On the Variational Stability
of a Class of
Nonlinear Parabolic Optimal Control Problems

N. S. Papageorgiou

Abstract. In this paper we study parametric optimal control problems governed by a nonlinear parabolic equation in divergence form. The parameter appears in all the data of the problem, including the partial differential operator. Using as tools the G-convergence of operators and the Γ-convergence of functionals, we show that the set-valued map of optimal pairs is upper-semicontinuous with respect to the parameter and the optimal value function responds continuously to changes of the parameter. Finally, in the case of semilinear systems we show that our framework can also incorporate systems with weakly convergent coefficients.

Keywords: G-convergence, Γ-convergence, evolution triples, monotone operators, compact embeddings, upper-semicontinuous multifunctions, semilinear systems

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1. Introduction

In this paper we study the dependence of a nonlinear optimal control problem on a parameter. The parameter appears in all the data of the problem, including the partial differential operator. First we establish the non-emptiness of the set of optimal "state-control" pairs and then we investigate how this set as well as the value of the problem respond to changes of the parameter. Such sensitivity analysis (also known in the literature as "variational stability") is important because it gives us information concerning the tolerances that are permitted in the specification of mathematical models, it suggests ways to solve parametric problems and can also give us valuable insight for the comuputational treatment of the problem.

Our tools are the G-convergence of operators and the Γ-convergence of functionals. Using these two convergence concepts, we derive continuous dependence results. Our approach follows that of Buttazzo and Dal Maso [2], who examined linear elliptic control systems and systems monitored by ordinary differential equations. Here we consider parabolic systems with nonlinear dynamics.

Γ-convergence is a convergence notion for sequences of functions specially designed in order to study convergence of solutions and values of corresponding minimization

N. S. Papageorgiou: Nat. Techn. Univ., Dept. Math., Zografou Campus, Athens 15780, Greece and Florida Inst. Techn., Dep. Math., 150 W. Univ. Blvd., Melbourne, FL 32901-6988, USA.

problems; i.e. it is a "variational convergence". Among variational convergences, Γ-convergence plays an important role for its nice compactness properties and for the powerful results it generates concerning the limits of integral functionals. In addition, Γ-convergence is closely related to the notion of G-convergence, which is used in the study of the convergence properties of the solutions of a sequence of elliptic and parabolic problems. Finally, almost all other variational convergences can be easily expressed in the language of Γ-convergence. Hence it seems appropriate to use these notions to study the variational stability (sensitivity) of optimal control problems.

2. Preliminaries

Let H be a separable Hilbert space of norm $|\cdot|$. Let X be a reflexive, separable Banach space with dual X^* such that $X \subseteq H \subseteq X^*$ with dense and compact injections. The norms of X and X^* will be denoted by $\|\cdot\|$ and $\|\cdot\|_*$, respectively. We will use $\langle x, x^* \rangle$ to denote the duality brackets between $x \in X$ and $x^* \in X^*$. This coincides with the inner product in H, whenever $x^* \in H$. Such a triple of spaces is usually known in the literature as "evolution triple" (see Zeidler [13]; the names "Gelfand triple" or "spaces in normal position" are also used). In concrete applications, evolution triples are generated by Sobolev spaces (see Section 3).

Let $T = [a, b]$ and define

$$W(T) = \Big\{ x \in L^2(T, X) : \dot{x} \in L^2(T, X^*) \Big\}.$$

In this definition, the derivative of x is understood in the sense of vector-valued distributions. Furnished with the norm

$$\|x\|_{W(T)} = \Big(\|x\|^2_{L^2(T,X)} + \|\dot{x}\|^2_{L^2(T,X^*)} \Big)^{1/2},$$

$W(T)$ becomes a Banach space which is separable and reflexive. Furthermore, $W(T)$ embeds continuously into $C(T, H)$ and compactly into $L^2(T, H)$. When X is a Hilbert space too, then so is $W(T)$ with inner product $(x, y)_{W(T)} = (x, y)_{L^2(T,X)} + (\dot{x}, \dot{y})_{L^2(T,X^*)}$. For further details we refer to Zeidler [13: Proposition 23.23, pp. 422-423 and p. 450]).

Following Kolpakov [6], we say that a sequence of operators $A_n : X \to X^*$ G-converges to an operator $A : X \to X^*$ if, for all $n \geq 1$, the inverse operators $A_n^{-1}, A^{-1} : X^* \to X$ are defined and, for every $x^* \in X^*$, $A_n^{-1} x^* \to A^{-1} x^*$ weakly in X (hence strongly in H). We will use the symbol G to indicate G-convergence. This is a nonlinear version of a convergence concept introduced first by Spagnolo [11] for linear parabolic and elliptic equations and which was later extended to abstract linear evolution equations by Zhikov, Kozlov and Oleinik [14].

Next, following Buttazzo and Dal Maso ([1: Chapter 5] and [2]), we introduce the notion of multiple sequential Γ-convergence. So let X_1 and X_2 be two topological spaces and let $f_n : X_1 \times X_2 \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ $(n \geq 1)$ be a sequence of functionals. We indicate by $Z(+)$ the *sup* operator and by $Z(-)$ the *inf* operator. Let $(x_1, x_2) \in X_1 \times X_2$

and denote by S_i the set of all sequences in X_i converging to x_i $(i = 1, 2)$. Finally let α_j $(j = 0, 1, 2)$ be one of the signs $+$ or $-$. We define

$$\Gamma_{\text{seq}}\big(\mathbb{N}^{\alpha_0}, X_1^{\alpha_2}, X_2^{\alpha_2}\big)f_n(x_1, x_2)$$
$$= Z(\alpha_1)_{(x_1^n)\in S_1} Z(\alpha_2)_{(x_2^n)\in S_2} Z(-\alpha_0)_{k\in\mathbb{N}} Z(\alpha_0)_{n\geq k} f_n(x_1^n, x_2^n).$$

So, for example,

$$\Gamma_{\text{seq}}\big(\mathbb{N}^+, X_1^-, X_2^+\big)f_n(x_1, x_2) = \inf_{x_1^n\to x_1} \sup_{x_2^n\to x_2} \limsup_{n\to\infty} f_n(x_1^n, x_2^n).$$

When the Γ_{seq}-limit is independent of the sign $+$ or $-$, associated with one of the spaces, then this sign is omitted. So, for example, if

$$\Gamma_{\text{seq}}\big(\mathbb{N}^+, X_1^-, X_2^+\big)f_n(x_1, x_2) = \Gamma_{\text{seq}}\big(\mathbb{N}^-, X_1^-, X_2^+\big)f_n(x_1, x_2),$$

then their common value will be indicated by $\Gamma_{\text{seq}}\big(\mathbb{N}, X_1^-, X_2^+\big)f_n(x_1, x_2)$.

If the topological spaces are first countable, then the above definition is equivalent to the original topological definition of the Γ-limits (see Dal Maso [3: Proposition 8.1/p. 86]). This is also the case in Banach spaces with a separable dual, equipped with the weak topology, and in reflexive Banach spaces again with the weak topology (see Dal Maso [3: Chapter 8]). The theory of Γ-convergence is an important tool in Optimal Control and in the Calculus of Variations, because the equicoercivity and the Γ-convergence of a sequence of functionals f_n to f, f not identically $+\infty$, imply the convergence of the minimizers $(x_n \to x)$ and of the corresponding minimal values $(f_n(x_n) \to f(x))$ (see Dal Maso [3: Theorem 7.19/ p. 80]). The interested reader can find a comprehensive introduction to the subject of Γ-convergence and its applications in the well-written monographs of Buttazzo [1] and Dal Maso [3].

Next we introduce our optimal control problem. So let $T = [0, r]$ and Z a bounded domain in $\mathbb{R}^N$ with smooth boundary $\Gamma = \partial Z$. Let Λ be a complete metric space of $\mathbb{R}^k$-valued, measurable functions defined on Z (the parameter space) with metric $d(\cdot, \cdot)$. We will be studying the following optimal control problem, parametrized by elements in Λ:

$$\left.\begin{array}{l} \displaystyle\int_0^r\!\!\int_Z L\big(t, z, x(t, z), u(t, z), \lambda(z)\big)dz\,dt \;\longrightarrow\; \inf = m(\lambda) \\[2mm] \text{such that} \\[2mm] \dfrac{\partial x}{\partial t} - \text{div}\big(a\big(z, Dx(t, z), \lambda(z)\big)\big) = f\big(t, z, x(t, z), \lambda(z)\big)\,u(t, z) \text{ a.e.} \\[2mm] x\big|_{T\times\Gamma} = 0 \quad\text{and}\quad x(0, z) = x_0(z) \;\; \big(x_0 \in L^2(Z)\big) \\[2mm] |u(t, z)| \leq \theta\big(t, z, \lambda(z)\big) \quad\text{with } u \text{ measurable} \end{array}\right\} \tag{1}$$

where $Dx = (D_1 x, \ldots, D_N x_N)$ denotes the gradient of x.

Throughout this paper the following hypotheses will be in effect.

H(a) $a(z, v, \lambda) = D_v \varphi(z, v, \lambda)$, where D_v denotes gradient with respect to the $v \in \mathbb{R}^N$ variable and $\varphi : Z \times \mathbb{R}^N \times \mathbb{R}^k \to \mathbb{R}$ is a function satisfying the following conditions:

1. $(z, \lambda) \to \varphi(z, v, \lambda)$ is measurable.

2. $v \to \varphi(z, v, \lambda)$ is convex.

3. $c_{1B} \|v\|^2 - c_{2B} \leq \varphi(z, v, \lambda(z)) \leq c_{3B}(1 + \|v\|^2)$ a.e. on Z for every $[v, \lambda] \in \mathbb{R}^N \times B$ with $B \subseteq \Lambda$ compact, $0 < c_{1B} \leq c_{3B} < \infty$ and $0 \leq c_{2B} < \infty$.

4. $\int_Z \Big(a\big(z, Dx_1(z), \lambda(z)\big) - a\big(z, Dx_2(z), \lambda(z)\big) \Big)\big(Dx_1(z) - Dx_2(z)\big) dz$
   $\geq \gamma_{1B} \|x_1 - x_2\|_{H_0^1(Z)}^2$ for every $x_1, x_2 \in H_0^1(Z)$ and every $\lambda \in B$, with $B \subseteq \Lambda$ compact, $\gamma_{1B} > 0, a(z, 0, \lambda) = 0$ and
   $\big\| a\big(\cdot, Dx(\cdot), \lambda(\cdot)\big) \big\|_{L^2(Z, \mathbb{R}^N)} \leq \gamma_{2B}\big(1 + \|x\|_{H_0^1(Z)}\big)$ for all $x \in H_0^1(Z)$ with $\gamma_{2B} > 0$.

5. $\varphi\big(z, v, \lambda_n(z)\big) \to \varphi\big(z, v, \lambda(z)\big)$ a.e. on Z when $\lambda_n \to \lambda$ in the metric space Λ.

H(f) $f : T \times Z \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is a function satisfying the following conditions:

1. $(t, z, \lambda) \to f(t, z, x, \lambda)$ is measurable.

2. $\big| f\big(t, z, x, \lambda(z)\big) - f\big(t, z, y, \lambda(z)\big) \big| \leq k_B(t, z)|x - y|$ a.e. for all $\lambda \in B, B \subseteq \Lambda$ compact, and $k_B \in L^1\big(T, L^\infty(Z)\big)$.

3. $\big| f\big(t, z, x, \lambda(z)\big) \big| \leq a_B(t, z) + b_B|x|$ a.e. for all $\lambda \in B, B \subseteq \Lambda$ compact, with $a_B \in L^2(T \times Z)$ and $b_B \geq 0$.

4. $f\big(t, \cdot, x, \lambda(\cdot)\big) \to_s f\big(t, \cdot, x, \lambda(\cdot)\big)$ in $L^2(Z)$ for almost all $t \in T$ when $\lambda_n \to \lambda$ in Λ.

Remark 1. We could have assumed that the controls are $\mathbb{R}^m$-valued $(m > 1)$, in which case $f = f(t, z, x, \lambda)$ is $\mathbb{R}^m$-valued and the right-hand side of the partial differential equation becomes $\big(f\big(t, z, x(z), \lambda(z)\big), u(z)\big)_{\mathbb{R}^m}$ with $(\cdot, \cdot)_{\mathbb{R}^m}$ denoting the Euclidean inner product in $\mathbb{R}^m$. However to simplify our notation, we have assumed that $m = 1$.

For a function $L = L(t, z, x, u, \lambda)$ let L^* denote the conjugate function with respect to u, i.e.

$$L^*(t, z, x, u^*, \lambda) = \sup_{u \in \mathbb{R}^k} \{(u^*, u)_{\mathbb{R}^k} - L(t, z, x, u, \lambda)\}$$

where $(\cdot, \cdot)_{\mathbb{R}^k}$ is the Euclidean inner product in $\mathbb{R}^k$.

H(θ) $\theta : T \times Z \times \mathbb{R}^k \to \mathbb{R}_+$ is a function satisfying the following conditions:

1. $\theta\big(\cdot, \cdot, \lambda(\cdot)\big) \in L^\infty(T \times Z)$ and $\sup\big\{ \big\| \theta\big(\cdot, \cdot, \lambda(\cdot)\big) \big\|_\infty : \lambda \in B \big\} < \infty$ for any $B \subseteq \Lambda$ compact.

2. $\theta\big(t, \cdot, \lambda_n(\cdot)\big) \to_s \theta\big(t, \cdot, \lambda(\cdot)\big)$ in $L^2(Z)$ for almost all $t \in T$, when $\lambda_n \to \lambda$ in Λ.

H(L) $L : T \times Z \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ is an integrand satisfying the following conditions:

1. $(t, z, \lambda) \to L(t, z, x, u, \lambda)$ is measurable.

2. $|u|^2 \leq L\big(t, z, x, u, \lambda(z)\big) \leq \gamma_{3B}\big(1 + |x|^2 + |u|^2\big)$ a.e. on $T \times Z$, for all $[x, u, \lambda] \in \mathbb{R} \times \mathbb{R} \times B, B \subseteq \Lambda$ compact, and with $\gamma_{3B} > 0$.

3. $\big| L\big(t, z, x, u, \lambda(z)\big) - L\big(t, z, y, u, \lambda(z)\big) \big| \leq \rho_B(|x - y|)\big(\ell + |x|^2 + |u|^2\big)$ for every $x, y \in \mathbb{R}$ such that $|x - y| < 1$ and every $\lambda \in B, B \subseteq \Lambda$ compact, where

$\rho_B : [0, 1) \to \mathbb{R}$ is increasing, continuous and $\rho_B(0) = 0$, and $\ell \geq 0$.

4. $u \to L(t, z, x, u, \lambda$ is convex.

5. $L^*\big(\cdot, \cdot, x, u^*, \lambda_n(\cdot)\big) \to L^*\big(\cdot, \cdot, x, u^*, \lambda(\cdot)\big)$ weakly in $L^1(T \times Z)$ for every $[x, u^*] \in \mathbb{R} \times \mathbb{R}$, when $\lambda_n \to \lambda$ in Λ.

Remark 2. This hypothesis which incorporates the quadratic cost functionals considered by Lions [7], will guarantee the $\Gamma_{\mathbf{seq}}$ -convergence of

$$J(x, u, \lambda_n) = \int\limits_0^b \int\limits_Z L\Big(t, z, x(t, z), u(t, z), \lambda_n(z)\Big) dz dt$$

as $\lambda_n \to \lambda$ in Λ. In particular, assume that the controls take values in $\mathbb{R}^m$ and consider the quadratic cost functional

$$J(x, u, \lambda) = \int\limits_0^b \int\limits_Z \big|x(t, z) - y_0\big(t, z, \lambda(z)\big)\big|^2 dz dt$$

$$+ \frac{1}{2} \int\limits_0^b \int\limits_Z \Big(N\big(t, z, \lambda(z)\big)u(t, z),\ u(t, z)\Big)_{\mathbb{R}^m} dz dt$$

with $y_0\big(\cdot, \cdot, \lambda(\cdot)\big) \in L^2(T \times Z)$ and $N\big(\cdot, \cdot, \lambda(\cdot)\big) \in L^1(T \times Z, \mathbb{R}^{m \times m})$ for every $\lambda(\cdot) \in \Lambda$. We assume that, for every $(t, z) \in T \times Z$, $N\big(t, z, \lambda(z)\big)$ is symmetric and positive definite and so $N\big(t, z, \lambda(z)\big)^{-1}$ exists. Furthermore, we assume that if $\lambda_n \to \lambda$, then

$$y_0\big(\cdot, \cdot, \lambda_n(\cdot)\big) \to y_0\big(\cdot, \cdot, \lambda(\cdot)\big) \qquad \text{and} \qquad y_0\big(\cdot, \cdot, \lambda_n(\cdot)\big)^2 \to y_0\big(\cdot, \cdot, \lambda(\cdot)\big)^2$$

weakly in $L^1(T \times Z)$ and

$$N\big(\cdot, \cdot, \lambda_n(\cdot)\big)^{-1} \to N\big(\cdot, \cdot, \lambda(\cdot)\big)^{-1}$$

weakly in $L^1(T \times Z, \mathbb{R}^{m \times m})$. So if we set

$$L(t, z, x, u, \lambda) = \big|x - y_0\big(t, z, \lambda(z)\big)\big|^2 + \frac{1}{2}\big(N\big(t, z, \lambda(z)\big)u, u\big)_{\mathbb{R}^m},$$

we have that

$$L^*(t, z, x, u^*, \lambda) = \frac{1}{2}\Big(N\big(t, z, \lambda(z)\big)^{-1}u^*, u^*\Big)_{\mathbb{R}^m} - \big|x - y_0(t, z, \lambda)\big|^2$$

and for this function L hypothesis H(L) is satisfied. This type of cost functionals was used by Lions [7].

3. Convergence of the costs and the constraints

We start with the Γ_{seq}-convergence of the costs, which can be obtained directly from the results of Buttazzo and Dal Maso [2]. Let $H = L^2(Z), X = H_0^1(Z)$ and $X^* = H^{-1}(Z)$. From the Sobolev embedding theorem, we know that (X, H, X^*) is an evolution triple. For this triple we will use the notation introduced in Section 2. Denote

$$J(x, u, \lambda) = \int_0^b \int_Z L\big(t, z, x(t, z), u(t, z), \lambda(z)\big) dz dt$$

for all $(x, u, \lambda) \in W(T) \times L^2(T, H) \times \Lambda$, where $W(T)$ is the Hilbert space introduced in Section 2.

Proposition 2. *If hypothesis* H(L) *holds and* $\lambda_n \to \lambda$ *in* Λ, *then*

$$\Gamma_{\text{seq}}\big(\mathbb{N}, w\text{-}W(T), w\text{-}L^2(T, H)^-\big) J(x, u, \lambda_n) = J(x, u, \lambda)$$

where $w\text{-}W(T)$ *and* $w\text{-}L^2(T, H)$ *denote the Hilbert spaces* $W(T)$ *and* $L^2(T, H) = L^2(T \times Z)$, *respectively, furnished with the weak topology.*

Proof. It goes exactly as the proof of Lemma 3.1 of Buttazzo and Dal Maso [2], with the independent variable $z \in Z$ replaced by $(t, z) \in T \times Z$, since our system is parabolic and not elliptic. Also recall that $W(T)$ embeds compactly into $L^2(T, H)$ (see Section 2), which allows us to consider $W(T)$ with the weak topology instead of $L^2(T, H)$ with the strong topology ∎

Next let $p : H_0^1(Z) \times H_0^1(Z) \times \Lambda \to \mathbb{R}$ be the Dirichlet form defined by

$$p(x, y, \lambda) = \int_Z a\big(z, Dx(z), \lambda(z)\big) Dy(z) \, dz.$$

From the Cauchy-Schwartz inequality, we have for all $\lambda \in B$, $B \subseteq \Lambda$ compact (cf. hypothesis H(a)/4):

$$|p(x, y, \lambda)| \le \big\|a(\cdot, Dx(\cdot), \lambda(\cdot))\big\|_{L^2(Z, \mathbb{R}^N)} \big\|Dy\big\|_{L^2(Z, \mathbb{R}^N)}$$
$$\le \gamma_{2B}\big(1 + \|x\|_{H_0^1(Z)}\big) \|y\|_{H_0^1(Z)}.$$

So there exists a generally nonlinear operator $A : X \times \Lambda \to X^*$ defined by

$$\langle A(x, \lambda), y \rangle = p(x, y, \lambda) \quad \text{for every } x, y, \in X \text{ and } \lambda \in \Lambda$$

where $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $\big(H_0^1(Z), H^{-1}(Z)\big)$ (see Section 2). Also let $\hat{f} : T \times H \times \Lambda \to H$ be the Nemitsky (superposition) operator corresponding to the function $F = f(t, z, x, \lambda)$, i.e. $\hat{f}(t, x, \lambda)(z) = f\big(t, z, x(z), \lambda(z)\big)$. Furthermore, let

$$U(t, \lambda) = \Big\{ u \in L^2(Z) : |u(z)| \le \theta(t, z, \lambda(z)) \text{ a.e.} \Big\}.$$

Clearly because of hypothesis H(θ), $U(t, \lambda) \subseteq L^\infty(Z)$ for every $(t, \lambda) \in T \times \Lambda$ and so given any $u \in U(t, \lambda)$, $(\hat{f}(t, x, \lambda)u)(\cdot) \in L^2(Z)$. Note that, for every $\lambda \in \Lambda$, the graph of the set-valued map $t \to U(t, \lambda)$ is given by

$$GrU(\cdot, \lambda) = \left\{ (t, u) \in T \times L^2(Z) \middle| \int_C |u(z)| \, dz \leq \int_C \theta(t, z, \lambda(z)) \, dz, \ C \in B(Z) \right\}$$

with $B(Z)$ being the Borel σ-field of Z. Let

$$\xi_1(u, C) = \int_C |u(z)| \, dz \qquad \text{and} \qquad \xi_2(u, C) = \int_C \theta(t, z, \lambda(z)) \, dz.$$

Then $u \to \xi_1(u, C)$ is continuous, while by Fubini's theorem $t \to \xi_2(t, C)$ is measurable. Recall that $B(Z)$ is countably generated and so we can find a countable field $\{C_n\}_{n \geq 1} \subseteq B(Z)$ which generates $B(Z)$, i.e. $\sigma(\{C_n\}_{n \geq 1}) = B(Z)$. Define

$$\xi_{1n}(u) = \xi_1(u, C_n) \qquad \text{and} \qquad \xi_{2n}(t) = \xi_2(u, C_n).$$

We have that

$$GrU(\cdot, \lambda) = \bigcap_{n \geq 1} \left\{ (t, u) \in T \times L^2(Z) : \xi_{1n}(u) \leq \xi_{2n}(t) \right\} \in B(T) \times B(L^2(Z))$$

with $B(T)$ and $B(L^2(Z))$ being the Borel σ-fields of T and $L^2(Z)$, respectively. So $t \to U(t, \lambda)$ is a measurable set-valued map (see Wagner [12: Theorem 4.2]) and by Aumann's selection theorem (see Wagner [12: Theorem 5.10]), it admits measurable selectors (i.e. maps $u : T \to L^2(Z)$ measurable such that $u(t) \in U(t, \lambda)$ for all $t \subset T$).

Then we can rewrite the dynamics of our optimal control problem (1) in the equivalent evolution equation form

$$\left.\begin{aligned} \dot{x}(t) + A(x(t), \lambda) &= \hat{f}(t, x(t), \lambda)u(t) \\ x(0) &= \hat{x}_0 \\ u(t) \in U(t, \lambda) \text{ a.e.,} \quad &u(\cdot) \text{ measurable} \end{aligned}\right\} \tag{2}$$

where $\hat{x}_0 = x_0(\cdot) \in L^2(Z)$.

By an admissible "state-control" pair of problem (2) (equivalently of problem (1)) we mean a pair $[x, u] \in W(T) \times L^2(T, H)$ satisfying problem (2) (equivalently the constraints of problem (1)). Given $\lambda \in \Lambda$, let $\Delta(\lambda) \subseteq W(T) \times L^2(T, H)$ be the set of admissible state-control pairs corresponding to this particular choice of the parameter. Let $\delta_{\Delta(\lambda)}(x, u)$ be the indicator function of $\Delta(\lambda)$, i.e.

$$\delta_{\Delta(\lambda)}(\dot{x}, u) = \begin{cases} 0 & \text{if } [x, u] \in \Delta(\lambda) \\ +\infty & \text{otherwise.} \end{cases}$$

Then problem (1) can be rewritten in the equivalent unconstrained form

$$m(\lambda) = \inf \left\{ J(x, u, \lambda) + \delta_{\Delta(\lambda)}(x, u) \right\}.$$

In the next proposition, we establish the Γ_{seq}-convergence of the sets of admissible state-control pairs.

Directly from the definition of Γ_{seq}-convergence, we have that if V_1 and V_2 are topological spaces and $B_n \subseteq V_1 \times V_2$ $(n \in \mathbb{N})$, then

$$\Gamma_{\text{seq}}(\mathbb{N}, V_1, V_2^-) \delta_{B_n}(x, y) = \delta_B(x, y)$$

if and only if the following two conditions hold:

(i) If $x_n \to x$ in V_1, $y_n \to y$ in V_2 and $(x_n, y_n) \in B_n$ for infinitely many n, then $(x, y) \in B$.

(ii) If $(x, y) \in B$ and $x_n \to x$ in V_1, then there exist $\{y_n\}_{n \geq 1} \subseteq V_2$ and $n_0 \geq 1$ such that $y_n \to y$ in V_2 and $(x_n, y_n) \in B$ for all $n \geq n_0$.

(See also Buttazzo and Dal Maso [2: p. 388].)

Remark 3. Let V_1 and V_2 are first countable and condition (ii) above be replaced by the following weaker condition:

(ii)′ If $(x, y) \in B$, then there exists a sequence $\{(x_n, y_n)\}_{n \geq 1} \subseteq V_1 \times V_2$ such that $(x_n, y_n) \in B_n$ for all $n \geq 1$ and $x_n \to x$ as well as $y_n \to y$ as $n \to \infty$.

Then we have a characterization of the classical Kuratowski convergence of sets (see Dal Maso [3: p. 41]).

We will use the above observation concerning the Γ_{seq}-convergence of indicator functions to establish the Γ_{seq}-limit of the sequence $\{\delta_{\Delta(\lambda_n)}(\cdot, \cdot)\}_{n \geq 1}$.

Proposition 4. *If hypotheses* H(a), H(f) *and* H(θ) *hold and* $\lambda_n \to \lambda$ *in* Λ, *then*

$$\Gamma_{\text{seq}}(\mathbb{N}, w - W(T)^-, w - L^2(T, H)) \delta_{\Delta(\lambda_n)}(x, u) = \delta_{\Delta(\lambda)}(x, u).$$

Proof. Let $[x, u] \in \Delta(\lambda)$ and $u_n \to u$ weakly in $L^2(T, H)$ with $u_n(t) \in U(t, \lambda_n)$ a.e. Let $x_n(\cdot) \in W(T)$ be the unique trajectory generated by the control $u_n(\cdot)$ (see Papageorgiou [8: Theorem 3.4]). Uniqueness follows from the (strong) monotonicity of $A(\cdot, \lambda_n)$ (cf. hypothesis H(a)/4) and the Lipschitzness of $\hat{f}(t, \cdot, \lambda_n) u_n$ (cf. hypothesis H(f)/2). So we have

$$\left.\begin{array}{c} \dot{x}_n(t) + A\big(x_n(t), \lambda_n\big) = \hat{f}\big(t, x_n(t), \lambda_n\big) u_n(t) \quad \text{a.e.} \\ x_n(0) = \hat{x}_0 \\ u_n(t) \in U(t, \lambda_n) \quad \text{a.e.} \end{array}\right\}$$

We will derive some a priori bounds for the x_n's. First let $B = \{\lambda_n, \lambda\}_{n \geq 1} \subseteq \Lambda$ compact. Then from hypothesis H(a)/4 we have for $\lambda' \in B$ with some constants $\hat{\gamma}_B, \hat{\gamma}'_B, \hat{\gamma}''_B > 0$

$$\langle A(x, \lambda') - A(y, \lambda'), x - y \rangle \geq \hat{\gamma}_B \|x - y\|^2$$

$$\langle A(x, \lambda'), x \rangle \geq \hat{\gamma}'_B \|x\|^2$$

$$\|A(x, \lambda)\|_* \leq \hat{\gamma}''_B (1 + \|x\|)$$

where $\| \cdot \|$ and $\| \cdot \|_*$ denotes the norm of $H_0^1(Z)$ and $H^{-1}(Z)$, respectively. Also $\langle \cdot, \cdot \rangle$ denotes the duality brackets for the pair $(X, X^*) = (H_0^1(Z), H^{-1}(Z))$, while in what follows by $(\cdot, \cdot)$ we will denote the inner product in $H = L^2(Z)$ and by $|\cdot|$ the corresponding norm. Recall that $\langle \cdot, \cdot \rangle \big|_{X \times H} = (\cdot, \cdot)$ (see Section 2). Then we have

$$\langle \dot{x}_n(t), x_n(t) \rangle + \langle A(x_n(t), \lambda_n), x_n(t) \rangle = (\hat{f}(t, x_n(t), \lambda_n) u_n(t), x_n(t)) \quad \text{a.e.}$$

which implies

$$\frac{1}{2} \frac{d}{dt} |x_n(t)|^2 + \hat{\gamma}_B' \|x_n(t)\|^2 \leq \|\hat{f}(t, x_n(t), \lambda_n) u_n(t)\|_* \|x_n(t)\| \quad \text{a.e.}$$

Applying on the right-hand side Cauchy's inequality with $\varepsilon > 0$, we get

$$\frac{1}{2} \frac{d}{dt} |x_n(t)|^2 + \hat{\gamma}_B' \|x_n(t)\|^2 \leq \frac{\varepsilon}{2} \|\hat{f}(t, x_n(t), \lambda_n) u_n(t)\|_*^2 + \frac{1}{2\varepsilon} \|x_n(t)\|^2 \quad \text{a.e.} \quad (3)$$

Let $\varepsilon = \frac{1}{2\hat{\gamma}_B'}$. We have

$$\frac{1}{2} \frac{d}{dt} |x_n(t)|^2 \leq \frac{1}{4\hat{\gamma}_B'} \|\hat{f}(t, x_n(t), \lambda_n) u_n(t)\|_*^2$$

which implies

$$|x_n(t)|^2 \leq |\hat{x}_0|^2 + \frac{1}{2\hat{\gamma}_B'} \int_0^t \|\hat{f}(s, x_n(s), \lambda_n) u_n(s)\|_*^2 ds$$

$$\leq |\hat{x}_0|^2 + \frac{1}{2\hat{\gamma}_B'} \int_0^t |\hat{f}(s, x_n(s), \lambda_n) u_n(s)|^2 ds$$

$$\leq |\hat{x}_0|^2 + \frac{1}{2\hat{\gamma}_B'} \left(\int_0^t |\hat{f}(s, x_n(s), \lambda_n)|^2 ds \right) \sup_{n \geq 1} \|\theta(\cdot, \cdot, \lambda_n)\|_\infty^2.$$

From hypothesis $H(\theta)/1$ we have that $\sup_{n \geq 1} \|\theta(\cdot, \cdot, \lambda_n)\|_\infty^2 < \infty$. So

$$|x_n(t)|^2 \leq |\hat{x}_0|^2 + \mu_B \int_0^t \left(2\hat{a}_B(s)^2 + 2b_B^2 |x_n(s)|^2 \right) ds$$

with $\mu_B > 0$, $\hat{a}_B(s) = \|a_B(s, \cdot)\|_2$ and $b_B > 0$ as in hypothesis $H(f)/3$. Invoking Gronwall's lemma, we deduce that there exists a constant $M_1 > 0$ such that for all $n \geq 1$ and all $t \in T$ we have

$$|x_n(t)| \leq M_1. \quad (4)$$

Next in inequality (3) above, let $\varepsilon = \frac{1}{\hat{\gamma}_B'}$. We get

$$\frac{1}{2} \frac{d}{dt} |x_n(t)|^2 + \frac{\hat{\gamma}_B'}{2} \|x_n(t)\|^2 \leq \frac{1}{2\hat{\gamma}_B'} |\hat{f}(t, x_n(t), \lambda_n) u_n(t)|^2 \quad \text{a.e.}$$

which implies

$$\hat{\gamma}'_B \int\limits_0^t \|x_n(s)\|^2 ds \le |\hat{x}_0|^2 + \frac{1}{\hat{\gamma}'_B} \sup_{n \ge 1} \|\theta(\cdot,\cdot,\lambda_n)\|_\infty^2 \int\limits_0^t \left(2\hat{a}_B(s)^2 + 2b_B^2 |x_n(s)|^2\right) ds.$$

Using bound (4) above, we deduce that there exists a constant $M_2 > 0$ such that

$$\|x_n\|_{L^2(T,X)} \le M_2 \qquad (n \ge 1). \tag{5}$$

Finally using hypotheses H(a)/4 and H(f)/3 as well as bounds (4) and (5) we conclude that there exists a constant $M_3 > 0$ such that

$$\|\dot{x}_n\|_{L^2(T,X^*)} \le M_3 \qquad (n \ge 1). \tag{6}$$

From (5) and (6) above we deduce that the sequence $\{x_n\}_{n\ge 1}$ is bounded in $W(T)$, hence relatively weakly sequentially compact. So by passing to a subsequence if necessary, we may assume that $x_n \to \hat{x}$ weakly in $W(T)$.

Next let $g(t) = \hat{f}(t, x(t), \lambda) u(t)$ and let $y(\cdot) \in W(T)$ be the unique solution of the evolution equation

$$\dot{y}_n(t) + A(t, y_n(t), \lambda_n) = g(t) \quad \text{a.e.}$$
$$y_n(0) = \hat{x}_0.$$

The existence and uniqueness of $y_n(\cdot) \in W(T)$ solving the above Cauchy problem is guaranteed by [13: Theorem 30.A/p. 771]. Let $\mathcal{U}$ be the family of all open subsets of Z and define the functional $\Phi : H_0^1(Z) \times \mathcal{U} \times \Lambda \to \mathbb{R}$ by

$$\Phi(x, Z', \lambda) = \int_{Z'} \varphi\big(z, Dx(z), \lambda(z)\big) \, dz.$$

Then from hypothesis H(a)/5 together with Dal Maso [3: Theorem 5.14/p. 51 and Proposition 8.10/p.93] we have that

$$\Gamma_{\text{seq}}\big(\mathbb{N}, w - H_0^1(Z')\big) \, \Phi(x, Z', \lambda_n) = \Phi(x, Z', \lambda).$$

Hence invoking Defranceschi [5: Theorem 3.2] we get that $A(\cdot, \lambda_n) \to_G A(\cdot, \lambda)$. Then Kolpakov [6: Theorem 1] tells us that $y_n \to x$ weakly in $W(T)$. Exploiting the monotonicity of the operator $A(\cdot, \lambda_n)$, we have

$$\langle \dot{x}_n(t) - \dot{y}_n(t), x_n(t) - y_n(t) \rangle$$
$$\le \Big(\hat{f}\big(t, x_n(t), \lambda_n\big) u_n(t) - g(t), x_n(t) - y_n(t)\Big) \quad \text{a.e.}$$
$$= \Big(\hat{f}\big(t, x_n(t), \lambda_n\big) u_n(t) - \hat{f}\big(t, x(t), \lambda_n\big) u_n(t), x_n(t) - y_n(t)\Big)$$
$$\quad + \Big(\hat{f}\big(t, x(t), \lambda_n\big) u_n(t) - \hat{f}\big(t, x(t), \lambda\big) u(t), x_n(t) - y_n(t)\Big) \quad \text{a.e.}$$

which implies

$$\frac{1}{2}|x_n(t) - y_n(t)|^2$$

$$\leq \int_0^t \Big(\hat{f}\big(s, x_n(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s), x_n(s) - y_n(s)\Big)\, ds$$

$$+ \int_0^t \Big(\hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda\big)\, u(s), x_n(s) - y_n(s)\Big)\, ds$$

$$\leq \int_0^t \Big|\hat{f}\big(s, x_n(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s)\Big|\, |x_n(s) - y_n(s)|\, ds$$

$$+ \int_0^t \Big(\hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda\big)u(s), x_n(s) - y_n(s)\Big)\, ds.$$

Observe that

$$\int_0^t \Big|\hat{f}\big(s, x_n(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s)\Big|\, |x_n(s) - y_n(s)|\, ds$$

$$\leq \sup_{n \geq 1} \|\theta(\cdot, \cdot, \lambda_n)\|_\infty \int_0^t \Big|\hat{f}\big(s, x_n(s), \lambda_n\big) - \hat{f}\big(s, x(s), \lambda_n\big)\Big|\, |x_n(s) - y_n(s)|\, ds$$

$$\leq \sup_{n \geq 1} \|\theta(\cdot, \cdot, \lambda_n)\|_\infty \int_0^t \hat{k}_B(s)|x_n(s) - y_n(s)|^2\, ds$$

with $\hat{k}_B(s) = \|k_B(s, \cdot)\|_\infty$. Also we have

$$\int_0^t \Big(\hat{f}\big(s, x(s), \lambda_n\big)\, u_n(s) - \hat{f}\big(s, x(s), \lambda\big)\, u(s), x_n(s) - y_n(s)\Big)\, ds$$

$$+ \int_0^t\!\!\int_Z \Big(f\big(s, z, x(s, z), \lambda_n(z)\big)\, u_n(s, z) - f\big(s, z, x(s, z), \lambda(z)\big)\Big)$$

$$\times u(s, z)\big(x_n(s, z) - y_n(s, z)\big)\, dz\, ds$$

$$= \int_0^t\!\!\int_Z f\big(s, z, x(s, z), \lambda_n(z)\big)\big(u_n(s, z) - u(s, z)\big)\big(x_n(s, z) - x(s, z)\big)\, dz\, ds$$

$$+ \int_0^t\!\!\int_Z \Big(f\big(s, z, x(s, z), \lambda_n(z)\big) - f\big(s, z, x(s, z), \lambda(z)\big)\Big)$$

$$\times u(s, z)\big(x_n(s, z) - y_n(s, z)\big)\, dz\, ds.$$

Since $W(T)$ embeds into $L^2(T, H)$ compactly, we have that $z_n \to_s \hat{z}$ and $y_n \to_s x$ in $L^2(T, H) = L^2(T \times Z)$. Also by hypothesis $u_n \to u$ weakly in $L^2(T \times Z)$ and because of hypothesis H(θ), we have $u_n \to u$ w^*-weakly in $L^\infty(T \times Z)$. So using hypothesis H(f), we get

$$\int\limits_0^t \int\limits_Z f\big(s, z, x(s, z), \lambda_n(z)\big)\big(u_n(s, z) - u(s, z)\big)\big(x_n(s, z) - y_n(s, z)\big)\, dz ds +$$

$$\int\limits_0^t \int\limits_Z \Big(f\big(s, z, x(s, z), \lambda_n(z)\big) - f\big(s, z, x(s, z), \lambda(z)\big)\Big) u(s, z)\big(x_n(s, z) - y_n(s, z)\big)\, dz ds$$

$$\longrightarrow 0$$

as $n \to \infty$ which implies

$$\int\limits_0^t \Big(\hat{f}\big(s, x(s), \lambda_n\big)u_n(s) - \hat{f}\big(s, x(s), \lambda\big)u(s),\ x_n(s) - y_n(s)\Big) ds \longrightarrow 0$$

as $n \to \infty$. Thus in the limit $n \to \infty$ we get

$$\big|\hat{x}(t) - x(t)\big|^2 \le 2 \sup_{n \ge 1} \big\|\theta(\cdot, \cdot, \lambda_n)\big\|_\infty \int\limits_0^t \hat{k}_B(s)\big|\hat{x}(s) - x(s)\big|^2 ds.$$

From Gronwall's lemma, we conclude that $x = \hat{z}$. Hence every subsequence of $\{x_n\}_{n \ge 1}$ has a further subsequence which weakly converges in $W(T)$ to x. Since $\{x_n\}_{n \ge 1}$ equipped with the relative weak-$W(T)$ topology is metrizable, we conclude that $x_n \to x$ weakly in $W(T)$ and $[x_n, y_n] \in \Delta(\lambda_n)$ $(n \ge 1)$. So we have established condition (ii) in the characterization of $\Gamma_{\text{seq}}\big(\mathbb{N}, w - W(T)^{-1}, w - L^2(T, H)\big)\delta_{\Delta(\lambda_n)}(\cdot, \cdot)$ provided earlier.

Next we will show that condition (i) is also valid, establishing this way the desired Γ_{seq}-convergence of the indicator maps $\delta_{\Delta(\lambda_n)}(\cdot, \cdot)$. So let $[x_n, u_n] \in \delta(\lambda_n)$ $(n \ge 1)$ and assume that $x_n \to x$ weakly in $W(T)$ and $u_n \to u$ weakly in $L^2(T, H)$. We will show that $[x, u] \in \Delta(\lambda)$. Let

$$g_n(t) = \hat{f}\big(t, x_n(t), \lambda_n\big) u_n(t) \qquad \text{and} \qquad g(t) = \hat{f}\big(t, x(t), \lambda\big) u(t).$$

We have already seen in the first part of the proof that, by passing to a subsequence if necessary, we may have $g_n \to g$ weakly in $L^2(T, H)$. Let $v_n \in W(T)$ be the unique solution of the evolution equation

$$\dot{v}_n(t) + A(v_n(t), \lambda_n) = g(t) \quad \text{a.e.}$$
$$v_n(0) = \hat{x}_0.$$

Since $A(\cdot, \lambda_n) \to_G A(\cdot, \lambda)$, from Kolpakov [6: Theorem 1] we know that $v_n \to v$ weakly in $W(T)$, with $v(\cdot) \in W(T)$ being the unique solution of the evolution equation

$$\dot{v}(t) + A(v(t), \lambda) = g(t) \quad \text{a.e.}$$
$$v(0) = \hat{x}_0.$$

Recalling that $W(T)$ embeds compactly into $L^2(T, H)$ and continuously into $C(T, H)$, by passing to a subsequence if necessary, we may assume that $x_n(t) \to x(t)$ and $v_n(t) \to v(t)$ in H for all $t \in T$ (in fact, using the results of Simon [9], we can actually show that $\{x_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ are relatively compact in $C(T, H)$). Exploiting the monotonicity of $A(\cdot, \lambda_n)$ we have

$$\langle \dot{x}_n(t) - \dot{v}_n(t), x_n(t) - v_n(t) \rangle \leq \big(g_n(t) - g(t), x_n(t) - v_n(t)\big) \quad \text{a.e.}$$

which implies

$$\frac{1}{2}\frac{d}{dt}|x_n(t) - v_n(t)|^2 \leq \big(g_n(t) - g(t), x_n(t) - v_n(t)\big) \quad \text{a.e.}$$

From here

$$\frac{1}{2}|x_n(t) - v_n(t)|^2$$

$$\leq \int_0^t \Big(\big(g_n(s) - g(s), x_n(s) - x(s)\big)$$

$$+ \big(g_n(s) - g(s), x(s) - v(s)\big) + \big(g_n(s) - g(s), v(s) - v_n(s)\big)\Big)ds,$$

thus $|x_n(t) - v_n(t)| \to 0$ and $|x(t) - v(t)| = 0$, i.e. $x = v$ follows. Therefore $[x, u] \in \Delta(\lambda)$. So we have established condition (i) and we can conclude that $\Gamma_{\text{seq}}\big(\mathbb{N}, w - W(T)^-, w - L^2(T, H)\big)\delta_{\Delta(\lambda_n)}(x, u) = \delta_{\Delta(\lambda)}(x, u)$ ∎

4. Main convergence theorem

In this section, using the auxiliary propositions of Section 3, we will examine the variational stability (sensitivity) of our optimal control problem (1).

Let $Q(\lambda)$ be the optimal state-control pairs corresponding to the parameter $\lambda \in \Lambda$, i.e.

$$Q(\lambda) = \Big\{[x, u] \in W(T) \times L^2(T, H) : [x, u] \text{ solves problem (1)}\Big\}.$$

Recall that if Y and Z are Hausdorff topological spaces, then a set-valued function $R : Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper-semicontinuous if, for all U open in Z, $R^+(U) = \{y \in Y : R(y) \subseteq U\}$ is open in Y. An upper-semicontinuous function $R(\cdot)$ with closed values has a closed graph, i.e. if $[y_\beta, z_\beta]$ is a net in $Y \times Z$, $z_\beta \in R(y_\beta)$ and $[y_\beta, z_\beta] \to [y, z]$, then we have $z \in R(y)$.

Theorem 5. *Let hypotheses* H(a), H(f), H(θ) *and* H(L) *hold. Then:*

(i) $Q(\lambda) \neq \emptyset$ *for all* $\lambda \in \Lambda$.

(ii) $m : \Lambda \to \mathbb{R}$ **is continuous.**

(iii) $\lambda \to Q(\lambda)$ **is upper-semicontinuous from** Λ **into the non-empty, weakly compact subsets of** $W(T) \times L^2(T, H)$.

Proof. From the a priori bounds established in the proof of Proposition 4, we know that, for every $\lambda \in \Lambda$, $\Delta(\lambda)$ is weakly sequentially compact in $W(T) \times L^2(T, H)$. Also, if $\lambda_n = \lambda$ $(n \geq 1)$ (constant sequence), we have from Proposition 2 that $J(\cdot, \cdot, \lambda)$ is the Γ-regularization of itself and so from Buttazzo [1: Proposition 1.3.1/p. 16] we get that $J(\cdot, \cdot, \lambda)$ is sequentially weakly lower semicontinuous on $W(T) \times L^2(T, H)$. So for every $\lambda \in \Lambda$ the problem $\inf \{J(x, u, \lambda) : [x, u] \in \Delta(\lambda)\}$ is solvable, i.e. $Q(\lambda) \neq \emptyset$.

Next let $\lambda_n \to \lambda$ in Λ. Combining Propositions 2 and 4 of this paper with Buttazzo and Dal Maso [2: Theorem 2.1], we get that

$$w - \limsup Q(\lambda_n) = \left\{ [x, u] \in W(T) \times L^2(T, H) \left| \begin{array}{l} [x, u] = w - \lim[x_{n_k}, u_{n_k}] \text{ with} \\ [x_{n_k}, u_{n_k}] \in Q(\lambda_{n_k}) \text{ and } n_k \uparrow \infty \end{array} \right. \right\}$$
$$\subseteq Q(\lambda).$$

But we saw in the proof of Proposition 3.1 that $\overline{\cup_{n \geq 1} Q(\lambda_n)}^w$ is a weakly compact subset of $W(T) \times L^2(T, H)$ (recall that the weak topology on this product space is the product of the weak topologies, i.e. $(W(T) \times L^2(T, H))_w = W(T)_w \times L^2(T, H)_w$). So from DeBlasi and Myjak [4: Remark 1.6] we get the desired upper-semicontinuity of the function $\lambda \to Q(\lambda)$.

Next let $\lambda_n \to \lambda$ in Λ and let $[x_n, u_n] \in Q(\lambda_n)$ $(n \geq 1)$. Then

$$m(\lambda_n) = J(x_n, u_n, \lambda_n)$$

and by passing to a subsequence if necessary, we may assume that $x_n \to x$ weakly in $W(T)$ and $u_n \to u$ weakly in $L^2(T, H)$. Then from Proposition 4 we have $[x, u] \in Q(\lambda)$. Also from Proposition 2 and the definition of Γ_{seq}-limits we have

$$J(x, u, \lambda) \leq \liminf J(x_n, u_n, \lambda_n) = \liminf \{\lambda_n\}$$

which implies

$$m(\lambda) \leq \liminf m(\lambda_n). \tag{7}$$

Next let $[x, u] \in Q(\lambda)$ and $\varepsilon > 0$. We have $m(\lambda) = J(x, u, \lambda)$ for $[x, u] \in \Delta(\lambda)$. From Propositions 2 and 4 and Buttazzo and Dal Maso [2: Corollary 2.1] we have that

$$\Gamma_{seq}(\mathbb{N}, w - W(T)^-, w - L^2(T, H)^-)(J(\cdot, \cdot, \lambda_n) + \delta_{\Delta(\lambda_n)}(\cdot, \cdot))(x, u)$$
$$= (J(\cdot, \cdot, \lambda) + \delta_{\Delta(\lambda)}(\cdot, \cdot))(x, u).$$

Then from the definition of Γ_{seq}-limits we know that we can find a sequence $\{[x_n, u_n]\}_{n \geq 1} \subseteq W(T) \times L^2(T, H)$ such that $x_n \to x$ weakly in $W(T)$, $u_n \to u$ weakly in $L^2(T, H)$ and

$$\limsup \{J(x_n, u_n, \lambda_n) + \delta_\Delta(\lambda_n)(x_n, u_n)\} \leq J(x, u, \lambda) + \delta_{\Delta(\lambda)}(x, u) + \varepsilon < \infty.$$

Hence for all n large enough we have that $[x_n, u_n] \in \Delta(\lambda_n)$ and so

$$\limsup m(\lambda_n) \leq \limsup J(x_n, u_n, \lambda_n) \leq m(\lambda) + \varepsilon.$$

Let $\varepsilon \downarrow$. We get that

$$\limsup m(\lambda_n) \leq m(\lambda). \tag{8}$$

From (7) and (8) above we deduce that $m(\lambda_n) \to m(\lambda)$ and so we have proved that the function $\lambda \to m(\lambda)$ is continuous ∎

5. Semilinear systems

In this section we consider systems with semilinear dynamics. The linearity of the partial differential operator $x \rightarrow A(x, \lambda)$ allows us to incorporate in the framework of this paper semilinear systems with weakly convergent coefficients (e.g. rapidly oscillating coefficients).

So let T and $Z \subseteq \mathbb{R}^N$ be as before. We consider the following optimal control problem:

$$
\left.
\begin{array}{l}
\displaystyle\int\limits_0^r \int\limits_Z L\big(t, z, x(t, z), u(t, z), \lambda(z)\big)\, dz dt \;\longrightarrow\; \inf = m(\lambda) \\[2ex]
\text{such that} \\[1ex]
\displaystyle\frac{\partial_x x}{\partial t} - \sum_{i,j=1}^{N} D_i\big(a_{ij}(z, \lambda(z))\, D_j x(t, z)\big) = f\big(t, z, x(t, z), \lambda(z)\big)\, u(t, z) \\[3ex]
\text{a.e. on } \; T \times Z \\[1ex]
x\big|_{T \times \Gamma} = 0, \qquad x(0, z) = x_0(z) \quad \text{a.e. on } \; Z \; \text{ with } \; x_0(\cdot) \in L^2(Z) \\[2ex]
|u(z)| \leq \theta\big(t, z, \lambda(z)\big) \quad \text{a.e.}
\end{array}
\right\}
\qquad (9)
$$

We will make the following hypothesis:

$H(a)_1$ The coefficients $a_{ij} = a_{ij}(z, \lambda)$ in problem (9) satisfies the following conditions:

   (i) $a_{ij}(\cdot, \lambda(\cdot)) \in L^\infty(Z)$ and $a_{ij}(\cdot, \lambda(\cdot)) = a_{ji}(\cdot, \lambda(\cdot))$ $(i, j = 1, \dots, N)$ for every $\lambda \in \Lambda$.

   (ii) $m_1 \|z\|^2 \leq \sum_{i,j=1}^{N} a_{ij}(z, \lambda(z)) z_i z_j \leq m_2 \|z\|^2$ for every $z \in \mathbb{R}^N$, where $0 < m_1 \leq m_2 < \infty$.

   (iii) $a_{ij}(\cdot, \lambda_n(\cdot)) \rightarrow_w a_{ij}(\cdot, \lambda(\cdot))$ $(i, j = 1, \dots, N)$ in $L^2(Z)$ as $\lambda_n \rightarrow \lambda$ in Λ.

   (iv) $\sum_{i=1}^{N} D_i a_{ij}(\cdot, \lambda_n(\cdot)) \rightarrow_s \sum_{i=1}^{N} D_i a_{ij}(\cdot, \lambda(\cdot))$ $(j = 1, \dots, N)$ in $H^{-1}(Z)$ as $\lambda_n \rightarrow \lambda$ in Λ.

In this case

$$
\varphi(z, x, \lambda) = \big(A(z, \lambda) x, x\big)_{\mathbb{R}^N} \quad \text{where} \quad A(z, \lambda) = \big(a_{ij}(z, \lambda)\big)_{i,j=1}^{N} \in \mathbb{R}^{N \times N}.
$$

Let $\Phi : H_0^1(Z) \times \mathcal{U} \times \Lambda \rightarrow \mathbb{R}$ be defined by

$$
\Phi(x, Z', \lambda) = \int_{Z'} \varphi\big(z, x(z), \lambda(z)\big)\, dz
$$

where as before (see the proof of Proposition 2) $\mathcal{U}$ is the collection of all open subsets of Z. Then because of hypothesis $H(a)_1$ we have that if $\lambda_n \rightarrow \lambda$, then $\Gamma_{seq}(\mathbb{N}, w -$

$H_0^1(Z)$) $\Phi(x, \lambda_n) = \Phi(x, \lambda)$ (see Dal Maso [3]). So if for $\lambda' \in \Lambda$ we define $\hat{A}(\lambda') \in \mathcal{L}(H_0^1(Z), H^{-1}(Z))$ by

$$\langle \hat{A}(\lambda')x, y \rangle = \int_Z \sum_{i,j=1}^N a_{ij}(z, \lambda'(z)) D_i x(z) D_j(z) \, dz,$$

then from Defranceschi [5: Theorem 3.2] we have that $\hat{A}(\lambda_n) \to_G \hat{A}(\lambda)$. Hence the proof of Proposition 4 goes through and therefore we can state the following result.

Theorem 6. *Let the hypotheses* H(a)$_1$, H(f), H(θ) *and* H(L) *hold. Then:*

(i) $Q(\lambda) \neq \emptyset$ *for all* $\lambda \in \Lambda$.

(ii) $m : \Lambda \to \mathbb{R}$ *is continuous.*

(iii) $\lambda \to Q(\lambda)$ *is upper-semicontinuous from* Λ *into the non-empty, weakly compact subsets of* $W(T) \times L^2(T, H)$.

As a simple illustration, let $Z \subseteq \mathbb{R}^2$ and assume that the sequence of partial differential operators of the approximating problems is $B_n = -\Delta - \frac{1}{2}\cos(nz_2)D_1^2$. Remark that $\{\frac{1}{2}\cos(nz_2)\}_{n \geq 1}$ is a sequence of C^∞-functions, which converges strongly in $H^{-1}(Z)$ but not in $L^2(Z)$ (recall that by the Riemann-Lebesgue lemma $\frac{1}{2}\cos(nz_2) \to 0$ weakly in $L^2(Z)$ and since $L^2(Z)$ embeds compactly into $H^{-1}(Z)$ we have $\frac{1}{2}\cos(nz_2) \to_s 0$ in $H^{-1}(Z)$). Then $B_n \to_G B = -\Delta$ and so Theorem 6 is applicable for systems monitored by parabolic partial differential equations involving these operators. In particular then we have convergence of the corresponding optimal values.

If $N = 1$, the situation is simpler. In this case the partial differential operator is $-\frac{d}{dz}\left(a(z, \lambda(z))\frac{dz}{dz}\right)$ and hypothesis H(a)$_1$ takes the following form.

H(a)$_2$ The coefficient $a = a(z, \lambda)$ in problem (9) satisfies the following conditions:

(i) $m_1 \leq a(z, \lambda(z)) \leq m_2$ for every $\lambda \in \Lambda$ and a.a. $z \in Z$, where $0 < m_1 \leq m_2 < \infty$.

(ii) $\frac{1}{a(\cdot, \lambda_n(\cdot))} \to_{w^*} \frac{1}{a(\cdot, \lambda(\cdot))}$ in $L^\infty(Z)$ as $\lambda_n \to \lambda$ in Λ.

Under this hypothesis, we know (cf. Dal Maso [3]) that

$$\Gamma_{\text{seq}}\left(\mathbb{N}, w - H_0^1(Z')\right)\Phi(x, Z', \lambda_n) = \Phi(x, Z', \lambda)$$

where $\Phi : H_0^1(Z) \times \mathcal{U} \times \Lambda \to \mathbb{R}$ is given by

$$\Phi(x, Z', \lambda') = \int_{Z'} a(z, \lambda'(z))\left(\frac{dx}{dz}\right)^2 dz.$$

So again Proposition 4 is valid and so we can state the following result.

Theorem 7. *Let hypotheses* $H(a)_2$, $H(f)$, $H(\theta)$ *and* $H(L)$ *hold. Then:*

(i) $Q(\lambda) \neq \emptyset$ *for all* $\lambda \in \Lambda$.

(ii) $\lambda \to m(\lambda)$ *is continuous.*

(iii) $\lambda \to Q(\lambda)$ *is upper-semicontinuous from* Λ *into the weakly compact subsets of* $W(T) \times L^2(T, H)$.

This is the case, for example, if $a_n(z) = 1 + e^{-nz}$ and $a(z) = 1$. Then $\frac{1}{a_n} \to_{w^*} \frac{1}{a} = 1$ in $L^\infty(Z)$. Note that $\|a_n - a\|_\infty = 1$. So $a_n \not\to a$ strongly in $L^\infty(Z)$.

Remark 8. This type of coefficient convergence was considered by Sokolowski [10].

Finally, we mention that the framework of this paper allows us also to treat optimal control problems with homogenization in the dynamics. In this case in the context of semilinear systems $a_{ij}^n(z) = a_{ij}(\gamma_n z)$ with a_{ij} periodic and $\gamma_n \to 0$ (see Dal Maso [3: Chapter 24]). Also we can investigate systems with controls in the coefficients (see Sokolowski [10]).

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