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A Necessary Condition to Regularity of a Boundary Point for a Degenerate Quasilinear Parabolic Equation

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Abstract. We shall study the behaviour of solutions of the equation

$$v(x)\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i\left(x,t,u,\frac{\partial u}{\partial x}\right) = a_0\left(x,t,u,\frac{\partial u}{\partial x}\right) \qquad ((x,t) \in Q_T = \Omega \times (0,T))$$

at a point $(x_0,t_0) \in S_T = \partial\Omega \times (0,T)$. Inded we establish a necessary condition to the regularity of a boundary point of the cylindrical domain Q_T extending the analogous result from paper [13] to the degenerate case. The degeneration is given by weights (depending on the space variable) from a suitable Muchenhoupt class. It is important to note that the coefficients of the equation depend on time too.

Keywords: *Degenerate nonlinear parabolic equations, regularity at boundary points*

AMS subject classification: Primary 35 K 65, secondary 35 B 65

1. Introduction

In the present paper we are concerned with the behavior near by the boundary of a cylindrical domain of solutions to a second order degenerate parabolic equation with coefficients which depend on time.

The Wiener condition to the regularity of a boundary point for a linear parabolic equation with measurable bounded coefficients is due to Lanconelli [9]. A Wiener-type sufficient condition for a quasilinear parabolic equation has been proved by Gariepy and Ziemer [6, 14] and a necessary condition by I. V. Skrypnik [13]. In [1] Biroli has extended the result of [14] to the parabolic degenerate case with a weight in the $A_{1+2/n}$ Muchenhoupt class (see in [1] about literature). In our paper we use the method of [13] for proving a necessary Wiener-type condition for such a problem.

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2. Notations, definitions and preliminary results

To begin we recall some facts from [2] about A_p weights. Let $0 < r \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$ $(n \geq 3)$, and put

$$B(x_0, r) = \left\{ x \in \mathbb{R}^n \,\middle|\, |x - x_0| < r \right\} \quad \text{and} \quad w(B(x_0, r)) = \int_{B(x_0,r)} w(x)\, dx.$$

We say that a non-negative and locally integrable function $w = w(x)$ in $\mathbb{R}^n$ is a *doubling weight* if there exists a constant $K_1 > 0$, independent of r and x_0, such that

$$w(B(x_0, 2r)) \leq K_1 w(B(x_0, r)). \tag{2.1}$$

Given $p \in (1, +\infty)$, we say the weight w belongs to A_p if there exists a constant $K_2 > 0$ such that, for all balls $B \subset \mathbb{R}^n$, we have

$$\left(\frac{1}{\operatorname{meas} B} \int_B w(x)\, dx \right) \left(\frac{1}{\operatorname{meas} B} \int_B w^{-1/(p-1)}(x)\, dx \right)^{p-1} \leq K_2. \tag{2.2}$$

We say $w \in D_\zeta$ if there exists a constant $K_3 > 0$ such that

$$w(B(x_0, r)) \leq K_3 \left(\frac{r}{s} \right)^{n\zeta} w(B(x_0, s)) \qquad \text{for all } \; s \in (0, r] \tag{2.3}$$

with K_3 independent of x_0, r and s.

We say the Poincaré inequality holds with weights w_1 and w_2, μ-average and exponent q $(q > 2)$ if there exists a constant $K_4 > 0$ such that

$$\begin{aligned} &\left(\frac{1}{w_2(B)} \int_B |F(x) - av_{B,\mu} F|^q w_2(x)\, dx \right)^{1/q} \\ &\qquad \leq K_4 \left(\operatorname{meas} B \right)^{1/n} \left(\frac{1}{w_1(B)} \int_B \left| \frac{\partial F(x)}{\partial x} \right|^2 w_1(x)\, dx \right)^{1/2} \end{aligned} \tag{2.4}$$

for every ball $B \subset \mathbb{R}^n$ and every $F \in \operatorname{Lip}(B)$ where

$$av_{B,\mu} F = \frac{1}{\mu(B)} \int_B F(x) \mu(x)\, dx.$$

As it follows from the result by Chanillo and Wheeden [2], the inequality (2.4) holds for $q > 0$ with $\mu = 1$ or $\mu = w_2$ whenever $w_1 \in A_2$ and

$$\left(\frac{\operatorname{meas} B_1}{\operatorname{meas} B_2} \right)^{1/n} \left(\frac{w_2(B_1)}{w_2(B_2)} \right)^{1/q} \leq K_5 \left(\frac{w_1(B_1)}{w_2(B_2)} \right)^{1/2} \tag{2.5}$$

for all balls B_1 and B_2 with $B_1 \subset B_2$ and with K_5 independent of the balls.

Also, if $v = v(x)$ is a weight, $w \in A_p(v)$ means an analogous inequality to (2.2) with dx and $\operatorname{meas} B$ replaced by $v(x)dx$ and $v(B)$, respectively. We also use the notation $A_\infty(v) = \bigcup_{p=1}^{\infty} A_p(v)$.

Remark 2.1. As it follows from the definition, if $w \in A_p$, then there exist constants $K_6 > 0$ and $\eta > 0$ such that

$$w(B(x_0,r)) \le K_6 \left(\frac{r}{s}\right)^{n\eta} w(B(x_0,s))$$

for $0 < r \le s$.

Let now Ω be a bounded domain in $\mathbb{R}^n$ and $Q_T = \Omega \times (0,T)$. We shall study the behavior of solutions of the equation

$$v(x)\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i \left(x,t,u,\frac{\partial u}{\partial x}\right) = a_0 \left(x,t,u,\frac{\partial u}{\partial x}\right) \qquad ((x,t) \in Q_T) \tag{2.6}$$

at a point $(x_0,t_0) \in S_T = \partial\Omega \times (0,T)$ under the assumptions that the functions $a_i = a_i(x,t,u,p)$ $(i = 0,\dots,n)$ are defined for $(x,t,u,p) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n$ and satisfy the following conditions:

(C1) For almost every (fixed) (x,t) the functions $a_i(x,t,u,p)$ are continuous with respect to u and p, and for all (u,p) they are measurable functions of (x,t); $a_i(x,t,0,0) = 0$ for $i = 0,\dots,n$.

(C2) For some constant $\nu_1 > 0$,

$$\sum_{i=1}^{n} \Big(a_i(x,t,u,p) - a_i(x,t,u,q)\Big)(p_i - q_i) \ge \nu_1\, w(x)|p-q|^2, \tag{2.7}$$

and for some constant $\nu_2 > 0$,

$$\begin{aligned} \Big|a_0(x,t,u,p) - a_0(x,t,v,q)\Big| &\le \nu_2\Big(v(x)|u-v| + v^{1/2}(x)w^{1/2}(x)|p-q|\Big) \\ \Big|a_i(x,t,u,p) - a_i(x,t,v,q)\Big| &\le \nu_2\Big(v^{1/2}(x)w^{1/2}(x)|u-v| + w(x)|p-q|\Big) \end{aligned} \tag{2.8}$$

for $i = 1,\dots,n$.

We will denote by $L^2(\Omega,w)$ the Banach space of all measurable functions f, defined on Ω, whose norm

$$\|f\|^2_{L^2(\Omega,w)} = \int_\Omega f^2(x)w(x)\,dx$$

is finite. $W_2^1(Q_T,v,w)$ will be the Banach space of functions f equipped with the norm

$$\begin{aligned} \|f\|^2_{W_2^1(Q_T,v,w)} &= \int_{Q_T} f^2(x,t)\big(v(x)+w(x)\big)\,dxdt \\ &+ \int_{Q_T} \left(\left|\frac{\partial f(x,t)}{\partial x}\right|^2 w(x) + \left|\frac{\partial f(x,t)}{\partial t}\right| v(x)\right) dxdt. \end{aligned} \tag{2.9}$$

We use also functions from the space $V_2(Q_T,v,w)$ endowed with the norm

$$\|f\|^2_{V_2(Q_T,v,w)} = \sup_{0<t<T} \int_\Omega f^2(x,t)v(x)\,dx + \int_{Q_T} \left|\frac{\partial f(x,t)}{\partial x}\right|^2 w(x)\,dxdt. \tag{2.10}$$

We will denote by $\mathring{W}_2^1(Q_T,v,w)$ and $\mathring{V}_2(Q_T,v,w)$ the spaces of functions belonging, respectively, to $W_2^1(Q_T,v,w)$ and $V_2(Q_T,v,w)$ and being equal to zero on S_T.

Definition 2.1. We say that a function $u \in V_2(Q_T, v, w)$ is a solution of the equation (2.6) if, for all functions $\psi = \psi(x,t)$ in $\overset{\circ}{W}{}_2^1(Q_T, v, w)$ vanishing at $t = 0$ and $t = T$, the identity

$$I_T(u, \psi) = 0$$

is satisfied, where

$$I_\tau(u,\psi) = \int_0^\tau\int_\Omega \Bigg\{ - v(x)u(x,t)\frac{\partial\psi}{\partial t} + \sum_{i=1}^n a_i\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial\psi}{\partial x_i} - a_0\left(x,t,u,\frac{\partial u}{\partial x}\right)\psi(x)\Bigg\}\,dx\,dt. \tag{2.11}$$

Definition 2.2. Let $f \in W_2^1(Q_T, v, w)$ and $g \in L_2(\Omega, v)$ for which

$$u(x,t) = f(x,t) \qquad ((x,t) \in S_T) \tag{2.12}$$

$$u(x,0) = g(x) \qquad (x \in \Omega). \tag{2.13}$$

We say that $u = u(x,t)$ in $V_2^1(Q_T, v, w)$ is a solution of the problem (2.6), (2.12), (2.13) if $u - f \in \overset{\circ}{V}_2(Q_T, v, w)$ and, moreover, for any $\psi \in \overset{\circ}{W}{}_2^1(Q_t, v, w)$ and $\tau \in (0,T)$,

$$\int_\Omega v(x)u(x,\tau)\psi(x,\tau)\,dx - \int_\Omega v(x)g(x)\psi(x,0)\,dx + I_\tau(u,\psi) = 0. \tag{2.14}$$

Definition 2.3. We say that $(x_0, t_0) \in S_T$ is a *regular boundary point* of the region Q_T for the equation (2.6) if for any its solution u, defined in Q_T, satisfying the condition

$$\phi(u - f) \in \overset{\circ}{V}_2(Q_T, v, w) \tag{2.15}$$

with $f \in C(\overline{Q}_T) \cap W_2^1(Q_T, v, w)$ and $\phi \in C^\infty(\mathbb{R}^{n+1})$ which is equal to one in a neighborhoud of (x_0, t_0), the equality

$$\lim_{(x,t)\to(x_0,t_0)} u(x,t) = f(x_0, t_0) \qquad ((x,t) \in Q_T) \tag{2.16}$$

holds.

For any set $E \subset \mathbb{R}^n$, let

$$\mathcal{M}(E) = \left\{\phi \in C_0^\infty(\mathbb{R}^n) \,\middle|\, \phi(x) \geq 1 \text{ for all } x \in E\right\}$$

and define

$$C_{2,w}(E) = \inf\left\{ \int_{\mathbb{R}^n} w(x)\left|\frac{\partial\phi}{\partial x}\right|^2 dx \,\middle|\, \phi \in \mathcal{M}(E)\right\}.$$

In the following we shall need two lemmata.

Lemma 2.1 (see [13]). *Let $\{\beta_i\}_{i\in\mathbb{N}}$ be a bounded numerical sequence such that*

$$\beta_i \le A\,\beta_{i+1}^{\delta}\,a^i \qquad (i \in \mathbb{N})$$

with positive constants A, a and $\delta \in (0,1)$. Then

$$\beta_1 \le C A^{1/(1-\delta)}$$

for a constant C depending only on δ and a.

Lemma 2.2 (see [5: Theorem 1.2]). *Let the Poincaré inequality hold with $w_1 = w$ and $w_2 = v$ or $w_1 = w$ and $w_2 = w$, $\mu = 1$ and $w \in A_{1+2/n}, w^{-1} \in A_{2-2/n}$. Then for an arbitrary function $u \in C_0^\infty(B(x_0,\mathbb{R}))$ one has*

$$\int_{B(x_0,\rho)} u^2(x)v(x)\,dx \le C_1\rho^2\frac{v(B(x_0,\rho))}{w(B(x_0,\rho))}\int_{B(x_0,R)}\left|\frac{\partial u}{\partial x}\right|^2 w(x)\,dx \tag{2.17}$$

or

$$\int_{B(x_0,\rho)} u^2(x)w(x)\,dx \le C_1\rho^2\int_{B(x_0,R)}\left|\frac{\partial u}{\partial x}\right|^2 w(x)\,dx \tag{2.18}$$

with a constant C_1 independent of u and $0 < \rho < R$.

3. Regularity at the boundary

In this section we prove our main result:

Theorem 3.1. *Let the functions a_i satisfy conditions* (C1) *and* (C2). *Suppose moreover the following:*

(i) $v, w \in A_2$.

(ii) *The Poincaré inequality holds for $w_1 = w_2 = w$ with $\mu = 1$ and some $q > 2$.*

(iii) *The Poincaré inequality holds for $w_1 = w$ and $w_2 = v$ with any $\mu = 1$ or $\nu = v$ and some $q > 2$.*

(iv) *The inequalities* (2.17) *and* (2.18) *hold.*

Then for $(x_0,t_0) \in \partial\Omega \times (0,T)$ to be a regular boundary point of the domain Q_T to the equation (2.6), *it is necessary that*

$$\int_0^1 \frac{C_{2,w}(B(x_0,r)\setminus\Omega)}{C_{2,w}(B(x_0,r))}\frac{dr}{r} = \infty. \tag{3.1}$$

Remark 3.1. Put, for any $\theta > 0$,

$$Q_r^\theta(x_0,t_0) = B\left(x_0,\frac{r}{2}\right) \times \left(t_0 - 2\theta r^2\frac{v(B(x_0,r))}{w(B(x_0,r))},\ t_0 - \theta r^2\frac{v(B(x_0,r))}{w(B(x_0,r))}\right)$$

and define, for any set $F \subset \mathbb{R}^{n+1}$,

$$\widetilde{\mathcal{M}}(F) = \left\{ \phi \in C_0^\infty(\mathbb{R}^{n+1}) \Big| \, \phi \geq 1 \text{ for all } (x,t) \in F \right\}$$

and

$$\Gamma_{v,w}(F) = \inf_{\psi \in \widetilde{\mathcal{M}}(F)} \left\{ \sup_t \int \phi^2(x,t) v(x)\, dx + \iint w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx dt \right\}.$$

As well as in [14] it can be proved that

$$\Gamma_{v,w}\left(Q_r^\theta(x_0,t_0) \setminus Q_T\right) \approx r^2 \frac{v(B(x_0,r))}{w(B(x_0,r))} C_{2,w}(B(x_0,r) \setminus \Omega).$$

Also we can easily obtain that

$$\Gamma_{v,w}\left(Q_r^\theta(x_0,t_0)\right) \approx v(B(x_0,r))$$

$$C_{2,w}(B(x_0,r)) \approx r^{-2} w(B(x_0,r))$$

so that condition (3.1) is equivalent to

$$\int_0^1 \frac{\Gamma_{v,w}\left(Q_r^\theta(x_0,t_0) \setminus Q_T\right)}{v(B(x_0,r))} \frac{dr}{r} = \infty. \tag{3.2}$$

From Theorem 3.1 and (3.2) we have that, in the case $v(x) = 1$, our necessary condition coincides with the sufficient one from [1].

Now we define auxiliary functions $u_k = u_k(x,t)$ $(k \in \mathbb{N})$ that will play a fundamental role in the proof of Theorem 3.1. For $k \in \mathbb{N}$ we define a numerical sequence $\{\alpha_k\}_{k \in \mathbb{N}}$ such that $\alpha_k \to 0$ when $k \to \infty$ and

$$\frac{\alpha_{k+1}}{\alpha_k} \leq K_7 = \left(\frac{1}{4} K_5^{-1} K_6^{-\frac{q-2}{2q}} \right)^{\frac{q}{n\eta(q-2)}}, \qquad K_7 < 1 \tag{3.3}$$

with the constant K_5 from inequality (2.5), K_6 and η from Remark 2.1 and some $q > 2$ for which the Poincaré inequality is valid. Let

$$E_k = B(x_0, \alpha_k) \setminus \Omega \qquad \text{and} \qquad E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}). \tag{3.4}$$

Let further $\mathcal{M}_k(E_k)$ and $\mathcal{M}^{(k)}(E^{(k)})$ be the subsets of $\mathcal{M}(E_k)$ and $\mathcal{M}(E^{(k)})$ consisting of functions with support contained in $B(x_0, \alpha_{k-1})$ and $B(x_0, \alpha_{k-1}) \setminus B(x_0, \alpha_{k+2})$, respectively.

Lemma 3.1. *There exists a constant $C_2 > 0$ such that*

$$\inf \left\{ \int_{R^n} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx \, \Big| \, \phi \in \mathcal{M}_k(E_k) \right\} \leq C_2 \, C_{2,w}(E_k) \tag{3.5}$$

$$\inf \left\{ \int_{R^n} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx \, \Big| \, \phi \in \mathcal{M}^{(k)}(E^{(k)}) \right\} \leq C_2 \, C_{2,w}(E^{(k)}). \tag{3.6}$$

Proof. We will prove only inequality (3.5), because (3.6) can be proved as well. Let $\chi_k \in C^\infty(\mathbb{R}^n)$ with

$$\chi_k(x) = 1 \quad \text{for } |x - x_0| \le \alpha_k, \qquad \text{and} \qquad \chi_k(x) = 0 \quad \text{for } |x - x_0| \ge \alpha_{k-1}$$

and

$$\left|\frac{\partial \chi_k}{\partial x}\right| \le \frac{1}{(1-K_7)\alpha_{k-1}}.$$

Then, for $\phi \in \mathcal{M}(E_k)$ with $\chi_k\phi \in \mathcal{M}_k(E_k)$, we have

$$\begin{aligned}
&\inf_{\psi \in \mathcal{M}_k(E_k)} \int_{R^n} w(x)\left|\frac{\partial \psi}{\partial x}\right|^2 dx \\
&\le C_3 \inf_{\phi \in \mathcal{M}(E_k)} \int_{B(x_0,\alpha_{k-1})} \left(\chi_k^2 \left|\frac{\partial \phi}{\partial x}\right|^2 w(x)dx + \left|\frac{\partial \chi_k}{\partial x}\right|^2 \phi^2(x) w(x)\right) dx \\
&\le C_4 \left(1 + \frac{1}{(1-K_7)^2}\right) \inf_{\phi \in \mathcal{M}(E_k)} \int_{B(x_0,\alpha_{k-1})} w(x)\left|\frac{\partial \phi}{\partial x}\right|^2 dx \\
&\le C_2 C_{2,w}(E_k)
\end{aligned}$$

and the assertion is proved ∎

Let $\operatorname{diam}\Omega$ be the diameter of the region Ω. From now on, putting $R = 2 + \operatorname{diam}\Omega$, B will denote the ball of radius R centered at x_0. We introduce a non-increasing function $\gamma \in C^\infty(\mathbb{R})$ with

$$0 \le \gamma(s) \le 1, \quad \gamma(s) = 0 \ \text{ for } s \ge 2, \quad \gamma(s) = 1 \ \text{ for } s \le 1, \quad \left|\frac{d\gamma(s)}{ds}\right| \le 2.$$

Further, let

$$h(x) = \gamma(|x - x_0|) \qquad \text{and} \qquad \lambda(t) = \gamma\left(\alpha_k^{-2} \frac{w(B(x_0,\alpha_k))}{v(B(x_0,\alpha_k))} |t - t_0|\right).$$

For a given point $(x_0, t_0) \in \partial\Omega \times (0,T)$ we can choose a number k_0 such that

$$\alpha_{k_0}^2 \frac{v(B(x_0,\alpha_{k_0}))}{w(B(x_0,\alpha_{k_0}))} < t_0.$$

And for $k > k_0, (x,t) \in Q_k = D_k \times (0,T)$ and $D_k = B \setminus \overline{E}_k$ we define the function $u_k = u_k(x,t)$ as the solution of equation (2.6) in Q_k satisfying

$$u_k(x,t) = h(x)\lambda_k(t) \qquad ((x,t) \in \partial D_k \times (0,T)) \tag{3.7}$$

$$u_k(x,0) = 0 \qquad (x \in D_k). \tag{3.8}$$

Extend then the function $u_k = u_k(x,t)$ to $B \times (0,T)$ by setting it equal to $\lambda_k = \lambda_k(t)$ for $(x,t) \in \overline{E}_k \times (0,T)$.

Lemma 3.2. *There exists a constant $K_8 > 0$ such that the function $u_k = u_k(x,t)$ satisfies*

$$\|u_k\|^2_{V_2(Q_k,v,w)} \le K_8\,\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\,C_{2,w}(E_k). \tag{3.9}$$

Proof. It is easy to show that for $h > 0$ and $0 < \tau < T - h$ the integral identity

$$\begin{aligned}\int_0^\tau\!\!\int_{D_k} \Big\{ v(x)\frac{\partial}{\partial t}[u_k(x,t)]_h\psi(x,t) + \sum_{i=1}^n \Big[a_i\Big(x,t,u_k,\frac{\partial u_k}{\partial x}\Big)\Big]_h \frac{\partial\psi(x,t)}{\partial x_i} \\ - \Big[a_0\Big(x,t,u_k,\frac{\partial u_k}{\partial x}\Big)\Big]_h \psi(x,t)\Big\}\,dx\,dt = 0\end{aligned} \tag{3.10}$$

holds for any $\psi \in \mathring{V}_2(Q_k,v,w)$ in which we use the notation

$$[g(x,t)]_h = \frac{1}{h}\int_t^{t+h} g(x,\tau)\,d\tau.$$

Let us put in (3.10) the function

$$\psi(x,t) = [u_k(x,t)]_h - \phi(x)\,[\lambda_k(t)]_h \qquad (\phi \in \mathcal{M}_k(E_k)). \tag{3.11}$$

In the inequality obtained from (3.10) by substituting (3.11), we integrate by parts the term containing $\frac{\partial}{\partial t}[u_k(x,t)]_h$, pass to the limit as $h \to 0$, and estimate using (2.7) and (2.8). Thus we get

$$\begin{aligned}&\int_{D_k} v(x)u_k^2(x,\tau)\,dx + \int_0^\tau\!\!\int_{D_k} w(x)\left|\frac{\partial u_k(x,t)}{\partial x}\right|^2 dx\,dt \\ &\le C_5\Big\{\int_0^\tau\!\!\int_{D_k}\Big(1+\Big|\frac{\partial\lambda_k}{\partial t}\Big|\Big)u_k^2(x,t)v(x)\,dx\,dt \\ &\quad + \int_{D_k}\Big[v(x)\phi^2(x) + \alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\Big|\frac{\partial\phi}{\partial x}\Big|^2 w(x)\Big]\,dx\Big\}.\end{aligned} \tag{3.12}$$

Using the Poincaré and Gronwall inequalities, from (3.12) we obtain

$$\begin{aligned}&\int_{D_k} v(x)u_k^2(x,\tau)dx + \int_0^\tau\!\!\int_{D_k} w(x)\left|\frac{\partial u_k(x,t)}{\partial x}\right|^2 dx\,dt \\ &\le C_6\,\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\int_{D_k}\left|\frac{\partial\phi}{\partial x}\right|^2 w(x)\,dx\end{aligned}$$

for $\tau \le T$. Hence, by Lemma 3.1, the estimate (3.9) follows ∎

Lemma 3.3. *There exists a constant* $K_9 > 0$ *such that, for* $k > k_0$,

$$m(k) = \inf\left\{|u_k(x,t)| \,\middle|\, |x-x_0| \le \alpha_{k+5},\ |t-t_0| \le \alpha_{k+5}^2 \frac{v(B(x_0,\alpha_{k+5}))}{w(B(x_0,\alpha_{k+5}))}\right\} \le K_9\left\{\alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))}\right\}^{1/2}. \tag{3.13}$$

Proof. Set

$$\tau(k) = \alpha_{k+5}^2 \frac{v(B(x_0,\alpha_{k+5}))}{w(B(x_0,\alpha_{k+5}))} \quad \text{and} \quad \tilde{u}_k(x,t) = \min\left[\frac{u_k(x,t)}{m(k)}, 1\right].$$

Then from the definition of capacity and doubling condition for w we get

$$w(B(x_0,\alpha_k))\alpha_k^{-2} \le C_7 C_{2,w}(B(x_0,\alpha_{k+5})) \le C_8 \tau^{-1}(k+5) \int_{t_0-\tau(k+5)}^{t_0+\tau(k+5)} \int_B w(x)\left|\frac{\partial \tilde{u}_k}{\partial x}\right|^2 dx dt. \tag{3.14}$$

Now from Lemma 3.2 it follows

$$\int_{t_0-\tau(k+5)}^{t_0+\tau(k+5)} \int_B w(x)\left|\frac{\partial \tilde{u}_k}{\partial x}\right|^2 dx dt \le C_9[m(k)]^{-2}\tau(k+5)\, C_{2,w}(E_k). \tag{3.15}$$

From (3.14) and (3.15) we have the thesis ∎

In the next sections the a priori estimates on which the proof of Theorem 3.1 is based will be proved.

Theorem 3.2. *Let all of the assumptions of Theorem* 3.1 *be held. Then there exists a constant* $K_{10} > 0$ *such that the solution* $u_k = u_k(x,t)$ *of problem* (2.6), (3.7), (3.8) *satisfies*

$$|u_k(x,t)| \le K_{10}\left\{\alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\right\} \tag{3.16}$$

for

$$(x,t) \notin \left\{(x,t) \in Q_k \,\middle|\, |x-x_0| < \alpha_{k-3},\ |t-t_0| < \alpha_{k-3} \frac{v(B(x_0,\alpha_{k-3}^2))}{w(B(x_0,\alpha_{k-3}))}\right\}.$$

Theorem 3.3. *Let all of the assumptions of Theorem* 3.1 *be held. Then there exists a constant* $K_{11} > 0$ *such that*

$$\left|u_k(x,t) - u_{k+1}(x,t)\right| \le K_{11}\left\{\alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\right\} \tag{3.17}$$

for

$$|x-x_0| \le \alpha_{k-4} \quad \text{and} \quad |t-t_0| \le \alpha_{k-4} \frac{v(B(x_0,\alpha_{k-4}))}{w(B(x_0,\alpha_{k-4}))}.$$

Let us remark that the Theorems 3.1 - 3.3 in the non-weighted case were proved in the paper [13].

Using Theorems 3.2 and 3.3 we can now prove Theorem 3.1.

Proof of Theorem 3.1. We construct a solution of equation (2.6) in Q_T satisfying condition (2.15) with function $f \in C(\overline{Q}_T) \cap W_2^1(Q_T, v, w)$ and discontinuous at (x_0, t_0) as soon as the inequality (3.1) is not satisfied.

Thus, let us assume the boundedness of the integral on the left-hand side of condition (3.1). Then using (3.3) and (2.5) we can easily show that

$$\sum_{k=1}^{\infty} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\} < \infty. \tag{3.18}$$

We can then find a number $k_1 \in \mathbb{N}$ such that

$$\sum_{k=k_1}^{\infty} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\} < \frac{1}{4K_{11}} \tag{3.19}$$

where K_{11} is the constant from Theorem 3.3. We will show that the function $u_{k_1} = u_{k_1}(x,t)$ defined above as the solution of the problem (2.6), (3.7), (3.8) is discontinuous at (x_0, t_0). Let $\delta > 0$ be an arbitrary number. By the convergence of the series (3.19) and estimate (3.13), we have $m(k) \to 0$ as $k \to 0$. In this way a number $k_2 = k_2(\delta)$ and a point $(x_\delta, t_\delta) \in Q_T$ can be chosen so that

$$|u_{k_2}(x_\delta, t_\delta)| \le \frac{1}{4} \qquad \text{and} \qquad |x_\delta - x_0|^2 + |t_\delta - t_0| < \delta^2. \tag{3.20}$$

From (3.17), (3.19) and (3.20) we have

$$|u_{k_1}(x_\delta, t_\delta)| \le |u_{k_2}(x_\delta, t_\delta)| + \sum_{k=k_1}^{k_2-1} |u_{k+1}(x_\delta, t_\delta) - u_k(x_\delta, t_\delta)| \le \frac{1}{2}.$$

So it follows that

$$\liminf_{(x,t)\to(x_0,t_0)} u_{k_1}(x,t) < 1 \qquad \big((x,t) \in Q_T\big).$$

This inequality proves the non-regularity of the boundary point (x_0, t_0) and thus Theorem 3.1 ∎

4. Pointwise estimates of the function $u_k = u_k(x,t)$

In this section we will use the following

Lemma 4.1 (see [7]). *Let the assumptions* (i) - (iii) *of Theorem* 3.1 *be satisfied. Then there exist constants* $C_{10} > 0$ *and* $h > 1$ *such that*

$$\begin{aligned} &\frac{1}{w(Q)} \iint_Q |u|^{2h} w(x)\,dxdt + \frac{1}{v(Q)} \iint_Q |u|^{2h} v(x)\,dxdt \\ &\le C_{10} \left\{ \sup_{t\in J} \frac{1}{v(B)} \int_B u^2(x,t) v(x)\,dx \right\}^{h-1} \\ &\quad \times \left\{ \sup_{t\in J} \frac{|J|}{v(B)} \int_B u^2(x,t) v(x)\,dx \right. \\ &\quad \left. + \frac{(\operatorname{meas} B)^{2/n}}{w(Q)} \iint_Q w(x) \left|\frac{\partial u}{\partial x}\right|^2 dxdt \right\} \end{aligned} \tag{4.1}$$

for any $u \in \overset{\circ}{V}_2(Q, v, w)$ *on* $Q = B \times J$, *with* J *being an interval and* B *a ball,* $|J| = \operatorname{meas} J$ *and*

$$w(Q) = |J| \int_B w(x)\,dx, \qquad v(Q) = |J| \int_B v(x)\,dx, \qquad v(B) = \int_B v(x)\,dx.$$

Let $\alpha_{k-3} \le \rho \le R$,

$$G(\rho) = \left\{ (x,t) \middle| \, |x - x_0| < \rho,\ |t - t_0| < \frac{v(B(x_0,\rho))}{w(B(x_0,\rho))}\rho^2 \right\},$$

$0 < \varepsilon < \rho$ and $\mu \ge \mu_k(\rho, \varepsilon)$ with

$$\mu_k(\rho,\varepsilon) = \sup\left\{ u_k(x,t) \middle| \, (x,t) \in [G(\rho+\varepsilon) \setminus G(\rho-\varepsilon)] \cap Q_k \right\}.$$

We then define the set

$$F(\rho,\mu) = Q_k \setminus G(\rho) \cup \left\{ (x,t) \in Q_k \cap G(\rho) \middle| \, u_k(x,t) \le \mu \right\}$$

and the function

$$u_k^{(\mu)}(x,t) = \begin{cases} u_k(x,t) & \text{for } (x,t) \in Q_k \setminus G(\rho) \\ \min\{u_k(x,t), \mu\} & \text{for } (x,t) \in Q_k \cap G(\rho). \end{cases}$$

Lemma 4.2. *For* $\alpha_{k-3} \le \rho \le R$, $0 < \varepsilon < \rho$ *and* $\mu \ge \mu_k(\rho,\varepsilon)$ *we have*

$$\begin{aligned} \sup_{0<t\le T} \int_{D_k} v(x) \left| u_k^{(\mu)}(x,t) \right|^2 dx + \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt \\ \le C_{11}\, \mu\, P_k(E_k, \rho) \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} P_k(E_k,\rho) = {} & \left[\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) \right]^{1/2} \\ & \times \left[\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) + \rho^2 \frac{v^2(B(x_0,\rho))}{w(B(x_0,\rho))} \right]^{1/2} \end{aligned} \tag{4.3}$$

and C_{11} *is a constant depending only on* $n, \nu_1\, \nu_2$ *and* T.

Proof. Into the integral identity (3.10) substitute the function

$$\psi(x,t) = \left[u_k^{(\mu)}(x,t) \right]_h - \phi(x) \lambda_{k,h}^{(\mu)}(t) \qquad \text{for } 0 < h < \varepsilon^2 \frac{v(B(x_0,\rho))}{w(B(x_0,\rho))}.$$

Here

$$\lambda_{k,h}^{(\mu)}(t) = \min\{[\lambda_k(t)]_h, \mu\} \qquad (\phi \in \mathcal{M}(E_k)).$$

After this substitution we transform the term containing $\frac{\partial [u_k(x,t)]_h}{\partial t}$, obtaining

$$\int_0^\tau\!\!\int_{D_k} v(x)\frac{\partial}{\partial t}[u_k(x,t)]_h \left\{ \left[u_k^{(\mu)}(x,t)\right]_h - \phi(x)\lambda_{k,h}^{(\mu)}(t)\right\} dxdt$$
$$= \int_{D_k} v(x)\left\{ \frac{1}{2}\left[u_k^{(\mu)}(x,\tau)\right]_h^2 + \mu\left([u_k(x,\tau)]_h - \left[u_k^{(\mu)}(x,\tau)\right]_h\right) - [u_k(x,\tau)]_h\phi(x)\lambda_{k,h}^{(\mu)}(\tau)\right\}dx$$
$$+ \int_0^\tau\!\!\int_{D_k} v(x)[u_k(x,t)]_h\phi(x)\frac{\partial \lambda_{k,h}^{(\mu)}(t)}{\partial t}\, dxdt.$$

Using this representation on passing to limit with respect to h in the integral identity, we get

$$\int_{D_k} v(x)\left\{ \frac{1}{2}\left[u_k^{(\mu)}(x,\tau)\right]^2 + \mu\left(u_k(x,\tau) - u_k^{(\mu)}(x,\tau)\right) - u_k(x,\tau)\phi(x)\lambda_k^{(\mu)}(\tau)\right\}dx$$
$$+ \int_0^\tau\!\!\int_{D_k} \left\{ \sum_{i=1}^n a_i\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right)\frac{\partial}{\partial x_i}\left[u_k^{(\mu)}(x,t) - \phi(x)\lambda_k^{(\mu)}(t)\right] \right. \tag{4.4}$$
$$- a_0\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right)\left[u_k^{(\mu)}(x,t) - \phi(x)\lambda_k^{(\mu)}(t)\right]$$
$$\left. + v(x)u_k(x,t)\phi(x)\frac{\partial\lambda_k^{(\mu)}(t)}{\partial t}\right\}dxdt$$
$$= 0.$$

Using inequalities (2.7), (2.8) and (3.9) we can estimate the left-hand side of (4.4) obtaining

$$\int_{D_k} v(x)\left[u_k^{(\mu)}(x,\tau)\right]_h^2 dx + \iint_{F(\rho,\mu)} w(x)\left|\frac{\partial u_k(x,t)}{\partial x}\right|^2 dxdt$$
$$\le C_{12}\left\{ \mu\,\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}C_{2,w}(E_k) + \iint_{F(\rho,\mu)} v(x)u_k^2(x,t)\,dxdt \right. \tag{4.5}$$
$$\left. + \int_0^\tau\!\!\int_{D_k}\left[v(x)|u_k(x,t)| + w^{1/2}(x)v^{1/2}(x)\left|\frac{\partial u_k(x,t)}{\partial x}\right|\right]\left|u_k^{(\mu)}(x,t)\right|dxdt\right\}.$$

We estimate the latter part on the right-hand side of (4.5) using the Hölder and Young inequalities, (3.9) and the observation that

$$u_k(x,t) = 0 \qquad \text{for } t \le t_0 - 2\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}.$$

We have

$$\begin{aligned}
&\int_0^{\tau}\!\!\int_{D_k} \left[v(x)\,|u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] \left| u_k^{(\mu)}(x,t) \right| dxdt \\
&\quad \le \mu \iint_{Q_k \cap G(\rho)} \left[v(x)|u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] dxdt \\
&\quad + \iint_{F(\rho,\mu)} \left[v(x)|u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] \left| u_k^{(\mu)}(x,t) \right| dxdt \\
&\quad \le \frac{C_{12}^{-1}}{2} \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dxdt \\
&\quad + C_{13} \Bigg\{ \iint_{F(\rho,\mu)} v(x) u_k^2(x,t)\, dxdt \\
&\quad + \mu \left[\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) \right]^{1/2} \left[\rho^2 \frac{v^2(B(x_0,\rho))}{w(B(x_0,\rho))} \right]^{1/2} \Bigg\}.
\end{aligned}$$

Hence from (4.5) it follows

$$\begin{aligned}
&\int_{D_k} \left[u_k^{(\mu)}(x,\tau) \right]^2 v(x)\, dx + \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dxdt \\
&\qquad \le C_{14} \left\{ \int_0^{\tau}\!\!\int_{D_k} v(x) \left[u_k^{(\mu)}(x,t) \right]^2 dxdt + \mu\, P_k(E_k,\rho) \right\}.
\end{aligned}$$

Now using the Gronwall inequality we obtain (4.2) ∎

Proof of Theorem 3.2. Let $\alpha_{k-3} \le \rho \le R$. We define two numerical sequences $\{\rho_{1,i}\}_{i\in\mathbf{N}}$ and $\{\rho_{2,i}\}_{i\in\mathbf{N}}$ by putting

$$\rho_{1,i}^2 = \frac{1+\alpha_i}{1+\alpha_0}\rho^2 \qquad \text{and} \qquad \rho_{2,i}^2 = \frac{1+2\alpha_0-\alpha_i}{1+\alpha_0}\rho^2$$

where α_i are defined by (3.3). Moreover consider infinitely differentiable functions $\chi = \chi_i(s)$ $(i \in \mathbb{N})$ on $\mathbb{R}$ such that

$$0 \le \chi_i(s) \le 1$$

$$\chi_i(s) = 1 \quad \text{on } [\rho_{1,i}^2, \rho_{2,i}^2] \quad \text{and} \quad \chi_i(s) = 0 \quad \text{on } \mathbb{R} \setminus [\rho_{1,i+1}^2, \rho_{2,i+1}^2]$$

$$\left| \frac{d\chi_i(s)}{ds} \right| \le \frac{1+\alpha_0}{1-K_7} \frac{1}{\alpha_i} \rho^{-2}.$$

Then let

$$\psi_i(x) = \chi_i\left(|x - x_0|^2\right) \qquad \text{and} \qquad \phi_i(t) = \chi_i\left(\frac{w(B(x_0,\rho))}{v(B(x_0,\rho))}|t - t_0|\right)$$

and substitute into (3.10) the function

$$\psi(x,t) = [u_k(x,t)]_h^{r+1}\,\psi_i^{s+2}(x)\phi_i^{s+2}(t)$$

for arbitrary non-negative numbers r and s. Integrating by parts the term containing $\frac{\partial}{\partial t}[u(x,t)]_h$, passing to the limit as $h \to 0$ and estimating using (2.7) and (2.8), we obtain

$$\begin{aligned}
&\int_{D_k} u_k^{r+2}(x,\tau)\psi_i^{s+2}(x)\phi_i^{s+2}(\tau)v(x)\,dx \\
&\quad + \int_0^\tau\!\!\int_{D_k} w(x)\left|\frac{\partial u_k}{\partial x}\right|^2 u_k^r(x,t)\psi_i^{s+2}(x)\phi_i^{s+2}(t)dxdt \\
&\le C_{15}\frac{\alpha_i^{-2}(r+s+1)^2}{\rho^2}\int_0^\tau\!\!\int_{D_k} u_k^{r+2}(x,t)\left[w(x) + \frac{w(B(x_0,\rho))}{v(B(x_0,\rho))}v(x)\right]\psi_i^s(x)\phi_i^s(t)\,dxdt.
\end{aligned}$$

Hence using Lemma 4.1 and applying the iterative technique (see, e.g, [13: Lemma 2.2]) to

$$m_i = m_i(\rho) = \sup\left\{ |u_k(x,t)| \;\middle|\; \begin{array}{l} (x,t) \in Q_k,\ \rho_{1,i} \le |x - x_0| \le \rho_{2,i} \\ \rho_{1,i}^2 \le \frac{w(B(x_0,\rho))}{v(B(x_0,\rho))}|t - t_0| \le \rho_{2,i}^2 \end{array}\right\} \tag{4.6}$$

we obtain the following estimate with some $h > 1$:

$$\begin{aligned}
[m_i]^2 &\le C_{16}\,\alpha_i^{-\frac{2h}{h-1}} \\
&\quad\times \int_0^\tau\!\!\int_{D_k} u_k^2(x,t)\left[\frac{w(x)}{\rho^2 v(B(x_0,\rho))} + \frac{w(B(x_0,\rho))v(x)}{\rho^2 v^2(B(x_0,\rho))}\right]\psi_i^2(x)\phi_i^2(t)\,dxdt.
\end{aligned} \tag{4.7}$$

Let us estimate the integral (4.7) applying Lemmata 2.2 and 4.2. We have

$$\begin{aligned}
&\int_0^\tau\!\!\int_{D_k} u_k^2(x,t)\left[\frac{w(x)}{\rho^2 v(B(x_0,\rho))} + \frac{w(B(x_0,\rho))v(x)}{\rho^2 v^2(B(x_0,\rho))}\right]\psi_i^2(x)\phi_i^2(t)\,dxdt \\
&\le \iint_{G(\rho_{i+1})\cap Q_k}\left|u_k^{(m_{i+1})}(x,t)\right|^2\left[\frac{w(x)}{\rho^2 v(B(x_0,\rho))} + \frac{w(B(x_0,\rho))v(x)}{\rho^2 v^2(B(x_0,\rho))}\right]dxdt \\
&\le C_{17}\frac{1}{v(B(x_0,\rho))}\iint_{F(\rho_{i+1},m_{i+1})} w(x)\left|\frac{\partial u_k(x,t)}{\partial x}\right|^2 dxdt \\
&\le C_{18}\frac{1}{v(B(x_0,\rho))}\,m_{i+1}P_k(E_k,\rho).
\end{aligned} \tag{4.8}$$

Here ρ_{i+1} is such that $\rho_{i+1} < \rho_{2,i+1}$, and $\chi_{i+1}(s) = 0$ for $s \geq \rho_{i+1}$.

Now, from (4.7) and (4.8) it follows that

$$[m_i]^2 \leq C_{19} \frac{\alpha_i^{-\frac{2h}{h-1}}}{v(B(x_0,\rho))} P_k(E_k,\rho) m_{i+1} \tag{4.19}$$

and further, by Lemma 2.1, we get

$$m_i(\rho) \leq C_{20} \frac{1}{v(B(x_0,\rho))} P_k(E_k,\rho). \tag{4.10}$$

Now for proving Theorem 3.2 it sufficies to show that for $\rho \geq \alpha_{k-3}$

$$\mu(\rho,\alpha_k) \leq C_{21} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} \right\}. \tag{4.11}$$

For $\rho = \alpha_{k-3}$, inequality (4.11) follows from (4.10). If for some $\rho > \alpha_{k-3}$ we had $m_1(\rho) \leq \mu(\alpha_{k-3},\alpha_k)$, then that ρ will satisfy (4.11). If however $m_1(\rho) > \mu(\alpha_{k-3},\alpha_k)$ for some ρ, then for all $i \in \mathbb{N}$ we have $m_{i+1}(\rho) > \mu(\alpha_{k-3},\alpha_k)$ and we operate a change in (4.8). In this case

$$F(\rho_{i+1}, m_{i+1}(\rho)) \subset F(\alpha_{k-3}, m_{i+1}(\rho))$$

and we obtain instead of (4.8)

$$\begin{aligned} \int_0^T \int_{D_k} u_k^2(x,t) \left[\frac{w(x)}{\rho^2 v(B(x_0,\rho))} + \frac{w(B(x_0,\rho)) v(x)}{\rho^2 v^2(B(x_0,\rho))} \right] \psi_i^2(x) \phi_i^2(t)\, dx dt \\ \leq C_{22} \frac{1}{v(B(x_0,\rho))} \iint_{F(\alpha_{k-3}, m_{i+1}(\rho))} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt. \end{aligned} \tag{4.12}$$

So as above we get the inequality

$$[m_i(\rho)]^2 \leq C_{23} \frac{\alpha_i^{-\frac{2h}{h-1}}}{v(B(x_0,\rho))} P_k(E_k,\alpha_{k-3}) m_{i+1}(\rho).$$

Hence, by Lemma 2.1, for a given ρ we have (4.11) and this completes the proof of Theorem 3.2 ∎

5. Integral estimates for the difference $u_k(x,t) - u_{k+1}(x,t)$

We shall need auxiliary functions $f_k = f_k(x)$ and $g_k = g_k(x)$ defined, respectively, as the solutions of the problems

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(w(x)\frac{\partial f_k}{\partial x_i}(x)\right) = 0 \quad \left(x \in D_k = B \setminus E_k,\ f_k - h \in \mathring{W}_2^1(D_k, w)\right) \tag{5.1}$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(w(x)\frac{\partial g_k}{\partial x_i}(x)\right) = 0 \quad \left(x \in D^{(k)} = B \setminus E^{(k)},\ g_k - h \in \mathring{W}_2^1(D^{(k)}, w)\right). \tag{5.2}$$

Lemma 5.1. *There exists a constant $C_{24} > 0$ such that*

$$\|f_k\|_{W_2^1(D_k,w)} \le C_{24}\, C_{2,w}(E_k) \tag{5.3}$$

$$\|g_k\|_{W_2^1(D^{(k)},w)} \le C_{24}\, C_{2,w}(E^{(k)}). \tag{5.4}$$

Proof. We will prove only the inequality (5.3), the proof of inequality (5.4) is analogous. In the definition of a weak solution for equation (5.3), choose as test function

$$\psi = f_k - \phi \qquad (\phi \in \mathcal{M}(E_k)).$$

We then obtain

$$\int_{D_k} w(x)\left|\frac{\partial f_k(x)}{\partial x}\right|^2 dx \le \int_{D_k} w(x)\left|\frac{\partial \phi}{\partial x}\right|^2 dx.$$

Now, using Lemma 3.1, we have (5.3) ∎

Let us denote

$$\delta_k = u_k - u_{k+1} \tag{5.5}$$

and

$$d_k = \sup\left\{ |\delta_k(x,t)| \;\middle|\; \begin{array}{l} (x,t) \in Q_k,\ |x - x_0| < \alpha_{k-3} \\ |t - t_0| < \alpha_{k-3}^2 \dfrac{v(B(x_0,\alpha_{k-3}))}{w(B(x_0,\alpha_{k-3}))} \end{array} \right\}. \tag{5.6}$$

For any function $f = f(x,t)$ and numbers A_1 and A_2 with $A_1 < A_2$, we define

$$[f(x,t)]_{\pm} = \max\left\{ \pm f(x,t), 0\right\}$$

$$[f(x,t)]_{(A_1,A_2)} = \max\left\{ \min\left[f(x,t), A_2\right], A_1\right\}.$$

Moreover we define for

$$\mu > 0 \qquad \text{and} \qquad t_k = t_0 + \alpha_{k-1}^2 \frac{v(B(x_0,\alpha_{k-1}))}{w(B(x_0,\alpha_{k-1}))}$$

the sets

$$F^{\pm}(\mu) = \left\{(x,t) \in B \times [0,t_k] \;\middle|\; \pm\left[\delta_k(x,t)\right]_{(-\mu,\mu)} \ge \mu\left[\bar{f}_k(x,t) + \bar{g}_k(x,t)\right]\right\},$$

$$F(\mu) = F^{+}(\mu) \cup F^{-}(\mu)$$

$$T(\mu) = \left\{(x,t) \in B \times [0,t_k] \;\middle|\; |\delta_k(x,t)| \le \mu\right\}$$

where

$$\bar{f}_k(x,t) = f_k(x)\gamma(\alpha_{k-1}^{-1}|x-x_0|)\{\lambda_{k-2}(t) - \lambda_{k+2}(t)\}$$

$$\bar{g}_k(x,t) = g_k(x)\Big\{\gamma\left(\alpha_{k-1}^{-1}|x-x_0|\right) - \gamma\left(\alpha_{k+2}^{-1}|x-x_0|\right)\Big\}\lambda_{k-2}(t)$$

and where γ and λ_k are the functions introduced in Section 3. Further, for an arbitrary subset $E \subset B \times (0,T)$, we denote by $\chi = \chi_E(x,t)$ its characteristic function.

Theorem 5.1. *For arbitrary $\mu > d_k$ there exists a constant $K_{12} > 0$ such that*

$$\begin{aligned}
&\sup_{0<t\le t_k} \int_B v(x)\left|[\delta_k(x,t)]_{(-\mu,\mu)}\right|^2 \chi_{F(\mu)}(x,t)\,dxdt \\
&\quad + \iint_{T(\mu)\cap F(\mu)} w(x)\left|\frac{\partial}{\partial x}\delta_k(x,t)\right|^2 dxdt \\
&\le K_{12}\,\mu\left\{\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\,C_{2,w}(E_k)\right\}^{1/2} \\
&\quad \times\left\{\alpha_k^2\frac{v^2(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} + C_{2,w}(E_k)\,\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\right\}^{1/2}.
\end{aligned} \tag{5.7}$$

Proof. Since $F(\mu) = F^+(\mu)\cup F^-(\mu)$, we prove the inequality (5.7) only for $F^+(\mu)$. Define for

$$h < \alpha_{k+2}^2\frac{v(B(x_0,\alpha_{k+2}))}{w(B(x_0,\alpha_{k+2}))}$$

the function

$$\psi(x,t) = \Big[[\delta_{k,h}(x,t)]_{(-\mu,\mu)} - \mu[\bar{f}_k(x,t)]_h - \mu[\bar{g}_k(x,t)]_h\Big]_+ \tag{5.8}$$

where $\delta_{k,h}(x,t) = [u_k(x,t)]_h - [u_{k+1}(x,t)]_h$. As well as in [13], using condition (3.3), we check that the function ψ defined by (5.8) belongs to the space $\mathring{V}_2(D_K \times (0,t_k), v, w)$. We plug ψ in the integral identity (3.10) for u_k and u_{k+1} obtaining

$$\begin{aligned}
&\int_0^T\!\!\int_B \Bigg\{v(x)\frac{\partial}{\partial t}\delta_{k,h}\psi(x,t) + \sum_{i=1}^n\Bigg(\left[a_i\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right)\right]_h \\
&\quad - \left[a_i\left(x,t,u_{k+1},\frac{\partial u_{k+1}}{\partial x}\right)\right]_h\Bigg)\frac{\partial\psi}{\partial x_i} \\
&\quad - \Bigg(\left[a_0\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right)\right]_h - \left[a_0\left(x,t,u_{k+1},\frac{\partial u_{k+1}}{\partial x}\right)\right]_h\Bigg)\psi\Bigg\}dxdt \\
&= 0.
\end{aligned} \tag{5.9}$$

We transform the first term under the integral sign in (5.9) in the following way:

$$\begin{aligned}
&\int_0^\tau\int_B v(x)\frac{\partial}{\partial t}\delta_{k,h}\psi(x,t)\,dx\,dt \\
&\quad = \frac{1}{2}\int_B v(x)\frac{\partial}{\partial t}\Bigg\{\Big[[\delta_{k,h}(x,\tau)]_{(-\mu,\mu)} - \mu\left[\bar{f}_k(x,\tau)+\bar{g}_k(x,\tau)\right]_h\Big]_+^2 \\
&\quad + 2\left(\delta_{k,h}(x,\tau) - [\delta_{k,h}(x,\tau)]_{(-\mu,\mu)}\right) \\
&\quad \times \Big[[\delta_{k,h}(x,\tau)]_{(-\mu,\mu)} - \mu\left[\bar{f}_k(x,\tau)+\bar{g}_k(x,\tau)\right]_h\Big]_+\Bigg\}dx \\
&\quad + \mu\int_0^\tau\int_B v(x)\frac{\partial}{\partial t}\left[\bar{f}_k(x,t)+\bar{g}_k(x,t)\right]_h \\
&\quad \times \Big[\delta_{k,h}(x,t) - \mu\left[\bar{f}_k(x,t)+\bar{g}_k(x,t)\right]_h\Big]_+ dx\,dt.
\end{aligned} \tag{5.10}$$

Using (5.10), passing to the limit as $h \to 0$ and applying (2.7), (2.8), (2.17), (2.18), (3.9), (5.3), (5.4), we obtain from (5.9)

$$\begin{aligned}
&\int_B v(x)\,[\delta_{k,h}(x,\tau)]^2_{(-\mu,\mu)}\,\chi_{F^+(\mu)}(x,\tau)dx \\
&\quad + \int_0^\tau\int_B w(x)\left|\frac{\partial\delta_k(x,t)}{\partial x}\right|^2\chi_{F^+(\mu)\cap T(\mu)}(x,t)\,dx\,dt \\
&\quad \le C_{25}\Bigg\{\mu\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}C_{2,w}(E_k) \\
&\quad + \int_0^\tau\int_B v(x)\delta_k^2(x,t)\chi_{F^+(\mu)\cap T(\mu)}(x,t)\,dx\,dt \\
&\quad + \int_0^\tau\int_B\left(v(x)\,|\delta_k(x,t)| + w^{1/2}(x)v^{1/2}(x)\left|\frac{\partial\delta_k(x,t)}{\partial x}\right|\right)\cdot \\
&\quad \times [\delta_{k,h}(x,t)]_{(-\mu,\mu)}\,\chi_{F^+(\mu)}(x,t)dx\,dt\Bigg\}.
\end{aligned} \tag{5.11}$$

We estimate the last integral of the right-hand side of (5.11) using the Hölder inequality, getting

$$\int_0^\tau\!\!\int_B \left(v(x)\,|\delta_k(x,t)| + w^{1/2}(x)v^{1/2}(x)\left|\frac{\partial \delta_k(x,t)}{\partial x}\right|\right)$$

$$\times\, [\delta_{k,h}(x,t)]_{(-\mu,\mu)}\, \chi_{F^+(\mu)}(x,t)\, dxdt$$

$$\leq \frac{1}{2C_{25}} \int_0^\tau\!\!\int_B w(x)\left|\frac{\partial \delta_k(x,t)}{\partial x}\right|^2 \chi_{F^+(\mu)\cap T(\mu)}(x,t)\, dxdt$$

$$+ C_{26}\int_0^\tau\!\!\int_B v(x)\delta_k^2(x,t)\chi_{F^+(\mu)\cap T(\mu)}(x,t)\, dxdt$$

$$+ C_{26}\,\mu\left\{\alpha_k^2\frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}\right\}^{1/2} \{v(F^+(\mu)\setminus T(\mu))\}^{1/2}$$

where

$$v(F^+(\mu)\setminus T(\mu)) = \iint_{F^+(\mu)\setminus T(\mu)} v(x)\, dxdt. \tag{5.12}$$

Since $\mu > d_k$, from (5.6) we have

$$F^+(\mu)\setminus T(\mu) \subset \left\{(x,t)\,\middle|\; |x-x_0| < \alpha_{k-3},\ |t-t_0| < \alpha_{k-3}^2\frac{v(B(x_0,\alpha_{k-3}))}{w(B(x_0,\alpha_{k-3}))}\right\}$$

and so

$$v(F^+(\mu)\setminus T(\mu)) \leq C_{27}\,\alpha_k^2\frac{v^2B(x_0,\alpha_k)}{w(B(x_0,\alpha_k))}. \tag{5.13}$$

Now applying the Gronwall inequality, from (5.11) - (5.13) we get (5.7) ∎

6. Proof of the Theorem 3.3

From the integral identity (3.10) for $u_k = u_k(x,t)$ and $u_{k+1} = u_{k+1}(x,t)$ it follows that, for an arbitrary function $\psi \in \overset{\circ}{V}_2(Q_K, v, w)$ and for $0 < \tau \leq T$,

$$\int_0^\tau\!\!\int_{D_k}\Bigg\{v(x)\frac{\partial}{\partial t}\delta_{k,h}(x,t)\psi(x,t)$$

$$+\sum_{i=1}^n\left[a_i\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right) - a_i\left(x,t,u_{k+1},\frac{\partial u_{k+1}}{\partial x}\right)\right]_h\frac{\partial\psi}{\partial x_i} \tag{6.1}$$

$$-\left[a_0\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right) - a_0\left(x,t,u_{k+1},\frac{\partial u_{k+1}}{\partial x}\right)\right]_h\psi(x,t)\Bigg\}dxdt$$

$$= 0.$$

Define the numerical sequences $\{\rho_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$ by the equalities

$$\rho_i = \alpha_{k+4}(1+\alpha_1-\alpha_i)$$
$$\sigma_i = \alpha_{k+4}^2 \frac{v(B(x_0,\alpha_{k+4}))}{w(B(x_0,\alpha_{k+4}))}(1+\alpha_1-\alpha_i).$$

Then define infinitely differentiable functions $\gamma_i = \gamma_i(x)$ $(x\in\mathbb{R}^n)$ and $\tilde{\gamma}_i = \tilde{\gamma}_i(t)$ $(t\in\mathbb{R})$ such that the following conditions are fulfilled:

a) $\gamma_i(x) = 1$ on the set $\{x\,|\,|x-x_0|\le\rho_i\}$, $\gamma_i(x)=0$ outside the set $\{x\,|\,|x-x_0|\le\rho_{i+1}\}$, $0\le\gamma_i(x)\le 1$ and $\left|\frac{\partial\gamma_i}{\partial x}\right|\le C_{28}\alpha_k^{-1}\alpha_i^{-1}$.

b) $\tilde{\gamma}_i(t)=1$ on the set $\{x\,|\,|t-t_0|\le\sigma_i\}$, $\tilde{\gamma}_i(x)=0$ outside the set $\{t\,|\,|t-t_0|\le\sigma_{i+1}\}$, $0\le\tilde{\gamma}_i(t)\le 1$ and $\left|\frac{\partial\tilde{\gamma}_i}{\partial t}\right|\le C_{29}\,\alpha_k^{-2}\alpha^{-1}\,\frac{w(B(x_0,\alpha_k))}{v(B(x_0,\alpha_k))}$.

Let us put in (6.1) the function $\psi(x,t) = [\delta_{k,h}(x,t)]^r\,\gamma_i^{s+2}\tilde{\gamma}_i^{s+2}$ with arbitrary positive numbers r and s. After integratig by parts in the term containing $\frac{\partial}{\partial t}\delta_{k,h}(x,t)$, passing to the limit as $h\to 0$ and using the inequalities (2.7) and (2.8), we get

$$\begin{aligned}
&\int_{D_k} v(x)\,|\delta_k(x,\tau)|\,\gamma_i^{s+2}(x)\,\tilde{\gamma}_i^{s+2}(\tau)\,dx \\
&\qquad + \int_0^\tau\!\!\int_{D_k} v(x)\,|\delta_k(x,t)|^r\left|\frac{\partial\delta_k(x,t)}{\partial x}\right|^2 \tilde{\gamma}_i^{s+2}(x)\,\tilde{\gamma}_i^{s+2}(t)\,dxdt \\
&\quad\le C_{30}\,(r+s+1)^2\alpha_i^{-2} \\
&\qquad\times\int_0^\tau\!\!\int_{D_k}\alpha_k^{-2}\left[v(x)\frac{w(B(x_0,\alpha_k))}{v(B(x_0,\alpha_k))}+w(x)\right]|\delta_k(x,t)|^{r+2}\,\tilde{\gamma}_i^s(x)\,\tilde{\gamma}_i^s(t)\,dxdt.
\end{aligned}\tag{6.2}$$

From (6.2), as well as in Section 4, we obtain with some $h>1$

$$\begin{aligned}
\mu^2(i) \le C_{31}\alpha_i^{-\frac{2h}{h-1}} &\iint_{R(i+1)} \delta_k^2(x,t)\tilde{\gamma}_i^2(x)\,\tilde{\gamma}_i^2(t)\,\alpha_k^{-2} \\
&\times\left[\frac{w(x)}{w(B(x_0,\alpha_k))}+\frac{w(B(x_0,\alpha_k))}{v^2\,(B(x_0,\alpha_k))}v(x)\right]dxdt
\end{aligned}\tag{6.3}$$

where

$$\mu(i) = \sup\left\{|\delta_k(x,t)|\,\Big|\,(x,t)\in R(i)\right\}$$

with

$$R(i) = \left\{(x,t)\in Q_k\,\Big|\,|x-x_0|\le\rho_i \ \text{ and } \ |t-t_0|\le\sigma_i\right\}.$$

Let us consider two possibilities:

1) $\mu(i+1)\le d_k$

2) $\mu(i+1)>d_k$.

If $\mu(i+1) \le d_k$, then from the definition of $\mu(i+1)$ we have

$$\sup\left\{ |\delta_k(x,t)| \,\middle|\, |x-x_0| < \alpha_{k+4},\ |t-t_0| < \alpha_{k+4}^2 \frac{v(B(x_0,\alpha_{k+4}))}{w(B(x_0,\alpha_{k+4}))} \right\} \le \mu(i+1) \le d_k$$

and (3.17) follows from (3.16) for u_k and the analogous estimate for u_{k+1}. If $\mu(i+1) > d_k$, then, as in [12], we can show that $R(i+1) \subset F(\mu(i+1))$ and using the inequality (5.7) we have

$$\begin{aligned}
&\iint_{R(i+1)} \delta_k(x,t)^2 \gamma_i^2(x)\,\tilde{\gamma}_i^2(t)\,\alpha_k^{-2} \left[v(x) \frac{w(B(x_0,\alpha_k))}{v^2(B(x_0,\alpha_k))} + \frac{w(x)}{v(B(x_0,\alpha_k))} \right] dxdt \\
&= \iint_{R(i+1)} \left| [\delta_k(x,t)]_{(-\mu(i+1),\mu(i+1))} \right|^2 \gamma_i^2(x)\,\tilde{\gamma}_i^2(t)\,\chi_{F(\mu(i+1))}\,\alpha_k^{-2} \\
&\quad \times \left[v(x) \frac{w(B(x_0,\alpha_k))}{v^2(B(x_0,\alpha_k))} + \frac{w(x)}{v(B(x_0,\alpha_k))} \right] dxdt \\
&\le C_{32}\,\mu(i+1) \frac{\alpha_i^{-\frac{2h}{h-1}}}{v(B(x_0,\alpha_k))} \left\{ \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) \right\}^{1/2} \\
&\quad \times \left\{ \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} \right\}^{1/2}.
\end{aligned}$$

Then applying Lemma 2.1, we obtain (3.17) and with this the proof of the theorem is complete ∎

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