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Weighted Inequalities for the Fractional Integral Operators on Monotone Functions

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Abstract. We give a characterization of weight functions u and v on $\mathbb{R}^n$ for which the fractional integral operator I_s of order s on $\mathbb{R}^n$ defined by $(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,dy$ sends all monotone functions which belong to the weighted Lebesgue space $L^p_v(\mathbb{R}^n)$ into the weighted Lebesgue space $L^q_u(\mathbb{R}^n)$. This characterization is done for all p and q with $1 < p < \infty$ and $0 < q < \infty$. The analogous Lorentz and Orlicz problems are also considered.

Keywords: *Weighted inequalities, fractional integral operators, Hardy operators*
AMS subject classification: 42 B 25

0. Introduction

The fractional integral operator I_s of order s $(0 < s < n)$ on $\mathbb{R}^n$ $(n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\})$ is defined by

$$(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,dy.$$

Let u and v be weight functions on $\mathbb{R}^n$ (i.e. non-negative locally integrable functions) and let $1 < p \le q < \infty$. Weighted inequalities of the form

$$\left(\int_{\mathbb{R}^n} (I_s f)^q(x)\,u(x)\,dx\right)^{\frac{1}{q}} \le C \left(\int_{\mathbb{R}^n} f^p(x)\,v(x)\,dx\right)^{\frac{1}{p}} \qquad \text{for all } f \ge 0 \qquad (0.0)$$

were studied by many authors (see the references in [8]). A characterization of weight functions u and v for which (0.0) holds was done by Sawyer and Wheeden [8]. In particular necessary (and sufficient for $1 < p < q$) conditions are

$$\left(\int_Q u(y)\,dy\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left[|Q|^{\frac{1}{n}} + |x_Q - y|\right]^{(s-n)p'} v^{-\frac{1}{p-1}}(y)\,dy\right)^{\frac{1}{p'}} \le c$$

$$\left(\int_Q v^{-\frac{1}{p-1}}(y)\,dy\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^n} \left[|Q|^{\frac{1}{n}} + |x_Q - y|\right]^{(s-n)q} u(y)\,dy\right)^{\frac{1}{q}} \le c \qquad .$$

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for all cubes Q, where $p' = \frac{p}{p-1}$, x_Q is the center of Q and $|Q|$ its Lebesgue measure. In these conditions, the cubes Q can be replaced by balls B, and particularly taking balls $B = B(0,R)$ centered at the origin and with radius R, it appears that necessary conditions for the above weighted inequality (0.0) are

$$\left(\int_{R<|y|} |y|^{(s-n)q} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_{|y|<R} v^{-\frac{1}{p-1}}(y)\,dy\right)^{\frac{1}{p'}} \leq c \qquad (0.1)$$

$$\left(\int_{|y|<R} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_{R<|y|} |y|^{(s-n)p'} v^{-\frac{1}{p-1}}(y)\,dy\right)^{1/p'} \leq c \qquad (0.1)^*$$

$$R^{s-n}\left(\int_{|y|<R} u(y)\,dy\right)^{\frac{1}{q}} \left(\int_{|y|<R} v^{-\frac{1}{p-1}}(y)\,dy\right)^{\frac{1}{p'}} \leq c \qquad (0.2)$$

for all $R > 0$, with a constant c not depending on R.

For the convenience, in the second formula, we write the star since the considered condition is known as the dual of the first one. Such a distinction will always be used throughout this paper when we deal with the dual of an inequality or a condition.

We emphasize that in these conditions we do not make use of integrations on arbitrary cubes, which are a brake for people who do computations. Thus (0.1) and its dual condition $(0.1)^*$ can be easily checked mainly for radial weight functions (which are often used in applications).

A function f satisfies the

Condition $\mathcal{RM}$

and we write $f \in \mathcal{RM}$ when $f(x) = \varphi(|x|)$ for some monotone function φ defined on $[0,\infty)$. We also write $f \in \mathcal{RD}$ and $f \in \mathcal{RI}$ if φ is a decreasing or increasing function, respectively.

In this paper we deal with the question of characterizing those weight functions u and v for which it is enough to test (0.0) for non-negative functions in $\mathcal{RM}$. Although (0.1), $(0.1)^*$ and (0.2) are no longer sufficient for (0.0) with general functions, we will prove in Corollary 1.2 that both (0.1) and $(0.1)^*$ are sufficient to ensure (0.0) for all non-negative functions in $\mathcal{RM}$. Moreover we are also able to get a similar result for the range of p and q with $q < p$. Since the technique we used is based on Hardy inequalities we can also deal with the analogous Lorentz and Orlicz problems.

Statements of results on I_s mapping L^p into L^q are given in Section 1. The next Section 2 is devoted to the Lorentz problem, and Section 3 yields the statements for the Orlicz setting. Proofs of all statements are given in Section 4.

1. Lebesgue spaces results

Instead of (0.0) we write $I_s : L^p_v \to L^q_u$, and when we only deal with non-negative functions in $\mathcal{RM}$, we denote the corresponding embedding by $I_s : L^p_v(\mathcal{RM}) \to L^q_u$.

Our first result is

Theorem 1.1. *Let* $0 < s < n$, $0 < p < \infty$ *and* $0 < q < \infty$. *Suppose that* $I_s : L^p_v \to L^q_u$. *Then there is a constant* $c > 0$ *such that*

$$\left(\int_{\mathbb{R}^n}\left[\int_{|y|<|x|} f(y)\,dy\right]^q u(x)|x|^{(s-n)q}dx\right)^{\frac{1}{q}} \le c\left(\int_{\mathbb{R}^n} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \quad (1.1)$$

$$\left(\int_{\mathbb{R}^n}\left[\int_{|x|<|y|} |y|^{(s-n)}f(y)\,dy\right]^q u(x)\,dx\right)^{\frac{1}{q}} \le c\left(\int_{\mathbb{R}^n} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \quad (1.1)^*$$

for all non-negative functions f. *Conversely, both inequalities* (1.1) *and* $(1.1)^*$ *imply that* $I_s : L^p_v(\mathcal{RM}) \to L^q_u$.

The proof of the theorem will be given in Section 4.

The inequalities (1.1) and $(1.1)^*$ are in fact forms of usual Hardy inequalities [5: p. 13]. With easy modifications of the classical proofs (by change of variables or by a direct method as Sawyer's proof) it is clear that if $1 < p \le q < \infty$, then inequality (1.1) or $(1.1)^*$ holds if and only if condition (0.1) or $(0.1)^*$ is satisfied, respectively. If $1 < p < \infty$ and $0 < q < p$, then (1.1) and $(1.1)^*$ is equivalent to

$$\int_{\mathbb{R}^n}\left[\left(\int_{|x|<|y|}|y|^{(s-n)q}u(y)\,dy\right)^{\frac{1}{q}} \times\left(\int_{|z|<|x|}v^{-\frac{1}{p-1}}(z)\,dz\right)^{1-\frac{1}{q}}\right]^{\theta} v^{-\frac{1}{p-1}}(x)\,dx < \infty \quad (1.2)$$

and

$$\int_{\mathbb{R}^n}\left[\left(\int_{|x|<|y|}|y|^{(s-n)p'}v^{-\frac{1}{p-1}}(y)\,dy\right)^{1-\frac{1}{p}} \times\left(\int_{|z|<|x|}u(z)\,dz\right)^{\frac{1}{p}}\right]^{\theta} u(x)\,dx < \infty, \quad (1.2)^*$$

respectively, where $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$ and $p' = \frac{p}{p-1}$.

Thus as a consequence of Theorem 1.1 we get

Corollary 1.2. *Let* $0 < s < n$ *and* $1 < p < \infty$. *Then, for* $p \le q$, *conditions* (0.1) *and* $(0.1)^*$ *together imply* $I_s : L^p(\mathcal{RM}) \to L^q_u$. *This embedding is also true for the range* $0 < q < p$ *whenever both* (1.2) *and* $(1.2)^*$ *are satisfied.*

Although for $1 < p \le q$ the conditions (0.1) and $(0.1)^*$ imply $I_s : L^p(\mathcal{RM}) \to L^q_u$, they are no longer necessary. To get the right necessary and sufficient conditions we can observe that (1.1) and its dual inequality $(1.1)^*$ with non-negative functions in $\mathcal{RM}$ are equivalent to

$$\left(\int_0^\infty \left[\int_0^r \varphi(\rho)\rho^{n-1}d\rho\right]^q r^{(s-n)q}\tilde{u}(r)\,dr\right)^{\frac{1}{q}} \le c\left(\int_0^\infty \varphi^p(r)\tilde{v}(r)\,dr\right)^{\frac{1}{p}} \tag{1.3}$$

and

$$\left(\int_0^\infty \left[\int_r^\infty \varphi(\rho)\rho^{s-1}d\rho\right]^q \tilde{u}(r)\,dr\right)^{\frac{1}{q}} \le c\left(\int_0^\infty \varphi^p(r)\,\tilde{v}(r)\,dr\right)^{\frac{1}{p}}, \tag{1.3*}$$

for all non-negative monotone functions φ, respectively. Here

$$\tilde{u}(r) = r^{n-1}\int_{S_{n-1}} u(r\omega)\,d\sigma(\omega) \qquad \text{and} \qquad \tilde{v}(r) = r^{n-1}\int_{S_{n-1}} v(r\omega)\,d\sigma(\omega),$$

S_{n-1} is the unit sphere of R^n and $d\sigma$ is the area measure on S_{n-1}. A key to get (1.3) and $(1.3)^*$ are Hardy inequalities for monotone functions like

$$\left(\int_0^\infty (A\psi)^q(r)\mu(r)\,dr\right)^{\frac{1}{q}} \le c\left(\int_0^\infty \psi^p(r)\,\nu(r)\,dr\right)^{\frac{1}{p}} \tag{1.4}$$

and

$$\left(\int_1^\infty (A^*\psi)^q\mu^*(r)\,dr\right)^{\frac{1}{q}} \le c\left(\int_1^\infty \psi^p(r)\,\nu^*(r)\,dr\right)^{\frac{1}{p}}, \tag{1.4*}$$

where A and its dual operator A^* are given by

$$(A\psi)(r) = \frac{1}{r}\int_0^r \psi(\rho)\,d\rho \qquad \text{and} \qquad (A^*\psi)(r) = \int_r^\infty \rho^{-1}\psi(\rho)\,d\rho.$$

Indeed a characterization of weight functions μ and ν for which (1.3) (and consequently for $(1.3)^*$) holds for decreasing functions ψ was done by Sawyer [7] and Stepanov [9]. The analogous problem for increasing functions was solved by Heinig and Stepanov [3].

For $1 < p \le q < \infty$ it is well known that inequality (1.4) for decreasing functions is equivalent together to

$$\left(\int_0^R \mu(r)\,dr\right)^{\frac{1}{q}} \le c_1\left(\int_0^R \nu(r)\,dr\right)^{\frac{1}{p}} \tag{1.5}$$

and

$$\left(\int_R^\infty r^{-q}\mu(r)\,dr\right)^{\frac{1}{q}}\left(\int_0^R \left[\int_0^r \nu(t)\,dt\right]^{-p'} r^{p'}\nu(r)\,dr\right)^{\frac{1}{p'}} \le c_2 \tag{1.6}$$

for all $R > 0$.

For $1 < q < p < \infty$ it is required that inequality (1.4) for decreasing functions holds if and only if together

$$\int_0^\infty \left[\left(\int_0^r \mu(t)\,dt\right)^{\frac{1}{p}}\left(\int_0^r \nu(t)\,dt\right)^{-\frac{1}{p}}\right]^\theta \mu(r)\,dr < \infty \tag{1.7}$$

and

$$\begin{aligned}\int_0^\infty &\left[\left(\int_r^\infty \rho^{-q}\mu(\rho)\,d\rho\right)^{\frac{1}{q}}\right.\\ &\times\left.\left(\int_0^r \rho^{p'}\left[\int_0^\rho \nu(t)\,dt\right]^{-p'}\nu(\rho)\,d\rho\right)^{\frac{1}{q'}}\right]^\theta r^{p'}\left[\int_0^r \nu(t)\,dt\right]^{-p'}\nu(r)\,dr < \infty\end{aligned} \tag{1.8}$$

where $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$. Results for the dual inequality $(1.4)^*$ can be found and deduced by results in [7, 9]. Analogous results for increasing functions can be seen in [3]. Consequently a characterization of the embedding $I_s : L^p_v(\mathcal{RM}) \to L^q_u$ can be reduced to express inequalities (1.3) and $(1.3)^*$ in terms of operators A and A^* like (1.4) and $(1.4)^*$. Thus our next result is the following

Theorem 1.3. *Let $1 < p < \infty$ and $0 < q < \infty$. Then the embedding $I_s : L^p_v(\mathcal{RM}) \to L^q_u$ is equivalent to the Hardy inequalities (with monotone functions) (1.4) and $(1.4)^*$ together, with weight functions μ, ν on $(0, +\infty)$ and μ^*, ν^* on $(1, +\infty)$ given by*

$$\mu(t) = t^{\frac{1}{n}[sq+1-n]}\tilde{u}(t^{\frac{1}{n}}) \qquad \text{and} \qquad \mu^*(t) = (\ln t)^{\frac{1}{s}-1}\tilde{u}((\ln t)^{\frac{1}{s}})t^{-1}$$

$$\nu(t) = t^{\frac{1}{n}[1-n]}\tilde{v}(t^{\frac{1}{n}}) \qquad\qquad \nu^*(t) = (\ln t)^{\frac{1}{s}-1}\tilde{v}((\ln t)^{\frac{1}{s}})t^{-1}$$

where $\tilde{u}$ and $\tilde{v}$ are defined as above.

As a consequence we can state the following two corollaries.

Corollary 1.4 (Decreasing functions with $1 < p \le q$). *Let $0 < s < n$, $1 < p \le q < \infty$ and $\int_{\mathbb{R}^n} v(x)\,dx = \infty$. Then the embedding $I_s : L^p_v(\mathcal{RD}) \to L^q_u$ (for decreasing functions) is equivalent to the four following conditions together:*

$$\left(\int_{|x|<R} |x|^{sq}u(x)\,dx\right)^{\frac{1}{q}} \le c_1\left(\int_{|x|<R} v(x)\,dx\right)^{\frac{1}{p}} \tag{1.9}$$

$$\left(\int_{|x|<R} |x|^{(s-n)q}u(x)\,dx\right)^{\frac{1}{q}} \times\left(\int_{|x|<R} |x|^{np'}\left[\int_{|y|<|x|} v(y)\,dy\right]^{-p'} v(x)\,dx\right)^{\frac{1}{p'}} \le c_2 \tag{1.10}$$

$$\left(\int_{|x|<R}(R^s-|x|^s)^q u(x)\,dx\right)^{\frac{1}{q}} \le c_3\left(\int_{|x|<R} v(x)\,dx\right)^{\frac{1}{p}} \tag{1.11}$$

$$\left(\int_{|x|<R} u(x)\,dx\right)^{\frac{1}{q}} \times\left(\int_{R<|x|}(|x|^s-R^s)^{p'}\left[\int_{|y|<|x|} v(y)\,dy\right]^{-p'} v(x)\,dx\right)^{\frac{1}{p'}} \le c_4 \tag{1.12}$$

for all $R>0$, where $c_1,\dots,c_4$ are non-negative constants not depending on R.

Corollary 1.5 (Decreasing functions with $1<q<p$). *Let $0<s<n$, $1<q<p<\infty$ and $\int_{\mathbb{R}^n} v(x)\,dx=\infty$. Then the embedding $I_s : L^p_v(\mathcal{RD}) \to L^q_u$ (for decreasing functions) is equivalent to the four following conditions together:*

$$\int_{\mathbb{R}^n}\left[\left(\int_{|y|<|x|}|y|^{sq}u(y)\,dy\right)^{\frac{1}{p}}\left(\int_{|z|<|x|}v(z)\,dz\right)^{-\frac{1}{p}}\right]^{\theta}|x|^{sq}u(x)\,dx<\infty \tag{1.13}$$

$$\int_{\mathbb{R}^n}\left[\left(\int_{|x|<|y|}|y|^{(s-n)q}u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{|z|<|x|}\left[\int_{|y|<|z|}v(y)\,dy\right]^{-p'}\right.\right.$$
$$\left.\left.\times|z|^{np'}v(z)\,dz\right)^{\frac{1}{p'}}\right]^{\theta}\left[\int_{|y|<|x|}v(y)\,dy\right]^{-p'}|x|^{np'}v(x)\,dx<\infty \tag{1.14}$$

$$\int_{\mathbb{R}^n}\left[\left(\int_{|y|<|x|}(|x|^s-|y|^s)^q u(y)\,dy\right)^{\frac{1}{q}}\left(\int_{|z|<|x|}v(z)\,dz\right)^{-\frac{1}{q}}\right]^{\theta}v(x)\,dx<\infty \tag{1.15}$$

$$\int_{\mathbb{R}^n}\left[\left(\int_{|z|<|x|}u(z)\,dz\right)^{\frac{1}{p}}\left(\int_{|x|<|y|}(|y|^s-|x|^s)^{p'}\right.\right.$$
$$\left.\left.\times\left[\int_{|z|<|y|}v(z)\,dz\right]^{-p'}v(y)\,dy\right)^{\frac{1}{p'}}\right]^{\theta}u(x)\,dx<\infty \tag{1.16}$$

where $\frac{1}{\theta}=\frac{1}{q}-\frac{1}{p}$.

The proof of Theorem 1.3 and Corollaries 1.4 and 1.5 will be given in Section 4. Analogous results for increasing functions are also possible by using Theorem 3 and other results in [3].

2. Lorentz spaces results

For $1 \le p \le \infty$ and $1 \le q < \infty$ we set

$$\|g\|^q_{L^{pq}_u} = q \int_0^\infty \left[\int_{\{y:\, |g(y)|>\lambda\}} u(x)\, dx \right]^{\frac{q}{p}} \lambda^{q-1}\, d\lambda$$

and for $1 \le p < \infty$ we set

$$\|g\|_{L^{p\infty}_u} = \sup_{\lambda>0} \lambda \left[\int_{\{y:\, |g(y)|>\lambda\}} u(x)\, dx \right]^{\frac{1}{p}}.$$

Let u, v and w_1, w_2 be weight functions on $\mathbb{R}^n$. In this section we deal with an analogy of inequality (0.0) which takes the form

$$\|w_2(I_s f)\|_{L^{q_1 q_2}_u} \le C \|w_1 f\|_{L^{p_1 p_2}_v} \qquad \text{for all functions } f \ge 0.$$

The consideration of such an inequality with four weight functions is useful in the Lorentz setting, since the weights cannot be combined as in the Lebesgue case (with expressions like $\int \left(|f(x)|u(x)\right)^p v(x)\, dx$).

In this section, we always assume

$$1 < p_1, q_1 < \infty \qquad \text{and} \qquad 1 \le p_2, q_2 \le \infty. \tag{2.0}$$

The above embedding is denoted as $I_s : L^{p_1 p_2}_v(w_1) \to L^{q_1 q_2}_u(w_2)$, and when we will limit onelelf to the case of non-negative functions in $\mathcal{RM}$, then we write $I_s : L^{p_1 p_2}_v(w_1)[\mathcal{RM}] \to L^{q_1 q_2}_u(w_2)$. Contrary to the Lebesgue case (i.e. $p_1 = p_2$ and $q_1 = q_2$) and as mentioned in [4], a characterization of $I_s : L^{p_1 p_2}_v(w_1) \to L^{q_1 q_2}_u(w_2)$ is not known in the literature, and until now it is still an open problem to obtain easy necessary and sufficient conditions for this embedding. However for $I_s : L^{p_1 p_2}_v(w_1)[\mathcal{RM}] \to L^{q_1 q_2}_u(w_2)$ we have the following

Theorem 2.1. *Let $0 < s < n$, and p_1, p_2 and q_1, q_2 as in (2.0). Suppose $I_s : L^{p_1 p_2}_v(w_1) \to L^{q_1 q_2}_u(w_2)$. Then there is a constant $c > 0$ such that*

$$\left\| w_2 |\cdot|^{s-n} \left(\int_{|y|<|\cdot|} f(y)\, dy \right) \right\|_{L^{p_2 q_2}_u} \le c \|w_1 f\|_{L^{p_1 q_1}_v} \tag{2.1}$$

$$\left\| w_2 \left(\int_{|\cdot|<|y|} |y|^{s-n} f(y)\, dy \right) \right\|_{L^{p_2 q_2}_u} \le c \|w_1 f\|_{L^{p_1 q_1}_v} \tag{2.1*}$$

for all non-negative functions f. Conversely inequalities (2.1) and (2.1)* together imply $I_s : L^{p_1 p_2}_v(w_1)[\mathcal{RM}] \to L^{q_1 q_2}_u(w_2)$.

The proof of Theorem 2.1 will be given in Section 4. Inequality (2.1) and its dual version (2.1)* can be seen as boundedness of generalized Hardy-type operators on Lorentz spaces. Such a problem was treated by Edmunds, Gurka and Pick [2]. With their results we can deduce the following

Proposition 2.2. *Let $0 < s < n$, and p_1, p_2 and q_1, q_2 as in (2.0) and satisfying the condition*

$$\max\{p_1, p_2\} \leq \min\{q_1, q_2\}. \tag{2.2}$$

Then condition (2.1) is equivalent to

$$\left\| w_2 |\cdot|^{s-n} \mathbf{1}_{R<|\cdot|} \right\|_{L_u^{p_2 q_2}} \left\| \frac{1}{w_1 v} \mathbf{1}_{|\cdot|<R} \right\|_{L_v^{p_1' q_1'}} \leq c \tag{2.3}$$

for all $R > 0$. Also condition $(2.1)^*$ is equivalent to

$$\left\| w_2 \mathbf{1}_{|\cdot|<R} \right\|_{L_u^{p_2 q_2}} \left\| \frac{1}{w_1 v} |\cdot|^{s-n} \mathbf{1}_{R<|\cdot|} \right\|_{L_v^{p_1' q_1'}} \leq c \tag{2.3*}$$

for all $R > 0$. Consequently conditions (2.3) and $(2.3)^*$ together imply the embedding $I_s : L_v^{p_1 p_2}(w_1)[\mathcal{RM}] \to L_u^{q_1 q_2}(w_2)$.

Recall that, for each measurable set E, $\mathbf{1}_E$ means its characteristic function.

The proof of Proposition 2.2 also will be given in Section 4. The conditions (2.3) and $(2.3)^*$ together are sufficient for the embedding $I_s : L_v^{p_1 p_2}(w_1)[\mathcal{RM}] \to L_u^{q_1 q_2}(w_2)$ and they are necessary for the embedding $I_s : L_v^{p_1 p_2}(w_1) \to L_u^{q_1 q_2}(w_2)$. Hardy inequalities results for monotone functions in the Lorentz setting are not largely studied in the literature, so we will limit our result to the above sufficient conditions.

3. Orlicz spaces results

Let u, v and w_1, w_2 be weight functions on $\mathbb{R}^n$. In this section we consider the Orlicz version of inequality (0.0) which is of the form

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\big(C_2 w_2(x)(I_s f)(x)\big) u(x)\, dx\right] \leq \Phi_1^{-1}\left[C_1 \int_{\mathbb{R}^n} \Phi_1\big(w_1(x) f(x)\big) v(x)\, dx\right]$$

for all non-negative functions f. This embedding is denoted as $I_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$, and when we will limit oneself to the case of non-negative functions in $\mathcal{RM}$, then we write $I_s : L_v^{\Phi_1}(w_1)[\mathcal{RM}] \to L_u^{\Phi_2}(w_2)$. Here Φ_1 and Φ_2 are φ-functions. Note that Φ is a φ-function if it is a non-negative increasing and continuous function on $[0, \infty)$ with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. A φ-function Φ is said to be subadditive if $\Phi(s + t) \leq \Phi(s) + \Phi(t)$ for all $s, t \geq 0$. Further, Φ is an N-function if it is a convex φ-function such that $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. The N-function complementary associated to Φ is defined by $\Phi^*(t) = \sup_{s\geq 0}\{st - \Phi(s)\}$. Such a function leads to define the Orlicz and Luxemburg norms

$$\|f\|_{\Phi,w} = \sup\left\{ \int |fg| w : \int \Phi^*(|g|) w \leq 1 \right\}$$

and

$$\|f\|_{(\Phi),w} = \inf\left\{ \lambda > 0 : \int \Phi(\lambda^{-1}|f|) w \leq 1 \right\},$$

respectively.

Except the case of $w_2 = 1$ and $\frac{1}{v} = w_1$ which was solved by Lai Qisheng [6] (see also an other particular case in [4]), a characterization of $I_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$ is still an open problem. However for $I_s : L_v^{\Phi_1}(w_1)[\mathcal{RM}] \to L_u^{\Phi_2}(w_2)$ we have the following

Theorem 3.1. *Let* $0 < s < n$, *let* Φ_1 *be an* N*-function and* Φ_2 *a* φ*-function. Suppose* $I_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$. *Then there are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\left(C_2 w_2(x)|x|^{s-n}\left[\int_{|y|<|x|} f(y)\,dy\right]\right)u(x)\,dx\right] \le \Phi_1^{-1}\left[C_1\int_{\mathbb{R}^n}\Phi_1\big(w_1(x)f(x)\big)v(x)\,dx\right] \qquad (3.1)$$

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\left(C_2 w_2(x)\left[\int_{|x|<|y|} |y|^{s-n}f(y)\,dy\right]\right)u(x)\,dx\right] \le \Phi_1^{-1}\left[C_1\int_{\mathbb{R}^n}\Phi_1\big(w_1(x)f(x)\big)v(x)\,dx\right]. \qquad (3.1)^*$$

for all non-negative functions f. *Conversely inequalities* (3.1) *and* (3.1)* *together imply* $I_s : L_v^{\Phi_1}(w_1)[\mathcal{RM}] \to L_u^{\Phi_2}(w_2)$.

The proof of Theorem 3.1 will be given in Section 4. Adapting the usual Lebesgue case, we obtain yet the following

Proposition 3.2. *Let* $0 < s < n$, *let* Φ_1 *be an* N*-function and* Φ_2 *a* φ*-function with* $\Phi_1\Phi_2^{-1}$ *subadditive. Then condition* (3.1) *is equivalent to*

$$\Phi_2^{-1}\left[\int_{R<|x|} \Phi_2\left(c_2 w_2(x)|x|^{s-n}\left\|\frac{1}{\varepsilon v w_1}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v}\right)u(x)\,dx\right] \le \Phi_1^{-1}[c_1\varepsilon^{-1}] \qquad (3.2)$$

for all $\varepsilon > 0$ *and* $R > 0$. *Also condition* (3.1)* *is equivalent to*

$$\Phi_2^{-1}\left[\int_{|x|<R} \Phi_2\left(c_2' w_2(x)\left\|\frac{1}{\varepsilon v w_1}|\cdot|^{s-n}\mathbf{1}_{R<|\cdot|}\right\|_{\Phi_1^*,\varepsilon v}\right)u(x)\,dx\right] \le \Phi_1^{-1}[c_1'\varepsilon^{-1}] \qquad (3.2)^*$$

for all $\varepsilon > 0$ *and* $R > 0$. *Consequently conditions* (3.2) *and* (3.2)* *together are sufficient for the embedding* $I_s : L_v^{\Phi_1}(w_1)[\mathcal{RM}] \to L_u^{\Phi_2}(w_2)$ *and necessary for the embedding* $I_s : L_v^{\Phi_1}(w_1) \to L_u^{\Phi_2}(w_2)$.

The proof of Proposition 3.2 will be given in Section 4. In view of a result of Bloom and Kerman [1] equivalent expressions which do not involve the Orlicz norm can be used instead of conditions (3.2) and (3.2)*.

4. Proofs of results

In this section the proofs of our results are collected. We begin by that of Theorem 1.1.

Proof of Theorem 1.1. The necessity of conditions (1.1) and (1.1)* can be easily obtained, since for all non-negative functions f

$$|x|^{s-n}\int_{|y|<|x|} f(y)\,dy \leq c\int_{|y-x|<2|x|} |x-y|^{s-n}f(y)\,dy \leq c\,(I_s f)(x) \tag{4.1}$$

and

$$\int_{|x|<|y|} |y|^{s-n}f(y)\,dy \leq c\int_{|y-x|<2|y|} |x-y|^{s-n}f(y)\,dy \leq c\,(I_s f)(x) \tag{4.2}$$

where $c>0$ is a constant which depends only on s and n.

For the converse, for all non-negative functions f we first have

$$(I_s f)(x) = A_1(x) + A_2(x) + A_3(x)$$

where

$$A_1(x) = \int_{|y|\leq \frac{1}{2}|x|} |x-y|^{s-n}f(y)\,dy$$
$$A_2(x) = \int_{\frac{3}{2}|x|\leq |y|} |x-y|^{s-n}f(y)\,dy$$
$$A_3(x) = \int_{\frac{1}{2}|x|\leq |y|<\frac{3}{2}|x|} |x-y|^{s-n}f(y)\,dy.$$

For $A_1(x)$ we observe that $\frac{1}{2}|x| \leq |x-y|$ whenever $|y| \leq \frac{1}{2}|x|$ and consequently

$$A_1(x) \leq c|x|^{s-n}\int_{|y|\leq\frac{1}{2}|x|} f(y)\,dy \leq c|x|^{s-n}\int_{|y|<|x|} f(y)\,dy.$$

Also since $\frac{1}{3}|y| \leq |x-y|$ whenever $\frac{3}{2}|x| \leq |y|$, we have

$$A_2(x) \leq c\int_{\frac{3}{2}|x|\leq|y|} |y|^{s-n}f(y)\,dy \leq c\int_{|x|<|y|} |y|^{s-n}f(y)\,dy$$

with also $c>0$ a constant depending on s and n. To estimate $A_3(x)$, the crucial point is that

$$\sup_{\frac{1}{2}|x|<|y|<2|x|} f(y) \leq C(n)\frac{1}{|x|^n}\int_{\frac{1}{4}|x|<|z|<4|x|} f(z)\,dz$$

for all non-negative functions $f \in \mathcal{RM}$ and $x \neq 0$. By this inequality we get

$$\begin{aligned}
A_3(x) &\leq \left(\sup_{\frac{1}{2}|x|<|z|<2|x|} f(z) \right) \int_{\frac{1}{4}|x|<|y|<4|x|} |x-y|^{(s-n)} dy \\
&\leq \left(\sup_{\frac{1}{2}|x|<|z|<2|x|} f(z) \right) \int_{|x-y|<5|x|} |x-y|^{(s-n)} dy \\
&\leq c \left(\sup_{\frac{1}{2}|x|<|z|<2|x|} f(z) \right) |x|^s \\
&\leq c_0 |x|^{(s-n)} \int_{\frac{1}{4}|x|<|z|<4|x|} f(z)\, dz \\
&\leq c_1 |x|^{(s-n)} \int_{\substack{\frac{1}{4}|x|<|z|<4|x| \\ |z|<|x|}} f(z)\, dz + c_2 \int_{\substack{\frac{1}{4}|x|<|z|<4|x| \\ |x|<|z|}} |z|^{(s-n)} f(y)\, dy \\
&\leq c_1 |x|^{(s-n)} \int_{|z|<|x|} f(z)\, dz + c_2 \int_{|x|<|z|} |z|^{(s-n)} f(z)\, dz
\end{aligned}$$

where $c, \ldots, c_2$ are non-negative constants which depend on s and n. With the above estimate we have proved that

$$(I_s f)(x) \leq C_1 |x|^{s-n} \int_{|y|<|x|} f(y)\, dy + C_2 \int_{|x|<|y|} |y|^{s-n} f(y)\, dy \tag{4.3}$$

for all non-negative functions $f \in R.\mathcal{M}.$ and consequently inequalities (1.1) and (1.1)* together imply $I_s : L_v^p(\mathcal{RM}) \to L_u^q$ ∎

Proof of Theorem 1.3. Note that by Theorem 1.1 the embedding $I_s : L_v^p(\mathcal{R}.M.) \to L_u^q$ is equivalent both to inequalities (1.3) and (1.3)*. To see the equivalence between (1.3) and (1.4) (resp. (1.3)* and (1.4)*) we will do a change of variable which preserves the monotonicity of the functions.

We first consider (1.3). Let $\Phi(t) = t^{\frac{1}{n}}$. Then it is clear that

$$\begin{aligned}
&\int_0^\infty \left[\int_0^r \rho^{n-1} \varphi(\rho)\, d\rho \right]^q r^{(s-n)q} \tilde{u}(r)\, dr \\
&\approx \int_0^\infty \left[\int_0^{r^n} \varphi(t^{\frac{1}{n}})\, dt \right]^q r^{(s-n)q} \tilde{u}(r)\, dr = \int_0^\infty [A\varphi \circ \Phi]^q (r^n) r^{sq} \tilde{u}(r)\, dr \\
&\approx \int_0^\infty [A(\varphi \circ \Phi)]^q (t)\, t^{\frac{1}{n}[sq+1-n]} \tilde{u}(t^{\frac{1}{n}})\, dt = \int_0^\infty [A(\varphi \circ \Phi)]^q (t) \mu(t)\, dt
\end{aligned}$$

with $\mu(t) = t^{\frac{1}{n}[sq+1-n]} \tilde{u}(t^{\frac{1}{n}})$. We also have

$$\begin{aligned}
\int_0^\infty \varphi^p(r) \tilde{v}(r)\, dr &= \int_0^\infty (\varphi \circ \Phi)^p (r^n) \tilde{v}(r)\, dr \\
&\approx \int_0^\infty (\varphi \circ \Phi)^p (t)\, \tilde{v}(t^{\frac{1}{n}})\, t^{\frac{1}{n}(1-n)} dt \\
&= \int_0^\infty (\varphi \circ \Phi)^p (t) \nu(t)\, dt
\end{aligned}$$

with $\nu(t) = t^{\frac{1}{n}(1-n)}\tilde{v}(t^{\frac{1}{n}})$. Consequently inequality (1.3) can be written as

$$\left(\int_0^\infty [A(\varphi\circ\Phi)]^q(t)\mu(t)\,dt\right)^{\frac{1}{q}} \leq c\left(\int_0^\infty (\varphi\circ\Phi)^p(t)\nu(t)\,dt\right)^{\frac{1}{p}}. \tag{4.4}$$

We are reduced to see the equivalence of this last one with inequality (1.4). The point is that Φ is an increasing and continuous function on $(0,\infty)$. Indeed suppose that (4.4) is true for all non-negative increasing (resp. decreasing) functions. Take an increasing (resp. decreasing) function ψ and define $\varphi = \psi\circ\Phi^{-1}$. This is an increasing (resp. decreasing) function and consequently by (4.4) the inequality (1.4) is true.

Conversely it is clear that (1.4) implies (4.4), since for each non-negative increasing (resp. decreasing) function φ also $\psi = \varphi\circ\Phi$ is an increasing (resp. decreasing) function.

Next we deal with inequality $(1.3)^*$. We set $\Theta(t) = (\ln t)^{1/s}$. It is clear that Θ is a non-negative continuous and increasing function on $(1,\infty)$. We have

$$\begin{aligned}
&\int_0^\infty \left[\int_r^\infty \rho^{s-1}\varphi(\rho)\,d\rho\right]^q \tilde{u}(r)\,dr \\
&\qquad\approx \int_0^\infty \left[\int_{\exp(r^s)}^\infty \varphi\big((\ln t)^{\frac{1}{s}}\big)t^{-1}\,dt\right]^q \tilde{u}(r)\,dr \\
&\qquad= \int_0^\infty [A^*\varphi\circ\Theta]^q\big(\exp(r^s)\big)\tilde{u}(r)\,dr \\
&\qquad\approx \int_1^\infty [A^*(\varphi\circ\Theta)]^q(t)(\ln t)^{\frac{1}{s}-1}\tilde{u}\big((\ln t)^{\frac{1}{s}}\big)t^{-1}\,dt \\
&\qquad= \int_1^\infty [A^*(\varphi\circ\Theta)]^q(t)\mu^*(t)\,dt
\end{aligned}$$

with $\mu^*(t) = (\ln t)^{\frac{1}{s}-1}t^{-1}\tilde{u}\big((\ln t)^{\frac{1}{s}}\big)t^{-1}$. On the other hand we also have

$$\begin{aligned}
\int_0^\infty \varphi^p(r)\tilde{v}(r)\,dr &\approx \int_1^\infty \varphi^p\big((\ln t)^{\frac{1}{s}}\big)(\ln t)^{\frac{1}{s}-1}\tilde{v}\big((\ln t)^{\frac{1}{s}}\big)t^{-1}\,dt \\
&= \int_1^\infty (\varphi\circ\Theta)^p(t)\nu^*(t)\,dt
\end{aligned}$$

with $\nu^*(t) = (\ln t)^{\frac{1}{s}-1}\tilde{v}\big((\ln t)^{\frac{1}{s}}\big)t^{-1}$. Consequently inequality $(1.3)^*$ can be written as

$$\left(\int_1^\infty [A^*(\varphi\circ\Theta)]^q(t)\mu^*(t)\,dt\right)^{\frac{1}{q}} \leq c\left(\int_1^\infty (\varphi\circ\Theta)^p(t)\nu^*(t)\,dt\right)^{\frac{1}{p}}.$$

The equivalence of this last inequality with condition $(1.4)^*$ can be seen as above ∎

Proof of Corollary 1.4. We have to prove that the Hardy inequality (1.4) (resp. $(1.4)^*$) with non-negative decreasing functions holds if and only if (1.9) and (1.10) together (resp. (1.11) and (1.12) together) are true.

As we have recalled in Section 1, inequality (1.4) with decreasing functions is equivalent to (1.5) and (1.6) together. Now with

$$\mu(t) = t^{\frac{1}{n}[sq+1-n]}\tilde{u}(t^{\frac{1}{n}}) \qquad \text{and} \qquad \nu(r) = t^{\frac{1}{n}[1-n]}\tilde{v}(t^{\frac{1}{n}})$$

we get the following:

$$\int_0^R \mu(t)\,dt = \int_0^R t^{\frac{1}{n}[sq+1-n]}\tilde{u}(t^{\frac{1}{n}})\,dt \approx \int_0^{R^{1/n}} r^{sq}\tilde{u}(r)\,dr \approx \int_{|x|<R^{1/n}} |x|^{sq}u(x)\,dx$$

$$\int_0^R \nu(t)\,dt = \int_0^R t^{\frac{1}{n}[1-n]}\tilde{v}(t^{\frac{1}{n}})\,dt \approx \int_0^{R^{1/n}} \tilde{v}(r)\,dr \approx \int_{|x|<R^{1/n}} v(x)\,dx$$

and

$$\begin{aligned}\int_R^\infty t^{-q}\mu(t)\,dt &= \int_R^\infty t^{\frac{1}{n}[(s-n)q+1-n]}\tilde{u}(t^{\frac{1}{n}})\,dt \\ &\approx \int_{R^{1/n}}^\infty r^{(s-n)q}\tilde{u}(r)\,dr \approx \int_{R^{1/n}<|x|} |x|^{(s-n)q}u(x)\,dx\end{aligned}$$

$$\begin{aligned}\int_0^R t^{p'}\left[\int_0^t \nu(\rho)\,d\rho\right]^{-p'}\nu(t)\,dt &= \int_0^R \left[\int_0^{t^{1/n}} \tilde{v}(\rho)\,d\rho\right]^{-p'} t^{\frac{1}{n}[np'+1-n]}\tilde{v}(t^{\frac{1}{n}})\,dt \\ &\approx \int_{|x|<R^{1/n}} \left[\int_{|y|<|x|} v(y)\,dy\right]^{-p'} |x|^{np'}v(x)\,dx.\end{aligned}$$

With these quantities, the conditions (1.5) and (1.6) are exactly (1.9) and (1.10).

Next we will prove the equivalence of (1.3)* with both conditions (1.11) and (1.12). Let $(Tg)(r) = \int_r^\infty t^{s-1}g(t)\,dt$ $(0 < s < n)$. Since we assume $\int_0^\infty \tilde{v}(r)\,dr \approx \int_{\mathbb{R}^n} v(x)\,dx = \infty$, by a result of Sawyer [7] the inequality (1.3)* is equivalent to

$$\begin{aligned}&\left(\int_0^\infty \left[\int_0^r (T^*g)(t)\,dt\right]^{p'}\left[\int_0^r \tilde{v}(t)\,dt\right]^{-p'}\tilde{v}(r)\,dr\right)^{\frac{1}{p'}} \\ &\qquad \le c\left(\int_0^\infty g^{q'}(r)\,(\tilde{u}(r))^{1-q'}\,dr\right)^{\frac{1}{q'}}\end{aligned} \tag{4.5}$$

for all non-negative functions g. First we have $(T^*h)(r) = r^{s-1}\int_0^r h(t)\,dt$. Indeed,

$$\begin{aligned}\int_0^\infty (Tg)(t)h(t)\,dt &= \int_0^\infty \left[\int_t^\infty \rho^{s-1}g(\rho)\,d\rho\right]h(t)\,dt \\ &= \int_0^\infty g(\rho)\left[\rho^{s-1}\int_0^\rho h(t)\,dt\right]d\rho.\end{aligned}$$

On the other hand, we have

$$\int_0^t (T^*h)(r)\,dr = \int_0^t \left[r^{s-1} \int_0^r h(\rho)\,d\rho \right] dr$$
$$= \int_0^t h(\rho) \left[\int_\rho^t r^{s-1}\,dr \right] d\rho \approx \int_0^t K(t,\rho)h(\rho)\,d\rho$$

with $K(t,\rho) = t^s - \rho^s$. Thus, inequality (4.5) is reduced to

$$\left(\int_0^\infty \left[\int_0^r K(r,\rho)h(\rho)\,d\rho \right]^{p'} \tilde{\nu}(r)\,dr \right)^{\frac{1}{p'}} \le c \left(\int_0^\infty h^{q'}(r)\tilde{\mu}(r)\,dr \right)^{\frac{1}{q'}} \tag{4.6}$$

for all non-negative functions h, where $\tilde{\nu}(r) = \left[\int_0^r \tilde{v}(t)\,dt \right]^{-p'} \tilde{v}(r)$ and $\tilde{\mu}(r) = \left(\tilde{u}(r)\right)^{1-q'}$. Such inequality was studied by Stepanov in [10] and is known to be equivalent both to

$$\text{(i)} \left(\int_R^\infty (r^s - R^s)^{p'} \tilde{\nu}(r)\,dr \right)^{\frac{1}{p'}} \left(\int_0^R \left(\tilde{\mu}(r)\right)^{1-q} dr \right)^{\frac{1}{q}} \le c_1$$

and

$$\text{(ii)} \left(\int_R^\infty \tilde{\nu}(r)\,dr \right)^{\frac{1}{p'}} \left(\int_0^R (R^s - r^s)^q \left(\tilde{\mu}(r)\right)^{1-q} dr \right)^{\frac{1}{q}} \le c_2$$

for all $R > 0$. Here we have

$$\int_0^R \left(\tilde{\mu}(r)\right)^{1-q} dr = \int_0^R \tilde{u}(r)\,dr \approx \int_{|x|<R} u(x)\,dx$$

and since $\int_0^\infty \tilde{v}(r)\,dr \approx \int_{\mathbb{R}^n} v(x)\,dx = \infty$, we have

$$\left(\int_R^\infty \tilde{\nu}(r)\,dr \right)^{\frac{1}{p'}} = \left(\int_R^\infty \left[\int_0^r \tilde{v}(t)\,dt \right]^{-p'} \tilde{v}(r)\,dr \right)^{\frac{1}{p'}}$$
$$\approx \left[\int_0^R \tilde{v}(t)\,dt \right]^{\frac{1-p'}{p'}} \approx \left[\int_{|x|<R} v(x)\,dx \right]^{-\frac{1}{p}}$$

$$\int_R^\infty (r^s - R^s)^{p'} \tilde{\nu}(r)\,dr \approx \int_{R<|x|} (|x|^s - R^s)^{p'} \left[\int_{|y|<|x|} v(y)\,dy \right]^{-p'} v(x)\,dx$$

$$\int_0^R (R^s - r^s)^q \left(\tilde{\mu}(r)\right)^{1-q} dr \approx \int_{|x|<R} (R^s - |x|^s)^q u(x)\,dx.$$

With these expressions, then conditions (i) and (ii) are exactly (1.12) and (1.11), respectively ∎

Proof of Corollary 1.5. As in the proof of Corollary 1.4, we have to get the equivalence of the Hardy inequality (1.4) (resp. (1.4)*) for non-negative decreasing functions with conditions (1.13) and (1.14) (resp. (1.15) and (1.16)).

As we have seen in Section 1 for the range $1 < q < p < \infty$, inequality (1.4) is equivalent both to (1.7) and (1.8) with

$$\mu(t) = t^{\frac{1}{n}[sq+1-n]}\tilde{u}(t^{\frac{1}{n}}) \qquad \text{and} \qquad \nu(r) = t^{\frac{1}{n}[1-n]}\tilde{v}(t^{\frac{1}{n}}).$$

Using the above computations of $\int_0^R \mu(r)\,dr$ and $\int_0^R \nu(r)\,dr$ and by making change of variables condition (1.7) takes the form as (1.13). Also using the above expressions, we see that condition (1.8) is the same as (1.14).

Again the key to obtain (1.4)* is the Hardy inequality (4.6) which is equivalent both to (for the range $1 < q < p < \infty$)

$$\text{(iii)} \int_0^\infty \left[\left(\int_t^\infty (r^s - t^s)^{p'} \tilde{\nu}(r)\,dr \right)^{\frac{1}{p'}} \left(\int_0^t (\tilde{\mu}(r))^{1-q} dr \right)^{\frac{1}{p}} \right]^\theta (\tilde{\mu}(t))^{1-q} dt$$

and

$$\text{(iv)} \int_0^\infty \left[\left(\int_t^\infty \tilde{\nu}(r)\,dr \right)^{\frac{1}{q'}} \left(\int_0^t (t^s - r^s)^q (\tilde{\mu}(r))^{1-q} dr \right)^{\frac{1}{q}} \right]^\theta \tilde{\nu}(t)\,dt < \infty$$

where $\frac{1}{\theta} = \frac{1}{q} - \frac{1}{p}$ and

$$\tilde{\mu}(r) = (\tilde{u}(r))^{1-q'}, \qquad \tilde{u}(r) = (\tilde{\mu}(r))^{1-q}, \qquad \tilde{\nu}(r) = \left[\int_0^r \tilde{v} \right]^{-p'} \tilde{v}(r).$$

The condition (iii) is the same as (1.16). Since $\int_R^\infty \tilde{\nu}(r)\,dr \approx \left[\int_0^R \tilde{v} \right]^{(1-p')}$ and $\frac{1-p'}{q'}\theta - p' = -\frac{1}{q}\theta$, (iv) yields the condition (1.15) ∎

Proof of Theorem 2.1. As in the proof of Theorem 1.1, by (4.1) and (4.2) the embedding $I_s : L_v^{p_1p_2}(w_1) \to L_u^{q_1q_2}(w_2)$ implies conditions (2.1) and (2.1)*. For the converse we have only to observe that by (4.3) both (2.1) and (2.1)* imply $I_s : L_v^{p_1p_2}(w_1)[\mathcal{RM}] \to L_u^{q_1q_2}(w_2)$ ∎

Proof of Proposition 2.2. Let a and b be measurable non-negative functions. The Hardy type operators we consider are

$$(\mathcal{H}f)(x) = a(x) \int_{|y|\le|x|} b(y)f(y)\,dy \quad \text{and} \quad (\mathcal{H}^*g)(x) = b(x) \int_{|x|\le|y|} a(y)g(y)\,dy.$$

We have the following

Lemma. *Let p_1, p_2 and q_1, q_2 reals satisfying conditions* (2.0) *and* (2.2). *Then*

$$\|\mathcal{H}f\|_{L^{p_2 q_2}_u} \leq c\|f\|_{L^{p_1 q_1}_v} \qquad \text{for all } f$$

if and only if

$$\sup_{R>0} \left\| a\mathbf{1}_{R<|\cdot|} \right\|_{L^{p_2 q_2}_u} \left\| \frac{1}{v}\, b\mathbf{1}_{|\cdot|<R} \right\|_{L^{p_1' q_1'}_v} < \infty. \tag{4.7}$$

Similarly

$$\|\mathcal{H}^* g\|_{L^{p_2 q_2}_v} \leq c\|g(\cdot)\|_{L^{p_1 q_1}_u} \qquad \text{for all } g$$

if and only if

$$\sup_{R>0} \left\| b\mathbf{1}_{|\cdot|<R} \right\|_{L^{p_2 q_2}_u} \left\| \frac{1}{v} a\mathbf{1}_{R<|\cdot|} \right\|_{L^{p_1' q_1'}_v} < \infty. \tag{4.7)*$$

The first part of this lemma can be obtained by adapting Theorem 3 in [2]. The second part can be deduced by the first one by using a duality argument.

In continuation of the proof of Proposition 2.2, now the equivalence of (2.1) and (2.3) is given by the first part of this Lemma by taking $a(x) = w_2(x)|x|^{s-n}$ and $b(y) = \frac{1}{w_1(y)}$. The next part with $b(x) = w_2(x)$ and $a(y) = \frac{1}{w_1(y)}|y|^{s-n}$ involves the equivalence of (2.1)* and (2.3)* ∎

Proof of Theorem 3.1. Since the proof is similar to that of Theorem 2.1, we leave any detail ∎

Proof of Proposition 3.2. As in the proof of Proposition 3.1, we are reduced to get the following

Lemma. *Let Φ_1 be an N-function and Φ_2 a φ-function with $\Phi_1\Phi_2^{-1}$ subadditive. Then*

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\big(C_2(\mathcal{H}f)(x)\big)u(x)\,dx\right] \leq \Phi_1^{-1}\left[C_1 \int_{\mathbb{R}^n} \Phi_1\big(f(x)\big)v(x)\,dx\right]$$

for all non-negative functions f is equivalent to

$$\Phi_2^{-1}\left[\int_{R<|x|} \Phi_2\left(c_2 a(x)\left\|\frac{b}{\varepsilon v}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v}\right)u(x)\,dx\right] \leq \Phi_1^{-1}[c_1\varepsilon^{-1}] \tag{4.8}$$

for all $\varepsilon > 0$ and $R > 0$. Also

$$\Phi_2^{-1}\left[\int_{\mathbb{R}^n} \Phi_2\big(C_2(\mathcal{H}^* g)(x)\big)u(x)\,dx\right] \leq \Phi_1^{-1}\left[C_1 \int_{\mathbb{R}^n} \Phi_1\big(g(x)\big)v(x)\,dx\right]$$

for all non-negative functions g is equivalent to

$$\Phi_2^{-1}\left[\int_{|x|<R} \Phi_2\left(c_2' b(x)\left\|\frac{a}{\varepsilon v}\mathbf{1}_{R<|\cdot|}\right\|_{\Phi_1^*,\varepsilon v}\right)u(x)\,dx\right] \leq \Phi_1^{-1}[c_1'\varepsilon^{-1}] \tag{4.8)*$$

for all $\varepsilon > 0$ *and* $R > 0$.

Proof. Since the proofs of both parts are similar, we only prove the first one and begin with the necessity of the condition (4.8). Clearly the inequality in the first part of the Lemma implies

$$\Phi_2^{-1}\left[\int_{R<|x|}\Phi_2\left(C_2a(x)\left[\int_{|y|<R}b(y)f(y)\,dy\right]\right)u(x)\,dx\right] \leq \Phi_1^{-1}\left[C_1\int_{|x|<R}\Phi_1\big(f(x)\big)v(x)\,dx\right] \tag{4.9}$$

for all non-negative functions f and all $R > 0$. Condition (4.8) is a consequence of this last inequality. Indeed, let $\varepsilon > 0$ and $R > 0$. Then by the definition of the Orlicz norm

$$S = \left\|\frac{b}{\varepsilon v}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v} = \int_{|x|<R} b(y)f(y)\,dy$$

for some non-negative function f with $\int_{|x|<R}\Phi_1\big(f(x)\big)\varepsilon v(x)\,dx \leq 1$ and consequently we obtain

$$\begin{aligned}
S &= \Phi_2^{-1}\left[\int_{R<|x|}\Phi_2\left(C_2a(x)\left\|\frac{b}{\varepsilon v}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v}\right)u(x)\,dx\right]\\
&= \Phi_2^{-1}\left[\int_{R<|x|}\Phi_2\left(C_2a(x)\int_{|y|<R}b(y)f(y)\,dy\right)u(x)\,dx\right]\\
&\leq \Phi_1^{-1}\left[C_1\int_{|x|<R}\Phi_1\big(f(x)\big)v(x)\,dx\right]\\
&\leq \Phi_1^{-1}[C_1\varepsilon^{-1}].
\end{aligned}$$

Now we deal with the sufficiency of the condition (4.8). Without loss on generality (and to simplify) we can only do the proof when $n = 1$ and on $(0,\infty)$. Let f be a non-negative function on $(0,\infty)$. If $\int_0^\infty b(y)f(y)\,dy < \infty$, then for some integer m, $\int_0^\infty b(y)f(y)\,dy \in [2^m, 2^{m+1}]$ and there is an increasing sequence of non-negative reals $(x_k)_{k=-\infty}^m$ so that

$$\begin{aligned}
2^k &= \int_0^{x_k} b(y)f(y)\,dy = \int_{x_k}^{x_{k+1}} b(y)f(y)\,dy \quad (k \leq m-1)\\
2^m &= \int_0^{x_m} b(y)f(y)\,dy.
\end{aligned}$$

Thus with $x_{m+1} = \infty$ we can write

$$(0,\infty) = \bigcup_{k=-\infty}^{m}[x_k, x_{k+1}).$$

When $\int_0^\infty b(y)f(y)\,dy = \infty$, the first identities also holds for all integer k, and the last one remains still valid. The main key for our proof is again the inequality (4.9) with $R = x_k$ and the function $f\mathbf{1}_{[x_{k-1},x_k)}$. We will postpone below the proof of (4.9). With these preliminaries we can obtain the conclusion as follow:

$$
\begin{aligned}
S &= \int_0^\infty \Phi_2(4^{-1}C_2(\mathcal{H}f)(x))u(x)\,dx \\
&\le \sum_{k\le m}\int_{x_k}^{x_{k+1}} \Phi_2\left(4^{-1}4C_2\,a(x)\left[\int_0^\infty b(y)\mathbf{1}_{[x_{k-1},x_k)}(y)f(y)\,dy\right]\right)u(x)\,dx \\
&\le \sum_{k\le m}\int_{x_k}^{\infty} \Phi_2\left(C_2\,a(x)\left[\int_0^{x_k} b(y)\mathbf{1}_{[x_{k-1},x_k)}(y)f(y)\,dy\right]\right)u(x)\,dx \\
&\le \sum_{k\le m}\Phi_1\Phi_2^{-1}\left[C_1\int_0^{x_k}\Phi_1\left(f(x)\mathbf{1}_{[x_{k-1},x_k)}(x)\right)v(x)\,dx\right] \\
&\le \Phi_1\Phi_2^{-1}\left[C_1\sum_{k\le m}\int_{x_{k-1}}^{x_k}\Phi_1(f(x))v(x)\,dx\right] \\
&\le \Phi_1\Phi_2^{-1}\left[C_1\int_0^\infty \Phi_1(f(x))v(x)\,dx\right].
\end{aligned}
$$

Finally we show how condition (4.8) implies (4.9). Indeed let $R > 0$. We can assume that

$$0 < \int_{|x|<R} \Phi_1(f(x))v(x)\,dx < \infty.$$

Thus if we choose an $\varepsilon = \varepsilon(f, R) > 0$ with

$$\int_{|x|<R} \Phi_1(f(x))\varepsilon v(x)\,dx = 1,$$

then $\|f\mathbf{1}_{|\cdot|<R}\|_{(\Phi_1),\varepsilon v} \le 1$. Consequently, by the Hölder inequality and condition (4.8), we get

$$
\begin{aligned}
\mathcal{S} &= \int_{R<|x|} \Phi_2\left[C_2\,a(x)\left(\int_{\mathbb{R}^n} bf\mathbf{1}_{|\cdot|<R}\right)\right]u(x)\,dx \\
&= \int_{R<|x|} \Phi_2\left[C_2\,a(x)\left\|\frac{b}{\varepsilon v}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v}\|f\mathbf{1}_{|\cdot|<R}\|_{(\Phi_1),\varepsilon v}\right]u(x)\,dx \\
&= \int_{R<|x|} \Phi_2\left[C_2\,a(x)\left\|\frac{b}{\varepsilon v}\mathbf{1}_{|\cdot|<R}\right\|_{\Phi_1^*,\varepsilon v}\right]u(x)\,dx \\
&\le \Phi_1\Phi_2^{-1}[C_1\varepsilon^{-1}] \\
&= \Phi_1\Phi_2^{-1}\left[C_1\int_{|x|<R}\Phi_1(f(x))v(x)\,dx\right]
\end{aligned}
$$

and the assertion is proved ∎

References

[1] Bloom, S. and R. Kerman: *Weighted L_Φ-integral inequalities for operators of Hardy type.* Studia Math. 110 (1994), 35 - 52.

[2] Edmunds, D., Gurka, P. and L. Pick: *Compactness of Hardy-type integral operators in weighted Banach functions spaces.* Studia Math. 109 (1994), 73 - 90.

[3] Heinig, H. and V. D. Stepanov: *Weighted Hardy inequalities for increasing functions.* Canad. J. Math. 45 (1993), 104 - 116.

[4] Kokilashvili, V. and M. Krbec: *Weighted Inequalities in Lorentz and Orlicz Spaces.* Singapore et al.: World Scientific 1991.

[5] Opic, B. and P. Gurka: *Hardy-Type Inequalities* (Pitman Research Notes in Mathematics Series: Vol. 211). Harlow: Longman Sci. & Techn. 1990.

[6] Qisheng, L.: *Two weight Φ-inequalities for the Hardy operator, Hardy-Littlewood maximal operator, and fractional integrals.* Proc. Amer. Math. Soc. 118 (1991), 129 - 141.

[7] Sawyer, E.: *Boundedness of classical operators on classical Lorentz spaces.* Studia Math. 96 (1990), 149 - 158.

[8] Sawyer, E. and R. L. Wheeden: *Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces.* Amer. J. Math. 114 (1992), 813 - 874.

[9] Stepanov, V. D.: *The weighted Hardy's inequality for non-increasing functions.* Trans. Amer. Math. Soc. 338 (1993), 173 - 186.

[10] Stepanov, V. D.: *Weighted norm inequalities of Hardy type for a class of integral operators.* J. London Math. Soc. 50 (1994), 105 - 120.

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