supp} \mathcal{F} f_0 \subseteq \{\xi : |\xi| \le c_1\} \\ &\operatorname{supp} \mathcal{F} f_k \subseteq \{\xi : c_2 2^k \le |\xi| \le c_1 2^k\} \qquad (k \in \mathbb{N}). \end{aligned}$$

Let $s \in \mathbb{R}$ *and* $0 < p, q \le \infty$.

(a) *Let* $q' = \frac{q}{q-1}$ *if* $1 < q \le \infty$ *and* $q' = \infty$ *if* $0 < q \le 1$. *If* $\left\|\{2^{ks} f_k\}\right\|_{l^q(L^p)} < \infty$, *then*

$$\sum_{k=0}^{N} f_k \longrightarrow \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(B^{s}_{p,q}, B^{-s'}_{p',q'}\right).$$

(b) *Let* $q' = \frac{q}{q-1}$ *if* $1 < q \le \infty$ *and* $1 \le p \le \infty$, *and let* $q' = \infty$ *if* $0 < q \le 1$ *or* $0 < p < 1$. *If* $\left\|\{2^{ks} f_k\}\right\|_{L^p(l^q)} < \infty$, *then*

$$\sum_{k=0}^{N} f_k \longrightarrow \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(F^{s}_{p,q}, F^{-s'}_{p',q'}\right).$$

Lemma 15. *Let* $p' = \frac{p}{p-1}$ *if* $1 < p \le \infty$ *and* $p' = \infty$ *if* $0 < p \le 1$, *and let* $s' = s - n \cdot \left(\max\{1, \frac{1}{p}\} - 1\right)$. *Let* $\{f_k\}_{k \in \mathbb{N}_0}$ *be a sequence of functions and* $c > 0$ *a constant such that*

$$\operatorname{supp} \mathcal{F} f_k \subseteq \{\xi : |\xi| \le c2^k\} \qquad (k \in \mathbb{N}_0).$$

Let $s \in \mathbb{R}$ *and* $0 < p, q \le \infty$.

(a) *Let* $q' = \frac{q}{q-1}$ *if* $1 < q \le \infty$ *and* $q' = \infty$ *if* $0 < q \le 1$. *If*

$$s > n \cdot \left(\max\{1, \tfrac{1}{p}\} - 1\right) \qquad \text{and} \qquad \left\|\{2^{ks} f_k\}\right\|_{l^q(L^p)} < \infty,$$

then

$$\sum_{k=0}^{N} f_k \longrightarrow \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(B^s_{p,q}, B^{-s'}_{p',q'}\right).$$

(b) *Let* $q' = \frac{q}{q-1}$ *if* $1 < q \le \infty$ *and* $1 \le p \le \infty$, *and let* $q' = \infty$ *if* $0 < q \le 1$ *or* $0 < p < 1$. *If*

$$s > n \cdot \left(\max\{1, \tfrac{1}{p}, \tfrac{1}{q}\} - 1\right) \qquad \text{and} \qquad \left\|\{2^{ks} f_k\}\right\|_{L^p(l^q)} < \infty,$$

then

$$\sum_{k=0}^{N} f_k \longrightarrow \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(F^s_{p,q}, F^{-s'}_{p',q'}\right).$$

We remark that both lemmata have to be modified in the $F^s_{\infty,q}$-case (compare [11: Lemma 13]). The relevant condition is in that case

$$\sup_{l,j} \left(2^{jn} \int_{Q_{l,j}} \sum_{k=j}^{\infty} 2^{ksq} |f_k|^q \, dx\right)^{1/q} < \infty. \tag{29}$$

Lemma 16. *Let* $p' = \frac{p}{p-1}$ *if* $1 < p \le \infty$ *and* $p' = \infty$ *if* $0 < p \le 1$. *Let* $\{f_k\}_{k \in \mathbb{N}_0}$ *be a sequence of functions and* $c > 0$ *a constant such that*

$$\operatorname{supp} \mathcal{F} f_k \subseteq \{\xi : |\xi| \le c2^k\} \qquad (k \in \mathbb{N}_0).$$

Let $s = n \cdot \left(\max\{1, \frac{1}{p}\} - 1\right)$ *and* $0 < p \le \infty$.

(a) *Let* $q' = \frac{q}{q-1}$ *if* $1 < q \le \infty$ *and* $q' = \infty$ *if* $0 < q \le 1$, *and let* $\gamma = \min\{1, p\}$. *If*

$$\left\|\{2^{ks} f_k\}\right\|_{l^\gamma(L^p)} < \infty,$$

then

$$\sum_{k=0}^{N} f_k \to \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(B^s_{p,\infty}, B^0_{p',1}\right).$$

(b) *Let* $0 < p < 1$. *If*

$$\left\|\{2^{ks} f_k\}\right\|_{L^p(l^\infty)} < \infty,$$

then

$$\sum_{k=0}^{N} f_k \to \sum_{k=0}^{\infty} f_k \qquad \text{in } \sigma\left(B^s_{p,\infty}, B^0_{\infty,1}\right).$$

The proofs of Lemmata 14 - 16 are a combination of the proofs of Lemmata 1 - 3 with that of Propostion 13.

Now we are in position to state and proof the main results of this section.

Theorem 17. *Let $a \in SB_\delta^m(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0 < \mu,\nu \le \infty$, $r > 0$, $(1-\delta)r \ge \frac{n}{\mu}$ and $1 \le \lambda \le \infty$. Let $0 < p,q \le \infty$ and $N > n \cdot \max\{\frac{1}{2}, \frac{1}{\lambda}, \frac{1}{p}\}$. As before let*

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p \le \infty \\ \infty & \text{if } 0 < p \le 1 \end{cases} \quad \text{and} \quad q' = \begin{cases} \frac{q}{q-1} & \text{if } 1 < q \le \infty \\ \infty & \text{if } 0 < q \le 1 \end{cases}$$

and let $s' = s - n \cdot (\max\{1, \frac{1}{p}\} - 1)$. Suppose that one of the following conditions holds:

(a) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - (1-\delta)r < s < r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$.

(b) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - (1-\delta)r < s = r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$ *and* $(1-\delta)r > \frac{n}{\mu}$, $\nu \le q \le \infty$.

(c) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - (1-\delta)r = s < r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$ *and* $(1-\delta)r > \frac{n}{\nu}$, $0 < q \le \min\{1,p\}$.

Then for any $f \in B_{p,q}^{s+m}$ and $g \in B_{p',q'}^{-s'}$ it holds that $(a(x,D)f, g) = (f, a(x,D)^* g)$.

Theorem 18. *Let $a \in SB_\delta^m(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0 < \mu,\nu \le \infty$, $r > 0$, $(1-\delta)r \ge \frac{n}{\mu}$ and $1 \le \lambda \le \infty$. Let $0 < p,q \le \infty$ and $N > n \cdot \max\{\frac{1}{2}, \frac{1}{\lambda}, 1 - \frac{1}{p}, \frac{1}{p} - \frac{1}{2}\}$. As before let*

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1 < p \le \infty \\ \infty & \text{if } 0 < p \le 1 \end{cases} \quad \text{and} \quad q' = \begin{cases} \frac{q}{q-1} & \text{if } 1 < q \le \infty \\ \infty & \text{if } 0 < q \le 1 \end{cases}$$

and let $s' = s - n \cdot (\max\{1, \frac{1}{p}\} - 1)$. Suppose that one of the following conditions holds:

(a) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - r < s < (1-\delta)r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$.

(b) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - r = s < (1-\delta)r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$ *and* $(1-\delta)r > \frac{n}{\mu}$, $\frac{1}{\nu} + \frac{1}{q} \ge \max\{1, \frac{1}{p}\}$.

(c) $n \cdot (\max\{1, \frac{1}{\mu} + \frac{1}{p}\} - 1) - r < s = (1-\delta)r - n \cdot \max\{\frac{1}{\mu} - \frac{1}{p}, 0\}$ *and* $(1-\delta)r > \frac{n}{\mu}$, $q = \infty$.

Then for any $f \in B_{p,q}^{s}$ and $g \in B_{p',q'}^{-s'+m}$ it holds that $(a(x,D)^* f, g) = (f, a(x,D) g)$.

It suffices to prove Theorem 17, the proof of Theorem 18 being similar.

Proof of Theorem 17. Observe that for $i = 1,2,3$ we have boundedness

$$a^{(i)}(x,D) : B_{p,q}^{s+m} \longrightarrow B_{p,q}^{s}$$

$$a^{(i)}(x,D)^* : B_{p',q'}^{-s'} \longrightarrow B_{p',q'}^{-s'-m}.$$

Now by the proof of Proposition 4, $a_N^{(i)}(x,D)$ and $a_N^{(i)}(x,D)^*$ are integral operators adjoint to each other. Let $f^N = \sum_{k=0}^N f_k$. Then

$$\begin{aligned}(a^{(i)}(x,D)f^N, g) &= (a_{N+4}^{(i)}(x,D)f^N, g^{N+8}) \\ &= (f^N, a_{N+4}^{(i)}(x,D)^* g^{N+8}) \\ &= (f^N, a^{(i)}(x,D)^* g).\end{aligned}$$

Since by Lemmata 14 - 16 $f^N \to f$ in $\sigma(B_{p,q}^{s+m}, B_{p',q'}^{-s'-m})$ and $a^{(i)}(x,D)f^N \to a^{(i)}(x,D)f$ in $\sigma(B_{p,q}^{s}, B_{p',q'}^{-s'})$, the theorem follows ∎

Theorems 17 - 18 have obvious counterparts for operators on Triebel spaces. For the appropriate symbol classes and the exact boundedness results needed see [11].

Theorem 19. *Let $a \in SF_\delta^m(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0<\mu,\nu\le\infty$, $r>0$, $0\le\delta<1$, $(1-\delta)r \ge \frac{n}{\mu}$ and $1\le\lambda\le\infty$. Let $0<p,q\le\infty$ and $N > n\cdot\max\{1,\frac{1}{p},\frac{1}{q}\}$. Let*

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1<p\le\infty \\ \infty & \text{if } 0<p\le 1 \end{cases} \quad \text{and} \quad q' = \begin{cases} \frac{q}{q-1} & \text{if } 1<q\le\infty \text{ and } 1\le p\le\infty \\ \infty & \text{if } 0<q\le 1 \text{ or } 0<p<1 \end{cases}$$

and let $s' = s - n\cdot\big(\max\{1,\frac{1}{p}\}-1\big)$. Suppose that one of the following conditions holds:

(a) $n\cdot\big(\max\{1,\frac{1}{\mu}+\frac{1}{p}\}-1\big)-(1-\delta)r < s < r - n\cdot\max\{\frac{1}{\mu}-\frac{1}{p},0\}$.

(b) $n\cdot\big(\max\{1,\frac{1}{\mu}+\frac{1}{p}\}-1\big)-(1-\delta)r < s = r - n\cdot\max\{\frac{1}{\mu}-\frac{1}{p},0\}$, $(1-\delta)r > \frac{n}{\mu}$ *and if $s=r$, then $\nu\le q\le\infty$.*

Then for any $f\in F_{p,q}^{s+m}$ and $g\in F_{p',q'}^{-s'}$ it holds that $\big(a(x,D)f,g\big) = \big(f,a(x,D)^*g\big)$.

Theorem 20. *Let $a \in SF_\delta^m(r,\mu,\nu;N,\lambda)$ be such that $m \in \mathbb{R}$, $0<\mu,\nu\le\infty$, $r>0$, $0\le\delta<1$, $(1-\delta)r \ge \frac{n}{\mu}$ and $1\le\lambda\le\infty$. Let $0<p,q\le\infty$ and $N > n\cdot\max\{1,\frac{1}{p},\frac{1}{q}\}$. Let*

$$p' = \begin{cases} \frac{p}{p-1} & \text{if } 1<p\le\infty \\ \infty & \text{if } 0<p\le 1 \end{cases} \quad \text{and} \quad q' = \begin{cases} \frac{q}{q-1} & \text{if } 1<q\le\infty \text{ and } 1\le p\le\infty \\ \infty & \text{if } 0<q\le 1 \text{ or } 0<p<1 \end{cases}$$

and let $s' = s - n\cdot\big(\max\{1,\frac{1}{p}\}-1\big)$. Suppose that one of the following conditions holds:

(a) $n\cdot\big(\max\{1,\frac{1}{\mu}+\frac{1}{p}\}-1\big)-r < s < (1-\delta)r - n\cdot\max\{\frac{1}{\mu}-\frac{1}{p},0\}$.

(b) $n\cdot\big(\max\{1,\frac{1}{\mu}+\frac{1}{p}\}-1\big)-r = s < (1-\delta)r - n\cdot\max\{\frac{1}{\mu}-\frac{1}{p},0\}$ *and* $(1-\delta)r > \frac{n}{\mu}$, *and if $s=-r$, then $0<q\le\nu'$.*

Then for any $f\in F_{p,q}^{s}$ and $g\in F_{p',q'}^{-s'+m}$ it holds that $\big(a(x,D)^*f,g\big) = \big(f,a(x,D)g\big)$.

There are also results in case $\mu=\infty$ and $\delta=1$. The values for the parameter s are in this case (see [8, 11])

$$n\cdot\big(\max\{1,\tfrac{1}{p},\tfrac{1}{q}\}-1\big) < s < r \quad \text{and} \quad n\cdot\big(\max\{1,\tfrac{1}{p},\tfrac{1}{q}\}-1\big)-r<s<0,$$

respectively. There are also results in the framework of Theorem 9. Note that assertions (a) and (d) resp. assertions (b) and (c) are dual statements. We leave the formulation of the results to the reader.

There is still another kind of dual results. We formulate it only for Triebel spaces, but there is an obvious counterpart for Besov spaces.

Proposition 21. *Let $a \in SB_\delta^m(r,\mu,\nu;N,\lambda)$ be such that $N > n$, $1 \le \lambda \le \infty$, $0 < \mu,\nu \le \infty$ and $(1-\delta)r \ge \frac{n}{\mu}$. Let $1 \le p,q \le \infty$ and suppose that*

$$s > n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - (1-\delta)r.$$

If $a(x,D) : F^{s+m}_{p,q} \to F^s_{p,q}$ is bounded, then so is $a(x,D)^* : F^{-s}_{p',q'} \to F^{-s-m}_{p',q'}$.

Proof. *Step* 1. The hypothesis implies the boundedness of

$$a^{(1)}(x,D),\ a^{(2)}(x,D) : F^{s+m}_{p,q} \longrightarrow F^s_{p,q}$$
$$a^{(1)}(x,D)^*,\ a^{(2)}(x,D)^* : F^{-s}_{p',q'} \longrightarrow F^{-s-m}_{p',q'}$$

and hence the boundedness of $a^{(3)}(x,D) : F^{s+m}_{p,q} \to F^s_{p,q}$. Let $f \in F^{s+m}_{p,q}$ and $g \in F^{-s}_{p',q'}$. Then as in the proof of Theorem 17

$$\left|(f, a^{(3)}_N(x,D)^* g)\right| = \left|(a^{(3)}_N(x,D)f, g)\right| \le C\|f\|_{F^{s+m}_{p,q}}\|g\|_{F^{-s}_{p',q'}}$$

which yields

$$\|a^{(3)}_N(x,D)^* g\|_{F^{-s-m}_{p',q'}} \le C\|g\|_{F^{-s}_{p',q'}}.$$

Step 2. We claim $a^{(3)}_N(x,D)^* g \to a^{(3)}(x,D)^* g$ in $\mathcal{S}'(\mathbb{R}^n)$, thus completing the proof of the proposition. If $f \in \mathcal{S}(\mathbb{R}^n)$, choose $h \in \mathcal{S}(\mathbb{R}^n)$ with compact spectrum such that $\|f-h\|_{F^{s+m}_{p,q}} < \varepsilon$. Then

$$\begin{aligned}
&\left|(f, a^{(3)}_N(x,D)^* g) - (f, a^{(3)}_M(x,D)^* g)\right| \\
&\quad \le \left|(h, a^{(3)}_N(x,D)^* g) - (h, a^{(3)}_M(x,D)^* g)\right| \\
&\qquad + \left|(f-h, a^{(3)}_N(x,D)^* g)\right| + \left|(f-h, a^{(3)}_M(x,D)^* g)\right| \\
&\quad \le C\varepsilon\|g\|_{F^{-s}_{p',q'}}
\end{aligned}$$

since the first summand vanishes if N and M are large enough. Thus $\left\{a^{(3)}_N(x,D)^* g\right\}_N$ is a Cauchy sequence in $\mathcal{S}'(\mathbb{R}^n)$ and therefore convergent to a limit which is by definition $a^{(3)}(x,D)^* g$. Hence $a^{(3)}(x,D)^*$ is bounded, too ∎

5. Remarks on multiplication properties

The theorems of Section 3 have immediate applications to multiplications. We first consider sufficient conditions. Note that if $a \in B^r_{\mu,\nu}$, then

$$f \longrightarrow a \cdot f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} a_j f_k$$

defines an operator $a(x,D)$ in $SB^0_0(r,\mu,\nu;N,\lambda)$ for any N and λ, which we decompose as usual into $a = a^{(1)} + a^{(2)} + a^{(3)}$.

Theorem 22. *Suppose that* $0 < p,q,\mu,\nu \le \infty$ *and* $r,s \in \mathbb{R}$. *Then*

$$B^r_{\mu,\nu} \cdot B^s_{p,q} \subseteq B^s_{p,q}$$

holds in the following two cases:

(a) **(i)** $r > \frac{n}{\mu}$.
(ii) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r \le s \le r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$.
(iii) *If* $s = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$, *then* $\nu \le q \le \infty$.
(iv) *If* $s = n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r$, *then* $\frac{1}{\nu} + \frac{1}{q} \ge \max\left\{1, \frac{1}{p}\right\}$.

(b) **(i)** $r = \frac{n}{\mu}$ *and* $0 < \mu < \infty$, $0 < \nu \le 1$.
(ii) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r < s < r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$.

Proof. *Step* 1. Assertions (a)/(i) - (a)/(iii) and (b) are clear by Theorem 7. The condition $0 < \nu \le 1$ in (b) is forced by $a \in L^\infty$.

Step 2. Let $s = n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r$ and $\frac{1}{\nu} + \frac{1}{q} \ge \max\left\{1, \frac{1}{p}\right\}$. Then analoguously to Step 5 in the proof of Theorem 7 we obtain

$$\left\|a^{(2)}(x,D)f\right\|_{B^{s+r-\frac{n}{\mu}}_{p,\infty}} \le C \left\| \sum_{k=0}^{\infty} \sum_{l=-3}^{3} a_{k+l} f_k \right\|_{B^{s+r-\frac{n}{\mu}}_{p,\infty}} \le C \left\|a\right\|_{B^r_{\mu,\nu}} \left\|f\right\|_{B^s_{p,q}}.$$

This yields (a)/(iv) ∎

Theorem 22 has a lot of forerunners. Let us mention only [1, 4, 14, 16, 19, 22]. In fact, most cases have been known for a long time. We remark that the inequality

$$\left\|a^{(2)}(x,D)^* f\right\|_{B^s_{p,q}} \le C \left\|a\right\|_{B^r_{\mu,\nu}} \left\|f\right\|_{B^{s+\frac{n}{\mu}-r}_{p,\min\{1,p\}}} \tag{30}$$

holds if $s = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$ and $\nu \le q \le \infty$. The proof is straightforward. For mixed multiplication there is the following immediate consequence of Theorem 9.

Theorem 23. *Suppose that* $0 < p, q, \mu, \nu \le \infty$, $r > \frac{n}{\mu}$ *and* $s \in \mathbb{R}$. *Then*

$$B^r_{\mu,\nu} \cdot F^s_{p,q} \subseteq F^s_{p,q}$$

holds if the following conditions are fulfilled:

(i) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r \le s \le r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$.

(ii) *If* $s = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$, *let* $\mu < p$ *and* $\nu \le p$.

(iii) *If* $s = n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r$, *let* $\frac{1}{\mu} + \frac{1}{p} > 1$ *and* $\frac{1}{\nu} + \frac{1}{p} \ge 1$.

The case $r > \frac{n}{\mu}$ of Theorem 22 is sharp. Counterexamples are scattered through the literature (see [1, 14, 16, 19] and especially [4: Theorem 4.2]). There is one exception, namely the case $0 < p < 1$ of (a)/(iv). The sharpness of this condition follows from the following

Proposition 24. *Let* $0 < \mu, \nu, p, q, \lambda \le \infty$ *and* $r, s \in \mathbb{R}$ *be such that* $\frac{1}{\mu} + \frac{1}{p} \ge 1, 0 < \lambda < 1$, $\frac{1}{\nu} + \frac{1}{q} < \frac{1}{\lambda}$ *and* $r + s = n\left(\frac{1}{\mu} + \frac{1}{p} - 1\right)$. *Then there exist* $a \in B^r_{\mu,\nu}$ *and* $f \in B^s_{p,q}$ *such that* $a^{(2)}(x, D)f \notin B^\tau_{\lambda,\infty}$ *for any* $\tau \in \mathbb{R}$.

Proof. *Step* 1. Let $\psi \in C_0^\infty(\mathbb{R}^+)$ be such that $\psi(\xi) = 1$ if $2 \le \xi \le 4$, $\psi \ge 0$ and $\operatorname{supp}\psi \subseteq \left[\frac{7}{4}, \frac{17}{4}\right]$. Define $\sigma(\xi) = \psi(\xi + 1)$ and extend both functions ψ and σ to $\mathbb{R}$: $\psi(\xi) = \psi(-\xi)$ and $\sigma(\xi) = \sigma(-\xi)$ if $\xi < 0$. Next extend them to $\mathbb{R}^n$:

$$\psi(\xi) = \prod_{i=1}^n \psi(\xi_i) \qquad \text{and} \qquad \sigma(\xi) = \prod_{i=1}^n \sigma(\xi_i).$$

It holds that

$$\int \psi(\xi - \eta)\sigma(\eta)\, d\eta = c \qquad \left(|\xi_i| \le \tfrac{1}{2};\ i = 1, \dots, n\right)$$

for some constant $c > 0$. Now let $\psi_j(\xi) = \psi(2^{-j}\xi)$ and $\sigma_j(\xi) = \sigma(2^{-j}\xi)$.

Step 2. For $R \ge 1$ large enough let $x_j = R(j, 0, \dots, 0)$, and choose $\lambda_0 < \nu$ and $\lambda_1 < q$ such that $\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{1}{\lambda_1}$. Define

$$a_j(x) = j^{-\frac{1}{\lambda_0}} 2^{8j\left(n\left(\frac{1}{\mu} - 1\right) - r\right)} \mathcal{F}^{-1}\psi_{8j}(x - x_j)$$

$$f_j(x) = j^{-\frac{1}{\lambda_1}} 2^{8j\left(n\left(\frac{1}{p} - 1\right) - s\right)} \mathcal{F}^{-1}\sigma_{8j}(x - x_j).$$

It holds that

$$a = \sum_{j=1}^\infty a_j \in B^r_{\mu,\nu}, \qquad f = \sum_{j=1}^\infty f_j \in B^s_{p,q}, \qquad a^{(2)}(x, D)f = \sum_{j=1}^\infty a_j f_j.$$

By Step 1, if $j \le k - 4$, then

$$\begin{aligned} &\mathcal{F}^{-1}\varphi_k * \left(2^{-jn}\mathcal{F}^{-1}\psi_j(\cdot - x_j)\mathcal{F}^{-1}\sigma_j(\cdot - x_j)\right) \\ &= \mathcal{F}^{-1}\left(\varphi_k(\xi)e^{-ix_j\cdot\xi}\int \psi(2^{-j}\xi - \eta)\sigma(\eta)\, d\eta\right) \\ &= c\,\mathcal{F}^{-1}\varphi_k(x - x_j). \end{aligned}$$

Step 3. Let $g_j(x) = a_j(x)f_j(x)$. Then, if $k < j$,

$$\mathcal{F}^{-1}\varphi_k * g_j(x) = cj^{-\frac{1}{\lambda}}\mathcal{F}^{-1}\varphi_k(x - x_j).$$

Suppose $\varphi_k(\xi) = \varphi(2^{-k}\xi)$, if $k \geq 1$, so that

$$\mathcal{F}^{-1}\varphi_k * g_j(x) = c2^{kn}j^{-\frac{1}{\lambda}}\mathcal{F}^{-1}\varphi(2^k(x - x_j)).$$

Observe that $|\mathcal{F}^{-1}\varphi(x)| \leq C(1 + |x|)^{-\frac{1}{\lambda}}$. Hence, if $|x - x_l| \leq 2^{-k}$, it follows that

$$\sum_{j>k,l} \left|\mathcal{F}^{-1}\varphi_k * g_j(x)\right| \leq C2^{kn} \sum_{j>k,l} j^{-\frac{1}{\lambda}}\left(2^kR|j - l|\right)^{-\frac{1}{\lambda}} \leq C2^{kn}(Rl)^{-\frac{1}{\lambda}}.$$

Since

$$\sum_{j=1}^{l-1} j^{-\frac{1}{\lambda}}(l - j)^{-\frac{1}{\lambda}} \leq Cl^{-\frac{1}{\lambda}}$$

we obtain if $|x - x_l| \leq 2^{-k}$

$$\sum_{k<j<l} \left|\mathcal{F}^{-1}\varphi_k * g_j(x)\right| \leq C2^{kn}(Rl)^{-\frac{1}{\lambda}}$$

and hence

$$\sum_{\substack{j>k \\ j\neq l}} \left|\mathcal{F}^{-1}\varphi_k * g_j(x)\right| \leq C2^{kn}(Rl)^{-\frac{1}{\lambda}}.$$

Now choose $R \geq 1$ so large that $CR^{-\frac{1}{\lambda}} \leq \frac{c}{4}|\mathcal{F}^{-1}\varphi(0)|$. Then there exists $0 < \delta \leq 1$ such that if $|x - x_l| \leq 2^{-k}\delta$, then

$$\sum_{\substack{j>k \\ j\neq l}} \left|\mathcal{F}^{-1}\varphi_k * g_j(x)\right| \leq \frac{c}{2}2^{kn}l^{-\frac{1}{\lambda}}\left|\mathcal{F}^{-1}\varphi(2^k(x - x_l))\right|$$

and hence

$$\left|\sum_{j>k} \mathcal{F}^{-1}\varphi_k * g_j(x)\right| \geq \frac{c}{2}2^{kn}l^{-\frac{1}{\lambda}}\left|\mathcal{F}^{-1}\varphi(2^k(x - x_l))\right|.$$

But this yields

$$\begin{aligned}\left\|\sum_{j>k} \mathcal{F}^{-1}\varphi_k * g_j(x)\right\|_{L^\lambda}^{\lambda} &\geq \left(\frac{c}{2}\right)^{\lambda} 2^{kn\lambda} \sum_{l=1}^{\infty} l^{-1} \int_{|x-x_l|\leq 2^{-k}\delta} \left|\mathcal{F}^{-1}\varphi(2^k(x - x_l))\right|^{\lambda} dx \\ &= \left(\frac{c}{2}\right)^{\lambda} 2^{-kn(1-\lambda)} \int_{|x|\leq\delta} \left|\mathcal{F}^{-1}\varphi(x)\right|^{\lambda} dx \sum_{l=1}^{\infty} l^{-1} \\ &= \infty.\end{aligned}$$

Consequently $a^{(2)}(x, D)f \notin B^{\tau}_{\lambda,\infty}$ for any $\tau \in \mathbb{R}$ ∎

Remark 25. Let $a_N = 2^{N\left(n\left(\frac{1}{\mu}-1\right)-r\right)}\mathcal{F}^{-1}\psi_N$ and $f_N = 2^{N\left(n\left(\frac{1}{p}-1\right)-s\right)}\mathcal{F}^{-1}\sigma_N$. Then the sequence $\{a_N\}_N$ is bounded in $B^r_{\mu,\nu}$ and $F^r_{\mu,\nu}$ for any $0 < \nu \le \infty$, and the sequence $\{f_N\}_N$ is bounded in $B^s_{p,q}$ and $F^s_{p,q}$ for any $0 < q \le \infty$. Let $r+s = n\left(\frac{1}{\mu}+\frac{1}{p}-1\right)$. Now if $k \le N-4$, then

$$\mathcal{F}^{-1}\big(\varphi_k\mathcal{F}(a_Nf_N)\big) = c\mathcal{F}^{-1}\varphi_k.$$

It holds that

$$\left\|\sup_k 2^{kn\left(\frac{1}{p}-1\right)}\left|\mathcal{F}^{-1}\varphi_k\right|\right\|_{L^p} = \|\delta\|_{F^{n(\frac{1}{p}-1)}_{p,\infty}} = \infty$$

$$\left(\sum_{k=0}^{\infty} 2^{kn\left(\frac{1}{p}-1\right)\rho}\left\|\mathcal{F}^{-1}\varphi_k\right\|^{\rho}_{L^p}\right)^{1/\rho} = \|\delta\|_{B^{n(\frac{1}{p}-1)}_{p,\rho}} = \infty \quad \text{if } \rho < \infty$$

where δ is the Dirac measure, and we conclude that the sequence $\{a_Nf_N\}_N$ is unbounded in $F^{n(\frac{1}{p}-1)}_{p,\infty}$ and $B^{n(\frac{1}{p}-1)}_{p,\rho}$, if $0 < p < \infty$ and $0 < \rho < \infty$ ∎

Now let us consider sharp estimates in case (b) of Theorem 22.

Porposition 26. *Let* $0 < \mu < \infty$ *and* $0 < \nu \le 1$. *The inclusion*

$$B^{n/\mu}_{\mu,\nu} \cdot B^s_{p,q} \subseteq B^s_{p,q}$$

holds in the following two cases:

(a) $n \cdot \left(\max\left\{1, \frac{1}{\mu}+\frac{1}{p}\right\}-1\right) - \frac{n}{\mu} < s = \frac{n}{\mu} - n\cdot\max\left\{\frac{1}{\mu}-\frac{1}{p}, 0\right\}$

and

(i) $0 < \mu \le p \le \infty$ *and* $\nu \le q \le 1$

or

(ii) $0 < p < \mu$ *and* $\frac{1}{p}-\frac{1}{\mu} \le \frac{1}{q} \le \frac{1}{\nu}$.

(b) $n \cdot \left(\max\left\{1, \frac{1}{\mu}+\frac{1}{p}\right\}-1\right) - \frac{n}{\mu} = s < \frac{n}{\mu} - n\cdot\max\left\{\frac{1}{\mu}-\frac{1}{p}, 0\right\}$

and

(i) $\mu' < p \le \infty$ *and* $\frac{1}{q} \le \frac{1}{\mu}+\frac{1}{p}$

or

(ii) $0 < p \le \mu'$, $q = \infty$ *and* $\nu \le p$.

Proof. *Step* 1. If $\mu \le p < \infty$, (a)/(i) is found in [19]. Therefore let $\mu < p = \infty$. Then $a^{(1)}(x,D) : B^0_{\infty,q} \to B^0_{\infty,q}$ and

$$\begin{aligned}\left\|a^{(2)}(x,D)f\right\|_{B^0_{\infty,q}} &\le C\left\|a^{(2)}(x,D)f\right\|_{B^{n/\mu}_{\mu,q}}\\ &\le C\left(\sum_{k=0}^{\infty}\sum_{l=-3}^{3} 2^{kn/\mu q}\left\|a_{k+l}f_k\right\|^q_{L^\mu}\right)^{1/q}\\ &\le C\|a\|_{B^{n/\mu}_{\mu,\nu}}\|f\|_{B^0_{\infty,q}}\end{aligned}$$

and

$$\|a^{(3)}(x,D)f\|_{B^0_{\infty,q}} \le C\left(\sum_{j=4}^{\infty}\left\|\sum_{k=0}^{j-4} f_k a_j\right\|^q_{L^\infty}\right)^{1/q} \le C\|a\|_{B^{n/\mu}_{\mu,\nu}}\|f\|_{B^0_{\infty,q}}.$$

Thus (a)/(i) is proved. For (a)/(ii) see [16: Theorem 4.3.1].

Step 2. (b)/(i) follows from (a)/(ii) by duality, the case $1 \le p \le \mu'$ of (b)/(ii) follows from (a)/(i). Now let $\nu \le p \le 1$. Then by Lemma 3/(a)

$$\begin{aligned}\|a^{(2)}(x,D)f\|_{B^{n(\frac{1}{p}-1)}_{p,\infty}} &\le C\left(\sum_{k=0}^{\infty}\sum_{l=-3}^{3} 2^{kn(1-p)}\|a_{k+l}f_k\|^p_{L^p}\right)^{1/p}\\ &\le C\|a\|_{B^{n/\mu}_{\mu,\nu}}\|f\|_{B^{n(\frac{1}{p}-1)}_{p,\infty}}.\end{aligned}$$

Since obviously $a^{(1)}(x,D)$ and $a^{(3)}(x,D)$ satisfy the desired estimates, (b)/(ii) follows ∎

All the conditions are not only sufficient, but also necessary. In (a)/(i) $\nu \le q \le 1$ is necessary (see [4: Theorem 4.2(5′) and (7)]). In (a)/(ii) $\frac{1}{p} - \frac{1}{\mu} \le \frac{1}{q} \le \frac{1}{\nu}$ is necessary (see [16: Theorem 4.3.2]). The case $p = \mu = \infty$ and $s = 0$ is not possible (see [16: Corollary 4.3.2]). By duality it follows that $\frac{1}{q} \le \frac{1}{\mu} + \frac{1}{p}$ in (b)/(i) and $q = \infty$ in case $1 \le p \le \mu'$ of (b)/(ii) are necessary. In case $0 < p < 1$ the condition $\frac{1}{\nu} + \frac{1}{q} \ge \frac{1}{p}$ is necessary by Proposition 24. By Remark 25 $q = \infty$ is necessary, hence $\nu \le p$, too. Observe also that by [16: Remark 4.3.4] in case $1 \le p < \infty$

$$B^0_{\infty,\nu} \cdot B^0_{p,q} \subseteq B^0_{p,q} \tag{31}$$

if and only if $0 < \nu \le 1$ and $p = q = 2$. Thus we have obtained a complete description of the inclusion in the case of Besov spaces.

Now we discuss the sharpness of Theorem 23. In (iii), if $\frac{1}{\mu} + \frac{1}{p} \ge 1$, then $\frac{1}{\nu} + \frac{1}{p} \ge 1$ is necessary, and if $\frac{1}{\mu} + \frac{1}{p} \le 1$, then $\frac{1}{\nu} + \frac{1}{q} \ge 1$ is necessary. In (ii), if $p \le \mu$, then $\nu \le q \le \infty$ is necessary. For all this see [4: Theorem 4.2]. Now let $\mu < p$ and suppose that $B^r_{\mu,\nu} \cdot F^s_{p,q} \subseteq F^s_{p,q}$ for $s = r - n(\frac{1}{\mu} - \frac{1}{p})$. Then if $p_1 > p$, then $B^r_{\mu,\nu} \cdot F^s_{p,q} \subseteq B^{s-n(\frac{1}{p}-\frac{1}{p_1})}_{p_1,p}$, and again by [4: Theorem 4.2] it follows $\nu \le p$ as a necessary condition for (ii). Again we see that the conditions in Theorem 23 are sharp.

For the cases not considered in Theorem 23 there is the following

Proposition 27. *Suppose that* $0 < p, q, \mu, \nu \le \infty$, $r > \frac{n}{\mu}$ *and* $s \in \mathbb{R}$. *Suppose also that* $\nu \le \mu$. *Then*

$$B^r_{\mu,\nu} \cdot F^s_{p,q} \subseteq F^s_{p,q}$$

holds in the following two cases:

(a) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - r \le s = r$, $p \le \mu$ *and* $\nu \le q$.

(b) $s = -r$, $\frac{1}{\mu} + \frac{1}{p} \le 1$ *and* $q \le \nu'$.

Proof. $\nu \le \mu$ implies $B^r_{\mu,\nu} \subseteq F^r_{\mu,\nu}$ and hence by [11: Theorem 21] the conclusion follows ∎

Again by [4: Theorem 4.2] the conditions $\nu \leq q$ in (i) and $q \leq \nu'$ in (ii) of Proposition 27 are necessary. The necessity of $\nu \leq \mu$ is deeper. The following proposition is the mainstep in disproving $\mu < \nu$ in Proposition 27.

Proposition 28. *Given $s \in \mathbb{R}$ there exists $f \in \mathcal{S}'(\mathbb{R}^n)$ with compact support such that $f \in B^s_{\mu,\nu}$ if $0 < \mu < \nu \leq \infty$ and $f \notin F^s_{p,\infty}$ if $0 < p \leq \infty$. Also $f \notin \left(\mathring{F}^{-s}_{p,q}\right)'$ if $1 \leq p \leq \infty$ and $0 < q \leq \infty$.*

Proof. *Step* 1. Let $\sigma > 1$,

$$\kappa_0 = 0 \qquad \text{and} \qquad \kappa_j = \sum_{l=1}^{j} l^{-1}\big(\ln(l+1)\big)^{-1}\big(\ln\ln(l+1)\big)^{-\sigma} \quad (j \in \mathbb{N}).$$

Since $\sigma > 1$ there exists $\kappa \in \mathbb{R}$ such that $\kappa_j \nearrow \kappa$. Next, if $j \in \mathbb{N}$, let

$$R_j = \Big\{x : \kappa_{j-1} \leq x_1 < \kappa_j \ \text{ and } \ 0 \leq x_l < 1 \ \ (l \geq 2)\Big\}.$$

In R_j there are contained N_j dyadic cubes of sidelenght 2^{-j} where

$$N_j \approx 2^{jn} j^{-1}\big(\ln(j+1)\big)^{-1}\big(\ln\ln(j+1)\big)^{-\sigma}.$$

Let $\varphi \in C_0^\infty\big((0,1)^n\big)$ and $\psi = \Delta^N \varphi$ where N is such that

$$2N > \max\Big\{s,\ n\cdot\big(\max\{1,\tfrac{1}{\mu}\} - 1\big) - s,\ n\cdot\big(\max\{1,\tfrac{1}{q}\} - 1\big) - s\Big\}.$$

Note that if $|\alpha| \leq 2N$, then the cancellation condition

$$\int x^\alpha \psi(x)\,dx = 0 \tag{32}$$

holds. Let $\{Q^{r,j}\}_{1\leq r\leq N_j}$ be an enumeration of the dyadic cubes contained in R_j, and $x^{r,j} = 2^{-j}l$ if $Q^{r,j} = Q_{l,j}$. Then define

$$f = \sum_{j=1}^{\infty}\sum_{r=1}^{N_j} 2^{-js}\ln(j+1)\big(\ln\ln(j+1)\big)^{\sigma}\psi\big(2^{j+1}(\cdot - x^{r,j})\big).$$

From

$$\operatorname{supp}\psi(2^{j+1}(\cdot - x^{r,j})) \subseteq Q^{r,j}$$

it follows that the support of f is compact.

Step 2. We prove that $f \in B^s_{\mu,\nu}$ if $0 < \mu < \nu \leq \infty$. Note that $2^{j(\frac{n}{\mu}-s)}\psi(2^{j+1}(\cdot - x^{r,j}))$ is an $(Q^{r,j}, s, \mu)$-atom in the sense of Frazier and Jawerth (see [21: p. 62]). Using the atomic decomposition of Besov spaces due to Frazier and Jawerth (see [21: Theorem 1.9.2]), we obtain

$$\begin{aligned}\|f\|_{B^s_{\mu,\nu}} &\le C\Bigg(\sum_{j=1}^{\infty}\Bigg(\sum_{r=1}^{N_j}\Big(2^{-j\frac{n}{\mu}}\ln(j+1)\big(\ln\ln(j+1)\big)^{\sigma}\Big)^{\mu}\Bigg)^{\nu/\mu}\Bigg)^{1/\nu}\\ &\le C\Bigg(\sum_{j=1}^{\infty} j^{-\frac{\nu}{\mu}}\big(\ln(j+1)\big)^{\nu(1-\frac{1}{\mu})}\big(\ln\ln(j+1)\big)^{\sigma\nu(1-\frac{1}{\mu})}\Bigg)^{1/\nu}\\ &< \infty\end{aligned}$$

since $\nu > \mu$.

Step 3. Due to the cancellation condition (32) we can apply [2: Theorem 3.7] to $b_{Q_{l,j}} = 2^{j\frac{n}{2}}\psi(2^{j+1}(\cdot - 2^{-j}l))$ which yields

$$\Big\|\Big\{|Q|^{-\frac{1}{2}-\frac{s}{n}}\langle g, b_Q\rangle\chi_Q\Big\}\Big\|_{L^p(l^q)} \le C\|g\|_{F^s_{p,q}}. \tag{33}$$

Since the $Q^{r,j}$ are disjoint it follows for any $0 < q \le \infty$ that

$$\begin{aligned}&\Big\|\Big\{2^{j(\frac{n}{2}+s)}\langle f, b_{Q^{r,j}}\rangle\chi_{Q^{r,j}}\Big\}\Big\|_{L^p(l^q)}\\ &\quad= \Big\|\Big\{\ln(j+1)\big(\ln\ln(j+1)\big)^{\sigma}\chi_{Q^{r,j}}\Big\}\Big\|_{L^p(l^q)}\\ &\quad\approx C\Bigg(\sum_{j=1}^{\infty} j^{-1}\big(\ln(j+1)\big)^{p-1}\big(\ln\ln(j+1)\big)^{\sigma(p-1)}\Bigg)^{1/p}\\ &\quad= \infty\end{aligned}$$

and hence $f \notin F^s_{p,\infty}$ by (33).

Step 4. Define

$$g = \sum_{j=1}^{\infty}\sum_{r=1}^{N_j} 2^{js}\big(\ln(j+1)\big)^{-1}\psi(2^{j+1}(\cdot - x^{r,j})).$$

Plainly

$$\begin{aligned}\langle f, g\rangle &= \sum_{j=1}^{\infty}\sum_{r=1}^{N_j}\big(\ln\ln(j+1)\big)^{\sigma}\int\psi\big(2^{j+1}(x - x^{r,j})\big)^2\,dx\\ &= c\sum_{j=1}^{\infty} 2^{-jn}\big(\ln\ln(j+1)\big)^{\sigma} N_j\\ &= \infty.\end{aligned}$$

Thus $f \notin (\mathring{F}^{-s}_{p,q})'$ if we can show $g \in \mathring{F}^{-s}_{p,q}$ for $1 \le p \le \infty$ and $0 < q \le \infty$. By [2: Theorem 3.5], in case $1 \le p < \infty$ we have

$$\begin{aligned}\|g\|_{F^{-s}_{p,q}} &\le C\left\|\left\{\left(\ln(j+1)\right)^{-1}\chi_{Q^{r,j}}\right\}\right\|_{L^p(l^q)} \\ &\le C\left(\sum_{j=1}^{\infty}\left(\ln(j+1)\right)^{-p}N_j 2^{-jn}\right)^{1/p} \\ &\le C\left(\sum_{j=1}^{\infty} j^{-1}\left(\ln(j+1)\right)^{-p-1}\right)^{1/p} \\ &< \infty\end{aligned}$$

and hence $g \in F^{-s}_{p,q}$, and in case $p = \infty$ we have

$$\|g\|_{F^{-s}_{\infty,q}} \le C\sup_P\left(|P|^{-1}\sum_{Q^{r,j}\subset P} 2^{-jn}\left(\ln(j+1)\right)^{-q}\right)^{1/q}.$$

Let $|P| = 2^{-kn}$ and $j \ge k$. If $P \subseteq R_j$, then at most $2^{(j-k)n}$ of the $Q^{r,j}$ are contained in P, and hence

$$\left(|P|^{-1}\sum_{Q^{r,j}\subset P} 2^{-jn}\left(\ln(j+1)\right)^{-q}\right)^{1/q} \le C\left(\ln(j+1)\right)^{-1}.$$

If $P \cap R_j \ne \emptyset$ for $j = j_0, \dots, j_1$, then

$$\sum_{j=j_0}^{j_1} j^{-1}\left(\ln(j+1)\right)^{-1}\left(\ln\ln(j+1)\right)^{-\sigma} \approx 2^{-k}.$$

Moreover, $P \cap R_j$ contains at most

$$2^{(j-k)(n-1)}2^j j^{-1}\left(\ln(j+1)\right)^{-1}\left(\ln\ln(j+1)\right)^{-\sigma}$$

cubes $Q^{r,j}$. Hence

$$\left(|P|^{-1}\sum_{Q^{r,j}\subset P} 2^{-jn}\left(\ln(j+1)\right)^{-q}\right)^{1/q} \le C\left(\ln(j_0+1)\right)^{-1}.$$

Thus we find $g \in F^{-s}_{\infty,q}$, and since our estimates show that

$$\sum_{j=1}^{M}\sum_{r=1}^{N_j} 2^{js}\left(\ln(j+1)\right)^{-1}\psi\left(2^{j+1}(\cdot - x^{r,j})\right) \longrightarrow g$$

as $M \to \infty$ in $F^{-s}_{\infty,q}$, we find $g \in \mathring{F}^{-s}_{\infty,q}$ ∎

The idea of the construction of f is taken from Step 5 of the proof of [16: Theorem 3.3.2]. Note that

$$(\mathring{F}^{-s}_{p,q})' = F^{s}_{p',q'} \tag{34}$$

if $1 \le p,q \le \infty$ (see [6]). Thus in that case Step 4 is immediate by Step 3. It is a conjecture that (34) holds in case $0 < q < 1$ too, but this is only known if $p = 1$ (see [19]). The difficulty is that one cannot use the Hahn-Banach theorem because the underlying spaces are not locally convex.

Now there is the announced

Proposition 29. *Suppose that* $0 < p,q,\mu,\nu \le \infty$ *and* $s \in \mathbb{R}$. *Suppose also that*

$$B^{r}_{\mu,\nu} \cdot F^{s}_{p,q} \subseteq F^{s}_{p,q}$$

holds and one of the following two conditions:

(a) $s = r$ *and* $p \le \mu$.

(b) $s = -r$ *and* $\frac{1}{\mu} + \frac{1}{p} \le 1$.

Then $\nu \le \mu$.

Proof. Let $\mu < \nu$, let $f \in B^{r}_{\mu,\nu}$ be as in Proposition 28 and choose $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ in a neighborhood of $\operatorname{supp} f$. Then $\varphi \cdot f = f \notin F^{r}_{p,q}$, hence $\nu \le \mu$ is necessary for (a). Now let $s = -r$ and $g \in F^{s}_{p,q}$ be the distribution constructed in Step 4 of the proof of Proposition 28. Then $\langle fg,\varphi\rangle = \langle f,g\rangle = \infty$ and thus $fg \notin S'(\mathbb{R}^n)$, which is a contradiction. Hence $\nu \le \mu$ is also necessary in case (b) ∎

To complete our characterization we need to consider the case $r = \frac{n}{\mu}$. Then $0 < \nu \le 1$ is necessary (see [4: Theorem 4.2]).

Proposition 30. *Suppose that* $0 < \mu < \infty$, $r = \frac{n}{\mu}$, $0 < \nu \le 1$, $0 < p,q \le \infty$ *and* $s \in \mathbb{R}$. *Then*

$$B^{r}_{\mu,\nu} \cdot F^{s}_{p,q} \subseteq F^{s}_{p,q}$$

holds if one of the following two conditions is fulfilled:

(a) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - \frac{n}{\mu} < s = \frac{n}{\mu} - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$

and

(i) $\mu \le p \le 1$, $\nu \le p$, *and* $\nu \le \min\{\mu, q\}$ *in case* $\mu = p$

or

(ii) $p < \mu$ *and* $\nu \le \min\{\mu, q\}$.

(b) $n \cdot \left(\max\left\{1, \frac{1}{\mu} + \frac{1}{p}\right\} - 1\right) - \frac{n}{\mu} = s < \frac{n}{\mu} - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$

and

(i) $0 < \mu \le 1$ *and* $p = \infty$

or

(ii) $\frac{1}{\mu} + \frac{1}{p} < 1$.

Proof. Since $B^{n/\mu}_{\mu,\nu} \hookrightarrow F^{n/p}_{p,q}$, (a)/(i) follows from the fact that $F^{n/p}_{p,q}$ is a multiplication algebra (see [1, 6]). Now let $p < \mu$ and $\frac{1}{p} = \frac{1}{\mu} + \frac{1}{p_2}$. Then

$$\begin{aligned} \|a^{(3)}(x,D)f\|_{F^{n/\mu}_{p,q}} &\le C\left\| \left(\sum_{j=4}^{\infty} 2^{jn/\mu q} \left| \sum_{k=0}^{j-4} f_k a_j \right|^q \right)^{1/q} \right\|_{L^p} \\ &\le C\|a\|_{F^{n/\mu}_{\mu,q}} \|f\|_{F^0_{p_2,1}} \\ &\le C\|a\|_{B^{n/\mu}_{\mu,\nu}} \|f\|_{F^{n/\mu}_{p,q}}. \end{aligned}$$

Now (a)/(ii) follows easily, and (b) is a matter of duality ∎

It is an easy consequence of Remark 25 that in case $\frac{1}{\mu} + \frac{1}{p} \ge 1$ and $0 < p < \infty$ of (b) the inclusion does not hold. By [4: Theorem 4.2(7)], $p \le 1$ is necessary for the case $\mu \le p$ of (a), and also $\nu \le p$ is necessary for (a)/(i). Also $\nu \le \min\{\mu, q\}$ for the case $p \le \mu$ of (a). The proof of [16: Theorem 4.3.2] shows that the case $p = \mu = \infty$ and $s = 0$ is not possible. And by [16: Remark 4.3.4], if $1 \le p < \infty$, then

$$B^0_{\infty,\nu} \cdot F^0_{p,q} \subseteq F^0_{p,q} \tag{35}$$

if and only if $0 < \nu \le 1$, $1 < p < \infty$ and $q = 2$, which completes our discussion.

These counterexamples also provide counterexamples for Theorems 7 - 9. In fact, in Theorem 7/(b) the condition $\nu \le q \le \infty$ is necessary, in Theorem 8/(b) the condition $\frac{1}{\nu} + \frac{1}{q} \ge \max\left\{1, \frac{1}{p}\right\}$, in Theorem 9/(b) the condition $\nu \le p$ and in Theorem 9/(c) the condititon $\frac{1}{\nu} + \frac{1}{p} \ge 1$. This follows by letting $a(x,D)$ be a multiplication operator. But much more can be said. Let $a \in B^r_{\mu,\infty}$ for $r \ge \frac{n}{\mu}$ and let

$$b(x,\xi) = \sum_{k=0}^{\infty} \sum_{j=k-3}^{k+3} 2^{k\delta r} a_j(x)\varphi_k(\xi) \ \in \ SB^0_\delta(r,\mu,\nu;N,\lambda) \tag{36}$$

where $(1-\delta)r \ge \frac{n}{\mu}$ and $0 < \nu \le \infty$, $N \in \mathbb{N}$ and $1 \le \lambda \le \infty$ are arbitrary. Observe simply

$$\|a_k\|_{L^\infty} \le C2^{k\frac{n}{\mu}} \|a_k\|_{L^\mu} \le C2^{k(\frac{n}{\mu}-r)} \|a\|_{B^r_{\mu,\infty}}.$$

Now let

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - r = s_1 < r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}$$

and $\frac{1}{q} < \max\left\{1, \frac{1}{p}\right\}$. Then there exist $a \in B^r_{\mu,\infty}$ and $f \in B^{s_1}_{p,q}$ such that $af \notin B^{s_1}_{p,q}$. Evidently $a^{(2)}(x,D)f \notin B^{s_1}_{p,q}$. Then suppose that

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - (1-\delta)r = s < r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}.$$

Then

$$h = \sum_{k=0}^{\infty} 2^{-k\delta r} f_k \in B^s_{p,q}$$

and since $s \geq s_1$

$$b(x, D)h = a^{(2)}(x, D)f \notin B^s_{p,q}.$$

Hence in Theorem 7/(c) the condition $0 < q \leq \min\{1, p\}$ is necessary. Similarly in Theorem 8/(c) the condition $q = \infty$ is necessary. In fact, suppose

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - r < s = (1-\delta)r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\},$$

$s_1 = r - n \cdot \max\left\{\frac{1}{\mu} - \frac{1}{p}, 0\right\}$ and $0 < q < \infty$. There exists $a \in B^r_{\mu,\infty}$ and $f \in B^{s_1}_{p,q}$ such that $af \notin B^{s_1}_{p,q}$. But then $a^{(2)}(x, D)^* f \notin B^{s_1}_{p,q}$. Since by lifting

$$\left\|a^{(2)}(x, D)^* f\right\|_{B^{s_1}_{p,q}} \sim \left\|b(x, D)^* f\right\|_{B^s_{p,q}}$$

we find $b(x, D)^* f \notin B^s_{p,q}$, and hence $q = \infty$ is necessary.

Finally we discuss briefly the sharpness of Theorem 9. Suppose that in Theorem 9/(a) we have

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - (1-\delta)r = s < r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}$$

and $\frac{1}{\mu} + \frac{1}{p} > 1$. Then $\frac{1}{\nu} + \frac{1}{p} \geq 1$ for any ν is necessary, and hence $0 < p \leq 1$. Thus if $1 < p \leq \infty$, then no boundedness result can hold. If $\frac{1}{\mu} + \frac{1}{p} \leq 1$, then $\nu \leq \mu$ is necessary by Proposition 29. But we must have $\nu = \infty$. Hence, if $0 < \mu < \infty$ and $1 < p \leq \infty$, then no boundedness result can hold whatever $0 < q \leq \infty$ is. For the case $\mu = \infty$ see [11: (36)].

Next, in Theorem 9/(b) let

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - (1-\delta)r < s = r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}.$$

If $\mu < p \leq \infty$, then $\nu \leq p$ is necessary, and if $p \leq \mu$, then by Proposition 29 we must have $\nu \leq \mu$. But in that case we have boundedness if $\nu \leq q \leq \infty$ (see [11: Theorem 14]), which is necessary, too.

Similarly, in Theorem 9/(c) let

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - r = s < (1-\delta)r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}.$$

If $\frac{1}{\mu} + \frac{1}{p} > 1$, then $\frac{1}{\nu} + \frac{1}{p} \geq 1$ is necessary. Let $\frac{1}{\mu} + \frac{1}{p} \leq 1$. Then by Proposition 29 we must have $\nu \leq \mu$, and here we have boundedness if $0 < q \leq \nu'$ (see [11: Theorem 14]), which is necessary, too.

Finally, in Theorem 9/(d) let

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - r < s = (1-\delta)r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\},$$

$0 < \mu < \infty$ and $\nu = \infty$. If $0 < p \leq \mu$, then no boundedness result can hold by Proposition 29. Let $\mu < p < \infty$. Then similarly to the necessity of $q = \infty$ in Theorem

8/(c) we prove that a boundedness result cannot hold (observe that $\nu \leq p$ is necessary for Theorem 23/(ii)). For the case $\mu = \infty$ see [11: (35)].

Summing up we have obtained a complete description of the conditions, when boundedness of pseudo-differential operators holds. This characterization extends without any difficulties to the symbol classes $SF_\delta^m(r,\mu,\nu;N,\lambda)$ studied in [11]. Let $0 < \mu < \infty$, $1 < p \leq \infty$, $(1-\delta)r \geq \frac{n}{\mu}$ and

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - (1-\delta)r = s < r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\}.$$

Then there exists $b_1 \in SF_\delta^0(r,\mu,\nu;\infty,\infty)$ such that $b_1(x,D)$ is unbounded on $F_{p,q}^s$. The point is that for b defined by (36) we have

$$b \in SF_\delta^0(r,\mu,\nu;\infty,\infty) \cap SB_\delta^0(r,\mu,\nu;\infty,\infty)$$

and hence our previous discussion yields the result. Similarly, if $0 < p, \mu < \infty$, $(1-\delta)r \geq \frac{n}{\mu}$ and

$$n \cdot \left(\max\left\{1, \tfrac{1}{\mu} + \tfrac{1}{p}\right\} - 1\right) - r < s = (1-\delta)r - n \cdot \max\left\{\tfrac{1}{\mu} - \tfrac{1}{p}, 0\right\},$$

then there exists $b_2 \in SF_\delta^0(r,\mu,\nu;\infty,\infty)$ such that $b_2(x,D)^*$ is unbounded on $F_{p,q}^s$. The other cases are already treated here and in [11].

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