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# A Numerically Rigorous Proof of Curve Veering in an Eigenvalue Problem for Differential Equations

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**Abstract.** We consider parameter-dependent self-adjoint eigenvalue problems for differential equations. Frequently the eigenvalue curves show the interesting phenomenon of curve veering. We propose a numerically rigorous procedure for proving this phenomenon in concrete situations.

**Keywords:** *Parameter-dependent eigenvalue problems, upper and lower bounds to eigenvalues, curve veering, interval arithmetic*

**AMS subject classification:** 49 R 05, 65 L 15, 65 L 60, 65 G 10

## 1. Introduction

Self-adjoint eigenvalue problems for ordinary or partial differential equations are very important in the sciences and in engineering. Frequently these problems depend on a system parameter, and one can observe the surprising phenomenon of curve veering (see Fig. 1). The curve veering phenomenon was studied by von Neumann and Wigner [25] as early as 1929 and can be seen for quite different problems, for example for vibrations of plates dependent on plate geometry [6, 19], for eigenfrequencies of a constant curvature ring dependent on eccentricity [22], for eigenfrequencies of a rotating circular string dependent on rotating speed or for the prediction of molecular geometry [15: pp. 265 and 310]. For all these problems we can ask the key question: are veerings in discretized (approximate) models representative for veerings in continuous models?

So far there have been only generic statements on curve veering, and the proof of this phenomenon for a concrete situation has been possible only in special cases. We will propose a procedure that allows the proof of curve veering in a concrete situation (for the continuous model) without requiring special properties (for example, monotonicity) of the eigenvalue curves. The procedure will be explained by means of an example.

We consider the natural bending vibrations of a free-standing blade of a turbine disc. The mathematical model we use to describe this problem [12] results in a parameter-dependent eigenvalue problem (the real parameter, $\Omega$, being the angular velocity) for a system of ordinary fourth-order differential equations.

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In this paper we will show how *verified* bounds of the form

$$p(\Omega) - \varepsilon \leq \lambda(\Omega) \leq p(\Omega) + \varepsilon \qquad \text{for all } \Omega \in [0, 30] \tag{1.1}$$

can be computed for the lowest eigenvalue curves $\lambda(\Omega)$. Here, $\varepsilon$ is a small positive number, and $p$ is an explicitly known function. The eigenvalue curves show the interesting phenomenon of curve veering (see Fig. 1); by means of the calculated bounds we can prove that the lowest eigenvalues curves do not *cross* each other.

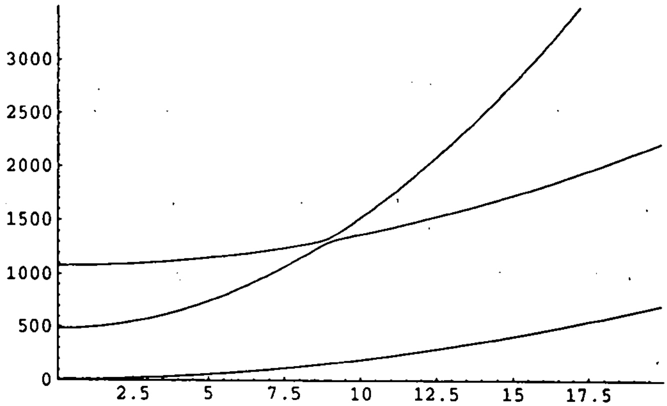


Figure 1: The lowest eigenvalues as function of the angular velocity $\Omega$

"Verified" means that rounding errors are rigorously controlled by the use of interval arithmetic. An advantage of our method is that it can be applied to eigenvalue problems for partial differential equations as well.

## 2. The eigenvalue problem

We consider an eigenvalue problem that results from the theoretical treatment of the vibrational behavior of turbine blades, an important subject in turbomachinery (see Irretier [12 - 14]). A considerable amount of work in this field deals with the computation of the eigenfrequencies of the blades. Our model problem (Irretier) takes into account all essential parameters such as the stagger angle $\alpha$ at the blade root ($x = 0$), the angle of the twist $\gamma x$ (the principal axes of each cross section are called $\eta$ and $\zeta$, they are related to $y$ and $z$ by the function of the twisting angle $\gamma x$; $x$ is the blade direction), the blade cross section $\Phi(x)$ and the rotation of the turbine with the angular velocity $\Omega$ (see Fig. 2).

The mathematical model results in the following eigenvalue problem for ordinary fourth-order differential equations:

$$\begin{aligned} (\Phi_z v'' + \Phi_{yz} w'')'' - \Omega^2(\Theta v')' - \Omega^2(v \cos\alpha - w \sin\alpha)\cos\alpha &= \lambda v \\ (\Phi_{yz} v'' + \Phi_y w'')'' - \Omega^2(\Theta w')' + \Omega^2(v \cos\alpha - w \sin\alpha)\sin\alpha &= \lambda w \end{aligned} \tag{2.1}$$

and the boundary conditions

$$\begin{aligned} v(0) = v'(0) = v''(1) = v'''(1) = 0 \\ w(0) = w'(0) = w''(1) = w'''(1) = 0 \end{aligned} \tag{2.2}$$

where

$$\Theta = \Theta(x) = \int_x^1 \Phi(\xi)(\epsilon + \xi)\, d\xi \tag{2.3}$$

$$\Phi_y = \Phi_y(x) = \Phi_\eta \cos^2(\gamma x) + \Phi_\zeta \sin^2(\gamma x) = -\tfrac{1}{2}(\Phi_\zeta - \Phi_\eta)\cos(2\gamma x) + \tfrac{1}{2}(\Phi_\zeta + \Phi_\eta) \tag{2.4}$$

$$\Phi_z = \Phi_z(x) = \Phi_\eta \sin^2(\gamma x) + \Phi_\zeta \cos^2(\gamma x) = \tfrac{1}{2}(\Phi_\zeta - \Phi_\eta)\cos(2\gamma x) + \tfrac{1}{2}(\Phi_\zeta + \Phi_\eta) \tag{2.5}$$

$$\Phi_{yz} = \Phi_{yz}(x) = (\Phi_\zeta - \Phi_\eta)\sin(\gamma x)\, \cos(\gamma x) = \tfrac{1}{2}(\Phi_\zeta - \Phi_\eta)\sin(2\gamma x). \tag{2.6}$$

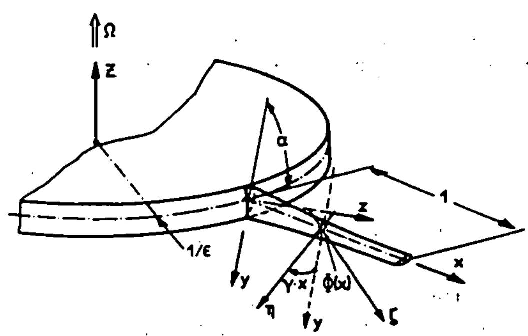


Figure 2: Notations

The (dimensionless) parameters have the following meaning:

| | |
|---|---|
| $x$ | Cartesian coordinate of the blade ($0 \leq x \leq 1$) |
| $v = v(x)$ | $y$ component of the eigenfunction (displacement) |
| $w = w(x)$ | $z$ component of the eigenfunction (displacement) |
| $\alpha$ | stagger angle at the blade root |
| $\gamma x$ | angle of the twist |
| $\Omega$ | angular velocity |
| $\Phi_y, \Phi_z, \Phi_{yz}$ | squares of the radii of gyration |
| $\Phi_\eta, \Phi_\zeta$ | squares of the local radii of gyration |
| $\Phi = \Phi(x)$ | blade of cross section |
| $\Omega^2\Theta(x)$ | normal force in the blade due to rotation |
| $\epsilon$ | ratio to the disc radius/blade length |
| $\lambda$ | eigenvalue (square of the eigenfrequency). |

In this paper we will restrict ourselves to a special case suggested by Prof. Irretier (Gesamthochschule Kassel):

- $\Phi = 1$ (constant blade cross section)
- $\Phi_\zeta = 87.1$ and $\Phi_\eta = 1$
- $\alpha = \frac{\pi}{2}$ and $\epsilon = 0.457$
- $0 \le \Omega \le 30$.

This means that we have to deal with a parameter-dependent eigenvalue problem (depending on the real parameter $\Omega$), which will be studied for some different values of $\gamma$, $0 < \gamma \le \frac{\pi}{12}$. Equations (2.1) then read as

$$\begin{aligned} (\Phi_z v'' + \Phi_{yz} w'')'' - \Omega^2(\Theta v')' &= \lambda v \\ (\Phi_{yz} v'' + \Phi_y w'')'' - \Omega^2(\Theta w')' - \Omega^2 w &= \lambda w. \end{aligned} \tag{2.7}$$

In our paper we will give numerical results and figures for $\gamma = \frac{\pi}{180}$. For $\gamma = 0$, equations (2.7) are decoupled and the eigenvalue curves $\lambda_2(\Omega)$ and $\lambda_3(\Omega)$ cross each other near $\Omega = 9$.

## 3. Inclusion method

Let $(H, (\cdot|\cdot))$ be an infinite dimensional Hilbert space with the inner product $(\cdot|\cdot)$ and the norm $\|\cdot\|$. Suppose that $V$ is a dense subspace of $H$ and that we have the inner product $[\cdot|\cdot]$ in $V$ such that $(V, [\cdot|\cdot])$ is a Hilbert space (the norm in V is denoted by $|\cdot|$). The embedding $V \hookrightarrow H$ is assumed to be compact.

We consider the right-definite eigenvalue problem

$$\text{Find } \lambda \in \mathbb{R} \text{ and } 0 \ne \varphi \in V \text{ such that } [\varphi|v] = \lambda(\varphi|v) \text{ for all } v \in V. \tag{3.1}$$

Problem (3.1) has a countable spectrum of eigenvalues, and the eigenvalues can be ordered by magnitude:

$$0 < \lambda_1 \le \lambda_2 \le \dots \qquad \text{and} \qquad \lim_{j\to\infty} \lambda_j = \infty. \tag{3.2}$$

The Rayleigh-Ritz procedure for calculating upper bounds is a discretization of the Poincaré principle

$$\lambda_j = \min_{\substack{E \subset V \\ \dim E = j}} \max_{0 \ne u \in E} \frac{[u|u]}{(u|u)} \qquad (j \in \mathbb{N}). \tag{3.3}$$

If we choose the linearly independent trial functions

$$u_1, \dots, u_n \in V \qquad (n \in \mathbb{N}), \tag{3.4}$$

we can reduce (3.3) to an $n$-dimensional subspace $V_n$ (the span of the chosen functions $u_1, \dots, u_n$) and obtain the values

$$\Lambda_1^{[n]} \le \Lambda_2^{[n]} \le \dots \le \Lambda_n^{[n]} \tag{3.5}$$

which are upper bounds to the following $\lambda_j$:

$$\lambda_j \leq \Lambda_j^{[n]} \qquad (j = 1, \dots, n). \tag{3.6}$$

We call $\Lambda_j^{[n]}$ a ***Rayleigh-Ritz bound*** for $\lambda_j$. If we form the real $(n \times n)$-matrices

$$A_0 = \Big((u_i|u_k)\Big)_{i,k=1,\dots,n} \qquad \text{and} \qquad A_1 = \Big([u_i|u_k]\Big)_{i,k=1,\dots,n}, \tag{3.7}$$

the Rayleigh-Ritz bounds are the eigenvalues of the matrix eigenvalue problem

$$A_1 x = \Lambda^{[n]} A_0 \qquad \big((\Lambda^{[n]}, x) \in \mathbb{R} \times \mathbb{R}^n\big). \tag{3.8}$$

The Rayleigh-Ritz bounds are monotonically decreasing in $n \in \mathbb{N}$.

The Lehmann-Goerisch procedure (see [16 - 18] and [5, 8, 10]) for calculating lower bounds can be understood as the discretization of a variational principle for characterizing the eigenvalues as well. This principle and a proof of the method is due to Zimmermann and Mertins [27].

Let $\rho \in \mathbb{R}$ be a spectral parameter such that for an $N \in \mathbb{N}$ the inequality

$$\lambda_N < \rho < \lambda_{N+1} \tag{3.9}$$

holds true. We express the first $N$ eigenvalues in the form

$$\lambda_{N+1-i} = \rho + \frac{1}{\sigma_i} \qquad (i = 1, \dots, N)$$

(assuming $\sigma_i < 0$). For $u \in V$, $w_u \in H$ denotes the uniquely determined solution of the equation

$$[u|v] = (w_u|v) \qquad \text{for all } v \in V,$$

the following $\sigma_i$ therefore are characterized by

$$\sigma_i = \inf_{\substack{E \subset V \\ \dim E = i}} \max_{0 \neq u \in E} \frac{[u|u] - \rho(u|u)}{(w_u|w_u) - 2\rho[u|u] + \rho^2(u,u)} \qquad (i = 1, \dots, N). \tag{3.10}$$

A negative upper bound for $\sigma_i$ results in a lower bound for $\lambda_{N+1-i}$. In order to discretize (3.10), we determine $w_1, \dots, w_n \in H$ such that

$$[u_i|v] = (w_i|v) \qquad \text{for all } \ v \in V, \tag{3.11}$$

then we define the matrix

$$A_2 = \Big((w_i|w_k)\Big)_{i,k=1,\dots,n} \tag{3.12}$$

and solve the matrix eigenvalue problem

$$\Big(A_1 - \rho A_0\Big)x = \tau\Big(A_2 - 2\rho A_1 + \rho^2 A_0\Big)x \qquad \big((\tau, x) \in \mathbb{R} \times \mathbb{R}^n\big). \tag{3.13}$$

If for $n \in \mathbb{N}$ the condition $\Lambda_N^{[n]} < \rho$ is fulfilled, then (3.13) has exactly $N$ negative eigenvalues

$$\tau_1 \leq \tau_2 \leq \ \dots \ \leq \tau_N < 0 \leq \ \dots \ \leq \tau_n.$$

These $\tau_i$ are upper bounds for our $\sigma_i$ ($\sigma_i \leq \tau_i$ for $i = 1, \dots, N$). We obtain the lower bounds

$$\Lambda_j^{\rho[n]} := \rho + \frac{1}{\tau_{N+1-j}} \leq \lambda_j \qquad (j = 1, \dots, N). \tag{3.14}$$

This discretization (3.13), (3.14) is the Lehmann-Goerisch procedure. We call $\Lambda_j^{\rho[n]}$ a *Lehmann-Goerisch bound* for $\lambda_j$.

## 4. Specification for our problem

In this section we define the function spaces and trial functions for our inclusion method and prove that the assumptions of the previous section are fulfilled.

Let $I = (0,1)$ be a real interval. As usual in the theory of Sobolev spaces, we use the notation $(L_2(I), (\cdot|\cdot)_\circ)$ and $(H^m(I), (\cdot|\cdot)_m)$ $(m \geq 1)$ for the Hilbert spaces and

$$\|u\|_\circ = \left(\int_0^1 u^2 dx\right)^{1/2} \qquad (u \in L_2(I))$$

$$\|u\|_m = \left(\sum_{0 \leq p \leq m} \|D^p u\|_\circ^2\right)^{1/2} \qquad (u \in H^m(I))$$

$$|||u|||_m = \|D^m u\|_\circ \qquad (u \in H^m(I))$$

for the norms and semi-norms, respectively. We define the quantities related to problem (2.7):

$$H = (L_2(I))^2, \tag{4.1}$$

the inner product in $H$:

$$(f|g) = \int_0^1 f_1 g_1\, dx + \int_0^1 f_2 g_2\, dx \qquad \text{for } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H \tag{4.2}$$

and

$$V = \left(\left\{ f \in H^2(I) \middle| f(0) = 0 \text{ and } f'(0) = 0 \right\}\right)^2 \tag{4.3}$$

$$\begin{aligned} [f|g]^*_\Omega = &\int_0^1 \left(\Phi_z f_1'' g_1'' + \Phi_{yz} f_2'' g_1'' + \Omega^2 \Theta f_1' g_1'\right) dx \\ &+ \int_0^1 \left(\Phi_{yz} f_1'' g_2'' + \Phi_y f_2'' g_2'' + \Omega^2 \Theta f_2' g_2' - \Omega^2 f_2 g_2\right) dx \\ &\text{for } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in V. \end{aligned} \tag{4.4}$$

$V$ is a closed subspace of the Hilbert space $(H^2(I))^2$ (with respect to the product topology). In order to have a bilinear form $[\cdot|\cdot]_\Omega$ which is monotonous in $\Omega$ we define

$$\begin{aligned} [f|g]_\Omega = &\int_0^1 \left(\Phi_z f_1'' g_1'' + \Phi_{yz} f_2'' g_1'' + \Omega^2 \Theta f_1' g_1' + \Omega^2 f_1 g_1\right) dx \\ &+ \int_0^1 \left(\Phi_{yz} f_1'' g_2'' + \Phi_y f_2'' g_2'' + \Omega^2 \Theta f_2' g_2'\right) dx \\ &\text{for } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in V. \end{aligned} \tag{4.5}$$

The eigenvalues of the problems

$$\begin{array}{l}\text{Find } \lambda^*(\Omega) \in \mathbb{R} \text{ and } 0 \neq \varphi^* \in V \text{ such that} \\ [\varphi^*|v]^*_\Omega = \lambda^*(\Omega)[\varphi^*|v] \text{ for all } v \in V\end{array} \tag{4.6}$$

and

$$\begin{array}{l}\text{Find } \lambda(\Omega) \in \mathbb{R} \text{ and } 0 \neq \varphi \in V \text{ such that} \\ [\varphi|v]_\Omega = \lambda(\Omega)(\varphi|v) \text{ for all } v \in V\end{array} \tag{4.7}$$

are related by $\lambda^*(\Omega) + \Omega^2 = \lambda(\Omega)$, hence [1] it is sufficient to know either $\lambda(\Omega)$ or $\lambda^*(\Omega)$.

For $f \in V$,

$$| f |_\Omega = \sqrt{[f|f]_\Omega}$$

denotes the norm [2] generated by $[\cdot|\cdot]_\Omega$. For $f = (f_1, f_2)^\top \in H^2(I) \times H^2(I)$ let

$$||f|| = \left(||f_1||_2^2 + ||f_2||_2^2\right)^{1/2} \qquad \text{and} \qquad |||f||| = \left(|||f_1|||_2^2 + |||f_2|||_2^2\right)^{1/2}.$$

These last two norms are equivalent in $V$.

Now we can formulate

**Theorem 4.1.** *$V$ is a dense subspace of $\left(H, (\cdot|\cdot)\right)$. For $0 \le \Omega \le 30$ and for $\gamma \in \mathbb{R}$, the embedding $\left(V, [\cdot|\cdot]_\Omega\right) \hookrightarrow \left(H, (\cdot|\cdot)\right)$ is compact.*

**Proof.** Since

$$C_0^\infty(I) \subseteq \left\{ f \in H^2(I) : f(0) = 0 \text{ and } f'(0) = 0 \right\} \subseteq L_2(I)$$

and $C_0^\infty(I)$ is a dense subspace of $L_2(I)$, $V$ is a dense subspace of $\left(H, (\cdot|\cdot)\right)$.

For all $r, s \in \mathbb{R}$ and $0 < \delta \in \mathbb{R}$ we have

$$-\delta r^2 - \frac{1}{\delta} s^2 \le 2rs \le \delta r^2 + \frac{1}{\delta} s^2. \tag{4.8}$$

If we use the notations

$$\begin{aligned} c_1 &= \max\left\{ \Phi_z(x) \Big| \; x \in [0,1] \right\} \\ c_2 &= \max\left\{ \Phi_y(x) \Big| \; x \in [0,1] \right\} \\ c_3 &= \max\left\{ \Phi_{yz}(x) \Big| \; x \in [0,1] \right\} \\ c_4 &= \max\left\{ \Omega^2\, \Theta(x) \Big| \; \Omega \in [0,30], x \in [0,1] \right\} \end{aligned}$$

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[1] Figure 1 shows the eigenvalue curves of problem (4.7) for $\gamma = \frac{\pi}{180}$.

[2] See Theorem 4.1 for a proof of the positive definiteness of $[\cdot|\cdot]_\Omega$.

and the right inequality (4.8) with $\delta = 1$ there follows for all $u \in V$

$$\mid u \mid_{\Omega}^2 \leq \int_0^1 \Big( (\Phi_z + \Phi_{yz}) u_1''^2 + (\Phi_y + \Phi_{yz}) u_2''^2 + \Omega^2 \, \Theta (u_1'^2 + u_2'^2) + \Omega^2 (u_1^2 + u_2^2) \Big) dx$$
$$\leq \max \Big\{ c_1 + c_3, c_2 + c_3, c_4, 900 \Big\} ||u||^2.$$

In order to prove the $V$-ellipticity of $[\cdot|\cdot]_\Omega$, we define

$$c = \frac{1}{2}(\Phi_\zeta - \Phi_\eta) = 43.055 \qquad \text{and} \qquad d = \frac{1}{2}(\Phi_\eta + \Phi_\zeta) = 44.055.$$

For any $u \in V$ and for $0 \leq \Omega \leq 30$ we obtain from (4.5)

$$\mid u \mid_{\Omega}^2 \geq \mid u \mid_0^2 = \int_0^1 u''^{\mathsf{T}} \begin{pmatrix} \Phi_z & \Phi_{yz} \\ \Phi_{yz} & \Phi_y \end{pmatrix} u'' dx \geq \int_0^1 \lambda_{\min}(Q) u''^{\mathsf{T}} u'' dx$$

where $Q = \begin{pmatrix} \Phi_z & \Phi_{yz} \\ \Phi_{yz} & \Phi_y \end{pmatrix}$. The characteristic polynomial of the matrix $Q$ is

$$P(\lambda) = \lambda^2 - (\Phi_z + \Phi_y)\,\lambda + \Phi_z \Phi_y - \Phi_{yz}^2$$

and therefore

$$\begin{aligned} \lambda_{\min}(Q) &= \frac{1}{2}(\Phi_z + \Phi_y) - \sqrt{\tfrac{1}{4}(\Phi_z + \Phi_y)^2 + \Phi_{yz}^2} \\ &= d - c\sqrt{\cos^2(2\gamma x) + \sin^2(2\gamma x)} \\ &= d - c \\ &= 1. \end{aligned}$$

This yields $\mid u \mid_{\Omega}^2 \geq \int_0^1 u''^{\mathsf{T}} u'' dx = |||u|||^2$. Hence the norms $\mid \cdot \mid_{\Omega}$, $||\cdot||$ and $|||\cdot|||$ are equivalent in $V$. Since the embedding $\big(H^2(I), (\cdot|\cdot)_2\big) \hookrightarrow \big(L_2(I), (\cdot|\cdot)_\circ\big)$ is compact, the embedding $\big(V, [\cdot|\cdot]_\Omega\big) \hookrightarrow \big(H, (\cdot|\cdot)\big)$ is compact ∎

In order to determine a spectral parameter $\rho$ (see (3.9)), we mention that the eigenvalues of our problem (4.7) are monotonous increasing functions in $\Omega$. Lower bounds for $\lambda(0)$ will be computed; we use the same notations as in the proof of Theorem 4.1.

We define an orthogonal, symmetric matrix

$$U = \begin{pmatrix} \cos\frac{\gamma}{2} & \sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & -\cos\frac{\gamma}{2} \end{pmatrix}.$$

Now we have

$$U \begin{pmatrix} \cos(2\gamma x) & \sin(2\gamma x) \\ \sin(2\gamma x) & -\cos(2\gamma x) \end{pmatrix} U^{\mathsf{T}} = \begin{pmatrix} \cos(2\gamma x - \gamma) & \sin(2\gamma x - \gamma) \\ \sin(2\gamma x - \gamma) & -\cos(2\gamma x - \gamma) \end{pmatrix}.$$

For any $u = (u_1, u_2)^\top \in V$ we have $v = (v_1, v_2)^\top := U u \in V$ and $u = Uv$. From the left inequality (4.8) we obtain

$$
\begin{aligned}
|u|_\Omega^2 &\geq |u|_0^2 \\
&= \int_0^1 \left( cu''^\top U^\top U \begin{pmatrix} \cos(2\gamma x) & -\sin(2\gamma x) \\ -\sin(2\gamma x) & -\cos(2\gamma x) \end{pmatrix} U^\top U u'' + du''^\top U^\top U u'' \right) dx \\
&= \int_0^1 \left( cv''^\top \begin{pmatrix} \cos(2\gamma x - \gamma) & -\sin(2\gamma x - \gamma) \\ -\sin(2\gamma x - \gamma) & -\cos(2\gamma x - \gamma) \end{pmatrix} v'' + dv''^\top v'' \right) dx \qquad (4.9) \\
&\geq \int_0^1 \Bigg( v_1''^2 \left( c\left( \cos(2\gamma x - \gamma) - \frac{1}{\delta} |\sin(2\gamma x - \gamma)| \right) + d \right) \\
&\qquad + v_2''^2 \left( c\left( -\cos(2\gamma x - \gamma) - \delta |\sin(2\gamma x - \gamma)| \right) + d \right) \Bigg) dx.
\end{aligned}
$$

We will now discuss the functions[3] $h_i : [0,1] \ni x \to h_i(x) \in \mathbb{R}$,

$$
\begin{aligned}
h_1(x) &= c\left( -\cos(2\gamma x - \gamma) - \delta |\sin(2\gamma x - \gamma)| \right) + d \\
h_2(x) &= c\left( \cos(2\gamma x - \gamma) - \frac{1}{\delta} |\sin(2\gamma x - \gamma)| \right) + d.
\end{aligned} \qquad (4.10)
$$

Let $b \in \mathbb{R}$, $1 \leq b < c + d$. We define

$$\delta_b = \frac{\sin\gamma}{\cos\gamma + \frac{d-b}{c}}.$$

For $\delta = \delta_b$, we obtain

$$h_2(1) = b \quad \text{and} \quad h_2(x) \geq b \qquad \text{for all } x \in [0,1] \text{ and } 0 < \gamma \leq \frac{\pi}{12}.$$

Furthermore let

$$x_m = \begin{cases} \frac{1}{2\gamma} \arctan \delta_b + \frac{1}{2} & \text{if } b \in [1, d) \\ 1 & \text{if } b \in [d, c+d) \end{cases}.$$

Then we have

$$h_1(x) \geq a := h_1(x_m) \qquad \text{for all } x \in [0,1] \text{ and } 0 < \gamma \leq \frac{\pi}{12}.$$

If $a > 0$, the choice of $b$ was reasonable and the eigenvalues $\mu$ of the problem

Find $\mu \in \mathbb{R}$ and $0 \neq \varphi = (\varphi_1, \varphi_2)^\top \in V$ such that

$$\int_0^1 \left( a\varphi_1'' v_1'' + b\varphi_2'' v_2'' \right) dx = \mu \int_0^1 \left( \varphi_1 v_1 + \varphi_2 v_2 \right) dx \quad \forall\, v = (v_1, v_2)^\top \in V \qquad (4.11)$$

[3] Without introducing the matrix $U$ we would obtain similar functions $h_1$ and $h_2$, but the lower bounds are worse.

yield lower bounds for the eigenvalues of (4.7). The eigenvalues of (4.11) can be computed from the solutions of the following two linear problems with constant coefficients:

$$a\varphi_1^{(IV)} = \mu^{(1)}\varphi_1 \quad \text{in } [0,1], \qquad \varphi_1(0) = \varphi_1'(0) = \varphi_1''(1) = \varphi_1'''(1) = 0 \tag{4.12}$$

$$b\varphi_2^{(IV)} = \mu^{(2)}\varphi_2 \quad \text{in } [0,1], \qquad \varphi_2(0) = \varphi_2'(0) = \varphi_2''(1) = \varphi_2'''(1) = 0. \tag{4.13}$$

If $\tau_i \in \mathbb{R}$, $0 < \tau_i < \tau_{i+1}$ for $i \in \mathbb{N}$, is a solution of

$$\cos\tau_i \cosh\tau_i + 1 = 0,$$

then the eigenvalues of (4.12) are $\mu_i^{(1)} = a\tau_i^4$ $(i \in \mathbb{N})$. The corresponding eigenfunctions are

$$\varphi_{1,i}(x) = (\cos\tau_i + \cosh\tau_i)(\sin\tau_i x - \sinh\tau_i x) - (\sin\tau_i + \sinh\tau_i)(\cos\tau_i x - \cosh\tau_i x).$$

Next we will explain how to construct the trial functions $u_i$. We consider the polynomials $\tilde{p}_i : [0,1] \to \mathbb{R}$,

$$\begin{aligned}
\tilde{p}_1(x) &= x^2\left(6 - 4x + x^2\right)\\
\tilde{p}_2(x) &= x^3\left(10 - 10x + 3x^2\right)\\
\tilde{p}_i(x) &= (1-x)^4 x^{i-1} \cdot \quad (i \ge 3)
\end{aligned}$$

which satisfy the boundary conditions (2.2). To avoid the well-known numerical problems with ill-conditioned matrices, we construct an orthogonal basis from the polynomials $\tilde{p}_i$ (orthogonal with respect to the $L_2$ inner product $(\cdot|\cdot)_0$) using the Gram-Schmidt process and the computer algebra program *Mathematica* (see [6, 26]). Besides the rounding error-free calculation of the functions $p_i$, we have the advantage that *Mathematica* can produce a C or C++ code for our polynomials. (In C++ a polynomial arithmetic combined with interval arithmetic can be used to compute the inner products without any analytical calculation.) We obtain[4)]

$$\begin{aligned}
p_1(x) &= \frac{x^2}{3}\left(6 - 4x + x^2\right)\\
p_2(x) &= \frac{x^2}{19}\left(-326 + 824x - 661x^2 + 182x^3\right)\\
p_3(x) &= \frac{x^2}{595}\left(37490 - 181120x + 305815x^2 - 218966x^3 + 57376x^4\right)\\
p_4(x) &= \frac{x^2}{17335}\Big(-2548170 + 19398020x - 54146415x^2 + 70839756x^3\\
&\qquad - 44146336x^4 + 10620480x^5\Big)\\
p_5(x) &= \frac{x^2}{143155}\Big(40512210 - 437785780x + 1790279235x^2 - 3625862604x^3\\
&\qquad + 3896636744x^4 - 2131724400x^5 + 468087750x^6\Big)
\end{aligned}$$

4) The polynomials fulfill the equation $p_i(1) = 1$.

$$p_6(x) = \frac{x^2}{8285}\Big( - 4034766 + 58114976\,x - 323567649\,x^2 + 923419434\,x^3 - 1482348280\,x^4 + 1354376928\,x^5 - 658061874\,x^6 + 132109516\,x^7\Big)$$

$$\vdots$$

Now we choose $n_1, n_2 \in \mathbb{N}$, set $n = n_1 + n_2$ and define

$$u_i = \begin{cases} \begin{pmatrix} p_i \\ 0 \end{pmatrix} & \text{for } i = 1, \ldots, n_1 \\ \begin{pmatrix} 0 \\ p_{i-n_1} \end{pmatrix} & \text{for } i = n_1 + 1, \ldots, n_1 + n_2 = n \end{cases}$$

as trial functions. For $v = (v_1, v_2)^\top \in C^4[0,1] \times C^4[0,1]$ we consider the differential operator

$$M\,v = \begin{cases} (\Phi_z v_1'' + \Phi_{yz} v_2'')'' - \Omega^2(\Theta v_1')' + \Omega^2 v_1 \\ (\Phi_{yz} v_1'' + \Phi_y v_2'')'' - \Omega^2(\Theta v_2')' \end{cases} \tag{4.14}$$

With the functions $w_i$ defined by

$$w_i = M u_i \qquad \text{for} \quad i = 1, \ldots, n$$

the equation $[u_i|v]_\Omega = (w_i|v)$ for all $v \in V$ is fulfilled. Now we can compute the parameter-dependent matrices

$$A_0(\Omega) = \Big((u_i|u_k)\Big)_{i,k=1,\ldots,n}$$

$$A_1(\Omega) = \Big([u_i|u_k]_\Omega\Big)_{i,k=1,\ldots,n}$$

$$A_2(\Omega) = \Big((w_i|w_k)\Big)_{i,k=1,\ldots,n}$$

and establish the parameter-dependent matrix eigenvalue problems for calculating upper Rayleigh-Ritz and lower Lehmann-Goerisch bounds.

## 5. Generalized temple quotients

In this section, we will consider the general matrix eigenvalue problem

$$A\,x = \Lambda\,B\,x\,, \tag{5.1}$$

for real $(n \times n)$-matrices $A$ and $B$, $A = A^\top$, $B = B^\top$, $B$ positive definite. Equation (5.1) has eigenpairs $(\Lambda_i, x_i) \in \mathbb{R} \times \mathbb{R}^n$ $(i = 1, \ldots, n)$. For $u, v \in \mathbb{R}^n$ we define the following inner products and bilinear form:

$$\{u|v\}_M = u^\top v \tag{5.2}$$

$$(u|v)_M = u^\top B v \tag{5.3}$$

$$[u|v]_M = u^\top A v = u^\top B\,B^{-1} A v = (u|B^{-1} A v)_M\,. \tag{5.4}$$

The eigenvectors are assumed to be orthogonal: $(x_i|x_k)_M = \delta_{i,k}$ $(i,k=1,\ldots,n)$. Then we have for all $u \in \mathbb{R}^n$

$$(u|u)_M = \sum_{i=1}^{m}(u|x_i)_M^2 \quad \text{and} \quad [u|u]_M = \sum_{i=1}^{m}\Lambda_i(u|x_i)_M^2 = (u|B^{-1}Au)_M. \tag{5.5}$$

For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we define

$$p(\Lambda) = (\alpha - \Lambda)(\beta - \Lambda) \quad \text{and} \quad P_u(\alpha,\beta) = \big((B^{-1}A-\alpha)u\big|(B^{-1}A-\beta)u\big)_M.$$

Now (5.5) yields

$$\begin{aligned}\sum_{i=1}^{n}p(\Lambda_i)(u|x_i)_M^2 &= \alpha\beta\sum_{i=1}^{n}(u|x_i)_M^2 - (\alpha+\beta)\sum_{i=1}^{n}\Lambda_i(u|x_i)_M^2 + \sum_{i=1}^{n}\Lambda_i^2(u|x_i)_M^2\\ &= \alpha\beta(u|u)_M - (\alpha+\beta)(B^{-1}Au|u)_M + (B^{-1}Au|B^{-1}Au)_M\\ &= P_u(\alpha,\beta).\end{aligned} \tag{5.6}$$

The next two theorems in this section are similar to those in [20] (see also [7]). For reasons of simplicity, we will provide only the results for matrices; the theorems can be proved for the more general case of compact self-adjoint operators (see [11]) as well.

**Theorem 5.1.** *Let $\alpha$, $\beta \in \mathbb{R}$ with $\alpha < \beta$. The following statements are equivalent:*

**a)** *The interval $[\alpha,\beta)$ contains at least one eigenvalue of $Ax = \Lambda Bx$.*

**b)** *There exists a vector $0 \neq u \in \mathbb{R}^n$ such that*

 **(i)** $(B^{-1}Au - \alpha u|B^{-1}Au - \beta u)_M \leq 0$

 **(ii)** $(B^{-1}Au - \beta u|u)_M \neq 0$.

**Proof.** We will show a) $\Rightarrow$ b). Let us assume $\Lambda_j \in [\alpha,\beta)$. From this there follows

$$P_{x_j}(\alpha,\beta) = \sum_{i=1}^{n}p(\Lambda_i)(x_j|x_i)_M^2 = p(\Lambda_j) = (\alpha-\Lambda_j)(\beta-\Lambda_j) \leq 0$$

and $(B^{-1}Ax_j - \beta x_j|x_j)_M = \Lambda_j - \beta \neq 0$.

To prove that b) $\Rightarrow$ a), we will assume that there is no eigenvalue of $Ax = \Lambda Bx$ in $[\alpha,\beta)$. We have

$$\big(B^{-1}Au - \alpha u\big|B^{-1}Au - \beta u\big)_M = P_u(\alpha,\beta) = \sum_{i=1}^{n}p(\Lambda_j)(u|x_i)_M^2 \leq 0. \tag{5.7}$$

Let $J = \{j \in \{1,\ldots,n\}|\ \Lambda_j = \beta\}$. Then $p(\Lambda_j) = 0$ for $j \in J$ and $p(\Lambda_j) > 0$ for $j \notin J$. For $j \notin J$, (5.7) implies that $(u|x_i)_M = 0$, hence $J = \emptyset$ cannot hold true, since $(u|x_i)_M = 0$ for $i = 1,\ldots,n$ contradicts $u \neq 0$. On the other hand,

$$(B^{-1}Au|u)_M = \sum_{i=1}^{m}\Lambda_i(u|x_i)_M^2 = \beta\sum_{i\in J}(u|x_i)_M^2 = \beta(u|u)_M,$$

that is, $(B^{-1}Au - \beta u|u)_M = 0$, a contradiction to (ii) ∎

**Remark 5.2.** In Theorem 5.1/a), we may choose the interval $(\alpha,\beta]$ instead of the interval $[\alpha,\beta)$. Then the condition b)/(ii) has to be replaced by $(B^{-1}Au-\alpha u|u)_M \neq 0$.

Now we will give a proof of Temple's inclusion theorem.

**Theorem 5.3.** *Let $\rho \in \mathbb{R}$, $0 \neq u \in \mathbb{R}^n$ and $v = B^{-1}Au$. We define the Schwarz constants*

$$a_{0,A,B} = (u|u)_M \tag{5.8}$$

$$a_{1,A,B} = [u|u]_M = (v|u)_M \tag{5.9}$$

$$a_{2,A,B} = (v|v)_M = [v|u]_M. \tag{5.10}$$

*We assume $a_{1,A,B} - \rho\, a_{0,A,B} \neq 0$. For $\rho \neq \pm\infty$ the Temple quotient is given by*

$$\tau_{A,B}(\rho) = \frac{a_{2,A,B} - \rho\, a_{1,A,B}}{a_{1,A,B} - \rho\, a_{0,A,B}}, \tag{5.11}$$

*or else by*

$$\tau_{A,B}(\pm\infty) = \frac{a_{1,A,B}}{a_{0,A,B}}. \tag{5.12}$$

*With these assumptions*

$$\left.\begin{array}{r} \rho < \tau_{A,B}(\rho) \\ \tau_{A,B}(\rho) < \rho \end{array}\right\} \text{ implies that the interval } \left\{\begin{array}{l} (\rho, \tau_{A,B}(\rho)] \\ [\tau_{A,B}(\rho), \rho) \end{array}\right.$$

*contains at least one eigenvalue of the eigenvalue problem $Ax = \Lambda Bx$.*

**Proof.** We consider the case $\tau_{A,B}(\rho) < \rho$ (the other one follows from Remark 5.2) and identify $\rho = \beta$ and $\tau_{A,B}(\rho) = \alpha$ in Theorem 5.1. The assumption $a_{1,A,B} - \rho\, a_{0,A,B} \neq 0$ corresponds to b)/(ii) in Theorem 5.1, furthermore we have

$$\begin{aligned} P_u(\alpha,\beta) &= a_{2,A,B} - (\alpha+\beta)\, a_{1,A,B} + \alpha\beta\, a_{0,A,B} \\ &= a_{2,A,B} - \alpha\, a_{1,A,B} - \beta\,(a_{1,A,B} - \alpha\, a_{0,A,B}) \\ &= 0. \end{aligned}$$

The case $\rho = \infty$ follows from taking limits ∎

If the assumptions of Theorem 5.3 are fulfilled, $(\rho, u)$ is not an eigenpair of $Ax = \Lambda Bx$, since $Au = \rho Bu$ implies $(B^{-1}Au|u)_M - \rho(u|u)_M = 0$, which contradicts $a_{1,A,B} - \rho\, a_{0,A,B} \neq 0$. This implies

$$a_{2,A,B} - 2\rho\, a_{1,A,B} + \rho^2 a_{0,A,B} = \left(B^{-1}Au - \rho u \middle| B^{-1}Au - \rho u\right)_M > 0.$$

Therefore $a_{1,A,B} - \rho\, a_{0,A,B} < 0$ if and only if $\tau_{A,B}(\rho) < \rho$, that is, if $\tau_{A,B}(\rho)$ is a lower eigenvalue bound, the denominator will be negative, if $\tau_{A,B}(\rho)$ is an upper bound, the denominator will be positive. Thus, the statement of Theorem 5.3 remains valid if the Schwarz constant $a_{2,A,B}$ is replaced by an upper bound $a_{2,A,B} \leq \tilde{a}_{2,A,B}$. This can be useful if the calculation of the *exact* solution of the linear system $Bv = Au$ is to be avoided or if it is impossible. An advantageous method for calculating a small $\tilde{a}_{2,A,B} \geq a_{2,A,B}$ without knowledge of the exact $v$ has been shown in [3].

**Theorem 5.4.** *Let $c \in \mathbb{R}$ with $0 < c \le \Lambda_{\min(B)}$ and $\tilde{v} \in \mathbb{R}^n$. Let*

$$\tilde{a}_{2,A,B} := \{\tilde{v}|Au\}_M - \{\tilde{v}|B\tilde{v} - Au\}_M + \frac{1}{c}\{B\tilde{v} - Au|B\tilde{v} - Au\}_M. \tag{5.13}$$

*Then $a_{2,A,B} \le \tilde{a}_{2,A,B}$.*

**Proof.** For $x \in \mathbb{R}^n$ the Cauchy-Schwarz inequality provides

$$c\{x|x\}_M \le \{x|Bx\}_M \le \left(\{x|x\}_M\right)^{1/2}\left(\{Bx|Bx\}_M\right)^{1/2}.$$

Thus, $\left(\{x|x\}\right)^{1/2} \le \frac{1}{c}\left(\{Bx|Bx\}\right)^{1/2}$. This implies

$$(x|x)_M = \{x|Bx\}_M \le \left(\{x|x\}_M\right)^{1/2}\left(\{Bx|Bx\}_M\right)^{1/2} \le \frac{1}{c}\{Bx|Bx\}_M$$

in turn and therefore

$$\left(\tilde{v} - v|\tilde{v} - v\right)_M \le \frac{1}{c}\left\{B\tilde{v} - Au|B\tilde{v} - Au\right\}_M.$$

The upper bound is obtained by

$$\begin{aligned}(v|v)_M &= (v|v)_M + (\tilde{v}|\tilde{v})_M - 2\{\tilde{v}|Bv\}_M - \left((\tilde{v}|\tilde{v})_M - 2\{\tilde{v}|Au\}_M\right)\\ &= (\tilde{v} - v|\tilde{v} - v)_M + \{\tilde{v}|Au\}_M - \{\tilde{v}|B\tilde{v} - Au\}_M\\ &\le \{\tilde{v}|Au\}_M - \{\tilde{v}|B\tilde{v} - Au\}_M + \frac{1}{c}\{B\tilde{v} - Au|B\tilde{v} - Au\}_M\end{aligned}$$

and the assertion is proved ∎

If we want to prove that an eigenvalue problem $Ax = \Lambda Bx$ has $n$ distinct eigenvalues $\Lambda_1 < \Lambda_2 < \ldots < \Lambda_n$, the following procedure based on Theorems 5.3 and 5.4 (see [4]) can be applied:

1. Calculate $0 < c \le \Lambda_{\min}(B)$.
2. Let $\rho := -\infty$ and $i := 1$.
3. Choose an appropriate $u \in \mathbb{R}^n$, let $v \approx B^{(-1)}Au$, and calculate $\tau_{A,B}(\rho)$ using $\tilde{a}_{2,A,B}$.
4. If $\tau_{A,B}(\rho) \le \rho$, then break off.
5. Set the interval $\overline{\Lambda}_i$ to $(\rho, \tau_{A,B}(\rho)]$.
6. If $i < n$, let $\rho := \tau_{A,B}(\rho)$ and $i := i + 1$, go to step 3.

If this procedure does *not* break off at step 4, then $\Lambda_i \in \overline{\Lambda}_i$ for disjoint intervals $\overline{\Lambda}_i$ $(i = 1, 2, \ldots, n)$ has been proved, that is, our matrix eigenvalue problem has no multiple eigenvalues. Furthermore, $\max(\overline{\Lambda}_i)$ can be a very precise upper bound to $\Lambda_i$. The quality of this upper bound evidently depends on the choice of the vector $u \in \mathbb{R}^n$. To obtain good bounds, $u$ has to be a good approximation to an eigenvector which belongs to $\Lambda_i$ [2, 3].

The same holds true if we start the procedure from above:

1. Calculate $0 < c \leq \Lambda_{\min}(B)$.
2. Let $\rho := \infty$ and $i := n$.
3. Choose an appropriate $u \in \mathbb{R}^n$, let $v \approx B^{(-1)}Au$, and calculate $\tau_{A,B}(\rho)$ using $\tilde{a}_{2,A,B}$.
4. If $\rho \leq \tau_{A,B}(\rho)$, then break off.
5. Set the interval $\underline{\Lambda}_i$ to $[\tau_{A,B}(\rho), \rho)$.
6. If $i > 1$, let $\rho := \tau_{A,B}(\rho)$ and $i := i - 1$, go to step 3.

Thus, sharper inclusions for the eigenvalues may be obtained by

$$\Lambda_i \in \left[\min(\underline{\Lambda}_i), \max(\overline{\Lambda}_i)\right] \qquad \text{for } i = 1, \ldots, n.$$

## 6. Application to parameter-dependent matrices

If the procedure based on Theorems 5.3 and 5.4 is applied to a parameter-dependent generalized matrix eigenvalue problem

$$Ax = \lambda Bx$$

with

$$A : [a, b] \ni \Omega \mapsto A(\Omega) \in \mathbb{R}^{n\times n} \qquad \text{and} \qquad B : [a, b] \ni \Omega \mapsto B(\Omega) \in \mathbb{R}^{n\times n},$$

$A(\Omega) = A^{\mathsf{T}}(\Omega)$, $B(\Omega) = B^{\mathsf{T}}(\Omega)$, $B(\Omega)$ positive definite for all $\Omega \in [a, b]$. Then $a_{0,A,B}$, $a_{1,A,B}$, $a_{2,A,B}$ and $\tilde{a}_{2,A,B}$ also depend on the parameter $\Omega$. Thus, $\tau_{A,B}(\rho) : [a, b] \ni \Omega \mapsto (\tau_{A,B}(\rho))(\Omega) \in \mathbb{R}$ is also a real function. Here the following question arises:

How can lower and upper bounds for $\tau_{A,B}(\rho)$ be calculated?

An idea that suggests itself is to calculate constant bounds for $\tau_{A,B}(\rho)$ over a given interval $[\alpha, \beta] \subseteq [a, b]$ by means of interval-analytic methods (that is, to bracket the range of the real function $\tau_{A,B}(\rho)([\alpha, \beta])$). This approach is unsatisfactory, since no intervals $[\alpha, \beta]$ with "reasonable" diameter can be chosen, if even one eigenvalue curve shows a gradient in $[\alpha, \beta]$ that differs substantially from zero. In order to calculate sharp bounds for an eigenvalue curve, this curve should be "flattened" in advance. This "flattening" can be achieved by means of a parameter-dependent spectral shift; however, it can generally be achieved only for one eigenvalue curve at a time.

To be more precise, we suggest the following procedure: First, we choose parameters $\alpha$ and $\beta$ such that $[\alpha, \beta] \subseteq [a, b]$. The discussion of numerical examples will clarify the issues that have to be taken into account for this choice. If in the $i$-th step bounds for

$\lambda_{i,A,B}$ are to be calculated, we will determine an interpolation polynomial $\tilde{p}_i$ for $\lambda_{i,A,B}$ in $[\alpha, \beta]$ and define

$$H_i(\Omega) = A(\Omega) - \tilde{p}_i(\Omega) \cdot B(\Omega). \tag{6.1}$$

The eigenvalues of $H_i x = \lambda B x$ and $Ax = \lambda Bx$ are closely related. In fact, we have $\lambda_{j,H_i,B}(\Omega) = \lambda_{j,A,B}(\Omega) - \tilde{p}_i(\Omega)$ for $j = 1, \ldots, m$, that is,

$$\lambda_{i,H_i,B} \approx 0 \qquad \text{in } [\alpha, \beta],$$

and the eigenvectors of both problems are identical. Next, we calculate $\tau_{H_i,B}(\rho - \tilde{p}_i)$ instead of $\tau_{A,B}(\rho)$ and determine bounds for the range of $\tau_{H_i,B}(\rho - \tilde{p}_i)$ by means of one of the well-known methods in interval mathematics [1, 21, 23]. The elements of this range are close to zero if $\beta - \alpha$ is sufficiently small:

$$-\underline{\varepsilon}_i \;\leq\; \left\{ \left(\tau_{H_i,B}(\rho - \tilde{p}_i)\right)(\Omega) \,\middle|\, \Omega \in [\alpha, \beta] \right\} \;\leq\; \overline{\varepsilon}_i. \tag{6.2}$$

This results in the bounds

$$\tilde{p}_i(\Omega) - \underline{\varepsilon}_i \;\leq\; \lambda_{i,A,B}(\Omega) \;\leq\; \tilde{p}_i(\Omega) + \overline{\varepsilon}_i \qquad \text{for all } \Omega \in [\alpha, \beta].$$

Further algorithmic details can be found in [4] where the special parameter-dependent matrix eigenvalue problem is treated.

If the quantity $c$ is not known a priori, a $c$ with $0 \leq c \leq \lambda_{\min}(B(\Omega))$ for all $\Omega \in [\alpha, \beta]$ can be determined by means of the proposed algorithm (applied to the special eigenvalue problem $B(\Omega)x = \lambda x$).

## 7. Numerical results

Now we will apply the procedure from Section 6 to determine parameter-dependent bounds for the eigenvalues of our problem (4.7). For this end we will first establish the parameter-dependent matrix eigenvalue problem

$$A_1(\Omega)\, x = \Lambda(\Omega)\, A_0(\Omega)\, x, \qquad \Lambda_i(\Omega) \leq \Lambda_{i+1}(\Omega) \quad \text{for } i = 1, \ldots, n-1$$

according to the Rayleigh-Ritz procedure. The upper bounds $p_{u,i}$ for $\Lambda_i$ are upper bounds for $\lambda_i$ as well,

$$\lambda_i(\Omega) \leq p_{u,i}(\Omega) \qquad \text{for all } \Omega \in [\alpha, \beta] \text{ and } i = 1, \ldots, n.$$

In order to calculate the lower bounds for the $\lambda_i$ according to the Lehmann-Goerisch procedure, we will consider the parameter-dependent matrix eigenvalue problem

$$\left(A_1(\Omega) - \rho A_0(\Omega)\right) x = \tau(\Omega) \left(A_2(\Omega) - 2\rho A_1(\Omega) + \rho^2 A_0(\Omega)\right) x,$$

from which we obtain the lower bounds $p_{l,i}$,

$$p_{l,i}(\Omega) \leq \lambda_i(\Omega) \qquad \text{for all } \Omega \in [\alpha, \beta] \text{ and } i = 1, \ldots, N.$$

If we define

$$p_i = \frac{1}{2}(p_{u,i} + p_{l,i}) \qquad \text{and} \qquad \varepsilon_i = \max\left\{ \frac{1}{2}(p_{u,i}(\Omega) - p_{l,i}(\Omega)) \Big| \Omega \in [\alpha, \beta] \right\},$$

we obtain

$$p_i(\Omega) - \varepsilon_i \leq \lambda_i(\Omega) \leq p_i(\Omega) + \varepsilon_i \qquad \text{for all } \Omega \in [\alpha, \beta] \text{ and } i = 1, \ldots, N,$$

that is, bounds of the form (1.1). If we want to prove a possible veering of the eigenvalue curves $i$ and $i+1$ in the interval $[\alpha, \beta]$, it is sufficient to show $p_{i+1}(\Omega) - p_i(\Omega) - \varepsilon_{i+1} - \varepsilon_i > 0$ in $[\alpha, \beta]$. Figure 3 shows the eigenvalue bounds $p_i \pm \varepsilon_i$ $(i = 2, 3)$ of our problem (4.7) for $\gamma = \frac{\pi}{180}, n_1 = n_2 = 10$ and $[\alpha, \beta] = [8.43, 9.53]$. Obviously there is curve veering. (It is easy to prove $p_3(\Omega) - p_2(\Omega) - \varepsilon_3 - \varepsilon_2 > 0$ in $[8.43, 9.53]$ using well-known interval analytic methods on the computer.)

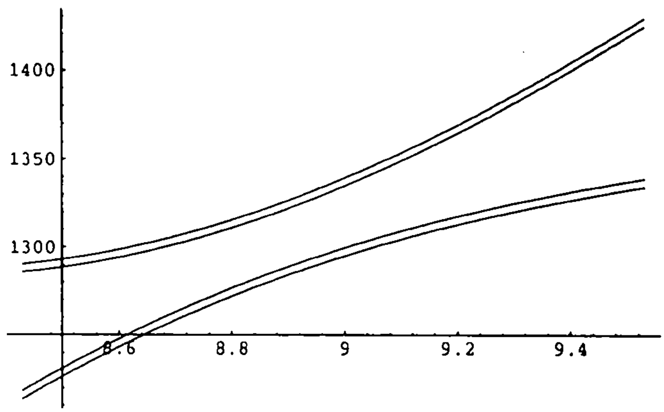


Figure 3: Verified bounds for eigenvalue curves two and three

In Table 1 we give the polynomials $p_i$ and $\varepsilon_i$ $(i = 1, 2, 3)$. For reasons of convenience, the coefficients of the polynomials are given as points and not as intervals. (Intervals would be the correct notation since we have to add two polynomials in order to compute the $p_i$, and we have to convert the binary representation into decimal representation.) A verified inclusion is obtained by rounding up and down the last given decimal figure by one.

| $p_i(\Omega) = \sum_{j=0}^{3} p_{i,2j}\Omega^{2j}$ | | | |
|---|---|---|---|
| | $i = 1$ | $i = 2$ | $i = 3$ |
| $p_{i,0}$ | 1.3543915E+01 | −2.9820202E+03 | 4.5513253E+03 |
| $p_{i,2}$ | 1.8494134E+00 | 1.3005260E+02 | −1.1690165E+02 |
| $p_{i,4}$ | −5.4841855E−04 | −1.3240800E+00 | 1.3241119E+00 |
| $p_{i,6}$ | 1.3686738E−06 | 4.5771071E−03 | −4.5774365E−03 |
| $\varepsilon_i$ | 0.0416131 | 2.34309 | 2.2665 |

Table 1: Bounds for $\lambda_i(\Omega)$ $(i = 1, 2, 3)$ and $\Omega \in [8.43, 9.53]$ of problem (4.7)

**Remark 7.1.** It is interesting to observe that the eigenfunctions change their character although the eigenvalues do not cross. Figure 4 shows the two components of the eigenelements which belong to the second and third eigenvalue for $\Omega = 5$ and for $\Omega = 13$.

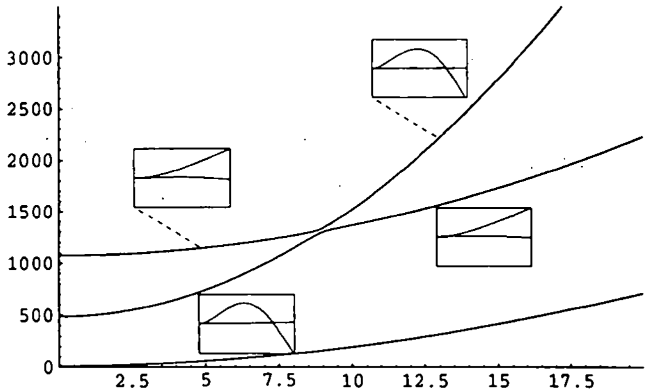


Figure 4: Eigenvalues and eigenfunctions of problem (4.7) for $\gamma = \frac{\pi}{180}$

Even more accurate bounds can be obtained if a smaller diameter of the parameter interval $[\alpha, \beta]$ is chosen. Then the interpolation polynomials approximate the eigenvalue curves more precisely. If we are interested in a parameter interval for which the eigenvalue curves under consideration do not show the curve veering phenomenon, a considerably wider parameter interval can be chosen. Table 2 shows the results for $\Omega \in [0,6]$ and $i = 1,2,3$.

| $p_i(\Omega) = \sum_{j=0}^{3} p_{i,2j}\Omega^{2j}$ | | | |
|---|---|---|---|
| | $i = 1$ | $i = 2$ | $i = 3$ |
| $p_{i,0}$ | 1.2295567E+01 | 4.8427217E+02 | 1.0773140E+03 |
| $p_{i,2}$ | 1.9065905E+00 | 1.0359157E+01 | 2.9175993E+00 |
| $p_{i,4}$ | −1.7416498E−03 | 9.7290060E−04 | 2.3429395E−05 |
| $p_{i,6}$ | 1.2769295E−05 | −1.2917310E−05 | 3.3329712E−06 |
| $\varepsilon_i$ | 0.103188 | 1.59546 | 0.0940387 |

Table 2: Bounds for $\Lambda_i(\Omega)$ $(i = 1,2,3)$ and $\Omega \in [0.0, 6.0]$ of problem (4.7)

To sum up: we have shown that we can prove the phenomenon of curve veering for a concrete situation without requiring special properties of the eigenvalue curves. The procedure is widely applicable since the inclusion theorems for self-adjoint eigenvalue problems exactly result in the class of matrix problems that we discussed in our paper, on the one hand, while, on the other hand, the power of the inclusion theorems has been proved by means of numerous parameter-independent eigenvalue problems for ordinary and partial differential equations (see [5, 6, 9, 27]).

It should also be emphasized that the use of computer algebra programs for orthogonalization allows to use classical trial functions (polynomials) without the usual numerical problems (see [6]). For further views on a combination of algebraic and numerical calculations see [24].

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