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On the Fundamental Solution of the Operator of Dynamic Linear Thermodiffusion

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Abstract. The fundamental matrix of the 5-by-5 system of partial differential operators describing linear thermodiffusion inside elastic media is – by a standard procedure – expressible through the fundamental solution of its determinant. This determinant is equal to the square of a wave operator multiplied by the so-called *operator of dynamic linear thermodiffusion*, which is of the fourth order with respect to the time variable. In this paper, we deduce, by means of a variant of Cagniard–de Hoop's method, a representation of the fundamental solution of this operator by simple definite integrals. This formula allows the explicit computation of thermal and diffusion effects which result from instantaneous point forces or heat sources.

Keywords: *Fundamental solutions, linear partial differential operators with constant coefficients, thermodiffusion, evolution operators*

AMS subject classification: 35 E 05, 35 K 22, 35 C 05, 73 B 30, 73 C 25, 80 A 20

1. Introduction and notations

A system of partial differential equations describing linear thermodiffusion in elastic solids has been presented first in 1961 by Ya. S. Podstrigač (cf. [15]). He considers the displacement, the temperature distribution and the concentration density inside the solid as unknown functions. In 1974, W. Nowacki (cf. [9]) gave a different version thereof by introducing chemical potential as an independent variable whereby concentration density can be expressed.

The purpose of this paper consists in deriving an explicit integral representation for "the" fundamental solution of the non-trivial irreducible factor of the determinant of this system. This factor is called *operator of dynamic linear thermodiffusion* and it is defined by

$$P(\partial) = \partial_t^4 + a\partial_t^3\Delta + b\partial_t^2\Delta^2 + c\partial_t^2\Delta + d\partial_t\Delta^2 + e\Delta^3. \tag{1}$$

In Section 2, we shall first discuss the connection of the fundamental matrix of the system $A(\partial)$ of thermodiffusion with the fundamental solution E_P of $P(\partial)$. We also show that $P(\partial)$ and $A(\partial)$ are quasihyperbolic and that, therefore, their fundamental solution

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and matrix, respectively, are uniquely determined if some natural growth condition is assumed. We then introduce dimensionless variables therewith reducing the number of five constants involved in the definition of $P(\partial)$ to three.

In Section 3, we deduce an integral representation for E_P by employing a variant of Cagniard–de Hoop's method. This new, short procedure replaces the more intricate method we had recourse to in [14: pp. 538 - 542] on dealing with the thermoelastic operator. Let us point out that, similarly to what has been done in [18], the fundamental matrix of $A(\partial)$ could be constructed explicitly starting from the representation of E_P given in Proposition 2.

Eventually, let us establish some terminology. As usual, the three-dimensional Euclidean space is written as $\mathbb{R}^3$. In matrix products, which are indicated by a dot, x is understood as a column vector and the raised letter T means matrix transposition, such that $x^T \cdot x$ is the square of the modulus $|x|$ of x, while $x \cdot x^T$ is a 3×3 matrix. The character I denotes the 3×3 unit matrix, and the bold face letters $\mathbf{u}$ and $\mathbf{F}$ are reserved for vector fields. We write A^{ad} for the adjoint matrix of the square matrix A, i.e., A^{ad}_{ij} is $(-1)^{i+j}$ times the determinant of the matrix resulting from A after deletion of the i-th column and of the j-th row. (Hence $AA^{\mathrm{ad}} = A^{\mathrm{ad}}A = \det(A)I$.) This definition also makes sense for a square matrix of partial differential operators $A(\partial)$. The Heaviside function is denoted by Y. We consider differential operators with constant coefficients only, and employ the differentiation symbols

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla^T = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right), \quad \partial = \begin{pmatrix} \partial_t \\ \nabla \end{pmatrix}$$

$$\Delta = \nabla^T \cdot \nabla, \quad \nabla \cdot \nabla^T = \text{grad div}.$$

We make constant use of the theory of distributions as it is explained in the textbooks [7] and [16] and we adopt their notations for the distribution spaces $\mathcal{D}'$ and $\mathcal{S}'$.

2. The Operator of dynamic linear thermodiffusion

For the reader's convenience, let us repeat first some elements from [4: p. 610]. The influence exerted by the action of external loads, heating and diffusion of matter inside a solid body results in a displacement $\mathbf{u} = \mathbf{u}(x,t)$, a temperature distribution $T = T(x,t)$ and a chemical potential $p = p(x,t)$. The relations which these functions fulfil in a space point $x \in \mathbb{R}^3$ and at the time t are called equations of thermodiffusion and have been investigated in various papers (cf. [2, 3, 9, 10, 15]). W. Nowacki gives the following partial differential equations for $\mathbf{u}, T$ and p (cf. [9: (1.1) - (1.3)/p. 205]):

$$\begin{aligned} \rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\cdot\nabla^T\cdot\mathbf{u} + \beta_1\nabla T + \beta_2\nabla p &= \mathbf{F} \\ \partial_t T - \kappa_1\Delta T + \eta_1\partial_t\nabla^T\cdot\mathbf{u} + \sigma_1\partial_t p &= Q \\ \partial_t p - \kappa_2\Delta p + \eta_2\partial_t\nabla^T\cdot\mathbf{u} + \sigma_2\partial_t T &= M \end{aligned} \tag{2}$$

where $\mathbf{F}$, Q and M correspond to the densities of the exterior force, of the heat generation, and of the diffusing mass, respectively. λ and μ denote the Lamé constants,

ρ the mass density and $\beta_i = (3\lambda + 2\mu)\alpha_i$ $(i = 1, 2)$, α_1 and α_2 being the coefficients of thermal and of diffusion expansion, respectively. Furthermore, κ_1 and κ_2 are the temperature and diffusion conductivity coefficients, respectively, σ_1 and σ_2 are the coefficients of thermodiffusion, $\eta_1 = \frac{\beta_1 T_0}{c_1 \rho}$ and $\eta_2 = \frac{\beta_2 p_0}{c_2 \rho}$ where c_1 and c_2 denote specific heat and potential per unit mass, respectively, and T_0 and p_0 stand for the temperature and potential at rest, respectively. All these constants are non-negative and the condition $\sigma_1\sigma_2 < 1$ is assumed throughout this paper. In [15], instead of p, the concentration function, which, up to a constant factor, is given by

$$p + \eta_2 \nabla^T \cdot \mathbf{u} + \sigma_2 T$$

is considered as an independent variable.

The five equations in (2) can be rewritten as $A(\partial)(\mathbf{u}, T, p)^T = (\mathbf{F}, Q, M)^T$ where the matrix $A(\partial)$ is defined by

$$A(\partial) = \begin{pmatrix} (\rho\partial_t^2 - \mu\Delta)I - (\lambda+\mu)\nabla \cdot \nabla^T & \beta_1 \nabla & \beta_2 \nabla \\ \eta_1 \partial_t \nabla^T & \partial_t - \kappa_1 \Delta & \sigma_1 \partial_t \\ \eta_2 \partial_t \nabla^T & \sigma_2 \partial_t & \partial_t - \kappa_2 \Delta \end{pmatrix}.$$

If $D(\partial) := \det A(\partial)$ and E_D denotes a fundamental solution of $D(\partial)$, then we obtain a (two-sided) fundamental matrix E_A of $A(\partial)$ in the form $E_A = A(\partial)^{\mathrm{ad}} E_D$ (cf. the procedure in [6: Section 3.8/pp. 94 – 95]). Before discussing uniqueness, let us compute D. This can be done most easily by exploiting the rotational symmetry of $A(\partial)$ (compare also [5: p. 625]):

$$\begin{aligned} &D(t,x) \\ &\quad = \det A(t, |x|, 0, 0) = \\ &\quad = \det \begin{pmatrix} \rho t^2 - (\lambda+2\mu)|x|^2 & 0 & 0 & \beta_1 |x| & \beta_2 |x| \\ 0 & \rho t^2 - \mu |x|^2 & 0 & 0 & 0 \\ 0 & 0 & \rho t^2 - \mu|x|^2 & 0 & 0 \\ \eta_1 |x| t & 0 & 0 & t - \kappa_1 |x|^2 & \sigma_1 t \\ \eta_2 |x| t & 0 & 0 & \sigma_2 t & t - \kappa_2 |x|^2 \end{pmatrix} \\ &\quad = \left(\rho t^2 - \mu|x|^2\right)^2 \Big\{ \rho(1-\sigma_1\sigma_2)t^4 - \rho(\kappa_1+\kappa_2)|x|^2 t^3 + \rho\kappa_1\kappa_2 |x|^4 t^2 \\ &\qquad + \Big[\beta_1\sigma_1\eta_2 + \beta_2\sigma_2\eta_1 - \beta_1\eta_1 - \beta_2\eta_2 - (\lambda+2\mu)(1-\sigma_1\sigma_2)\Big] |x|^2 t^2 \\ &\qquad + \Big[\kappa_1\beta_2\eta_2 + \kappa_2\beta_1\eta_1 + (\lambda+2\mu)(\kappa_1+\kappa_2)\Big] |x|^4 t - (\lambda+2\mu)\kappa_1\kappa_2|x|^6 \Big\}. \end{aligned}$$

Hence

$$D(\partial) = \det A(\partial) = \rho(1-\sigma_1\sigma_2)(\rho\partial_t^2 - \mu\Delta)^2 P(\partial)$$

where $P(\partial)$ is the thermodiffusion operator defined in (1) and

$$\begin{aligned} a &= -\frac{\kappa_1+\kappa_2}{1-\sigma_1\sigma_2} \\ b &= \frac{\kappa_1\kappa_2}{1-\sigma_1\sigma_2} \\ c &= \frac{\beta_1\sigma_1\eta_2+\beta_2\sigma_2\eta_1-\beta_1\eta_1-\beta_2\eta_2-(\lambda+2\mu)(1-\sigma_1\sigma_2)}{\rho(1-\sigma_1\sigma_2)} \\ d &= \frac{\kappa_1\beta_2\eta_2+\kappa_2\beta_1\eta_1+(\lambda+2\mu)(\kappa_1+\kappa_2)}{\rho(1-\sigma_1\sigma_2)} \\ e &= -\frac{(\lambda+2\mu)\kappa_1\kappa_2}{\rho(1-\sigma_1\sigma_2)}. \end{aligned} \tag{3}$$

Note that $a<0,\ b>0,\ d>0$ and $e<0$.

The following proposition shows that, under these conditions, the operator $P(\partial)$ is quasihyperbolic in the t-direction (cf. [13: p. 442]). Therefore, the system $A(\partial)$ is also quasihyperbolic in this direction (cf. [14: p. 530]) and has a unique (two-sided) fundamental matrix E_A such that $\mathrm{e}^{-\sigma t}E_A$ is a (5×5)-matrix of temperate distributions for all real values of σ above some bound σ_0. Furthermore, if E_W denotes the unique fundamental solution of the iterated wave operator $W(\partial)=\rho(1-\sigma_1\sigma_2)(\rho\partial_t^2-\mu\Delta)^2$ with support in the half-space $\{(t,x)\in\mathbb{R}^4 : t\geq 0\}$, i.e.,

$$E_W=\frac{1}{8\pi(\rho\mu)^{3/2}(1-\sigma_1\sigma_2)}Y\left(t-\sqrt{\frac{\rho}{\mu}}|x|\right),$$

and if E_P denotes the unique fundamental solution of $P(\partial)$ such that $\mathrm{e}^{-\sigma t}E_P$ is temperate for all large σ, then $E_A=A(\partial)^{\mathrm{ad}}E_W * E_P$. In this sense, the construction of E_A is reduced to that of E_P. We shall focus in this paper on the latter task only.

Proposition 1. *Let $a<0,\ b>0,\ c,d\in\mathbb{C}$ and $e<0$. Then the thermodiffusion operator* (1) *is quasihyperbolic in the t-direction, i.e.,*

$$\exists\sigma_0\in\mathbb{R}:\ \forall\sigma>\sigma_0:\ \forall(\tau,\xi)\in\mathbb{R}^4:\quad P(\sigma+\mathrm{i}\tau,\mathrm{i}\xi)\neq 0. \tag{4}$$

In particular, $P(\partial)$ possesses a unique fundamental solution $E_P\in\mathcal{D}'(\mathbb{R}^4)$ such that

$$\exists\sigma_0\in\mathbb{R}:\ \forall\sigma>\sigma_0:\quad \mathrm{e}^{-\sigma t}E_P\in\mathcal{S}'(\mathbb{R}^4). \tag{5}$$

Moreover we have $\operatorname{supp}E_P\subset\{(t,x)\in\mathbb{R}^4 : t\geq 0\}$.

Proof. Similarly as in [14: p. 532], we have to analyze the behaviour of the roots of the polynomial

$$P(\sigma+\mathrm{i}\tau,\mathrm{i}\xi)=\zeta^4-a\zeta^3\rho^2+b\zeta^2\rho^4-c\zeta^2\rho^2+d\zeta\rho^4-e\rho^6=:Q(\zeta,\rho) \tag{6}$$

with the abbreviations $\zeta = \sigma + i\tau$ and $\rho = |\xi|$ (not to be confounded with the density of mass also denoted by ρ). For $\rho \to \infty$, the expansion $\zeta = A\rho^2 + B\rho + C + \dots$ yields

$$\begin{aligned}
&\left(A^4 - aA^3 + bA^2\right)\rho^8 + \\
&\left(4A^3B - 3aA^2B + 2bAB\right)\rho^7 + \\
&\left(6A^2B^2 + 4A^3C - 3aAB^2 - 3aA^2C + 2bAC + bB^2 - cA^2 + dA - e\right)\rho^6 + \\
&\dots \;=\; 0.
\end{aligned}$$

Since $a < 0$ and $b > 0$, the two roots $A_{1,2}$ of the equation $A^2 - aA + b = 0$ have negative real part. This implies that the corresponding two solutions $\zeta_{1,2}(\rho)$ of $Q(\zeta, \rho) = 0$ (where Q is defined in (6)) also have a negative real part if ρ is a large positive number. On the other hand, for $A = 0$, we infer from $b > 0$ and $e < 0$ that $B = \pm i\sqrt{-\frac{e}{b}}$. Hence there is no branch point over $\rho = \infty$ on the Riemannian surface over the ρ-plane which is defined by $Q(\zeta, \rho) = 0$, and the other two solutions $\zeta_{3,4}(\rho)$ have the expansion

$$\zeta_{3,4}(\rho) = \pm i\sqrt{-\frac{e}{b}}\,\rho + \text{const} + O\left(\frac{1}{\rho}\right) \qquad \text{for } \rho \to \infty.$$

This shows that the real part of $\zeta_{3,4}(\rho)$ is bounded for large positive ρ. The same is evidently true for all roots if ρ remains bounded, and hence the condition (4) is satisfied. Since this implies by [13: Prop. 1/p. 442] that there exists a unique fundamental solution E_P of $P(\partial)$ satisfying (5) (and having its support in the half-space where $t \geq 0$), the proof is complete ∎

In order to simplify further elaboration, we introduce dimensionless variables (comp. [14: p. 529]): For $\gamma > 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$, we abbreviate by $E_{\gamma,\varepsilon}$ the unique fundamental solution of the operator

$$\begin{aligned}
P_{\gamma,\varepsilon}(\partial) &= (\partial_t - \gamma\Delta)(\partial_t - \gamma^{-1}\Delta)(\partial_t^2 - \Delta) - \varepsilon_1\partial_t^2\Delta + \varepsilon_2\partial_t\Delta^2 \\
&= \partial_t^4 - (\gamma + \gamma^{-1})\partial_t^3\Delta + \partial_t^2\Delta^2 - (1 + \varepsilon_1)\partial_t^2\Delta + (\gamma + \gamma^{-1} + \varepsilon_2)\partial_t\Delta^2 - \Delta^3
\end{aligned}$$

which fulfills condition (5). Linear transformations of the co-ordinates yield $t_0^{-1} r_0^{-3} u^{-1} E_{\gamma,\varepsilon}\left(\frac{t}{t_0}, \frac{x}{r_0}\right)$ as a fundamental solution of the operator

$$\begin{aligned}
&t_0^4 u\,\partial_t^4 - (\gamma + \gamma^{-1})t_0^3 r_0^2 u\,\partial_t^3\Delta + t_0^2 r_0^4 u\,\partial_t^2\Delta^2 \\
&\qquad - (1 + \varepsilon_1)t_0^2 r_0^2 u\,\partial_t^2\Delta + (\gamma + \gamma^{-1} + \varepsilon_2)t_0 r_0^4 u\,\partial_t\Delta^2 - r_0^6 u\,\Delta^3
\end{aligned}$$

for $r_0, t_0, u > 0$. Therefore, putting

$$t_0 = \sqrt{\frac{\kappa_1\kappa_2}{1 - \sigma_1\sigma_2}\frac{\rho}{\lambda + 2\mu}}, \qquad r_0 = \sqrt{\frac{\kappa_1\kappa_2}{1 - \sigma_1\sigma_2}}\sqrt{\frac{\rho}{\lambda + 2\mu}}, \qquad u = t_0^{-4}$$

and

$$\gamma = \frac{\kappa_1 + \kappa_2 + \sqrt{(\kappa_1 - \kappa_2)^2 + 4\kappa_1\kappa_2\sigma_1\sigma_2}}{2\sqrt{\kappa_1\kappa_2}\sqrt{1-\sigma_1\sigma_2}}$$

$$\varepsilon_1 = \frac{\beta_1\eta_1 + \beta_2\eta_2 - \beta_1\sigma_1\eta_2 - \beta_2\sigma_2\eta_1}{(1-\sigma_1\sigma_2)(\lambda + 2\mu)} \tag{7}$$

$$\varepsilon_2 = \frac{\kappa_1\beta_2\eta_2 + \kappa_2\beta_1\eta_1}{\sqrt{\kappa_1\kappa_2}\sqrt{1-\sigma_1\sigma_2}(\lambda + 2\mu)}$$

we obtain a representation of E_P in terms of $E_{\gamma,\varepsilon}$:

$$E_P(t,x) = \left(\frac{\rho}{\lambda + 2\mu}\right)^{3/2} E_{\gamma,\varepsilon}\left(\sqrt{\frac{1-\sigma_1\sigma_2}{\kappa_1\kappa_2}}\frac{\lambda+2\mu}{\rho}\,t, \sqrt{\frac{1-\sigma_1\sigma_2}{\kappa_1\kappa_2}}\sqrt{\frac{\lambda+2\mu}{\rho}}\,x\right).$$

We point out that γ, ε_1, ε_2, $\frac{t}{t_0}$, $\frac{x}{r_0}$ and $E_{\gamma,\varepsilon}$ are dimensionless quantities. In analogy to [17: p.41] and [11: p. 207], the quantities t_0^{-1} and r_0 could be called characteristic frequency and characteristic length, respectively. γ is a measure of the interdependence of thermal and diffusion effects. Note that we always have $\gamma > 1$ and that $\gamma = 1$ would correspond to the limit case defined by $\sigma_1\sigma_2 = 0$ and $\kappa_1 = \kappa_2$. On the other hand, ε is a measure of the mutual interaction between strain, heat transfer and diffusion. Under physical conditions, ε_1 and ε_2 are small positive constants.

3. An integral representation for $E_{\gamma,\varepsilon}$

In the sequel, we shall consider only $P_{\gamma,\varepsilon}(\partial)$ and its fundamental solution $E_{\gamma,\varepsilon}$. For convenience, we also use the constants a, b, c, d and e corresponding to the definition in (1). Hence we set $b = -e = 1$ and

$$a = -(\gamma + \gamma^{-1}), \qquad c = -(1+\varepsilon_1), \qquad d = \gamma + \gamma^{-1} + \varepsilon_2 \tag{8}$$

with γ and $\varepsilon_1, \varepsilon_2$ as in (7). Applying the Fourier-Laplace transform as in [14: p. 538], we obtain, for $\sigma > \sigma_0$ and σ_0 as in (4),

$$\begin{aligned} E_{\gamma,\varepsilon} &= \frac{1}{(2\pi)^4}\lim_{k\to\infty}\int_{-k}^{+k} e^{(\sigma+i\tau)t}\,d\tau \int_{\mathbb{R}^3} \frac{e^{ix^T\cdot\xi}}{P_{\gamma,\varepsilon}(\sigma+i\tau, i\xi)}\,d\xi \\ &= \lim_{k\to\infty}\frac{1}{2\pi^2|x|}\frac{1}{2\pi i}\int_{\sigma-ik}^{\sigma+ik} e^{\zeta t}\,d\zeta \int_0^\infty \frac{\rho\sin(\rho|x|)\,d\rho}{Q(\zeta,\rho)} \\ &= \frac{1}{2\pi^2|x|}\mathcal{L}^{-1}_{\zeta\to t}\Big(H(\zeta,-i|x|) - H(\zeta,i|x|)\Big) \end{aligned}$$

where $Q(\zeta,\rho) = P_{\gamma,\varepsilon}(\zeta, i\rho, 0, 0)$ (compare (6)),

$$H(\zeta,z) = \frac{1}{2i}\int_0^\infty \frac{\rho\, e^{-z\rho}\,d\rho}{Q(\zeta,\rho)} \qquad \text{for } \operatorname{Re} z \geq 0 \text{ and } \operatorname{Re}\zeta > \sigma_0$$

and

$$\mathcal{L}^{-1}_{\zeta\to t}(G(\zeta)) = \frac{1}{2\pi \mathrm{i}} \lim_{k\to\infty} \int\limits_{\sigma-\mathrm{i}k}^{\sigma+\mathrm{i}k} \mathrm{e}^{\zeta t} G(\zeta)\, d\zeta \qquad (\sigma > \sigma_0).$$

Let us suppose first that $z > 0$. If ζ is real with $\zeta > \sigma_0$, then the substitution $\rho = \zeta s$ yields

$$H(\zeta, z) = \frac{\zeta^2}{2\mathrm{i}} \int\limits_0^\infty \frac{s\, \mathrm{e}^{-z\zeta s}\, ds}{Q(\zeta, \zeta s)}. \tag{9}$$

By analytic continuation, the validity of (9) extends to complex ζ with $\mathrm{Re}\,\zeta > \sigma_0$, once we have checked that $Q(\zeta, \zeta s)$ does not vanish for $\mathrm{Re}\,\zeta > \sigma_0$ and $0 < s < \infty$. Because of

$$\begin{aligned} Q(\zeta, \zeta s) &= \zeta^4 \Big(1 - a\zeta s^2 + \zeta^2 s^4 - cs^2 + d\zeta s^4 + \zeta^2 s^6\Big) \\ &= \zeta^4 s^4 (s^2+1)\big(\zeta + f_1(s)\big)\big(\zeta + f_2(s)\big) \end{aligned}$$

with

$$f_{1,2}(s) = \frac{ds^2 - a \pm \sqrt{(ds^2-a)^2 - 4(1-cs^2)(s^2+1)}}{2s^2(s^2+1)} \tag{10}$$

and since $1 - cs^2$ and $ds^2 - a$ are positive, we indeed have $\mathrm{Re}\, f_{1,2}(s) > 0$ and hence $\zeta + f_i(s) \neq 0$ for $\mathrm{Re}\,\zeta > 0$ and $i = 1, 2$.

We now interchange the inverse Laplace transform with the integral in (9) and obtain from [12: p. 217] the following (still for $z > 0$ only):

$$\begin{aligned} \mathcal{L}^{-1}_{\zeta\to t}(H(\zeta, z)) &= \frac{1}{2\mathrm{i}} \int\limits_0^\infty \mathcal{L}^{-1}_{\zeta\to t}\bigg(\frac{\zeta^2 \mathrm{e}^{-z\zeta s}}{Q(\zeta,\zeta s)}\bigg) s\, ds \\ &= \frac{Y(t)}{2\mathrm{i}} \int\limits_0^{t/z} \mathcal{L}^{-1}_{\zeta\to t-zs}\bigg(\frac{\zeta^2}{Q(\zeta,\zeta s)}\bigg) s\, ds \\ &= \frac{Y(t)}{2\mathrm{i}} \int\limits_0^{t/z} \mathcal{L}^{-1}_{\zeta\to t-zs}\bigg(\frac{1}{\zeta^2\big(\zeta+f_1(s)\big)\big(\zeta+f_2(s)\big)}\bigg) \frac{ds}{s^3(s^2+1)} \\ &= \frac{Y(t)}{2\mathrm{i}} \int\limits_0^{t/z} \bigg[\frac{f_1(s)f_2(s)(t-zs) - f_1(s) - f_2(s)}{f_1(s)^2 f_2(s)^2} \\ &\qquad + \frac{1}{f_1(s) - f_2(s)}\bigg(\frac{\mathrm{e}^{-f_2(s)(t-zs)}}{f_2(s)^2} - \frac{\mathrm{e}^{-f_1(s)(t-zs)}}{f_1(s)^2}\bigg)\bigg] \frac{ds}{s^3(s^2+1)}. \end{aligned}$$

Finally, we come back to $E_{\gamma,\varepsilon}$ by analytic continuation with respect to z. For the then arising two integrals over s from 0 to $\frac{\mathrm{i}t}{|x|}$ and to $-\frac{\mathrm{i}t}{|x|}$, respectively, we can use as integration paths semicircles in the right half-plane (compare Fig. 1). The substitution $s \mapsto -s$ in the second integral and the formulae

$$f_1(s) + f_2(s) = \frac{ds^2 - a}{s^2(s^2+1)} \qquad \text{and} \qquad f_1(s) f_2(s) = \frac{1 - cs^2}{s^4(s^2+1)}$$

yield

$$
\begin{aligned}
E_{\gamma,\varepsilon}(t,x) = \frac{Y(t)}{4\pi^2 \mathrm{i}|x|} \int_C \Bigg[& \frac{f_1(s)f_2(s)(t+\mathrm{i}|x|s) - f_1(s) - f_2(s)}{f_1(s)^2 f_2(s)^2} \\
& + \frac{1}{f_1(s)-f_2(s)} \left(\frac{\mathrm{e}^{-f_2(s)(t+\mathrm{i}|x|s)}}{f_2(s)^2} - \frac{\mathrm{e}^{-f_1(s)(t+\mathrm{i}|x|s)}}{f_1(s)^2} \right) \Bigg] \frac{ds}{s^3(s^2+1)}
\end{aligned}
\tag{11}
$$

where C is the circle through 0 and $\mathrm{i}\frac{t}{|x|}$ which is symmetric with respect to the imaginary s-axis and oriented in the counterclockwise direction (compare Fig. 2).

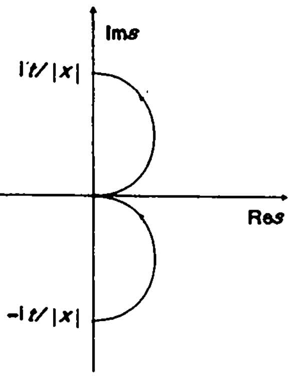

Figure 1

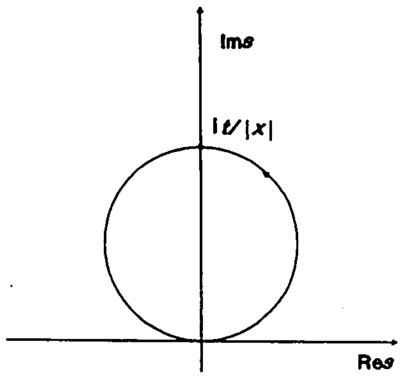

Figure 2

The algebraic function

$$
R(s) = \sqrt{(ds^2 - a)^2 - 4(1 - cs^2)(s^2+1)}
\tag{12}
$$

defines a Riemannian surface $p: \ S \longrightarrow \overline{\mathbb{C}}$ over the s-plane. Its branch points are the roots of the polynomial

$$
q(s) = (d^2 + 4c)s^4 + \big(4(c-1) - 2ad\big)s^2 + a^2 - 4
$$

and its genus g is either 0 or 1, i.e., S is homeomorphic to a ball or to a torus, respectively (compare [1: Section 17.15/p. 128]). Since $a < -2$, $g = 0$ occurs if $d^2 + 4c = 0$ or if the discriminant of q, that is

$$
\big(4(c-1) - 2ad\big)^2 - 4(d^2+4c)(a^2-4) = 16\big[\varepsilon_1^2 + \varepsilon_2^2 - \varepsilon_1\varepsilon_2(\gamma + \gamma^{-1})\big]
$$

vanishes. For simplicity, let us concentrate on the case of $\varepsilon_1^2 + \varepsilon_2^2 < \varepsilon_1\varepsilon_2(\gamma + \gamma^{-1})$. In this case, both functions $f_1(s)$ and $f_2(s)$ are real-valued on the real s-axis.

Since the integrand in (11) remains invariant under a sign change of R, the integral in (11) can be written as one over the closed contour $p^{-1}(C)$ on S. Let ω denote the corresponding complex one-form on S, i.e.,

$$
\omega(u) = \left(\frac{1}{f(u)^2} - \frac{t+\mathrm{i}|x|s}{f(u)} + \frac{(t+\mathrm{i}|x|s)^2}{2} - \frac{\mathrm{e}^{-f(u)(t+\mathrm{i}|x|s)}}{f(u)^2} \right) \frac{ds}{sR(u)}
$$

where $s = p(u)$ and $f(u)$ denotes the single-valued function $\frac{ds^2-a+R(u)}{2s^2(s^2+1)}$ on S. Because $\frac{ds}{R(u)}$ is holomorphic on S (compare [1: Section 17.15/p. 128]), this is also true for ω in the branch points of S. Furthermore, ω has essential singularities in the points over $s = 0$ and $s = \pm i$ and it is analytic at the points on S above $s = \pm i\frac{1}{\sqrt{1+\varepsilon_1}}$ and $s = \infty$, respectively. Indeed, e.g., for $s \to \infty$, we have $f_j(s) = c_j s^{-2} + O(s^{-3})$ for $j = 1, 2$ and some constants c_j, and the power series of the exponential function readily leads to the estimate $\omega(u) = O(s^{-2})\,ds$ for $s = p(u) \to \infty$. As in [14: p. 540], we now distinguish two cases. For $t > |x|$, we can homotopically deform C into the real line without touching the essential singularities $\pm i$. Hence, inside this cone, $E_{\gamma,\varepsilon}$ is given by

$$E^{(1)}_{\gamma,\varepsilon}(t,x) = \frac{1}{2\pi^2|x|}\int_0^\infty \left\{ \frac{s^3|x|}{1-cs^2} + \frac{e^{-f_1(s)t}\sin\left(sf_1(s)|x|\right)}{f_1(s)^2R(s)} - \frac{e^{-f_2(s)t}\sin\left(sf_2(s)|x|\right)}{f_2(s)^2R(s)} \right\}\frac{ds}{s}. \tag{13}$$

In the case of $0 < t < |x|$, however, we also have to consider the residues of ω at $\pm i$. Hence in this case, it holds $E_{\gamma,\varepsilon} = E^{(1)}_{\gamma,\varepsilon} + E^{(2)}_{\gamma,\varepsilon}$ with

$$E^{(2)}_{\gamma,\varepsilon}(t,x) = \frac{1}{2\pi|x|}\mathrm{Res}_{s=i}\left\{ \left(\frac{e^{-f_1(s)(t+i|x|s)}}{f_1(s)^2} - \frac{e^{-f_2(s)(t+i|x|s)}}{f_2(s)^2} \right) \times \frac{1}{(f_1(s)-f_2(s))s^3(s^2+1)} \right\}. \tag{14}$$

(Let us note that the above residue at the essential singularity $s = i$ can of course be represented by a curve integral. There does not seem to exist a "canonical" contour though, and we resorted in [14: Formula (14)/p. 542] in the case of the thermo*elastic* operator to develop this residue into a power series in the variables t and $|x|$.)

In conclusion, we have proven the following

Proposition 2. *Let $\gamma > 1$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ satisfy $\varepsilon_1^2 + \varepsilon_2^2 < \varepsilon_1\varepsilon_2(\gamma + \gamma^{-1})$ and let $E_P = E_{\gamma,\varepsilon}$ be the unique fundamental solution of*

$$\begin{aligned} P_{\gamma,\varepsilon}(\partial) &:= (\partial_t - \gamma\Delta)(\partial_t - \gamma^{-1}\Delta)(\partial_t^2 - \Delta) - \varepsilon_1\partial_t^2\Delta + \varepsilon_2\partial_t\Delta^2 \\ &= \partial_t^4 - (\gamma+\gamma^{-1})\partial_t^3\Delta + \partial_t^2\Delta^2 - (1+\varepsilon_1)\partial_t^2\Delta + (\gamma+\gamma^{-1}+\varepsilon_2)\partial_t\Delta^2 - \Delta^3 \end{aligned}$$

fulfilling (5). *Define a, c and d by* (8), *$f_{1,2}$ by* (10) *and R by* (12), *respectively. Then $E_{\gamma,\varepsilon}$ is a locally integrable function and*

$$E_{\gamma,\varepsilon}(t,x) = Y(t)E^{(1)}_{\gamma,\varepsilon}(t,x) + Y(t)Y(|x|-t)E^{(2)}_{\gamma,\varepsilon}(t,x)$$

where $E^{(1)}_{\gamma,\varepsilon}$ and $E^{(2)}_{\gamma,\varepsilon}$ are given in (13) *and* (14), *respectively.*

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