A Necessary Condition to Regularity of a Boundary Point for a Degenerate Quasilinear Parabolic Equation

S. **Leonardi and I. I. Skrypnik**

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\n
$$
v(x)\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i \left(x, t, u, \frac{\partial u}{\partial x}\right) = a_0 \left(x, t, u, \frac{\partial u}{\partial x}\right) \qquad \left((x, t) \in Q_T = \Omega \times (0, T)\right)
$$
\nwith $t \in Q_T$ and $t \in Q_T$.

at a point $(x_0, t_0) \in S_T = \partial \Omega \times (0, T)$. Inded we establish a necessary condition to the regularity of a boundary point of the cylindrical domain Q_T extending the analogous result from paper [13] to the degenerate case. The degeneration is given by weights (depending on the space variable) from a suitable Muchenhoupt class. It is important to note that the coefficients of the equation depend on time too.

Keywords: *Degenerate nonlinear parabolic equations, regularity at boundary points* AMS **subject classification:** Primary 35 K 65, secondary 35 B 65

1. Introduction

In the present paper we are concerned with the behavior near by the boundary of a cylindrical domain of solutions to a second order degenèráte parabolic equation with coefficients which depend on time.

The Wiener condition to the regularity of a boundary point for a linear parabolic equation with measurable bounded coefficients is due to Lanconelli [9]. A Wiener-type sufficient condition for a quasilinear parabolic equation has been proved by Gariepy and Ziemer $[6, 14]$ and a necessary condition by I. V. Skrypnik $[13]$. In $[1]$ Biroli has extended the result of [14] to the parabolic degenerate case with a weight in the $A_{1+2/n}$ Muchenhoupt class (see in $[1]$ about literature). In our paper we use the method of $[13]$ for proving a necessary Wiener-type condition for such a problem. The Wiener condition to

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extended the result of [14] to

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This paper was written while the second author was visiting the Department of Mathematics

2. Notations, definitions and preliminary results

To begin we recall some facts from [2] about A_p weights. Let $0 \lt r \in \mathbb{R}$ and $x_0 \in$ \mathbb{R}^n ($n \geq 3$), and put

Notations, definitions and preliminary results

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$$
A_p
$$
 weights. Let $0 < r \in \mathbb{R}$ and $a \geq 3$), and put

\n
$$
B(x_0, r) = \left\{ x \in \mathbb{R}^n \, \middle| \, |x - x_0| < r \right\}
$$
 and $w(B(x_0, r)) = \int_{B(x_0, r)} w(x) \, dx$.

\nwith at a non-negative and locally integrable function $w = w(x)$ in \mathbb{R}^n is a $d\alpha$.

We say that a non-negative and locally integrable function $w = w(x)$ in \mathbb{R}^n is a *doubling weight* if there exists a constant $K_1 > 0$, independent of r and x_0 , such that

$$
w(B(x_0, 2r)) \leq K_1 w(B(x_0, r)).
$$
\n(2.1)

such that, for all balls $B \subset \mathbb{R}^n$, we have

$$
\mathbb{R}^{n} (n \geq 3), \text{ and put}
$$
\n
$$
B(x_{0}, r) = \left\{ x \in \mathbb{R}^{n} \middle| |x - x_{0}| < r \right\} \quad \text{and} \quad w(B(x_{0}, r)) = \int_{B(x_{0}, r)} w(x) dx.
$$
\nWe say that a non-negative and locally integrable function $w = w(x)$ in \mathbb{R}^{n} is a doubling weight if there exists a constant $K_{1} > 0$, independent of r and x_{0} , such that\n
$$
w(B(x_{0}, 2r)) \leq K_{1}w(B(x_{0}, r)). \tag{2.1}
$$
\nGiven $p \in (1, +\infty)$, we say the weight w belongs to A_{p} if there exists a constant $K_{2} > 0$ such that, for all balls $B \subset \mathbb{R}^{n}$, we have\n
$$
\left(\frac{1}{\text{meas } B} \int_{B} w(x) dx \right) \left(\frac{1}{\text{meas } B} \int_{B} w^{-1/(p-1)}(x) dx \right)^{p-1} \leq K_{2}. \tag{2.2}
$$
\nWe say $w \in D_{\zeta}$ if there exists a constant $K_{3} > 0$ such that\n
$$
w(B(x_{0}, r)) \leq K_{3} \left(\frac{r}{s} \right)^{n_{\zeta}} w(B(x_{0}, s)) \quad \text{for all } s \in (0, r] \tag{2.3}
$$
\nwith K_{3} independent of x_{0} , r and s .\nWe say the Poincaré inequality holds with variable m ; and m , μ converges and $\sin \phi$.

We say $w \in D_{\zeta}$ if there exists a constant $K_3 > 0$ such that

$$
w(B(x_0,r)) \leq K_3 \left(\frac{r}{s}\right)^{n\zeta} w(B(x_0,s)) \quad \text{for all} \quad s \in (0,r]
$$
 (2.3)

with K_3 independent of x_0 , r and s.

We say the Poincaré inequality holds with weights w_1 and w_2 , μ -average and exponent *q* (*q* > 2) if there exists a constant $K_4 > 0$ such that
 $\left(\frac{1}{1 - \int |F(x) - g(y)|^2} F_{\text{max}}(x) dx\right)^{1/q}$

$$
D_{\zeta}
$$
 if there exists a constant $K_3 > 0$ such that
\n
$$
w(B(x_0, r)) \le K_3 \left(\frac{r}{s}\right)^{n_{\zeta}} w(B(x_0, s)) \quad \text{for all } s \in (0, r]
$$
\n(2.3)
\nlependent of x_0, r and s.
\nthe Poincaré inequality holds with weights w_1 and w_2, μ -average and expo-
\n2) if there exists a constant $K_4 > 0$ such that
\n
$$
\left(\frac{1}{w_2(B)} \int_B |F(x) - av_{B,\mu}F|^q w_2(x) dx\right)^{1/q}
$$
\n
$$
\le K_4 (\text{meas } B)^{1/n} \left(\frac{1}{w_1(B)} \int_B \left|\frac{\partial F(x)}{\partial x}\right|^2 w_1(x) dx\right)^{1/2}
$$
\n(2.4)
\n11 $B \subset \mathbb{R}^n$ and every $F \in \text{Lip}(B)$ where
\n
$$
av_{B,\mu}F = \frac{1}{\mu(B)} \int_B F(x) \mu(x) dx.
$$

\nIf from the result by Chainilo and Wheeler [2], the inequality (2.4) holds for
\n $u = 1$ or $\mu = w_2$ whenever $w_1 \in A_2$ and
\n
$$
\left(\frac{\text{meas } B_1}{\text{meas } B_2}\right)^{1/n} \left(\frac{w_2(B_1)}{w_2(B_2)}\right)^{1/q} \le K_5 \left(\frac{w_1(B_1)}{w_2(B_2)}\right)^{1/2}
$$
\n(2.5)
\n B_1 and B_2 with $B_1 \subset B_2$ and with K_5 independent of the balls.
\n $v = v(x)$ is a weight, $w \in A_p(v)$ means an analogous inequality to (2.2) with

for every ball $B \subset \mathbb{R}^n$ and every $F \in \text{Lip}(B)$ where

$$
av_{B,\mu}F=\frac{1}{\mu(B)}\int_B F(x)\mu(x)\,dx.
$$

As it follows from the result by Chanillo and Wheeden [2], the inequality (2.4) holds for $q > 0$ with $\mu = 1$ or $\mu = w_2$ whenever $w_1 \in A_2$ and $av_{B,\mu}F = \frac{1}{\mu(B)} \int_B F(x)\mu(x) dx.$

As it follows from the result by Chanillo and Wheeden [2], the inequality (2.4
 $q > 0$ with $\mu = 1$ or $\mu = w_2$ whenever $w_1 \in A_2$ and
 $\left(\frac{\text{meas } B_1}{\text{meas } B_2}\right)^{1/n} \left(\frac{w_2(B_1)}{w_2(B_2)}\$

$$
\mathbb{R}^n \text{ and every } F \in \text{Lip}(B) \text{ where}
$$
\n
$$
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$$
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\n
$$
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$$
\n
$$
\left(\frac{\text{meas } B_1}{\text{meas } B_2}\right)^{1/n} \left(\frac{w_2(B_1)}{w_2(B_2)}\right)^{1/q} \le K_5 \left(\frac{w_1(B_1)}{w_2(B_2)}\right)^{1/2} \tag{2.5}
$$

Also, if $v = v(x)$ is a weight, $w \in A_p(v)$ means an analogous inequality to (2.2) with *dx* and meas *B* replaced by $v(x)dx$ and $v(B)$, respectively. We also use the notation $A_{\infty}(v) = \bigcup_{p=1}^{\infty} A_p(v).$

Remark 2.1. As it follows from the definition, if $w \in A_p$, then there exist constants $K_6 > 0$ and $\eta > 0$ such that

$$
w(B(x_0,r))\leq K_6\left(\frac{r}{s}\right)^{n\eta}w(B(x_0,s))
$$

for $0 < r \leq s$.

behavior of solutions of the equation

$$
0 \text{ and } \eta > 0 \text{ such that}
$$
\n
$$
w(B(x_0, r)) \le K_6 \left(\frac{r}{s}\right)^{n\eta} w(B(x_0, s))
$$
\n
$$
0 < r \le s.
$$
\nLet now Ω be a bounded domain in \mathbb{R}^n and $Q_T = \Omega \times (0, T)$. We shall study the
\navior of solutions of the equation

\n
$$
v(x)\frac{\partial u}{\partial t} - \sum_{i=1}^n \frac{\partial}{\partial x_i} a_i \left(x, t, u, \frac{\partial u}{\partial x}\right) = a_0 \left(x, t, u, \frac{\partial u}{\partial x}\right) \qquad ((x, t) \in Q_T) \qquad (2.6)
$$
\npoint $(x_0, t_2) \in S_T = \partial\Omega \times (0, T)$ under the assumptions that the functions $a_i \in \mathbb{R}^n$.

at a point $(x_0, t_0) \in S_T = \partial\Omega \times (0, T)$ under the assumptions that the functions $a_i =$ the following conditions:

- *a_i*(*x,t,u,p*) (*i* = 0,...,*n*) are defined for $(x, t, u, p) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n$ and satisfy the following conditions:

(C1) For almost every (fixed) (x, t) the functions $a_i(x, t, u, p)$ are continuous with respec (C1) For almost every (fixed) (x,t) the functions $a_i(x,t,u,p)$ are continuous with
- (C2) For some constant $\nu_1 > 0$,

respect to
$$
u
$$
 and p , and for all (u, p) they are measurable functions of (x, t) ;
\n $a_i(x, t, 0, 0) = 0$ for $i = 0, ..., n$.
\nFor some constant $\nu_1 > 0$,
\n
$$
\sum_{i=1}^n \Big(a_i(x, t, u, p) - a_i(x, t, u, q) \Big) (p_i - q_i) \ge \nu_1 w(x) |p - q|^2, \qquad (2.7)
$$

and for some constant $\nu_2 > 0$,

$$
\left| a_0(x, t, u, p) - a_0(x, t, v, q) \right| \le \nu_2 \left(v(x) |u - v| + v^{1/2}(x) w^{1/2}(x) |p - q| \right)
$$
\n
$$
\left| a_i(x, t, u, p) - a_i(x, t, v, q) \right| \le \nu_2 \left(v^{1/2}(x) w^{1/2}(x) |u - v| + w(x) |p - q| \right)
$$
\nfor $i = 1, ..., n$.
\ndenote by $L^2(\Omega, w)$ the Banach space of all measurable functions f , defined on
\nwe norm\n
$$
\|f\|_{L^2(\Omega, w)}^2 = \int_{\Omega} f^2(x) w(x) dx
$$
\n
$$
W_2^1(O_T, v, w) \text{ will be the Banach space of functions } f \text{ equipped with the norm}
$$

We will denote by $L^2(\Omega, w)$ the Banach space of all measurable functions $f,$ defined on Ω , whose norm

$$
||f||_{L^2(\Omega,w)}^2 = \int_{\Omega} f^2(x)w(x) dx
$$

is finite. $W_2^1(Q_T, v, w)$ will be the Banach space of functions f equipped with the norm

$$
\left| a_i(x, t, u, p) - a_i(x, t, v, q) \right| \leq \nu_2 \left(v^{1/2}(x) w^{1/2}(x) |u - v| + w(x) |p - q| \right)
$$

for $i = 1, ..., n$.
Il denote by $L^2(\Omega, w)$ the Banach space of all measurable functions f, defined on
use norm

$$
||f||_{L^2(\Omega, w)}^2 = \int_{\Omega} f^2(x) w(x) dx
$$

e. $W_2^1(Q_T, v, w)$ will be the Banach space of functions f equipped with the norm

$$
||f||_{W_2^1(Q_T, v, w)}^2 = \int_{Q_T} f^2(x, t) (v(x) + w(x)) dx dt
$$

$$
+ \int_{Q_T} \left(\left| \frac{\partial f(x, t)}{\partial x} \right|^2 w(x) + \left| \frac{\partial f(x, t)}{\partial t} \right| v(x) \right) dx dt.
$$

e also functions from the space $V_2(Q_T, v, w)$ endowed with the norm

$$
||f||_{V_2(Q_T, v, w)}^2 = \sup_{0 < t < T} \int_{\Omega} f^2(x, t) v(x) dx + \int_{Q_T} \left| \frac{\partial f(x, t)}{\partial x} \right|^2 w(x) dx dt.
$$
 (2.10)
Il denote by $\tilde{W}_2^1(Q_T, v, w)$ and $\tilde{V}_2(Q_T, v, w)$ the spaces of functions belonging,
tively, to $W_2^1(Q_T, v, w)$ and $V_2(Q_T, v, w)$ and being equal to zero on S_T .

We use also functions from the space $V_2(Q_T, v, w)$ endowed with the norm

$$
||f||_{V_2(Q_T,v,w)}^2 = \sup_{0 < t < T} \int_{\Omega} f^2(x,t)v(x) \, dx + \int_{Q_T} \left| \frac{\partial f(x,t)}{\partial x} \right|^2 w(x) \, dx \, dt. \tag{2.10}
$$

We will denote by $\check{W}_2^1(Q_T, v, w)$ and $\check{V}_2(Q_T, v, w)$ the spaces of functions belonging, respectively, to $W_2^1(Q_T, v, w)$ and $V_2(Q_T, v, w)$ and being equal to zero on S_T .

Definition 2.1. We say that a function $u \in V_2(Q_T, v, w)$ is a solution of the equa-**162** S. Leonardi and I. I. Skrypnik
 Definition 2.1. We say that a function $u \in V_2(Q_T, v, w)$ is a solution of the equation (2.6) if, for all functions $\psi = \psi(x, t)$ in $\mathring{W}_2^1(Q_T, v, w)$ vanishing at $t = 0$ and $t = T$, the identity

$$
I_T(u,\psi)=0
$$

is satisfied, where

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\n**efinition 2.1.** We say that a function
$$
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$$
 is a solution of the equa-
\n.6) if, for all functions $\psi = \psi(x, t)$ in $\mathring{W}_2^1(Q_T, v, w)$ vanishing at $t = 0$ and $t = T$,
\n
$$
I_T(u, \psi) = 0
$$
\n
\n**field, where**
\n
$$
I_T(u, \psi) = \iint_{0}^{n} \left\{ -v(x)u(x, t) \frac{\partial \psi}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \right\} \frac{\partial u}{\partial x} + \sum_{i=1}^{n} a_i \left(x, t, u, \frac{\partial u}{\partial x} \right) \frac{\partial \psi}{\partial x_i} - a_0 \left(x, t, u, \frac{\partial u}{\partial x} \right) \psi(x) \right\} dx dt.
$$
\n
\n**efficientation 2.2.** Let $f \in W_2^1(Q_T, v, w)$ and $g \in L_2(\Omega, v)$ for which
\n
$$
u(x, t) = f(x, t) \qquad ((x, t) \in S_T) \qquad (2.12)
$$
\n
$$
u(x, 0) = g(x) \qquad (x \in \Omega). \qquad (2.13)
$$
\n
$$
u(t, t) = u(t, t) \qquad (u(t, t) \in S_T) \qquad (2.14)
$$
\n
$$
u(t, t) = u(t, t) \qquad (u(t, t) \in S_T) \qquad (2.15)
$$
\n
$$
u(t, t) = u(t, t) \qquad (u(t, t) \in V_2^1(Q_T, v, w) \text{ is a solution of the problem (2.6), (2.12), (2.13)}
$$
\n
$$
f \in V_2(Q_T, v, w) \text{ and, moreover, for any } \psi \in \mathring{W}_2^1(Q_t, v, w) \text{ and } \tau \in (0, T),
$$

Definition 2.2. Let $f \in W_2^1(Q_T, v, w)$ and $g \in L_2(\Omega, v)$ for which

$$
u(x,t) = f(x,t) \qquad ((x,t) \in S_T) \tag{2.12}
$$

$$
u(x,0) = g(x) \qquad (x \in \Omega). \tag{2.13}
$$

We say that $u = u(x,t)$ in $V_2^1(Q_T, v, w)$ is a solution of the problem $(2.6), (2.12), (2.13)$ if $u - f \in V_2(Q_T, v, w)$ and, moreover, for any $\psi \in \mathring{W}_2^1(Q_t, v, w)$ and $\tau \in (0, T)$,

$$
u(x,t) = f(x,t) \qquad ((x,t) \in S_T) \tag{2.12}
$$
\n
$$
u(x,0) = g(x) \qquad (x \in \Omega). \tag{2.13}
$$
\n
$$
u = u(x,t) \text{ in } V_2^1(Q_T, v, w) \text{ is a solution of the problem } (2.6), (2.12), (2.13)
$$
\n
$$
2(Q_T, v, w) \text{ and, moreover, for any } \psi \in \mathring{W}_2^1(Q_t, v, w) \text{ and } \tau \in (0,T),
$$
\n
$$
\int_{\Omega} v(x)u(x,\tau)\psi(x,\tau) dx - \int_{\Omega} v(x)g(x)\psi(x,0) dx + I_\tau(u,\psi) = 0. \tag{2.14}
$$
\n
$$
\text{ion 2.3. We say that } (x_0, t_0) \in S_T \text{ is a regular boundary point of the region}
$$
\n
$$
\text{quation (2.6) if for any its solution } u, \text{ defined in } Q_T, \text{ satisfying the condition}
$$
\n
$$
\phi(u - f) \in \mathring{V}_2(Q_T, v, w) \tag{2.15}
$$
\n
$$
(\overline{Q}_T) \cap W_2^1(Q_T, v, w) \text{ and } \phi \in C^\infty(\mathbb{R}^{n+1}) \text{ which is equal to one in a neighborhood.}
$$

Definition 2.3. We say that $(x_0, t_0) \in S_T$ is a *regular boundary point* of the region Q_T for the equation (2.6) if for any its solution *u*, defined in Q_T , satisfying the condition

> $\phi(u-f) \in V_2(Q_T, v, w)$ (2.15)

with $f\in C(\overline{Q}_T)\cap W_2^1(Q_T,v,w)$ and $\phi\in C^\infty(\mathbb{R}^{n+1})$ which is equal to one in a neighborhoud of (x_0, t_0) , the equality if for any its solution u , defined in Q_T , satisfying
 $\phi(u - f) \in V_2(Q_T, v, w)$
 $Q_T, v, w)$ and $\phi \in C^\infty(\mathbb{R}^{n+1})$ which is equal to or

equality
 $\lim_{\mathbf{u} \to (\mathbf{x}_0, t_0)} u(x, t) = f(x_0, t_0) \qquad ((x, t) \in Q_T)$

$$
\lim_{(x,t)\to(x_0,t_0)} u(x,t) = f(x_0,t_0) \qquad ((x,t)\in Q_T)
$$
\n(2.16)

holds.

For any set $E \subset \mathbb{R}^n$, let

$$
\mathcal{M}(E) = \left\{ \phi \in C_0^{\infty}(\mathbb{R}^n) \middle| \phi(x) \ge 1 \text{ for all } x \in E \right\}
$$

and define

$$
C_{2,w}(E)=\inf\left\{\int_{\mathbb{R}^n}w(x)\left|\frac{\partial\phi}{\partial x}\right|^2dx\right|\phi\in\mathcal{M}(E)\right\}.
$$

In the following we shall need two lemmata.

Lemma 2.1 (see [13]). Let $\{\beta_i\}_{i\in\mathbb{N}}$ be a bounded numerical sequence such that

A Condition to Regular
et
$$
\{\beta_i\}_{i \in \mathbb{N}}
$$
 be a bounded nu
 $\beta_i \leq A \beta_{i+1}^{\delta} a^i$ $(i \in \mathbb{N})$

with positive constants A, a and $\delta \in (0,1)$ *. Then*

 $\bigcap_{1} \bigotimes C A^{1/(1-\delta)}$

for a constant C depending only on S and a.

Lemma 2.2 (see [5: Theorem 1.2]). Let the Poincaré inequality hold with $w_1 = w$ *and* $w_2 = v$ or $w_1 = w$ and $w_2 = w$, $\mu = 1$ and $w \in A_{1+2/n}, w^{-1} \in A_{2-2/n}$. Then for *an arbitrary function* $u \in C_0^{\infty}(B(x_0, \mathbb{R}))$ *one has u wequality hold with* $w_1 = w$, $w^{-1} \in A_{2-2/n}$. Then for $\left(\frac{u}{x}\right)^2 w(x) dx$ (2.17) *Pa*
 *u*₁ = *w* and *w*₂ = *w*,
 *u*²(*x*)*v*(*x*) *dx* $\le C_1 \rho^2$; $Poincaré~ineq$
 $\begin{aligned} \n\eta \in A_{1+2/n}, \n\omega \n\end{aligned}$
 $\int_{B(x_0,R)} \left| \frac{\partial u}{\partial x} \right|$ *which* the *Poincaré* ineq
 w(*B(xo,p)*) *Let the Poincaré ineq*
 w(*B(xo,p)*) *Dne has*
 w(*B(xo,p)*) $\int_{B(x_0,R)} \left| \frac{\partial u}{\partial x} \right|$
 w(*B(xo,p)*) $\int_{B(x_0,R)} \left| \frac{\partial u}{\partial x} \right|$ *v* $\beta_1 \leq CA^{1/(1-\delta)}$
 ug only on δ *and a.*

Theorem 1.2]). Let the Poincaré inequality hold with $w_1 = w$
 $ud w_2 = w, \mu = 1$ and $w \in A_{1+2/n}, w^{-1} \in A_{2-2/n}$. Then for
 $C_0^{\infty}(B(x_0, \mathbb{R}))$ one has
 $dx \leq C_1 \rho^2 \frac{v(B(x_0, \rho$

$$
\int_{B(x_0,\rho)} u^2(x)v(x)\,dx \leq C_1\rho^2 \frac{v(B(x_0,\rho))}{w(B(x_0,\rho))} \int_{B(x_0,R)} \left|\frac{\partial u}{\partial x}\right|^2 w(x)\,dx \tag{2.17}
$$

or

$$
\int_{B(x_0,\rho)} u^2(x)v(x) dx \le C_1 \rho^2 \frac{\langle v \rangle_{L(f_0,\rho)}}{w(B(x_0,\rho))} \int_{B(x_0,R)} \left| \frac{\partial u}{\partial x} \right|^{2} w(x) dx \qquad (2.17)
$$
\nor

\n
$$
\int_{B(x_0,\rho)} u^2(x)w(x) dx \le C_1 \rho^2 \int_{B(x_0,R)} \left| \frac{\partial u}{\partial x} \right|^{2} w(x) dx \qquad (2.18)
$$
\nwith a constant C_1 independent of u and $0 < \rho < R$.

3. Regularity at the boundary

In this section we prove our main result:

Theorem 3.1. *Let the functions ai satisfy conditions (Cl) and (C2). Suppose moreover the following:*

(i) $v, w \in A_2$.

(ii) The Poincaré inequality holds for $w_1 = w_2 = w$ with $\mu = 1$ and some $q > 2$.

(iii) The Poincaré inequality holds for $w_1 = w$ and $w_2 = v$ with any $\mu = 1$ or $\nu = v$ and some $q > 2$.

(iv) The inequalities (2.17) and (2.18) hold.

Then for $(x_0,t_0) \in \partial \Omega \times (0,T)$ *to be a regular boundary point of the domain* Q_T *to the equation (2.6), it is necessary that*

ality holds for
$$
w_1 = w_2 = w
$$
 with $\mu = 1$ and some $q > 2$.
\nality holds for $w_1 = w$ and $w_2 = v$ with any $\mu = 1$ or $\nu = v$
\n17) and (2.18) hold.
\nT) to be a regular boundary point of the domain Q_T to the
\ny that
\n
$$
\int_0^1 \frac{C_{2,w}(B(x_0,r) \setminus \Omega)}{C_{2,w}(B(x_0,r))} \frac{dr}{r} = \infty.
$$
\n(3.1)
\nany $\theta > 0$,
\n
$$
\int_0^1 \times \left(t_0 - 2\theta r^2 \frac{v(B(x_0,r))}{w(B(x_0,r))} \right), \quad t_0 - \theta r^2 \frac{v(B(x_0,r))}{w(B(x_0,r))} \right)
$$

Remark 3.1. Put, for any $\theta > 0$,

$$
m e q > 2.
$$

\n) The inequalities (2.17) and (2.18) hold.
\n
$$
or (x_0, t_0) \in \partial\Omega \times (0, T)
$$
 to be a regular boundary point of the domain Q:
\n
$$
n (2.6), it is necessary that
$$
\n
$$
\int_{0}^{1} \frac{C_{2,w}(B(x_0, r) \setminus \Omega)}{C_{2,w}(B(x_0, r))} \frac{dr}{r} = \infty.
$$
\n
$$
maxk 3.1. Put, for any $\theta > 0$,
$$
\n
$$
Q_r^{\theta}(x_0, t_0) = B\left(x_0, \frac{r}{2}\right) \times \left(t_0 - 2\theta r^2 \frac{v(B(x_0, r))}{w(B(x_0, r))}, t_0 - \theta r^2 \frac{v(B(x_0, r))}{w(B(x_0, r))}\right)
$$

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and define, for any set $F \subset \mathbb{R}^{n+1}$,

$$
\widetilde{\mathcal{M}}(F) = \left\{ \phi \in C_0^{\infty}(\mathbb{R}^{n+1}) \middle| \phi \ge 1 \text{ for all } (x, t) \in F \right\}
$$

and

5. Leonardi and 1. 1. Skrypnik
\ne, for any set
$$
F \,\subset \mathbb{R}^{n+1}
$$
,
\n
$$
\widetilde{\mathcal{M}}(F) = \left\{ \phi \in C_0^{\infty}(\mathbb{R}^{n+1}) \middle| \phi \ge 1 \text{ for all } (x, t) \in F \right\}
$$
\n
$$
\Gamma_{v,w}(F) = \inf_{\psi \in \widetilde{\mathcal{M}}(F)} \left\{ \sup_t \int \phi^2(x, t) v(x) \, dx + \iint w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx dt \right\}.
$$
\n
$$
\text{sin [14] it can be proved that}
$$
\n
$$
\Gamma_{v,w} \left(Q_r^{\theta}(x_0, t_0) \setminus Q_T \right) \approx r^2 \frac{v(B(x_0, r))}{w(B(x_0, r))} C_{2,w}(B(x_0, r) \setminus \Omega).
$$
\nwe can easily obtain that\n
$$
\Gamma_{v,w} \left(Q_\tau^{\theta}(x, t_0) \right) \approx v(B(x_0, r))
$$

As well as in [14] it can be proved that

$$
\Gamma_{v,w}\left(Q_r^{\theta}(x_0,t_0)\setminus Q_T\right)\approx r^2\frac{v(B(x_0,r))}{w(B(x_0,r))}C_{2,w}(B(x_0,r)\setminus\Omega).
$$

Also we can easily obtain that

$$
t_0 \setminus (Q_T) \approx r^2 \frac{\sqrt{w(B(x_0, r))}}{w(B(x_0, r))} C_{2,w}(R)
$$

ain that

$$
\Gamma_{v,w} (Q_r^{\theta}(x_0, t_0)) \approx v(B(x_0, r))
$$

$$
C_{2,w}(B(x_0, r)) \approx r^{-2} w(B(x_0, r))
$$

so that condition (3.1) is equivalent to

$$
w(B(x_0, r)) = 2, w(-e_0, y_0) + 2, w(-e_0, y_0)
$$
\n
$$
\text{ain that}
$$
\n
$$
\Gamma_{v,w} (Q_r^{\theta}(x_0, t_0)) \approx v(B(x_0, r))
$$
\n
$$
C_{2,w}(B(x_0, r)) \approx r^{-2}w(B(x_0, r))
$$
\n
$$
\int_{0}^{1} \frac{\Gamma_{v,w} (Q_r^{\theta}(x_0, t_0) \setminus Q_T)}{v(B(x_0, r))} \frac{dr}{r} = \infty. \qquad (3.2)
$$
\n
$$
\text{and } (3.2) \text{ we have that, in the case } v(x) = 1, \text{ our necessary}
$$
\n
$$
\text{the sufficient one from [1]}.
$$
\n
$$
\text{y functions } u_k = u_k(x, t) \quad (k \in \mathbb{N}) \text{ that will play a fundamental}
$$
\n
$$
\text{em 3.1. For } k \in \mathbb{N} \text{ we define a numerical sequence } \{\alpha_k\}_{k \in \mathbb{N}} \to \infty \text{ and}
$$
\n
$$
K_7 = \left(\frac{1}{4}K_5^{-1}K_6^{-\frac{t-2}{2q}}\right)^{\frac{q}{n\eta(q-2)}}, \qquad K_7 < 1 \qquad (3.3)
$$
\n
$$
\text{inequality } (2.5), K_6 \text{ and } \eta \text{ from Remark 2.1 and some } q > 2
$$
\n
$$
\text{quality is valid. Let}
$$

From Theorem 3.1 and (3.2) we have that, in the case $v(x) = 1$, our necessary condition coincides with the sufficient one from [1].

Now we define auxiliary functions $u_k = u_k(x,t) \, \left(k \in \mathbb{N} \right)$ that will play a fundamental role in the proof of Theorem 3.1. For $k \in \mathbb{N}$ we define a numerical sequence $\{\alpha_k\}_{k\in\mathbb{N}}$ such that $\alpha_k \to 0$ when $k \to \infty$ and $\int_{0}^{1} \frac{\Gamma_{v,w} \left(Q_r^{\theta}(x_0, t_0) \setminus Q_T\right) dr}{v(B(x_0, r))} dr$

n 3.1 and (3.2) we have that, in the

s with the sufficient one from [1].

auxiliary functions $u_k = u_k(x, t)$ ($k \in \mathbb{N}$ Theorem 3.1. For $k \in \mathbb{N}$ we define

when 1].

(*k*) $(k \in \text{define}$
 $\frac{4}{n\pi(\pi-2)}$ [1].
 t) $(k \in \mathbb{N})$ that will play a fundame

e define a numerical sequence $\{\alpha_k\}$
 $\int_{\frac{\pi}{2}}^{\frac{q}{n-(q-2)}}$, $K_7 < 1$

and η from Remark 2.1 and some q
 $E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}).$

bsets of $\mathcal{M}(E_k)$ and $\mathcal{$

$$
\frac{\alpha_{k+1}}{\alpha_k} \le K_7 = \left(\frac{1}{4}K_5^{-1}K_6^{-\frac{q-2}{2q}}\right)^{\frac{q}{n\eta(q-2)}}, \qquad K_7 < 1 \tag{3.3}
$$
\nant K_5 from inequality (2.5), K_6 and η from Remark 2.1 and some $q > 2$

\nPoincaré inequality is valid. Let

\n
$$
E_k = B(x_0, \alpha_k) \setminus \Omega \qquad \text{and} \qquad E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}). \tag{3.4}
$$
\nLet

\n
$$
E_k = B(x_0, \alpha_k) \setminus \Omega \qquad \text{and} \qquad E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}). \tag{3.4}
$$

with the constant K_5 from inequality (2.5), K_6 and η from Remark 2.1 and some $q>2$ for which the Poincaré inequality is valid. Let

$$
E_k = B(x_0, \alpha_k) \setminus \Omega \quad \text{and} \quad E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}). \tag{3.4}
$$

Let further $\mathcal{M}_k(E_k)$ and $\mathcal{M}^{(k)}(E^{(k)})$ be the subsets of $\mathcal{M}(E_k)$ and $\mathcal{M}(E^{(k)})$ consisting of functions with support contained in $B(x_0, \alpha_{k-1})$ and $B(x_0, \alpha_{k-1})\setminus B(x_0, \alpha_{k+2})$, respectively.

Lemma 3.1. *There exists a constant* $C_2 > 0$ *such that*

$$
E_k = B(x_0, \alpha_k) \setminus \Omega \quad \text{and} \quad E^{(k)} = E_k \setminus B(x_0, \alpha_{k+1}). \tag{3.4}
$$
\n
$$
d_k(E_k) \text{ and } \mathcal{M}^{(k)}(E^{(k)}) \text{ be the subsets of } \mathcal{M}(E_k) \text{ and } \mathcal{M}(E^{(k)}) \text{ consist-}
$$
\nas with support contained in $B(x_0, \alpha_{k-1})$ and $B(x_0, \alpha_{k-1}) \setminus B(x_0, \alpha_{k+2})$,

\n3.1. There exists a constant $C_2 > 0$ such that

\n
$$
\inf \left\{ \int_{R^n} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx \middle| \phi \in \mathcal{M}_k(E_k) \right\} \leq C_2 C_{2,w}(E_k) \tag{3.5}
$$
\n
$$
\inf \left\{ \int_{R^n} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx \middle| \phi \in \mathcal{M}^{(k)}(E^{(k)}) \right\} \leq C_2 C_{2,w}(E^{(k)}). \tag{3.6}
$$

$$
\inf \left\{ \int_{R^n} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx \, \middle| \, \phi \in \mathcal{M}^{(k)}(E^{(k)}) \right\} \le C_2 C_{2,w}(E^{(k)}). \tag{3.6}
$$

Proof. We will prove only inequality (3.5), because (3.6) can be proved as well. Let $\chi_k \in C^\infty(\mathbb{R}^n)$ with

A Condition to Regularity of a Boundary Point
\nProof. We will prove only inequality (3.5), because (3.6) can be proved as
\n
$$
\chi_k \in C^{\infty}(\mathbb{R}^n)
$$
 with
\n $\chi_k(x) = 1$ for $|x - x_0| \le \alpha_k$, and $\chi_k(x) = 0$ for $|x - x_0| \ge \alpha_{k-1}$

$$
\left|\frac{\partial \chi_k}{\partial x}\right| \leq \frac{1}{(1-K_7)\alpha_{k-1}}.
$$

Proof. We will prove only inequality (3.5), because (3.6) can be proved a
\nLet
$$
\chi_k \in C^{\infty}(\mathbb{R}^n)
$$
 with
\n
$$
\chi_k(x) = 1 \text{ for } |x - x_0| \le \alpha_k, \text{ and } \chi_k(x) = 0 \text{ for } |x - x_0| \ge \alpha_{k-1}
$$
\nand
\n
$$
\left| \frac{\partial \chi_k}{\partial x} \right| \le \frac{1}{(1 - K_7)\alpha_{k-1}}.
$$
\nThen, for $\phi \in \mathcal{M}(E_k)$ with $\chi_k \phi \in \mathcal{M}_k(E_k)$, we have
\n
$$
\lim_{\psi \in \mathcal{M}_k(E_k)} \int_{R^n} w(x) \left| \frac{\partial \psi}{\partial x} \right|^2 dx
$$
\n
$$
\le C_3 \inf_{\phi \in \mathcal{M}(E_k)} \int_{B(x_0, \alpha_{k-1})} \left(\chi_k^2 \left| \frac{\partial \phi}{\partial x} \right|^2 w(x) dx + \left| \frac{\partial \chi_k}{\partial x} \right|^2 \phi^2(x) w(x) \right) dx
$$
\n
$$
\le C_4 \left(1 + \frac{1}{(1 - K_7)^2} \right) \inf_{\phi \in \mathcal{M}(E_k)} \int_{B(x_0, \alpha_{k-1})} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx
$$
\n
$$
\le C_2 C_{2,\omega}(E_k)
$$

and the assertion is proved \blacksquare

Let diam Ω be the diameter of the region Ω . From now on, putting $R = 2 + \text{diam } \Omega$, *B* will denote the ball of radius *R* centered at x_0 . We introduce a non-increasing function $\gamma \in C^\infty(\mathbb{R})$ with

$$
0\leq \gamma(s)\leq 1, \quad \gamma(s)=0 \;\;\text{for}\; s\geq 2, \quad \gamma(s)=1 \;\;\text{for}\; s\leq 1, \quad \left|\frac{d\gamma(s)}{ds}\right|\leq 2.
$$

Further, let

and

let
\n
$$
h(x) = \gamma(|x - x_0|) \quad \text{and} \quad \lambda(t) = \gamma\left(\alpha_k^{-2} \frac{w(B(x_0, \alpha_k))}{v(B(x_0, \alpha_k))} |t - t_0|\right).
$$
\n
$$
\text{ven point } (x_0, t_0) \in \partial\Omega \times (0, T) \text{ we can choose a number } k_0 \text{ such that}
$$
\n
$$
\alpha_{k_0}^2 \frac{v(B(x_0, \alpha_{k_0}))}{w(B(x_0, \alpha_{k_0}))} < t_0.
$$
\n
$$
k > k_0, (x, t) \in Q_k = D_k \times (0, T) \text{ and } D_k = B \setminus \overline{E}_k \text{ we define the function}
$$
\n
$$
(x, t) \text{ as the solution of equation (2.6) in } Q_k \text{ satisfying}
$$
\n
$$
u_k(x, t) = h(x)\lambda_k(t) \quad ((x, t) \in \partial D_k \times (0, T))
$$
\n
$$
u_k(x, 0) = 0 \quad (x \in D_k). \tag{3.8}
$$

For a given point $(x_0, t_0) \in \partial\Omega \times (0, T)$ we can choose a number k_0 such that

$$
\alpha_{k_0}^2 \frac{v(B(x_0,\alpha_{k_0}))}{w(B(x_0,\alpha_{k_0}))} < t_0.
$$

And for $k > k_0$, $(x, t) \in Q_k = D_k \times (0, T)$ and $D_k = B \setminus \overline{E}_k$ we define the function $u_k = u_k(x, t)$ as the solution of equation (2.6) in Q_k satisfying

$$
u_k(x,t) = h(x)\lambda_k(t) \qquad ((x,t) \in \partial D_k \times (0,T)) \tag{3.7}
$$

$$
u_k(x,0) = 0 \qquad \qquad (x \in D_k). \tag{3.8}
$$

*u*_c $x_0(t) = x_0(t)$ and $\lambda(t) = \gamma \left(\alpha_k^{-2} \frac{\alpha_k(x_0, \alpha_k)}{\upsilon(B(x_0, \alpha_k))} |t - t_0| \right).$
 $a, t_0) \in \partial\Omega \times (0, T)$ we can choose a number k_0 such that
 $\alpha_{k_0}^2 \frac{\upsilon(B(x_0, \alpha_{k_0}))}{\upsilon(B(x_0, \alpha_{k_0}))} < t_0.$
 $t) \in Q_k = D_k \times (0, T)$ and D_k Extend then the function $u_k = u_k(x,t)$ to $B \times (0,T)$ by setting it equal to $\lambda_k = \lambda_k(t)$ for $(x, t) \in \overline{E}_k \times (0, T)$.

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Lemma 3.2. There exists a constant $K_8 > 0$ such that the function $u_k = u_k(x,t)$ *satisfies*

nd I. I. Skrypnik
\nThere exists a constant
$$
K_8 > 0
$$
 such that the function $u_k = u_k(x, t)$
\n
$$
||u_k||_{V_2(Q_k, v, w)}^2 \le K_8 \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k).
$$
\n(3.9)
\ny to show that for $h > 0$ and $0 < \tau < T - h$ the integral identity

Proof. It is easy to show that for $h > 0$ and $0 < r < T - h$ the integral identity

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\nLemma 3.2. There exists a constant
$$
K_8 > 0
$$
 such that the function $u_k = u_k(x, t)$
\nsatisfies
\n
$$
||u_k||_{V_2(Q_k, v, w)}^2 \le K_8 \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k).
$$
\n(3.9)
\nProof. It is easy to show that for $h > 0$ and $0 < \tau < T - h$ the integral identity
\n
$$
\iint_{0}^{T} \left\{ v(x) \frac{\partial}{\partial t} [u_k(x, t)]_h \psi(x, t) + \sum_{i=1}^{n} \left[a_i \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \right]_h \frac{\partial \psi(x, t)}{\partial x_i} - \left[a_0 \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \right]_h \psi(x, t) \right\} dx dt = 0
$$
\nholds for any $\psi \in V_2(Q_k, v, w)$ in which we use the notation

$$
\left[a_0\left(x,t,u_k,\frac{\partial u_k}{\partial x}\right)\right]_h \psi(x,t) \Big\} dx dt = 0
$$

$$
\hat{V}_2(Q_k,v,w) \text{ in which we use the not at}
$$

$$
[g(x,t)]_h = \frac{1}{h} \int_t^{t+h} g(x,\tau) d\tau.
$$

0) the function

$$
(x,t) = [u_k(x,t)]_h - \phi(x) [\lambda_k(t)]_h \qquad (0)
$$
obtained from (3.10) by substituting (3.1)

Let us put in (3.10) the function

$$
\in \mathring{V}_2(Q_k, v, w) \text{ in which we use the notation}
$$
\n
$$
[g(x, t)]_h = \frac{1}{h} \int_t^{t+h} g(x, \tau) d\tau.
$$
\n.10) the function

\n
$$
\psi(x, t) = [u_k(x, t)]_h - \phi(x) [\lambda_k(t)]_h \qquad (\phi \in \mathcal{M}_k(E_k)). \tag{3.11}
$$
\nto obtained from (3.10) by substituting (3.11), we integrate by parts the

In the inequality obtained from (3.10) by substituting (3.11), we integrate by parts the term containing $\frac{\partial}{\partial t} \left[u_k(x,t)\right]_h$, pass to the limit as $h \to 0$, and estimate using (2.7) and (2.8). Thus we get (3.10) b
ass to t

ut in (3.10) the function
\n
$$
\psi(x,t) = [u_k(x,t)]_h - \phi(x) [\lambda_k(t)]_h \qquad (\phi \in M_k(E_k)). \qquad (3.11)
$$
\n
$$
\text{nequality obtained from (3.10) by substituting (3.11), we integrate by parts the\ntraining $\frac{\partial}{\partial t} [u_k(x,t)]_h$, pass to the limit as $h \to 0$, and estimate using (2.7) and
\nhus we get
\n
$$
\int_{D_k} v(x) u_k^2(x,\tau) dx + \int_0^{\tau} \int_{D_k} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_5 \left\{ \int_0^{\tau} \int_{D_k} \left(1 + \left| \frac{\partial \lambda_k}{\partial t} \right| \right) u_k^2(x,t) v(x) dx dt \right. \qquad (3.12)
$$
\n
$$
+ \int_{D_k} \left[v(x) \phi^2(x) + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \left| \frac{\partial \phi}{\partial x} \right|^2 w(x) \right] dx \right\}.
$$
\ne Poincaré and Gronwall inequalities, from (3.12) we obtain
\n
$$
\int_{D_k} v(x) u_k^2(x,\tau) dx + \int_0^{\tau} \int_{D_k} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_6 \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{\left| \frac{\partial \phi}{\partial x} \right|^2} \int_0^{\infty} \int_0^{\infty} w(x) dx
$$
$$

Using the Poincaré and Gronwall inequalities, from (3.12) we obtain

Using the Poincaré and Gronwall inequalities, from (3.12) we obtain\n
$$
\int_{D_k} v(x) u_k^2(x,\tau) dx + \int_0^{\tau} \int_{D_k} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_6 \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} \int_{D_k} \left| \frac{\partial \phi}{\partial x} \right|^2 w(x) dx
$$
\nfor $\tau \leq T$. Hence, by Lemma 3.1, the estimate (3.9) follows

Lemma 3.3. *There exists a constant* $K_9 > 0$ *such that, for* $k > k_0$,

A Condition to Regularity of a Boundary Point 167
\nLemma 3.3. There exists a constant
$$
K_9 > 0
$$
 such that, for $k > k_0$,
\n
$$
m(k) = \inf \left\{ |u_k(x,t)| \left| |x - x_0| \le \alpha_{k+5}, |t - t_0| \le \alpha_{k+5}^2 \frac{v(B(x_0, \alpha_{k+5}))}{w(B(x_0, \alpha_{k+5}))} \right\} \right\}
$$
\n
$$
\le K_9 \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} \right\}^{1/2}.
$$
\nProof. Set
\n
$$
\tau(k) = \alpha_{k+5}^2 \frac{v(B(x_0, \alpha_{k+5}))}{w(B(x_0, \alpha_{k+5}))} \quad \text{and} \quad \tilde{u}_k(x,t) = \min \left[\frac{u_k(x,t)}{m(k)}, 1 \right].
$$
\n
$$
\text{in from the definition of capacity and doubling condition for } w \text{ we get}
$$
\n
$$
w(B(x_0, \alpha_k))\alpha_k^{-2} \le C_7 C_{2,w}(B(x_0, \alpha_{k+5}))
$$
\n
$$
\le C_8 \tau^{-1}(k+5) \int_0^{t_0 + r(k+5)} \int_B w(x) \left| \frac{\partial \tilde{u}_k}{\partial x} \right|^2 dx dt. \tag{3.14}
$$

Proof. Set

,
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$$
\tau(k) = \alpha_{k+5}^2 \frac{v(B(x_0, \alpha_{k+5}))}{w(B(x_0, \alpha_{k+5}))} \quad \text{and} \quad \tilde{u}_k(x,t) = \min\left[\frac{u_k(x,t)}{m(k)}, 1\right].
$$

$$
\leq K_{9} \left\{ \alpha_{k}^{2} \frac{C_{2,w}(E_{k})}{w(B(x_{0}, \alpha_{k}))} \right\}^{1/2}.
$$
\n(3.13)

\nProof. Set

\n
$$
\tau(k) = \alpha_{k+5}^{2} \frac{v(B(x_{0}, \alpha_{k+5}))}{w(B(x_{0}, \alpha_{k+5}))} \quad \text{and} \quad \tilde{u}_{k}(x, t) = \min \left[\frac{u_{k}(x, t)}{m(k)}, 1 \right].
$$
\nThen from the definition of capacity and doubling condition for w we get

\n
$$
w(B(x_{0}, \alpha_{k})) \alpha_{k}^{-2} \leq C_{7} C_{2,w}(B(x_{0}, \alpha_{k+5}))
$$
\n
$$
\leq C_{8} \tau^{-1}(k+5) \int_{t_{0}-\tau(k+5)}^{t_{0}+\tau(k+5)} \int_{B} w(x) \left| \frac{\partial \tilde{u}_{k}}{\partial x} \right|^{2} dx dt.
$$
\nNow from Lemma 3.2 it follows

\n
$$
\int_{t_{0}-\tau(k+5)}^{t_{0}+\tau(k+5)} \int_{B} w(x) \left| \frac{\partial \tilde{u}_{k}}{\partial x} \right|^{2} dx dt \leq C_{9}[m(k)]^{-2} \tau(k+5) C_{2,w}(E_{k}).
$$
\n(3.15)

Now from Lemma 3.2 it follows

$$
\leq C_8 \tau^{-1} (k+5) \int_{B} w(x) \left| \frac{\partial \tau}{\partial x} \right| dx dt.
$$

\n
$$
\int_{t_0 - r(k+5)}^{t_0 + r(k+5)} \int_{B} w(x) \left| \frac{\partial \tilde{u}_k}{\partial x} \right|^2 dx dt \leq C_9 [m(k)]^{-2} \tau (k+5) C_{2,w}(E_k).
$$
(3.15)
\nand (3.15) we have the thesis
\nnext sections the a priori estimates on which the proof of Theorem 3.1 is
\ne proved.
\n
$$
m \text{ 3.2. } Let all of the assumptions of Theorem 3.1 be held. Then there exists\nK_{10} > 0 such that the solution $u_k = u_k(x, t)$ of problem (2.6), (3.7), (3.8)
\n
$$
|u_k(x, t)| \leq K_{10} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\}
$$
(3.16)
\n
$$
d \int_{B} |x + t| \leq C_8 \tau^{-1} \left\{ \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\}
$$
$$

From (3.14) and (3.15) we have the thesis \blacksquare

In the next sections the a priori estimates on which the proof of Theorem 3.1 is based will be proved.

Theorem 3.2. *Let all of the assumptions of Theorem* 3.1 *be held. Then there exists a* constant $K_{10} > 0$ such that the solution $u_k = u_k(x,t)$ of problem (2.6) , (3.7) , (3.8) *satisfies* sections the a priori estimates on which
wed.
2. Let all of the assumptions of Theore
> 0 such that the solution $u_k = u_k(x,$
 $u_k(x,t)| \le K_{10} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \right\}$

$$
|u_k(x,t)| \le K_{10} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} \right\}
$$
(3.16)

for

EXECUTE: Let all of the solutions of Theorem 3.1 to find the factor
$$
u_k = u_k(x, t)
$$
 of problem (2.6), (3.7).

\nis

\n
$$
|u_k(x, t)| \leq K_{10} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\}
$$
\n(x, t) $\notin \left\{ (x, t) \in Q_k \, \middle| \, |x - x_0| < \alpha_{k-3}, |t - t_0| < \alpha_{k-3} \frac{v(B(x_0, \alpha_{k-3}))}{w(B(x_0, \alpha_{k-3}))} \right\}$

\ntheorem 3.3. Let all of the assumptions of Theorem 3.1 be held. Then there

\nthat $K_{11} > 0$ such that

\n
$$
|u_k(x, t) - u_{k+1}(x, t)| \leq K_{11} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\}
$$

a constant $K_{11} > 0$ *such that*

Theorem 3.3. Let all of the assumptions of Theorem 3.1 be held. Then there exists
\n*nstant*
$$
K_{11} > 0
$$
 such that
\n
$$
|u_k(x,t) - u_{k+1}(x,t)| \leq K_{11} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0,\alpha_k))} + \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} \right\}
$$
\n
$$
|x - x_0| \leq \alpha_{k-4} \quad and \quad |t - t_0| \leq \alpha_{k-4} \frac{v(B(x_0,\alpha_{k-4}))}{w(B(x_0,\alpha_{k-4}))}.
$$
\nLet us remark that the Theorems 3.1 - 3.3 in the non-weighted case were proved in

for

$$
|x-x_0|\leq \alpha_{k-4} \qquad \text{and} \qquad |t-t_0|\leq \alpha_{k-4}\frac{v(B(x_0,\alpha_{k-4}))}{w(B(x_0,\alpha_{k-4}))}.
$$

Let us remark that the Theorems 3.1 - 3.3 in the non-weighted case were proved in the paper [13].

Using Theorems 3.2 and 3.3 we can now prove Theorem 3.1.

Proof of Theorem 3.1. We construct a solution of equation (2.6) in Q_T satisfying condition (2.15) with function $f \in C(\overline{Q}_T) \cap W_2^1(Q_T,v,w)$ and discontinuous at (x_0,t_0) as soon as the inequality (3.1) is not satisfied. and I. I. Skrypnik

vectors a solution of equals the function $f \in C(\overline{Q}_T) \cap W_2^1(Q_T, v, w)$ is

uality (3.1) is not satisfied.

12 Lume the boundedness of the integral on the

3.3) and (2.5) we can easily show that
 $\sum_{k=$ **1.** I. Skrypnik
 a 3.1. We construct a solution of equation (2.6) in Q_T satisfying
 and $f \in C(\overline{Q}_T) \cap W_2^1(Q_T, v, w)$ and discontinuous at (x_0, t_0)
 y (3.1) is not satisfied.

the boundedness of the integral on *we construct a solution of equa*

on $f \in C(\overline{Q}_T) \cap W_2^1(Q_T, v, w)$ and
 m s is not satisfied.

boundedness of the integral on the

(2.5) we can easily show that
 $\frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))}$

Thus, let us assume the boundedness of the integral on the left-hand side of condition (3.1) . Then using (3.3) and (2.5) we can easily show that

k=i

We can then find a number $k_1 \in \mathbb{N}$ such that

equality (3.1) is not satisfied.
\n
$$
(3.3) \text{ and } (2.5) \text{ we can easily show that}
$$
\n
$$
\sum_{k=1}^{\infty} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\} < \infty. \tag{3.18}
$$
\na number $k_1 \in \mathbb{N}$ such that\n
$$
\sum_{k=k_1}^{\infty} \left\{ \alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, \alpha_k))} + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\} < \frac{1}{4K_{11}} \tag{3.19}
$$

where K_{11} is the constant from Theorem 3.3. We will show that the function $u_{k_1} =$ $u_{k_1}(x,t)$ defined above as the solution of the problem (2.6), (3.7), (3.8) is discontinuous at (x_0, t_0) . Let $\delta > 0$ be an arbitrary number. By the convergence of the series (3.19) and estimate (3.13), we have $m(k) \to 0$ as $k \to 0$. In this way a number $k_2 = k_2(\delta)$ and a point $(x_{\delta}, t_{\delta}) \in Q_T$ can be chosen so that *l* $\left\{\n\begin{aligned}\n &\sum_{k=1}^{\infty} \binom{1}{k} \frac{w(B(x_0, a))}{w(B(x_0, a))} \\
 &\sum_{k=k_1}^{\infty} \left\{\n\alpha_k^2 \frac{C_{2,w}(E_k)}{w(B(x_0, a))}\n\end{aligned}\n\right\}$ above as the solution $\delta > 0$ be an arbitrary i.13), we have $m(k) \to Q_T$ can be chosen s $|u_{k_2}(x_{\delta}, t_{\delta$ $\left(\frac{k}{2\kappa}\right) + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} < \infty.$ (3.18)

uch that
 $\left(\frac{k}{k}\right) + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} < \frac{1}{4K_{11}}$ (3.19)

corem 3.3. We will show that the function $u_{k_1} =$

n of the problem (2.6), (3.7), (3.

$$
|u_{k_2}(x_{\delta},t_{\delta})| \leq \frac{1}{4} \quad \text{and} \quad |x_{\delta}-x_0|^2 + |t_{\delta}-t_0| < \delta^2. \quad (3.20)
$$

From (3.17), (3.19) and (3.20) we have

$$
|u_{k_2}(x_{\delta}, t_{\delta})| \ge \frac{1}{4} \quad \text{and} \quad |x_{\delta} - x_0| + |t_{\delta} - t_0| < \delta.
$$
\n
$$
|u_{k_1}(x_{\delta}, t_{\delta})| \le |u_{k_2}(x_{\delta}, t_{\delta})| + \sum_{k=k_1}^{k_2-1} |u_{k+1}(x_{\delta}, t_{\delta}) - u_k(x_{\delta}, t_{\delta})| \le \frac{1}{2}.
$$
\nso that

\n
$$
\liminf_{(x,t) \to (x_0, t_0)} u_{k_1}(x, t) < 1 \quad ((x, t) \in Q_T).
$$
\nallow prove the non-regularity of the boundary point (x, t) and $(x, t) \in Q_T$.

So it follows that

$$
\liminf_{x,t)\to(x_0,t_0)}u_{k_1}(x,t)<1\qquad ((x,t)\in Q_T).
$$

This inequality proves the non-regularity of the boundary point (x_0, t_0) and thus Theorem 3.1

4. Pointwise estimates of the function $u_k = u_k(x,t)$

In this section we will use the following

Lemma 4.1 (see [7]). *Let the assumptions (i) - (iii) of Theorem* 3.1 *be satisfied. Then there exist constants* $C_{10} > 0$ *and* $h > 1$ *such that*

estimates of the function
$$
u_k = u_k(x, t)
$$

\nwill use the following
\n(see [7]). Let the assumptions (i) - (iii) of Theorem 3.1 be satisfied.
\nconstants $C_{10} > 0$ and $h > 1$ such that
\n
$$
\frac{1}{w(Q)} \iint_Q |u|^{2h} w(x) dx dt + \frac{1}{v(Q)} \iint_Q |u|^{2h} v(x) dx dt
$$
\n
$$
\leq C_{10} \left\{ \sup_{t \in J} \frac{1}{v(B)} \int_B u^2(x, t) v(x) dx \right\}^{h-1}
$$
\n
$$
\times \left\{ \sup_{t \in J} \frac{|J|}{v(B)} \int_B u^2(x, t) v(x) dx + \frac{(\text{meas } B)^{2/h}}{w(Q)} \iint_Q w(x) \left| \frac{\partial u}{\partial x} \right|^2 dx dt \right\}
$$
\n(4.1)

for any $u \in V_2(Q,v,w)$ *on* $Q = B \times J$ *, with J being an interval and B a ball,* $|J| =$ *measJ and*

A Condition to Regularity of a Boundary Point
\n
$$
u \in \dot{V}_2(Q, v, w)
$$
 on $Q = B \times J$, with J being an interval and B a ba
\nand
\n $w(Q) = |J| \int_B w(x) dx$, $v(Q) = |J| \int_B v(x) dx$, $v(B) = \int_B v(x) dx$.
\n $\alpha_{k-3} \le \rho \le R$,

Let $\alpha_{k-3} \leq \rho \leq R$,

$$
\rho \leq R,
$$

\n
$$
G(\rho) = \left\{ (x,t) \middle| \ |x-x_0| < \rho, \ |t-t_0| < \frac{v(B(x_0,\rho))}{w(B(x_0,\rho))} \rho^2 \right\},
$$

 $0 < \varepsilon < \rho$ and $\mu \geq \mu_k(\rho, \varepsilon)$ with

$$
w(B(x_0, \rho))'
$$

$$
\left\{\begin{array}{l}w(B(x_0, \rho))'\end{array}\right\}
$$

$$
\mu_k(\rho, \varepsilon) = \sup \left\{u_k(x, t) \Big| (x, t) \in [G(\rho + \varepsilon) \setminus G(\rho - \varepsilon)] \cap Q_k\right\}.
$$

We then define the set

$$
F(\rho,\mu)=Q_k\setminus G(\rho)\cup \left\{(x,t)\in Q_k\cap G(\rho)\Big|\,u_k(x,t)\leq \mu\right\}
$$

and the function

$$
\mu \ge \mu_k(\rho, \varepsilon) \text{ with}
$$
\n
$$
(\rho, \varepsilon) = \sup \left\{ u_k(x, t) \middle| (x, t) \in \left[G(\rho + \varepsilon) \setminus G(\rho - \varepsilon) \right] \cap \mathcal{A} \right\}
$$
\nthe set

\n
$$
F(\rho, \mu) = Q_k \setminus G(\rho) \cup \left\{ (x, t) \in Q_k \cap G(\rho) \middle| u_k(x, t) \le \mu \right\}
$$
\nin

\n
$$
u_k^{(\mu)}(x, t) = \begin{cases} u_k(x, t) & \text{for } (x, t) \in Q_k \setminus G(\rho) \\ \min \{ u_k(x, t), \mu \} & \text{for } (x, t) \in Q_k \cap G(\rho). \end{cases}
$$

Lemma 4.2. *For* $\alpha_{k-3} \leq \rho \leq R$, $0 < \varepsilon < \rho$ and $\mu \geq \mu_k(\rho, \varepsilon)$ we have

$$
u_k^{(\mu)}(x,t) = \begin{cases} u_k(x,t) & \text{for } (x,t) \in Q_k \setminus G(\rho) \\ \min\{u_k(x,t),\mu\} & \text{for } (x,t) \in Q_k \cap G(\rho). \end{cases}
$$

4.2. For $\alpha_{k-3} \le \rho \le R$, $0 < \varepsilon < \rho$ and $\mu \ge \mu_k(\rho, \varepsilon)$ we have

$$
\sup_{0 < t \le T} \int_{D_k} v(x) |u_k^{(\mu)}(x,t)|^2 dx + \int_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$

$$
\le C_{11} \mu P_k(E_k, \rho)
$$

$$
P_k(E_k, \rho) = \left[\alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) \right]^{1/2}
$$

$$
\times \left[\alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) + \alpha_k^2 \frac{v^2(B(x_0, \rho))}{w(B(x_0, \rho))} \right]^{1/2}
$$
(4.3)

where

a 4.2. For
$$
\alpha_{k-3} \leq \rho \leq R
$$
, $0 < \varepsilon < \rho$ and $\mu \geq \mu_k(\rho, \varepsilon)$ we have
\n
$$
\sup_{0 < t \leq T} \int_{D_k} v(x) \left| u_k^{(\mu)}(x, t) \right|^2 dx + \int_{F(\rho, \mu)} w(x) \left| \frac{\partial u_k(x, t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_{11} \mu P_k(E_k, \rho)
$$
\n
$$
P_k(E_k, \rho) = \left[\alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) \right]^{1/2}
$$
\n
$$
\times \left[\alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) + \rho^2 \frac{v^2(B(x_0, \rho))}{w(B(x_0, \rho))} \right]^{1/2}
$$
\na constant depending only on $n, \nu_1 \nu_2$ and T .
\nInto the integral identity (3.10) substitute the function
\n
$$
(x, t) = \left[u_k^{(\mu)}(x, t) \right]_h - \phi(x) \lambda_{k,h}^{(\mu)}(t) \quad \text{for } 0 < h < \varepsilon^2 \frac{v(B(x_0, \rho))}{w(B(x_0, \rho))}.
$$
\n
$$
\frac{u(k)}{(\mu)} = \frac{u(k)}{(\mu)} \left(\frac{u(k)}{(\mu)} \right)^2 + \frac{u(k)}{(\mu)} \left(\frac{u(k)}{(\mu
$$

and C_{11} is a constant depending only on $n, \nu_1 \nu_2$ and T .

Proof. Into the integral identity (3.10) substitute the function

is a constant depending only on
$$
n, \nu_1 \nu_2
$$
 and T .
\n**f.** Into the integral identity (3.10) substitute the function
\n
$$
\psi(x,t) = \left[u_k^{(\mu)}(x,t) \right]_h - \phi(x) \lambda_{k,h}^{(\mu)}(t) \quad \text{for } 0 < h < \varepsilon^2 \frac{v(B(x_0,\rho))}{w(B(x_0,\rho))}.
$$
\n
$$
\lambda_{k,h}^{(\mu)}(t) = \min \left\{ \left[\lambda_k(t) \right]_h, \mu \right\} \quad (\phi \in \mathcal{M}(E_k)).
$$

Here

$$
\lambda_{k,h}^{(\mu)}(t) = \min\left\{[\lambda_k(t)]_h, \mu\right\} \qquad \left(\phi \in \mathcal{M}(E_k)\right).
$$

After this substitution we transform the term containing $\frac{\partial [u_k(x,t)]_h}{\partial t}$, obtaining

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\nAfter this substitution we transform the term containing
$$
\frac{\partial [u_k(x,t)]_h}{\partial t}
$$
, obtain
\n
$$
\int_0^r \int_{D_k} v(x) \frac{\partial}{\partial t} [u_k(x,t)]_h \left\{ [u_k^{(\mu)}(x,t)]_h - \phi(x) \lambda_{k,h}^{(\mu)}(t) \right\} dx dt
$$
\n
$$
= \int_{D_k} v(x) \left\{ \frac{1}{2} \left[u_k^{(\mu)}(x,\tau) \right]_h^2 + \mu \left([u_k(x,\tau)]_h - \left[u_k^{(\mu)}(x,\tau) \right]_h \right) - [u_k(x,\tau)]_h \phi(x) \lambda_{k,h}^{(\mu)}(\tau) \right\} dx
$$
\n
$$
+ \int_0^r \int_{D_k} v(x) [u_k(x,t)]_h \phi(x) \frac{\partial \lambda_{k,h}^{(\mu)}(t)}{\partial t} dx dt.
$$
\nUsing this representation on passing to limit with respect to *h* in the integral
\nwe get

Using this representation on passing to limit with respect to *h* in the integral identity, we get

Using this representation on passing to limit with respect to *h* in the integral identity,
\nwe get
\n
$$
\int_{D_k} v(x) \left\{ \frac{1}{2} \left[u_k^{(\mu)}(x,\tau) \right]^2 + \mu \left(u_k(x,\tau) - u_k^{(\mu)}(x,\tau) \right) - u_k(x,\tau) \phi(x) \lambda_k^{(\mu)}(\tau) \right\} dx
$$
\n
$$
+ \int_0^{\tau} \int_{D_k} \left\{ \sum_{i=1}^n a_i \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \frac{\partial}{\partial x_i} \left[u_k^{(\mu)}(x,t) - \phi(x) \lambda_k^{(\mu)}(t) \right] - a_0 \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \left[u_k^{(\mu)}(x,t) - \phi(x) \lambda_k^{(\mu)}(t) \right] + v(x) u_k(x,t) \phi(x) \frac{\partial \lambda_k^{(\mu)}(t)}{\partial t} \right\} dx dt
$$
\n
$$
= 0.
$$
\nUsing inequalities (2.7), (2.8) and (3.9) we can estimate the left-hand side of (4.4) obtaining
\n
$$
\int_{D_k} v(x) \left[u_k^{(\mu)}(x,\tau) \right]_h^2 dx + \iint_{E(x)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$

Using inequalities (2.7) , (2.8) and (3.9) we can estimate the left-hand side of (4.4)

ng inequalities (2.7), (2.8) and (3.9) we can estimate the left-hand side of (4.4)
\n
$$
\int_{D_k} v(x) \left[u_k^{(\mu)}(x,\tau) \right]_h^2 dx + \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_{12} \left\{ \mu \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) + \iint_{F(\rho,\mu)} v(x) u_k^2(x,t) dx dt + \int_0^{\tau} \int_{D_k} \left[v(x) |u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \right] \frac{\partial u_k(x,t)}{\partial x} \right] \left| u_k^{(\mu)}(x,t) \right| dx dt \right\}.
$$
\n(4.5)

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We estimate the latter part on the right-hand side of (4.5) using the Hölder and Young inequalities, (3.9) and the observation that

A Condition to Regularity of a Bot
atter part on the right-hand side of (4.5) using the
and the observation that

$$
u_k(x,t) = 0 \quad \text{for} \quad t \le t_0 - 2\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}.
$$

We have

A Condition to Regularity of a Boundary Point
\nestimate the latter part on the right-hand side of (4.5) using the Hölder and Y
\nualities, (3.9) and the observation that
\n
$$
u_k(x,t) = 0 \quad \text{for } t \le t_0 - 2\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))}.
$$
\nhave
\n
$$
\int_0^t \int_{D_k} \left[v(x) |u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] \left| u_k^{(\mu)}(x,t) \right| dx dt
$$
\n
$$
\le \mu \iint_{Q_k \cap G(\rho)} \left[v(x) |u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] dx dt
$$
\n
$$
+ \iint_{F(\rho,\mu)} \left[v(x) |u_k(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right| \right] \left| u_k^{(\mu)}(x,t) \right| dx dt
$$
\n
$$
\le \frac{C_{12}^{-1}}{2} \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
+ C_{13} \left\{ \iint_{F(\rho,\mu)} v(x) u_k^2(x,t) dx dt
$$
\n
$$
+ \mu \left[\alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) \right]^{1/2} \left[\rho^2 \frac{v^2(B(x_0,\rho))}{w(B(x_0,\rho))} \right]^{1/2} \right\}.
$$
\n
$$
\text{or from (4.5) it follows}
$$

Hence from (4.5) it follows

$$
(4.5) \text{ it follows}
$$
\n
$$
\int_{D_k} \left[u_k^{(\mu)}(x,\tau) \right]^2 v(x) dx + \iint_{F(\rho,\mu)} w(x) \left| \frac{\partial u_k(x,t)}{\partial x} \right|^2 dx dt
$$
\n
$$
\leq C_{14} \left\{ \int_0^{\tau} \int_{D_k} v(x) \left[u_k^{(\mu)}(x,t) \right]^2 dx dt + \mu P_k(E_k, \rho) \right\}.
$$
\nThe Gronwall inequality we obtain (4.2) **I**

\nif Theorem 3.2. Let $\alpha_{k-3} \leq \rho \leq R$. We define two numerical

\n
$$
\{ \rho_{2,i} \}_{i \in \mathbb{N}} \text{ by putting}
$$
\n
$$
\rho_{1,i}^2 = \frac{1 + \alpha_i}{1 + \alpha_0} \rho^2 \qquad \text{and} \qquad \rho_{2,i}^2 = \frac{1 + 2\alpha_0 - \alpha_i}{1 + \alpha_0} \rho^2
$$
\nthe defined by (3.3). Moreover, consider infinitely differentiable

\n
$$
i \in \mathbb{N} \text{ on } \mathbb{R} \text{ such that}
$$

Now using the Gronwall inequality we obtain (4.2)

Proof of Theorem 3.2. Let $\alpha_{k-3} \leq \rho \leq R$. We define two numerical sequences $\{\rho_{1,i}\}_{i\in\mathbb{N}}$ and $\{\rho_{2,i}\}_{i\in\mathbb{N}}$ by putting

around linequality we obtain (4.2) **II**

\ntheorem 3.2. Let
$$
\alpha_{k-3} \leq \rho \leq R
$$
. We define two numbers:

\n $\rho_{1,i}^2 = \frac{1 + \alpha_i}{1 + \alpha_0} \rho^2$ and $\rho_{2,i}^2 = \frac{1 + 2\alpha_0 - \alpha_i}{1 + \alpha_0} \rho^2$

where α_i are defined by (3.3). Moreover consider infinitely differentiable functions $\chi = \chi_i(s)$ $(i \in \mathbb{N})$ on R such that

$$
0 \le \chi_{i}(s) \le 1
$$

\n
$$
\chi_{i}(s) = 1 \text{ on } [\rho_{1,i}^{2}, \rho_{2,i}^{2}] \text{ and } \chi_{i}(s) = 0 \text{ on } \mathbb{R} \setminus [\rho_{1,i+1}^{2}, \rho_{2,i+1}^{2}]
$$

\n
$$
\left| \frac{d\chi_{i}(s)}{ds} \right| \le \frac{1 + \alpha_{0}}{1 - K_{7}} \frac{1}{\alpha_{i}} \rho^{-2}.
$$

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Then let

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t

$$
\psi_i(x) = \chi_i \left(\left| x - x_0 \right|^2 \right) \quad \text{and} \quad \phi_i(t) = \chi_i \left(\frac{w(B(x_0, \rho))}{v(B(x_0, \rho))} \left| t - t_0 \right| \right)
$$
stitute into (3.10) the function

and substitute into (3.10) the function

$$
\psi(x,t) = [u_k(x,t)]_h^{r+1} \psi_i^{s+2}(x) \phi_i^{s+2}(t)
$$

for arbitrary non-negative numbers *r* and *s.* Integrating by parts the term containing $[u(x,t)]_h$, passing to the limit as $h \to 0$ and estimating using (2.7) and (2.8), we obtain

$$
\psi(x,t) = [u_k(x,t)]_h^{r+1} \psi_i^{s+2}(x)\phi_i^{s+2}(t)
$$
\nor arbitrary non-negative numbers r and s . Integrating by parts the term contain $\frac{\partial}{\partial t} [u(x,t)]_h$, passing to the limit as $h \to 0$ and estimating using (2.7) and (2.8), we obtain

\n
$$
\int_{D_k} u_k^{r+2}(x,\tau)\psi_i^{s+2}(x)\phi_i^{s+2}(\tau)v(x) dx
$$
\n
$$
+ \int_0^r \int_{D_k} w(x) \left| \frac{\partial u_k}{\partial x} \right|^2 u_k^r(x,t)\psi_i^{s+2}(x)\phi_i^{s+2}(t) dx dt
$$
\n
$$
\leq C_{15} \frac{\alpha_i^{-2}(r+s+1)^2}{\rho^2} \int_0^r \int_{D_k} u_k^{r+2}(x,t) \left[w(x) + \frac{w(B(x_0,\rho))}{v(B(x_0,\rho))} v(x) \right] \psi_i^s(x)\phi_i^s(t) dx dt.
$$
\nence using Lemma 4.1 and applying the iterative technique (see, e.g., [13: Lemma 2.2])

\n
$$
m_i = m_i(\rho) = \sup \left\{ |u_k(x,t)| \left| \begin{array}{l} (x,t) \in Q_k, \rho_{1,i} \leq |x-x_0| \leq \rho_{2,i} \\ \rho_{1,i}^2 \leq \frac{w(B(x_0,\rho))}{v(B(x_0,\rho))} |t-t_0| \leq \rho_{2,i}^2 \\ \end{array} \right\}
$$
\n(4.6)

Hence using Lemma 4.1 and applying the iterative technique (see, e.g, [13: Lemma 2.2]) to

$$
\rho^2 \qquad \int_{0}^{1} J_{D_k} u_k^{(x, t)} \left[\frac{u(x)}{u(x)} + v(B(x_0, \rho)) \frac{v(x)}{u(x)} \right] \varphi_i(x) \varphi_i(t) dx dt.
$$
\nng Lemma 4.1 and applying the iterative technique (see, e.g., [13: Lemma 2.2])

\n
$$
m_i = m_i(\rho) = \sup \left\{ \left| u_k(x, t) \right| \left| \begin{array}{l} (x, t) \in Q_k, \ \rho_{1,i} \le |x - x_0| \le \rho_{2,i} \\ \rho_{1,i}^2 \le \frac{w(B(x_0, \rho))}{v(B(x_0, \rho))} |t - t_0| \le \rho_{2,i}^2 \end{array} \right. \right\}
$$
\n(4.6)

\nthe following estimate with some $h > 1$:

to
\n
$$
m_{i} = m_{i}(\rho) = \sup \left\{ |u_{k}(x,t)| \left| \begin{array}{l} (x,t) \in Q_{k}, \rho_{1,i} \leq |x - x_{0}| \leq \rho_{2,i} \\ \rho_{1,i}^{2} \leq \frac{w(B(x_{0},\rho))}{v(B(x_{0},\rho))} |t - t_{0}| \leq \rho_{2,i}^{2} \end{array} \right\} \right\}
$$
\n(4.6)
\nwe obtain the following estimate with some $h > 1$:
\n
$$
[m_{i}]^{2} \leq C_{16} \alpha_{i}^{-\frac{2\hbar}{\hbar - 1}} \times \int_{0}^{t} \int_{R_{k}} u_{k}^{2}(x,t) \left[\frac{w(x)}{\rho^{2}v(B(x_{0},\rho))} + \frac{w(B(x_{0},\rho))v(x)}{\rho^{2}v^{2}(B(x_{0},\rho))} \right] \psi_{i}^{2}(x) \phi_{i}^{2}(t) dx dt.
$$
\n(4.7)
\nLet us estimate the integral (4.7) applying Lemma 2.2 and 4.2. We have
\n
$$
\int_{0}^{t} \int_{D_{k}} u_{k}^{2}(x,t) \left[\frac{w(x)}{\rho^{2}v(B(x_{0},\rho))} + \frac{w(B(x_{0},\rho))v(x)}{\rho^{2}v^{2}(B(x_{0},\rho))} \right] \psi_{i}^{2}(x) \phi_{i}^{2}(t) dx dt
$$
\n(4.7)

Let us estimate the integral (4.7) applying Lemmata 2.2 and 4.2. We have

$$
+\int_{0}^{t} \int_{D_{k}} w(x) \left| \frac{\partial u_{k}}{\partial x} \right|^{2} u_{k}^{r}(x, t) \psi_{i}^{s+2}(x) \phi_{i}^{s+2}(t) dx dt
$$
\n
$$
\leq C_{15} \frac{\alpha_{i}^{-2}(r+s+1)^{2}}{\rho^{2}} \int_{0}^{t} \int_{D_{k}} u_{k}^{r+2}(x, t) \left[w(x) + \frac{w(B(x_{0}, \rho))}{v(B(x_{0}, \rho))} v(x) \right] \psi_{i}^{s}(x) \phi_{i}^{s}(t) dx dt.
$$
\n
$$
\text{ce using Lemma 4.1 and applying the iterative technique (see, e.g., [13: Lemma 2.2])}
$$
\n
$$
m_{i} = m_{i}(\rho) = \sup \left\{ |u_{k}(x, t)| \left| \begin{array}{l} (x, t) \in Q_{k}, \rho_{1, i} \leq |x - x_{0}| \leq \rho_{2, i} \\ \rho_{1, i}^{2} \leq \frac{w(B(x_{0}, \rho))}{v(B(x_{0}, \rho))} |t - t_{0}| \leq \rho_{2, i}^{2} \end{array} \right\} \right. \tag{4.6}
$$
\n
$$
\text{obtain the following estimate with some } h > 1:
$$
\n
$$
m_{i}^{12} \leq C_{16} \alpha_{i}^{-\frac{2}{k-1}}
$$
\n
$$
\times \int_{0}^{t} \int_{D_{k}} u_{k}^{2}(x, t) \left[\frac{w(x)}{\rho^{2}v(B(x_{0}, \rho))} + \frac{w(B(x_{0}, \rho))v(x)}{\rho^{2}v^{2}(B(x_{0}, \rho))} \right] \psi_{i}^{2}(x) \phi_{i}^{2}(t) dx dt. \tag{4.7}
$$
\n
$$
\text{us estimate the integral (4.7) applying Lemma 2.2 and 4.2. We have}
$$
\n
$$
\int_{0}^{t} \int_{D_{k}} u_{k}^{2}(x, t) \left[\frac{w(x)}{\rho^{2}v(B(x_{0}, \rho))} + \frac{w(B(x_{0}, \rho))v(x)}{\rho^{2}v^{2}(B(x_{0}, \rho))} \right] \psi_{i}^{2}(x) \phi_{i}^{2}(t) dx dt \right. \tag{4.8}
$$

Here ρ_{i+1} is such that $\rho_{i+1} < \rho_{2,i+1}$, and $\chi_{i+1}(s) = 0$ for $s \geq \rho_{i+1}$.

Now, from (4.7) and (4.8) it follows that

A Condition to Regularity of a Boundary Point 173
\n
$$
p_{i+1} < p_{2,i+1}, \text{ and } \chi_{i+1}(s) = 0 \text{ for } s \ge p_{i+1}.
$$
\nIf (4.8) it follows that\n
$$
[m_i]^2 \le C_{19} \frac{\alpha_i^{-\frac{2\lambda}{\hbar - 1}}}{\nu(B(x_0, \rho))} P_k(E_k, \rho) m_{i+1}
$$
\n
$$
2.1, \text{ we get}
$$
\n
$$
m_i(\rho) \le C_{20} \frac{1}{\nu(B(x_0, \rho))} P_k(E_k, \rho).
$$
\n
$$
(4.10)
$$
\n
$$
m_i^2 \ge C_{19} \frac{1}{\nu(B(x_0, \rho))} P_k(E_k, \rho).
$$

and further, by Lemma 2. 1, we get

$$
m_i(\rho) \le C_{20} \frac{1}{v(B(x_0, \rho))} P_k(E_k, \rho).
$$
 (4.10)

Now for proving Theorem 3.2 it sufficies to show that for $\rho \ge \alpha_{k-3}$

A Condition to Regularity of a Boundary Point 173
\nh that
$$
\rho_{i+1} < \rho_{2,i+1}
$$
, and $\chi_{i+1}(s) = 0$ for $s \ge \rho_{i+1}$.
\n4.7) and (4.8) it follows that
\n
$$
[m_i]^2 \le C_{19} \frac{\alpha_i^{-\frac{2\lambda}{\lambda-1}}}{\nu(B(x_0,\rho))} P_k(E_k,\rho) m_{i+1}
$$
\n(4.19)
\nLemma 2.1, we get
\n
$$
m_i(\rho) \le C_{20} \frac{1}{\nu(B(x_0,\rho))} P_k(E_k,\rho).
$$
\n(4.10)
\n; Theorem 3.2 it suffices to show that for $\rho \ge \alpha_{k-3}$
\n
$$
\mu(\rho,\alpha_k) \le C_{21} \left\{ \alpha_k^2 \frac{C_{2,\omega}(E_k)}{\omega(B(x_0,\alpha_k))} + \alpha_k^2 \frac{\nu(B(x_0,\alpha_k))}{\omega(B(x_0,\alpha_k))} \right\}.
$$
\n(4.11)
\ninequality (4.11) follows from (4.10). If for some $\rho > \alpha_{k-3}$ we had
\n s,α_k), then that ρ will satisfy (4.11). If however $m_1(\rho) > \mu(\alpha_{k-3},\alpha_k)$

For $\rho = \alpha_{k-3}$, inequality (4.11) follows from (4.10). If for some $\rho > \alpha_{k-3}$ we had $m_1(\rho) \leq \mu(\alpha_{k-3}, \alpha_k)$, then that ρ will satisfy (4.11). If however $m_1(\rho) > \mu(\alpha_{k-3}, \alpha_k)$ for some ρ , then for all $i \in \mathbb{N}$ we have $m_{i+1}(\rho) > \mu(\alpha_{k-3}, \alpha_k)$ and we operate a change in (4.8). In this case

$$
F(\rho_{i+1},m_{i+1}(\rho))\subset F(\alpha_{k-3},m_{i+1}(\rho))
$$

and we obtain instead of (4.8)

A Condition to Regularity of a Boundary Point 173
\n
$$
i+1
$$
 is such that $\rho_{i+1} < \rho_{2,i+1}$, and $\chi_{i+1}(s) = 0$ for $s \ge \rho_{i+1}$.
\n
$$
[m_i]^2 \le C_{19} \frac{\alpha_i^{-\frac{2b}{h-1}}}{\nu(B(x_0, \rho))} P_k(E_k, \rho) m_{i+1}
$$
\n
$$
[m_i]^2 \le C_{19} \frac{\alpha_i^{-\frac{2b}{h-1}}}{\nu(B(x_0, \rho))} P_k(E_k, \rho) m_{i+1}
$$
\n
$$
m_i(\rho) \le C_{20} \frac{1}{\nu(B(x_0, \rho))} P_k(E_k, \rho).
$$
\n(4.10)
\n
$$
[m_i(\rho) \le C_{21} \left\{ \alpha_k^2 \frac{C_{2,\omega}(E_k)}{\omega(B(x_0, \alpha_k))} + \alpha_k^2 \frac{\nu(B(x_0, \alpha_k))}{\omega(B(x_0, \alpha_k))} \right\}.
$$
\n(4.11)
\n
$$
= \alpha_{k-3}, \text{ inequality (4.11) follows from (4.10). If for some $\rho > \alpha_{k-3}$ we had $\le \mu(\alpha_{k-3}, \alpha_k)$, then that ρ will satisfy (4.11). If however $m_i(\rho) > \mu(\alpha_{k-3}, \alpha_k)$ and we operate a change ρ , then for all $i \in \mathbb{N}$ we have $m_{i+1}(\rho) > \mu(\alpha_{k-3}, \alpha_k)$ and we operate a change $F(\rho_{i+1}, m_{i+1}(\rho)) \subset F(\alpha_{k-3}, m_{i+1}(\rho))$ obtain instead of (4.8)
\n
$$
\int \int_{0} u_k^2(x, t) \left[\frac{\omega(x)}{\rho^2 v(B(x_0, \rho))} + \frac{\omega(B(x_0, \rho)) v(x)}{\rho^2 v^2(B(x_0, \rho))} \right] \psi_i^2(x) \phi_i^2(t) dx dt
$$
\n
$$
\le C_{22} \frac{1}{v(B(x_0, \rho))} \int \int_{F(\alpha_{k-3}, m_{i+1}(\rho))} \omega(x) \left| \frac{\partial u_k(x, t)}{\partial x}
$$
$$

So as above we get the inequality

$$
[m_i(\rho)]^2 \leq C_{23} \frac{\alpha_i^{-\frac{2h}{h-1}}}{v(B(x_{0},\rho))} P_k(E_k,\alpha_{k-3}) m_{i+1}(\rho).
$$

Hence, by Lemma 2.1, for a given ρ we have (4.11) and this completes the proof of Theorem 3.2

5. Integral estimates for the difference $u_k(x,t) - u_{k+1}(x,t)$

We shall need auxiliary functions $f_k = f_k(x)$ and $g_k = g_k(x)$ defined, respectively, as the solutions of the problems

5. Leonard and 1.1. Skryphik
\n**Integral estimates for the difference**
$$
u_k(x, t) - u_{k+1}(x, t)
$$

\n**hall need auxiliary functions** $f_k = f_k(x)$ **and** $g_k = g_k(x)$ **defined, respectively, as**
\n**olutions of the problems**
\n
$$
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(w(x) \frac{\partial f_k}{\partial x_i}(x) \right) = 0 \quad \left(x \in D_k = B \setminus E_k, f_k - h \in \mathring{W}_2^1(D_k, w) \right) \tag{5.1}
$$

S. Leonardi and I. I. Skrypnik
\nIntegral estimates for the difference
$$
u_k(x, t) - u_{k+1}(x, t)
$$

\nshall need auxiliary functions $f_k = f_k(x)$ and $g_k = g_k(x)$ defined, respectively, as
\nolutions of the problems
\n
$$
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(w(x) \frac{\partial f_k}{\partial x_i}(x) \right) = 0 \quad \left(x \in D_k = B \setminus E_k, f_k - h \in \mathring{W}_2^1(D_k, w) \right) \tag{5.1}
$$
\n
$$
\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(w(x) \frac{\partial g_k}{\partial x_i}(x) \right) = 0 \quad \left(x \in D^{(k)} = B \setminus E^{(k)}, g_k - h \in \mathring{W}_2^1(D^{(k)}, w) \right) . (5.2)
$$
\nLemma 5.1. There exists a constant $C_{24} > 0$ such that
\n
$$
||f_k||_{W_2^1(D_k, w)} \leq C_{24} C_{2,w}(E_k) \tag{5.3}
$$
\n
$$
||g_k||_{W_2^1(D^{(k)}, w)} \leq C_{24} C_{2,w}(E^{(k)}).
$$
\nProof. We will prove only the inequality (5.3), the proof of inequality (5.4) is
\ngous. In the definition of a weak solution for equation (5.3), choose as test function

Lemma 5.1. *There exists a constant* $C_{24} > 0$ *such that*

$$
||f_k||_{W_2^1(D_k, w)} \le C_{24} C_{2,w}(E_k)
$$
\n(5.3)

$$
||g_k||_{W_2^1(D^{(k)},w)} \leq C_{24} C_{2,w}(E^{(k)}).
$$
\n(5.4)

Proof. We will prove only the inequality (5.3), the proof of inequality (5.4) is analogous. In the definition of a weak solution for equation (5.3), choose as test function $\begin{aligned} k \| w_2^1(D^{(k)}, w) \leq k \end{aligned}$
 hly the inequal
 h = *f_k* - *φ*

$$
\psi = f_k - \phi \qquad (\phi \in \mathcal{M}(E_k)).
$$

We then obtain

Contract

15k = Uk - Uk+1

Now, using Lemma 3.1, we have (5.3)

Let us denote

$$
\delta_k = u_k - u_{k+1} \tag{5.5}
$$

and

 $\sim 10^7$

The definition of a weak solution for equation (5.3), choose as test function
\n
$$
\psi = f_k - \phi \qquad (\phi \in M(E_k)).
$$
\n
$$
\int_{D_k} w(x) \left| \frac{\partial f_k(x)}{\partial x} \right|^2 dx \le \int_{D_k} w(x) \left| \frac{\partial \phi}{\partial x} \right|^2 dx.
$$
\n
$$
\text{mma } 3.1 \text{, we have (5.3)} \blacksquare
$$
\n
$$
\delta_k = u_k - u_{k+1} \qquad (5.5)
$$
\n
$$
d_k = \sup \left\{ \left| \delta_k(x, t) \right| \begin{array}{l} (x, t) \in Q_k, |x - x_0| < \alpha_{k-3} \\ |t - t_0| < \alpha_{k-3}^2 \frac{v(B(x_0, \alpha_{k-3}))}{w(B(x_0, \alpha_{k-3}))} \end{array} \right\}.
$$
\n
$$
\text{on } f = f(x, t) \text{ and numbers } A_1 \text{ and } A_2 \text{ with } A_1 < A_2 \text{, we define}
$$
\n
$$
\left| \begin{array}{l} f(x, t) = \max_{k=1}^{\infty} \{ f(x, t) \} 0 \end{array} \right|
$$

For any function $f = f(x, t)$ and numbers A_1 and A_2 with $A_1 < A_2$, we define

$$
[f(x,t)]_{\pm} = \max \{ \pm f(x,t), 0 \}
$$

$$
[f(x,t)]_{(A_1,A_2)} = \max \{ \min [f(x,t), A_2], A_1 \}.
$$

the for

$$
\mu > 0 \quad \text{and} \quad t_k = t_0 + \alpha_{k-1}^2 \frac{\nu(B(x_0, \alpha_{k-1})}{\nu(B(x_0, \alpha_{k-1})}))
$$

Moreover we define for

$$
\mu > 0
$$
 and $t_k = t_0 + \alpha_{k-1}^2 \frac{v(B(x_0, \alpha_{k-1}))}{w(B(x_0, \alpha_{k-1}))}$

the sets

For any function
$$
f = f(x, t)
$$
 and numbers A_1 and A_2 with $A_1 < A_2$, we define
\n
$$
[f(x, t)]_{\pm} = \max \left\{ \pm f(x, t), 0 \right\}
$$
\n
$$
[f(x, t)]_{(A_1, A_2)} = \max \left\{ \min [f(x, t), A_2], A_1 \right\}.
$$
\nMoreover we define for
\n
$$
\mu > 0 \quad \text{and} \quad t_k = t_0 + \alpha_{k-1}^2 \frac{\nu(B(x_0, \alpha_{k-1}))}{\nu(B(x_0, \alpha_{k-1}))}
$$
\nthe sets
\n
$$
F^{\pm}(\mu) = \left\{ (x, t) \in B \times [0, t_k] \right\} \pm \left[\delta_k(x, t) \right]_{(-\mu, \mu)} \ge \mu \left[\bar{f}_k(x, t) + \bar{g}_k(x, t) \right] \right\},
$$
\n
$$
F(\mu) = F^+(\mu) \cup F^-(\mu)
$$
\n
$$
T(\mu) = \left\{ (x, t) \in B \times [0, t_k] \middle| \left| \delta_k(x, t) \right| \le \mu \right\}
$$

 $\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L}}(\mathcal{L}^{\mathcal{L$

where

 $\frac{1}{2}$

 $\ddot{}$

$$
\begin{aligned} \bar{f}_k(x,t) &= f_k(x)\gamma\big(\alpha_{k-1}^{-1}|x-x_0|\big)\big\{\lambda_{k-2}(t) - \lambda_{k+2}(t)\big\} \\ \bar{g}_k(x,t) &= g_k(x)\Big\{\gamma\big(\alpha_{k-1}^{-1}|x-x_0|\big) - \gamma\big(\alpha_{k+2}^{-1}|x-x_0|\big)\Big\}\lambda_{k-2}(t) \end{aligned}
$$

 $\Delta \sim 10^{11}$ and $\Delta \sim 10^{11}$ and $\Delta \sim 10^{11}$ and $\Delta \sim 10^{11}$

and where γ and λ_k are the functions introduced in Section 3. Further, for an arbitrary

Theorem 5.1. For arbitrary $\mu > d_k$ there exists a constant $K_{12} > 0$ such that

$$
J_{k}(x,t) = J_{k}(x)\gamma(\alpha_{k-1}|x-x_{0}|)\{\lambda_{k-2}(t) - \lambda_{k+2}(t)\}
$$
\n
$$
\bar{g}_{k}(x,t) = g_{k}(x)\Big\{\gamma(\alpha_{k-1}^{-1}|x-x_{0}|) - \gamma(\alpha_{k+2}^{-1}|x-x_{0}|)\Big\}\lambda_{k-2}(t)
$$
\nand where γ and λ_{k} are the functions introduced in Section 3. Further, for an arbitrary
\nsubset $E \subset B \times (0,T)$, we denote by $\chi = \chi_{E}(x,t)$ its characteristic function.
\n**Theorem 5.1.** For arbitrary $\mu > d_{k}$ there exists a constant $K_{12} > 0$ such that\n
$$
\sup_{0 \le t \le t_{k}} \int_{B} v(x) \Big|\left[\delta_{k}(x,t)\right|_{(-\mu,\mu)}\Big|^{2} \chi_{F_{(\mu)}}(x,t) dx dt + \int_{T(\mu) \cap F(\mu)} w(x) \Big|\frac{\partial}{\partial x} \delta_{k}(x,t)\Big|^{2} dx dt + \int_{T(\mu) \cap F(\mu)} w(x) \Big|\frac{\partial}{\partial x} \delta_{k}(x,t)\Big|^{2} dx dt
$$
\n
$$
\leq K_{12} \mu \Big\{\alpha_{k}^{2} \frac{v(B(x_{0}, \alpha_{k}))}{w(B(x_{0}, \alpha_{k}))} C_{2,w}(E_{k})\Big\}^{1/2} \times \Big\{\alpha_{k}^{2} \frac{v(B(x_{0}, \alpha_{k}))}{w(B(x_{0}, \alpha_{k}))} + C_{2,w}(E_{k}) \alpha_{k}^{2} \frac{v(B(x_{0}, \alpha_{k}))}{w(B(x_{0}, \alpha_{k}))}\Big\}^{1/2}.
$$
\nProof. Since $F(\mu) = F^{+}(\mu) \cup F^{-}(\mu)$, we prove the inequality (5.7) only for $F^{+}(\mu)$.
\nDefine for\n
$$
h < \alpha_{k+2}^{2} \frac{v(B(x_{0}, \alpha_{k+2}))}{w(B(x_{0}, \alpha_{k+2}))}
$$
\nthe function\n
$$
\psi(x,t) = \Big[\delta_{k,h}(x,t)\Big|_{k} - \frac{\left[u_{k}(x,t)\right|_{k} \Big|_{k} \Delta_{k} \text{ and } \frac{\left[u_{k}(x,t)\right|_{k}}{\left
$$

Proof. Since $F(\mu) = F^+(\mu) \cup F^-(\mu)$, we prove the inequality (5.7) only for $F^+(\mu)$.
Define for

$$
h < \alpha_{k+2}^2 \frac{v(B(x_0, \alpha_{k+2}))}{w(B(x_0, \alpha_{k+2}))}
$$

the function

$$
\psi(x,t) = \left[[\delta_{k,h}(x,t)]_{(-\mu,\mu)} - \mu [\bar{f}_k(x,t)]_h - \mu [\bar{g}_k(x,t)]_h \right]_+ \tag{5.8}
$$

where $\delta_{k,h}(x,t) = [u_k(x,t)]_h - [u_{k+1}(x,t)]_h$. As well as in [13], using condition (3.3), we We plug ψ in the integral identity (3.10) for u_k and u_{k+1} obtaining

$$
\psi(x,t) = \left[[\delta_{k,h}(x,t)]_{(-\mu,\mu)} - \mu[\bar{f}_k(x,t)]_h - \mu[\bar{g}_k(x,t)]_h \right]_+ \qquad (5.8)
$$
\nwhere $\delta_{k,h}(x,t) = [u_k(x,t)]_h - [u_{k+1}(x,t)]_h$. As well as in [13], using condition (3.3), we check that the function ψ defined by (5.8) belongs to the space $\tilde{V}_2(D_K \times (0,t_k), v, w)$. We plug ψ in the integral identity (3.10) for u_k and u_{k+1} obtaining\n
$$
\int_0^{\tau} \int_B \left\{ v(x) \frac{\partial}{\partial t} \delta_{k,h} \psi(x,t) + \sum_{i=1}^n \left(\left[a_i \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \right]_h \right. \right. \\ \left. - \left[a_i \left(x, t, u_{k+1}, \frac{\partial u_{k+1}}{\partial x} \right) \right]_h \right\} \frac{\partial \psi}{\partial x_i} - \left(\left[a_0 \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) \right]_h - \left[a_0 \left(x, t, u_{k+1}, \frac{\partial u_{k+1}}{\partial x} \right) \right]_h \right) \psi \right\} dx dt
$$
\n
$$
= 0.
$$
\n(5.9)

We transform the first term under the integral sign in (5.9) in the following way:

nsform the first term under the integral sign in (5.9) in the following way:
\n
$$
\int_{0}^{T} \int_{B} v(x) \frac{\partial}{\partial t} \delta_{k,h} \psi(x,t) dx dt
$$
\n
$$
= \frac{1}{2} \int_{B} v(x) \frac{\partial}{\partial t} \left\{ \left[\left[\delta_{k,h}(x,\tau) \right]_{(-\mu,\mu)} - \mu \left[\bar{f}_{k}(x,\tau) + \bar{g}_{k}(x,\tau) \right]_{h} \right]_{+}^{2} + 2 \left(\delta_{k,h}(x,\tau) - \left[\delta_{k,h}(x,\tau) \right]_{(-\mu,\mu)} \right)
$$
\n
$$
\times \left[\left[\delta_{k,h}(x,\tau) \right]_{(-\mu,\mu)} - \mu \left[\bar{f}_{k}(x,\tau) + \bar{g}_{k}(x,\tau) \right]_{h} \right]_{+} \right\} dx
$$
\n
$$
+ \mu \int_{0}^{T} \int_{B} v(x) \frac{\partial}{\partial t} \left[\bar{f}_{k}(x,t) + \bar{g}_{k}(x,t) \right]_{h}
$$
\n
$$
\times \left[\delta_{k,h}(x,t) - \mu \left[\bar{f}_{k}(x,t) + \bar{g}_{k}(x,t) \right]_{h} \right]_{+} dx dt.
$$
\n(5.10)

Using (5.10), passing to the limit as $h \to 0$ and applying (2.7), (2.8), (2.17), (2.18), $(3.9), (5.3), (5.4),$ we obtain from (5.9)

$$
\times \left[\delta_{k,h}(x,t) - \mu \left[\bar{f}_k(x,t) + \bar{g}_k(x,t) \right]_h \right]_+ dx dt.
$$
\n
\npassing to the limit as $h \to 0$ and applying (2.7), (2.8), (2.17), (2.18),
\n5.4), we obtain from (5.9)\n
\n
$$
\int_B v(x) \left[\delta_{k,h}(x,\tau) \right]_{(-\mu,\mu)}^2 \chi_{F^+(\mu)}(x,\tau) dx
$$
\n
$$
+ \int_0^{\tau} \int_B w(x) \left| \frac{\partial \delta_k(x,t)}{\partial x} \right|^2 \chi_{F^+(\mu) \cap T(\mu)}(x,t) dx dt
$$
\n
$$
\leq C_{25} \left\{ \mu \alpha_k^2 \frac{v(B(x_0,\alpha_k))}{w(B(x_0,\alpha_k))} C_{2,w}(E_k) + \int_0^{\tau} \int_B v(x) \delta_k^2(x,t) \chi_{F^+(\mu) \cap T(\mu)}(x,t) dx dt + \int_0^{\tau} \int_B \left(v(x) \left| \delta_k(x,t) \right| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial \delta_k(x,t)}{\partial x} \right| \right) \right.
$$
\n
$$
\times \left[\delta_{k,h}(x,t) \right]_{(-\mu,\mu)} \chi_{F^+(\mu)}(x,t) dx dt
$$
\n(5.11)

We estimate the last integral of the right-hand side of (5.11) using the Hölder inequality,

getting

A Condition to Regularity of a boundary Point
\n
$$
\int_{0}^{T} \int_{B} \left(v(x) |\delta_{k}(x,t)| + w^{1/2}(x) v^{1/2}(x) \left| \frac{\partial \delta_{k}(x,t)}{\partial x} \right| \right)
$$
\n
$$
\times [\delta_{k,h}(x,t)]_{(-\mu,\mu)} \times F^{+}(\mu)(x,t) dx dt
$$
\n
$$
\leq \frac{1}{2C_{25}} \int_{0}^{T} \int_{B} w(x) \left| \frac{\partial \delta_{k}(x,t)}{\partial x} \right|^{2} \times F^{+}(\mu) \cap T(\mu)(x,t) dx dt
$$
\n
$$
+ C_{26} \int_{0}^{T} \int_{B} v(x) \delta_{k}^{2}(x,t) \times F^{+}(\mu) \cap T(\mu)(x,t) dx dt
$$
\n
$$
+ C_{26} \mu \left\{ \alpha_{k}^{2} \frac{v(B(x_{0}, \alpha_{k}))}{w(B(x_{0}, \alpha_{k}))} \right\}^{1/2} \left\{ v(F^{+}(\mu) \setminus T(\mu)) \right\}^{1/2}
$$
\n
$$
v(F^{+}(\mu) \setminus T(\mu)) = \iint_{F^{+}(\mu) \setminus T(\mu)} v(x) dx dt.
$$
\n5.12)

where

$$
v(F^+(\mu)\setminus T(\mu))=\iint_{F^+(\mu)\setminus T(\mu)} v(x)\,dxdt.\tag{5.12}
$$

Since $\mu > d_k$, from (5.6) we have

$$
\mu > d_k, \text{ from (5.6) we have}
$$
\n
$$
F^+(\mu) \setminus T(\mu) \subset \left\{ (x, t) \middle| \left| x - x_0 \right| < \alpha_{k-3}, \left| t - t_0 \right| < \alpha_{k-3}^2 \frac{\upsilon(B(x_0, \alpha_{k-3}))}{\upsilon(B(x_0, \alpha_{k-3}))} \right\}
$$
\n
$$
\upsilon(F^+(\mu) \setminus T(\mu)) \le C_{27} \alpha_k^2 \frac{\upsilon^2 B(x_0, \alpha_k)}{\upsilon(B(x_0, \alpha_k))}.
$$

and so

 \cdot

$$
v(F^+(\mu)\setminus T(\mu))\leq C_{27}\,\alpha_k^2\frac{v^2B(x_0,\alpha_k)}{w(B(x_0,\alpha_k))}.\tag{5.13}
$$

Now applying the Gronwall inequality, from *(5.11) - (5.13)* we get (5.7) *I*

6. Proof of the Theorem 3.3

From the integral identity (3.10) for $u_k = u_k(x,t)$ and $u_{k+1} = u_{k+1}(x,t)$ it follows that,

6. Proof of the Theorem 3.3
\nFrom the integral identity (3.10) for
$$
u_k = u_k(x, t)
$$
 and $u_{k+1} = u_{k+1}(x, t)$ it follows that,
\nfor an arbitrary function $\psi \in V_2(Q_K, v, w)$ and for $0 < \tau \leq T$,
\n
$$
\int_{0}^{t} \int_{D_k} \left\{ v(x) \frac{\partial}{\partial t} \delta_{k,h}(x, t) \psi(x, t) + \sum_{i=1}^{n} \left[a_i \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) - a_i \left(x, t, u_{k+1}, \frac{\partial u_{k+1}}{\partial x} \right) \right]_h \frac{\partial \psi}{\partial x_i} - \left[a_0 \left(x, t, u_k, \frac{\partial u_k}{\partial x} \right) - a_0 \left(x, t, u_{k+1}, \frac{\partial u_{k+1}}{\partial x} \right) \right]_h \psi(x, t) \right\} dx dt
$$
\n
$$
= 0.
$$
\n(6.1)

Define the numerical sequences $\{\rho_i\}_{i\in\mathbb{N}}$ and $\{\sigma_i\}_{i\in\mathbb{N}}$ by the equalities

1. I. Skrypnik
\nequences
$$
\{\rho_i\}_{i \in \mathbb{N}}
$$
 and $\{\sigma_i\}_{i \in \mathbb{N}}$ by the equa-
\n
$$
\rho_i = \alpha_{k+4} (1 + \alpha_1 - \alpha_i)
$$
\n
$$
\sigma_i = \alpha_{k+4}^2 \frac{v(B(x_0, \alpha_{k+4}))}{w(B(x_0, \alpha_{k+4}))} (1 + \alpha_1 - \alpha_i).
$$
\nifferentiable functions $\gamma_i = \gamma_i(x)$ $(x \in \mathbb{R}^n)$

Then define infinitely differentiable functions $\gamma_i = \gamma_i(x)$ $(x \in \mathbb{R}^n)$ and $\tilde{\gamma}_i = \tilde{\gamma}_i(t)$ $(t \in \mathbb{R})$ such that the following conditions are fulfilled: *a*) *n* define infinitely differentiable functions $\gamma_i = \gamma_i(x)$ ($x \in \mathbb{R}^n$) and $\tilde{\gamma}_i = \tilde{\gamma}_i(t)$ ($t \in \mathbb{R}^n$) that the following conditions are fulfilled:
 a) $\gamma_i(x) = 1$ on the set $\{x | |x - x_0| \le \rho_i\}$, γ_i

such that the following conditions are fulfilled:
 a) $\gamma_i(x) = 1$ on the set $\{x | |x - x_0| \le \rho_i\},\ \rho_{i+1}\},\ 0 \le \gamma_i(x) \le 1$ and $\left|\frac{\partial \gamma_i}{\partial x}\right| \le C_{28}\alpha_k^{-1}\alpha_i^{-1}.$

b) $\tilde{\gamma}_i(t) = 1$ on the set $\{x \mid |t-t_0| \leq \sigma_i\}, \ \tilde{\gamma}_i(x) = 0$ outside the set $\{t \mid |t-t_0| \leq \sigma_{i+1}\},$ Then define infinitely differentiable fun
such that the following conditions are
a) $\gamma_i(x) = 1$ on the set $\{x | |x - x_0|$
 $\rho_{i+1}\}, \ 0 \leq \gamma_i(x) \leq 1$ and $\left| \frac{\partial \gamma_i}{\partial x} \right| \leq C_{28}$
b) $\tilde{\gamma}_i(t) = 1$ on the set $\{x | |t - t_0|$

Let us put in (6.1) the function $\psi(x, t) = [\delta_{k,h}(x, t)]^r \gamma_i^{s+2} \tilde{\gamma}_i^{s+2}$ with arbitrary positive numbers r and *s*. After integratig by parts in the term containing $\frac{\partial}{\partial t} \delta_{k,h}(x,t)$, passing to the limit as $h \to 0$ and using the inequalities (2.7) and (2.8), we get

1),
$$
0 \le \gamma_i(x) \le 1
$$
 and $\left| \frac{\partial \gamma_i}{\partial x} \right| \le C_{28} \alpha_k^{-1} \alpha_i^{-1}$.
\nb) $\tilde{\gamma}_i(t) = 1$ on the set $\{x | |t - t_0| \le \sigma_i\}$, $\tilde{\gamma}_i(x) = 0$ outside the set $\{t | |t - t_0| \le \sigma_{i+1}\}$,
\nb) $\tilde{\gamma}_i(t) \le 1$ and $\left| \frac{\partial \tilde{\gamma}_i}{\partial t} \right| \le C_{29} \alpha_k^{-2} \alpha^{-1} \frac{w(B(x_0, \alpha_k))}{v(B(x_0, \alpha_k))}$.
\n1 us put in (6.1) the function $\psi(x, t) = [\delta_{k,h}(x, t)]^T \gamma_i^{s+2} \tilde{\gamma}_i^{s+2}$ with arbitrary positive
\nthers r and s. After integrating by parts in the term containing $\frac{\partial}{\partial t} \delta_{k,h}(x, t)$, passing
\nthe limit as $h \to 0$ and using the inequalities (2.7) and (2.8), we get
\n
$$
\int_{D_k} v(x) |\delta_k(x, \tau)| \gamma_i^{s+2}(x) \tilde{\gamma}_i^{s+2}(\tau) dx
$$
\n
$$
+ \int_{0}^{t} \int_{D_k} v(x) |\delta_k(x, t)|^r \left| \frac{\partial \delta_k(x, t)}{\partial x} \right|^2 \tilde{\gamma}_i^{s+2}(x) \tilde{\gamma}_i^{s+2}(t) dx dt
$$
\n
$$
\le C_{30} (r + s + 1)^2 \alpha_i^{-2}
$$
\n
$$
\times \int_{0}^{t} \int_{D_k} \alpha_k^{-2} \left[v(x) \frac{w(B(x_0, \alpha_k))}{v(B(x_0, \alpha_k))} + w(x) \right] |\delta_k(x, t)|^{r+2} \tilde{\gamma}_i^s(x) \tilde{\gamma}_i^s(t) dx dt.
$$
\n(6.2) as well as in Section 4, we obtain with some $h > 1$
\n
$$
\mu^2(i) \le C_{31} \alpha_i^{-\frac{2h}{h-1}} \int_{R(i+1)} \delta_k^2(x, t) \tilde{\gamma}_i^2(x) \tilde{\gamma}_i^2(t) \alpha
$$

From (6.2) , as well as in Section 4, we obtain with some $h > 1$

$$
+ s + 1)^{2} \alpha_{i}^{-2}
$$
\n
$$
\alpha_{k}^{-2} \left[v(x) \frac{w(B(x_{0}, \alpha_{k}))}{v(B(x_{0}, \alpha_{k}))} + w(x) \right] |\delta_{k}(x, t)|^{r+2} \tilde{\gamma}_{i}^{s}(x) \tilde{\gamma}_{i}^{s}(t) dx dt.
$$
\n
$$
= \text{all as in Section 4, we obtain with some } h > 1
$$
\n
$$
\mu^{2}(i) \leq C_{31} \alpha_{i}^{-\frac{2h}{h-1}} \iint_{R(i+1)} \delta_{k}^{2}(x, t) \tilde{\gamma}_{i}^{2}(x) \tilde{\gamma}_{i}^{2}(t) \alpha_{k}^{-2}
$$
\n
$$
\times \left[\frac{w(x)}{w(B(x_{0}, \alpha_{k}))} + \frac{w(B(x_{0}, \alpha_{k}))}{v^{2}(B(x_{0}, \alpha_{k}))} v(x) \right] dx dt
$$
\n
$$
u(i) = \sup \left\{ |\delta_{k}(x, t)| | (x, t) \in R(i) \right\}
$$
\n
$$
(6.3)
$$

where

$$
\mu(i) = \sup \left\{ |\delta_k(x,t)| \Big| (x,t) \in R(i) \right\}
$$

with

$$
R(i) = \left\{ (x, t) \in Q_k \, \middle| \, |x - x_0| \leq \rho_i \text{ and } |t - t_0| \leq \sigma_i \right\}
$$

Let us consider two possibilities:

1) $\mu(i+1) \leq d_k$ 2) $\mu(i + 1) > d_k$. If $\mu(i + 1) \leq d_k$, then from the definition of $\mu(i + 1)$ we have

A Condition to Regularity of a Boundary Point
\n
$$
\mu(i+1) \leq d_k
$$
, then from the definition of $\mu(i+1)$ we have
\n
$$
\sup \left\{ |\delta_k(x,t)| \middle| |x - x_0| < \alpha_{k+4}, |t - t_0| < \alpha_{k+4}^2 \frac{v(B(x_0, \alpha_{k+4}))}{w(B(x_0, \alpha_{k+4}))} \right\} \leq \mu(i+1) \leq d_k
$$

and (3.17) follows from (3.16) for u_k and the analogous estimate for u_{k+1} . If $\mu(i+1)$ *>* d_k , then, as in [12], we can show that $R(i + 1) \subset F(\mu(i + 1))$ and using the inequality (5.7) we have

$$
\left\{ |\delta_k(x,t)| \left| |x - x_0| < \alpha_{k+4}, |t - t_0| < \alpha_{k+4}^2 \frac{v(B(x_0, \alpha_{k+4}))}{w(B(x_0, \alpha_{k+4}))} \right\} \leq \mu(i+1) \leq
$$
\n
$$
\left\{ |\delta_k(x,t)| \left| |x - x_0| < \alpha_{k+4}, |t - t_0| < \alpha_{k+4}^2 \frac{v(B(x_0, \alpha_{k+4}))}{w(B(x_0, \alpha_{k+4}))} \right\} \leq \mu(i+1) \leq
$$
\n
$$
\therefore
$$
\n
$$
\therefore
$$
17) follows from (3.16) for u_k and the analogous estimate for u_{k+1} . If $\mu(i-1)$ en, as in [12], we can show that $R(i+1) \subset F(\mu(i+1))$ and using the inequality
$$
\int \int_{R(i+1)} \delta_k(x,t)^2 \gamma_i^2(x) \tilde{\gamma}_i^2(t) \alpha_k^{-2} \left[v(x) \frac{w(B(x_0, \alpha_k))}{v^2(B(x_0, \alpha_k))} + \frac{w(x)}{v(B(x_0, \alpha_k))} \right] dx dt
$$
\n
$$
= \iint_{R(i+1)} \left| [\delta_k(x,t)]_{(-\mu(i+1),\mu(i+1))} \right|^2 \gamma_i^2(x) \tilde{\gamma}_i^2(t) \chi_{F(\mu(i+1))} \alpha_k^{-2} \times \left\{ v(x) \frac{w(B(x_0, \alpha_k))}{v^2(B(x_0, \alpha_k))} + \frac{w(x)}{v(B(x_0, \alpha_k))} \right\} dx dt
$$
\n
$$
\leq C_{32} \mu(i+1) \frac{\alpha_i^{-\frac{2}{h-1}}}{v(B(x_0, \alpha_k))} \left\{ \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) \right\}^{1/2}
$$
\n
$$
\times \left\{ \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} C_{2,w}(E_k) + \alpha_k^2 \frac{v(B(x_0, \alpha_k))}{w(B(x_0, \alpha_k))} \right\}^{1/2}
$$
\n
$$
\times
$$

Then applying Lemma 2.1, we obtain (3.17) and with this the proof of the theorem is complete I

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