Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 1, 223–244

On the Condition of Orthogonal Polynomials via Modified Moments

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Abstract. We consider the condition of orthogonal polynomials, encoded by the coefficients of their three-term recurrence relation, if the measure is given by modified moments (i.e. integrals of certain polynomials forming a basis). The results concerning various polynomial bases are illustrated by simple examples of generating (possibly shifted) Chebyshev polynomials of first and second kind.

Keywords: *Orthogonal polynomials, recurrence relations, modified moments, polynomial bases*

AMS subject classification: Primary 42 C 05, secondary 33 A 65

1. Introduction

Let σ be a given positive measure with infinite support $S(\sigma)$. Then there uniquely exists a family $\{\tilde{\pi}_j\}_{j\in\mathbb{N}_0}$ of monic polynomials $\tilde{\pi}_j$ with

$$\int \tilde{\pi}_l(x)\,\tilde{\pi}_j(x)\,d\sigma(x) = 0 \quad (l < j) \qquad \text{and} \qquad \int \tilde{\pi}_j^2(x)\,d\sigma(x) > 0.$$

They satisfy a three-term recurrence relation

$$\tilde{\pi}_{j+1}(x) = (x - \alpha_j)\tilde{\pi}_j(x) - \beta_j \tilde{\pi}_{j-1}(x)$$

if we set $\tilde{\pi}_{-1} = 0$. We follow the usual convention

$$\beta_0 = \int 1\,d\sigma(x) \tag{1}$$

which has no meaning for the recurrence relation, but unifies some other formulas (see equation (14) below).

It is well-known that the generation of the orthogonal polynomials (or, equivalently, of the coefficients α_j and β_j) from ordinary moments

$$\mu_k = \int x^k\,d\sigma(x) \qquad (k \in \mathbb{N}_0)$$

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ISSN 0232-2064 / \$ 2.50

is severely ill-conditioned (see the classical paper of W. Gautschi [3]). More promising there are modified moments

$$m_k = \int p_k(x)\, d\sigma(x) \qquad (k \in \mathbb{N}_0) \tag{2}$$

with some properly chosen polynomials p_k (e.g. Chebyshev polynomials). Of course, we should be able to calculate or estimate the condition of the map

$$K_n : [m_0, \dots, m_{2n-1}]^\top \longrightarrow [\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}]^\top.$$

In [3] and [5], the map G_n from modified moments to the vector of nodes and weights of a Gauss quadrature rule is considered, instead, and from this estimates are given for the condition of our original map K_n. Fortunately, a more direct approach is possible.

2. Maps and norms

If we denote by $m \in \mathbb{R}^{2n}$ the vector of the first $2n$ modified moments $[m_0, \dots, m_{2n-1}]^\top$ and by ρ the vector of the recursion coefficients $[\alpha_0, \dots, \alpha_{n-1}, \beta_0, \dots, \beta_{n-1}]^\top$, then we are interested in the condition of the map $K_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$K_n(m) = \rho.$$

In order to compare our results with those of W. Gautschi [3 - 5], we introduce the vector

$$\gamma = \left[\tau_1, \dots, \tau_n, \sigma_1, \dots, \sigma_n\right]^\top$$

of nodes and weights of the Gauss-Christoffel quadrature rule

$$\int q(x)\, d\sigma(x) = \sum_{k=1}^{n} \sigma_k\, q(\tau_k)$$

for any polynomial q of degree less or equal $2n-1$, and denote by G_n the map $G_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$G_n(m) = \gamma.$$

The sensitivity of the nonlinear maps K_n and G_n can be measured by the norm of the Fréchet derivatives K_n' and G_n', which are linear maps (the Jacobians of K_n and G_n) from $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$. We consider here only two norms of matrices $A = (a_{j,k})_{j,k=0}^{2n-1}$: The

$$\text{sup-norm} \quad \|A\|_\infty = \max_{0 \le j \le 2n-1} \sum_{k=0}^{2n-1} |a_{jk}|$$

and the

$$\text{Frobenius norm} \quad \|A\|_F = (\operatorname{Tr} A^\top A)^{1/2} = \left(\sum_{j,k=0}^{2n-1} a_{jk}^2 \right)^{1/2}.$$

We mention here that the condition number "cond" defined in [3: Formula 2.1] for the map $\gamma = G_n(m)$ in our notations will be

$$\operatorname{cond}(G_n, m) = \frac{\|m\|}{\|\gamma\|} \|G_n'(m)\| \tag{3}$$

(compare [3: Formula 2.2]). The partial derivatives constituting the Jacobian K_n' will be evaluated in the following section.

3. Calculation of some partial derivatives

We assume that $(p_j)_{j=0}^N$ is any polynomial basis in the space $\mathcal{P}_N$ of all polynomials of degree less or equal N, i.e. there is a representation

$$q(x) = \sum_{j=0}^{N} c_j(q)\, p_j(x) \tag{4}$$

for all polynomials $q \in \mathcal{P}_N$, with some linear functionals c_j. Note that for the degree of the polynomials we do *not* assume $\deg(p_j) = j$, i.e. the case of Bernstein polynomials or Lagrange interpolation polynomials is included! The case of ordinary moments is included as well, here we have $p_j(x) = x^j$ and $c_j(q) = \frac{q^{(j)}(0)}{j!}$ (remember that we are in a finite-dimensional space, where all norms are equivalent and all linear operations are continuous). For the following, we need partial derivatives of integrals.

Lemma 1. *Let $q \in \mathcal{P}_N$ be some polynomial depending on $m_0, \ldots, m_N$, with continuous partial derivatives in some neighbourhood of the point $[m_0, \ldots, m_N]^{\mathsf{T}}$, and let* (2) *and* (4) *hold. Then*

$$\frac{\partial}{\partial m_k} \int q(x)\, d\sigma(x) = c_k(q) + \int \frac{\partial q}{\partial m_k}\, d\sigma(x). \tag{5}$$

Proof. From (2) and (4) we have immediately

$$\int q(x)\, d\sigma(x) = \sum_{j=0}^{N} c_j(q)\, m_j$$

and, differentiating,

$$\begin{aligned} \frac{\partial}{\partial m_k} \int q(x)\, d\sigma(x) &= c_k(q) + \sum_{j=0}^{N} \frac{\partial c_j(q)}{\partial m_k}\, m_j \\ &= c_k(q) + \sum_{j=0}^{N} c_j\left(\frac{\partial q}{\partial m_k}\right) m_j = c_k(q) + \int \frac{\partial q}{\partial m_k}\, d\sigma(x). \end{aligned}$$

The interchange of partial derivatives and the functionals c_j is justified, since the latter are continuous ∎

Now we consider the *orthonormal* polynomials π_j $(j \in \mathbb{N}_0)$, satisfying the relations

$$\int \pi_l(x)\, \pi_j(x)\, d\sigma(x) = 0 \quad (l < j) \qquad \text{and} \qquad \int \pi_j^2(x)\, d\sigma(x) = 1. \tag{6}$$

¿From the algorithm given in [5] it is clear that α_j and β_j for $j \le \frac{N-1}{2}$ (and thus all $\tilde{\pi}_j$ for $j \le \frac{N+1}{2}$) are just rational functions of $m_0, \ldots, m_N$. Normalization inserts square-roots of positive quantities, and consequently the π_j are differentiable functions of $m_0, \ldots, m_N$

in some sufficiently small neighbourhood of $[m_0, \dots, m_N]^\top$. Differentiating the relations (6) we obtain first, with the help of (5),

$$\begin{aligned} 0 &= \frac{\partial}{\partial m_k} \int \pi_l(x)\,\pi_j(x)\,d\sigma(x) \\ &= c_k(\pi_l\pi_j) + \int \frac{\partial \pi_l(x)}{\partial m_k}\,\pi_j(x)\,d\sigma(x) + \int \pi_l(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x) \\ &= c_k(\pi_l\pi_j) + \int \pi_l(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x) \end{aligned}$$

for $l < j$, i.e.

$$\int \pi_l(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x) = -c_k(\pi_l\pi_j). \tag{7}$$

The second relation in (6) gives

$$0 = \frac{\partial}{\partial m_k} \int \pi_j^2(x)\,d\sigma(x) = c_k(\pi_j^2) + 2\int \pi_j(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x)$$

and this means

$$\int \pi_j(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x) = -\frac{1}{2}\,c_k(\pi_j^2). \tag{8}$$

Thus we have the following

Corollary 1. *Under our assumptions* (2), (4) *and* (6) *the formula*

$$\frac{\partial \pi_j}{\partial m_k} = -\sum_{l<j} c_k(\pi_l\pi_j)\,\pi_l - \frac{1}{2}\,c_k(\pi_j^2)\,\pi_j \qquad \textit{for } 2j \le N$$

holds.

Though this is interesting, it is not our main goal. We consider now the dependence of the coefficients α_j and β_j on the moments m_k. The three-term recurrence relation for the orthonormal polynomials π_j reads

$$\beta_{j+1}^{1/2}\,\pi_{j+1}(x) = (x-\alpha_j)\,\pi_j(x) - \beta_j^{1/2}\,\pi_{j-1}(x) \qquad (j \in \mathbb{N}_0). \tag{9}$$

Differentiating with respect to m_k, we arrive at

$$\begin{aligned} &\frac{\partial \beta_{j+1}^{1/2}}{\partial m_k}\,\pi_{j+1}(x) + \beta_{j+1}^{1/2}\,\frac{\partial \pi_{j+1}(x)}{\partial m_k} \\ &\quad = -\frac{\partial \alpha_j}{\partial m_k}\,\pi_j(x) + (x-\alpha_j)\frac{\partial \pi_j(x)}{\partial m_k} - \frac{\partial \beta_j^{1/2}}{\partial m_k}\,\pi_{j-1}(x) - \beta_j^{1/2}\frac{\partial \pi_{j-1}(x)}{\partial m_k}. \end{aligned} \tag{10}$$

Since

$$(x-\alpha_j)\,\pi_{j+1}(x) = \beta_{j+2}^{1/2}\,\pi_{j+2}(x) + (\alpha_{j+1}-\alpha_j)\,\pi_{j+1}(x) + \beta_{j+1}^{1/2}\,\pi_j(x)$$

and the orthogonality of π_{j+1} and π_{j+2} with any polynomial of lower degree, multiplying (10) by π_{j+1}, integrating and using (8) we have

$$\frac{\partial \beta_{j+1}^{1/2}}{\partial m_k} - \frac{1}{2}\beta_{j+1}^{1/2}\, c_k(\pi_{j+1}^2) = -\frac{1}{2}\beta_{j+1}^{1/2}\, c_k(\pi_j^2).$$

This immediately gives

$$\frac{\partial \beta_{j+1}^{1/2}}{\partial m_k} = \frac{1}{2}\beta_{j+1}^{1/2}\, c_k(\pi_{j+1}^2 - \pi_j^2)$$

and (after multiplication by $2\beta_{j+1}^{1/2}$)

$$\frac{\partial \beta_{j+1}}{\partial m_k} = \beta_{j+1}\, c_k(\pi_{j+1}^2 - \pi_j^2). \tag{11}$$

Multiplying (10) by π_j and integrating we get

$$-\beta_{j+1}^{1/2}\, c_k(\pi_j \pi_{j+1}) = -\frac{\partial \alpha_j}{\partial m_k} - \beta_j^{1/2}\, c_k(\pi_{j-1}\pi_j) \tag{12}$$

since in virtue of (9) we have $(x-\alpha_j)\,\pi_j(x) = \beta_{j+1}^{1/2}\,\pi_{j+1}(x) + \beta_j^{1/2}\,\pi_{j-1}(x)$ and thus by relation (7)

$$\int (x-\alpha_j)\,\pi_j(x)\,\frac{\partial \pi_j(x)}{\partial m_k}\,d\sigma(x) = -\beta_j^{1/2}\, c_k(\pi_{j-1}\pi_j).$$

The derivation of (11) is not valid for $j=-1$. The equation, however, is true: We have by (1) and (5)

$$\frac{\partial \beta_0}{\partial m_k} = c_k(1) = \beta_0\, c_k\left(\frac{1}{\beta_0}\right) = \beta_0\, c_k(\pi_0^2 - \pi_{-1}^2)$$

since $\pi_0^2 = \frac{1}{\beta_0}$ and $\pi_{-1}^2 = 0$. This is (11) for $j=-1$. Writing j instead of $j+1$ in relation (11) we finally obtain the following

Theorem 1. *Let (2),(4) and (6) be satisfied. Then the partial derivatives of the coefficients in the three-term recurrence relation α_j and β_j with respect to m_k evaluate as*

$$\frac{\partial \alpha_j}{\partial m_k} = \beta_{j+1}^{1/2}\, c_k(\pi_j\pi_{j+1}) - \beta_j^{1/2}\, c_k(\pi_{j-1}\pi_j) \qquad \text{for } 2j+1 \le N \tag{13}$$

and

$$\frac{\partial \beta_j}{\partial m_k} = \beta_j\, c_k(\pi_j^2 - \pi_{j-1}^2) \qquad \text{for } 2j \le N. \tag{14}$$

The Jacobian K_j' consists of the partial derivatives

$$\frac{\partial \alpha_j}{\partial m_k} \quad \text{and} \quad \frac{\partial \beta_j}{\partial m_k} \qquad \begin{pmatrix} j = 0,\ldots,n-1 \\ k = 0,\ldots,2n-1 \end{pmatrix}$$

so it is clear that we must have a polynomial basis with $N \geq 2n-1$. Now we can introduce the notations

$$\psi_{2j}(x) = \beta_j\big(\pi_j^2(x) - \pi_{j-1}^2(x)\big) \tag{15}$$

$$\psi_{2j+1}(x) = \beta_{j+1}^{1/2}\,\pi_j(x)\,\pi_{j+1}(x) - \beta_j^{1/2}\,\pi_{j-1}(x)\,\pi_j(x) \tag{16}$$

for $j = 0,\ldots,n-1$. Then ψ_j is a polynomial of degree j, and obviously $\|K_n'\|_\infty = \|\Psi\|_\infty$ and $\|K_n'\|_F = \|\Psi\|_F$, where $\Psi = (\psi_{jk})_{j,k=0,\ldots,2n-1}$ with $\psi_{jk} = c_k(\psi_j)$. These norms can be estimated (or evaluated exactly in some special cases) for various choices of polynomial bases $(p_j)_{n=0,\ldots,N}$. We first reconsider ordinary moments, investigate the (somewhat exotic) bases of Bernstein polynomials and Lagrange interpolation polynomials, and are finally concerned with the practically important case, where the p_j are orthonormal polynomials with respect to some other measure s.

To evaluate the norms, we just have to find a way to express

$$a_\infty(q) = \sum_{k=0}^{N} |c_k(q)| \qquad \text{and} \qquad a_2(q) = \sum_{k=0}^{N} c_k^2(q)$$

for all $q \in \mathcal{P}_N$. Then evidently we will have

$$\|K_n'\|_\infty = \|\Psi\|_\infty = \max_{0\leq j\leq 2n-1} a_\infty(\psi_j) \tag{17}$$

and

$$\|K_n'\|_F = \|\Psi\|_F = \left(\sum_{j=0}^{2n-1} a_2(\psi_j)\right)^{1/2}. \tag{18}$$

Since the coordinate functionals c_k are linear, one easily observes that $a_\infty(\cdot)$ and $a_2(\cdot)^{1/2}$ are norms in $\mathcal{P}_N$, i.e. they must satisfy the norm inequalities

$$a_\infty(q+r) \leq a_\infty(q) + a_\infty(r) \qquad \text{and} \qquad a_2(q+r)^{1/2} \leq a_2(q)^{1/2} + a_2(r)^{1/2}$$

for $q, r \in \mathcal{P}_N$.

4. Some examples

In order to illustrate our results, we consider Chebyshev polynomials of first and second kind.

Example 4.1 (*Chebyshev polynomials of first kind*). These polynomials are the orthogonal polynomials with respect to the measure

$$d\sigma(x) = \frac{dx}{\pi\sqrt{1-x^2}} \qquad \text{for} \quad x \in [-1,1]$$

(observe that we use the normalization $\sigma([-1,1]) = 1$, i.e. $\beta_0 = 1$). In this case, we have $\alpha_j = 0$ by symmetry, $\beta_j = \frac{1}{2}$ for $j = 1$ and $\beta_j = \frac{1}{4}$ for $j > 1$. The orthonormal polynomials will be

$$\pi_j = \begin{cases} 1 & \text{for } j = 0 \\ \sqrt{2}\,T_j & \text{for } j > 0. \end{cases}$$

Then the polynomials ψ_j defined above can be calculated easily using the well-known identities for Chebyshev polynomials (see [1: Formula 22.7.24]):

$$\psi_{2j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{1}{2}T_2 & \text{for } j = 1 \\ \frac{1}{4}(T_{2j} - T_{2j-2}) & \text{for } j > 1 \end{cases} \tag{19}$$

and

$$\psi_{2j+1} = \begin{cases} T_1 & \text{for } j = 0 \\ \frac{1}{2}(T_{2j+1} - T_{2j-1}) & \text{for } j > 0. \end{cases} \tag{20}$$

To compare our results with those of W. Gautschi [3], we consider shifted Chebyshev polynomials as well (we will mark any quantity related to the shifted polynomials with an asterisk).

Example 4.1* (*Shifted Chebyshev polynomials of first kind*). The measure here is defined by

$$d\sigma^*(x) = \frac{dx}{\pi\sqrt{x(1-x)}} \qquad \text{for } \ x \in [0,1].$$

Again it has total weight 1, and the orthonormal polynomials are connected with the above via

$$\pi_j^*(x) = \pi_j(2x-1) = \begin{cases} 1 & \text{for } j = 0 \\ \sqrt{2}\,T_j^*(x) & \text{for } j > 0 \end{cases}$$

where the common notation for shifted Chebyshev polynomials of first kind is used (see [1: Formula 22.5.14]): $T_n^*(x) = T_n(2x-1)$. The recurrence coefficients of these shifted polynomials can be readily determined from their scaling properties:

$$\alpha_j^* = \frac{\alpha_j + 1}{2} = \frac{1}{2} \qquad \text{and} \qquad \beta_j^* = \frac{1}{2^2}\beta_j = \begin{cases} \frac{1}{8} & \text{for } j = 1 \\ \frac{1}{16} & \text{for } j > 1 \end{cases}$$

(this is essentially [2: Chapter I/Exercise 4.4(a)]). Of course, we have $\beta_0^* = 1$. From (15) and (16) we obtain

$$\psi_{2j}^*(x) = \frac{1}{4}\,\psi_{2j}(2x-1) \ \ (j \ge 1) \quad \text{and} \quad \psi_{2j+1}^*(x) = \frac{1}{2}\,\psi_{2j+1}(2x-1) \ \ (j \ge 0),$$

respectively; clearly it must be $\psi_0^* = 1$. Thus, the explicit formulas will read as

$$\psi_{2j}^* n = \begin{cases} 1 & \text{for } j = 0 \\ \frac{1}{8}T_2^* & \text{for } j = 1 \\ \frac{1}{16}(T_{2j}^* - T_{2j-2}^*) & \text{for } j > 1 \end{cases} \tag{21}$$

and

$$\psi^*_{2j+1} = \begin{cases} \frac{1}{2}\, T^*_1 & \text{for } j = 0 \\ \frac{1}{4}\,(T^*_{2j+1} - T^*_{2j-1}) & \text{for } j > 0. \end{cases} \tag{22}$$

Example 4.2 (*Chebyshev polynomials of second kind*). These polynomials are the orthogonal polynomials with respect to the measure

$$d\sigma(x) = \frac{2}{\pi}\sqrt{1-x^2}\,dx \qquad \text{for } x \in [-1,1]$$

(again we use the normalization $\sigma([-1,1]) = 1$, i.e. $\beta_0 = 1$). In this case, we have $\alpha_j = 0$ by symmetry, $\beta_j = \frac{1}{4}$ for $j > 0$, and the orthonormal polynomials will be $\pi_j = U_j$. The polynomials ψ_j can be calculated easily using the well-known identities for Chebyshev polynomials of second kind (see [1: Formula 22.7.25])

$$\psi_{2j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{1}{4}\, U_{2j} & \text{for } j > 0 \end{cases} \tag{23}$$

and

$$\psi_{2j+1} = \frac{1}{2}\, U_{2j+1}. \tag{24}$$

As in the preceding subsection, we will consider shifted polynomials.

Example 4.2* (*Shifted Chebyshev polynomials of second kind*). The measure is defined by

$$d\sigma^*(x) = \frac{8}{\pi}\sqrt{x(1-x)}\,dx \qquad \text{for } x \in [0,1].$$

The explicit formulas for our polynomials ψ^*_j can be obtained analogously as above:

$$\psi^*_{2j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{1}{16}\, U^*_{2j} & \text{for } j > 0 \end{cases} \tag{25}$$

and

$$\psi^*_{2j+1} = \frac{1}{4}\, U^*_{2j+1}, \tag{26}$$

where U^*_n denotes the shifted Chebyshev polynomials of second kind: $U^*_n(x) = U_n(2x-1)$ (see [1: Formula 22.5.15]).

5. Ordinary moments

For our special choice $p_j(x) = x^j$ and $c_j(q) = \frac{q^{(j)}(0)}{j!}$ the following lemma holds.

Lemma 2. *Let $q \in \mathcal{P}_N$ be a polynomial with real coefficients. Then the auxiliary quantities $a_\infty(q)$ and $a_2(q)$ introduced in Section 3 can be written down explicitly:*

(i) *If the coefficients of q have alternating sign, then $a_\infty(q) = |q(-1)|$.*

(ii) *If q contains only odd (or only even) powers of x with coefficients of alternating sign, then $a_\infty(q) = |q(i)|$.*

(iii) *The quantity $a_2(q)$ can be expressed as $a_2(q) = \frac{1}{2\pi}\int_0^{2\pi} |q(e^{i\phi})|^2\, d\phi$.*

Proof. (i) We have $\varepsilon\,(-1)^j c_j(q) \geq 0$ with some $\varepsilon = \pm 1$, and from this there follows

$$a_\infty(q) = \sum_{j=0}^{N} |c_j(q)| = \sum_{j=0}^{N} \varepsilon\,(-1)^j c_j(q) = \varepsilon \sum_{j=0}^{N} c_j(q)\, p_j(-1) = \varepsilon\, q(-1).$$

But, since $a_\infty(q)$ is non-negative and $|\varepsilon| = 1$, we must have $\varepsilon\, q(-1) = |q(-1)|$.

(ii) We have $c_{2j}(q) = 0$ for $2j \leq N$, and again $\varepsilon\,(-1)^j c_{2j+1}(q) \geq 0$ for $2j+1 \leq N$ with some $\varepsilon = \pm 1$. Consequently, from $(-1)^j = (-i)\, i^{2j+1} = (-i)\, p_{2j+1}(i)$ we obtain

$$a_\infty(q) = \sum_{j=0}^{[\frac{N-1}{2}]} |c_{2j+1}(q)| = \sum_{j=0}^{[\frac{N-1}{2}]} \epsilon(-1)^j c_{2j+1}(q) = (-i)\,\varepsilon \sum_{j=0}^{N} c_j(q)\, p_j(i) = (-i)\,\varepsilon\, q(i).$$

As above, we have $|(-i)\,\varepsilon| = 1$, and the equation $(-i)\,\varepsilon\, q(i) = |q(i)|$ follows. The case of only even coefficients with alternating sign can be dealt with analogously.

(iii) The formula for $a_2(q)$ is an immediate consequence of the orthogonality of the functions $e^{ij\phi}$ ∎

We are now ready to give explicit results for the simple Examples 4.1 – 4.2* above.

Theorem 2. *For our examples the sup-norm of the Jacobian K_n' can be calculated exactly.*

(i) *Let the measure σ be defined as in Example 4.1 (Chebyshev polynomials of first kind). Then the sup-norm of K_n' will be*

$$\|K_n'\|_\infty = 4\, U_{n-2}(3) \quad \text{for } n \geq 2. \tag{27}$$

In the case of shifted Chebyshev polynomials of first kind this norm evaluates as

$$\|K_n'\|_\infty = 4\, U_{2n-3}(3) \quad \text{for } n \geq 2. \tag{28}$$

(ii) *For Chebyshev polynomials of second kind (Example 4.2) we have the result*

$$\|K_n'\|_\infty = U_{n-1}(3) \quad \text{for } n \geq 2. \tag{29}$$

Finally, in the case of shifted Chebyshev polynomials of second kind (Example 4.2) we obtain the formula*

$$\|K_n'\|_\infty = \frac{1}{4}\,U_{2n-1}(3) \qquad \text{for } n \geq 2. \tag{30}$$

Proof. First, we prove equation (27). According to equation (17), we have to investigate the functionals $a_\infty(\psi_j)$. The fact that the Chebyshev polynomials $T_n = T_n(x)$ and $U_n = U_n(x)$ contain only odd or even powers of x with coefficients of alternating sign, is well known (see the explicit expressions [1: Formula 22.3.6] and [1: Formula 22.3.7] or [6: Formula 4.15] and [6: Formula 4.16]). For the polynomials $T_{j+1} - T_{j-1}$ this follows from the identity

$$T_{j+1}(x) - T_{j-1}(x) = 2\,(x^2 - 1)\,U_{j-1}(x) \tag{31}$$

(this is [1: Equation 22.7.25] with $n = j$ and $m = 1$). Thus, the polynomials ψ_j satisfy the assumptions of Lemma 2, and we have $a_\infty(\psi_j) = |\psi_j(i)|$. From equations (17), (19), (20) and (31) we can see that $\|K_n'\|_\infty$ will be the maximum of

$$\max_{0\leq j\leq 2} |\psi_j(i)| = \frac{3}{2}, \qquad \max_{2\leq j\leq n-1} |U_{2j-2}(i)|, \qquad 2\max_{1\leq j\leq n-1} |U_{2j-1}(i)|.$$

We will need the formulas for Chebyshev polynomials of first and second kinds (see [6: Formulas 4.13 and 4.14]):

$$T_n(x) = \frac{1}{2}\Big((x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\Big) \tag{32}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\Big((x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}\Big) \tag{33}$$

We see easily that there always holds $|U_j(i)| \geq |U_{j-1}(i)|$. Consequently, the maximal value will be $2\max_{1\leq j\leq n-1} |U_{2j-1}(i)| = 2\,|U_{2n-3}(i)|$ if only this is not less than $\frac{3}{2}$. To convert the result to real expressions, we need the identity

$$U_{2j-1}(x) = 2x\,U_{j-1}(2x^2 - 1). \tag{34}$$

This equation follows at once from the trigonometric form of the definition $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ (see [1: Formula 22.3.16]), since we have

$$U_{2j-1}(\cos\theta) = \frac{\sin 2j\theta}{\sin\theta} = \frac{\sin 2\theta}{\sin\theta}\cdot\frac{\sin(j\cdot 2\theta)}{\sin 2\theta} = 2\cos\theta\,U_{j-1}(\cos 2\theta).$$

Setting $x = i$, we obtain $U_{2n-3}(i) = 2iU_{n-2}(-3)$ and $\|K_n'\|_\infty = 2\,|2iU_{n-2}(-3)| = 4U_{n-2}(3)$ (this value is greater than $\frac{3}{2}$ for all $n \geq 2$).

The equation (28) can be shown analogously. First, we have to check the assumptions of Lemma 2. Indeed, the polynomial U_j^* has coefficients of alternating sign: From (34) we see

$$U_j^*(x) = U_j(2x - 1) = \frac{U_{2j+1}(\sqrt{x})}{2\sqrt{x}}.$$

The same is true for the polynomials $T^*_{j+1} - T^*_{j-1}$, since (31) gives (with $2x-1$ instead of x)

$$T^*_{j+1}(x) - T^*_{j-1}(x) = 8\,(x^2 - x)\,U^*_{j-1}(x). \tag{35}$$

Hence, all polynomials ψ^*_j have coefficients of alternating sign, and $a_\infty(\psi^*_j) = |\psi^*_j(-1)|$. Again, from equations (17), (21), (22) and (35) we obtain $\|K'_n\|_\infty = |\psi^*_{2n-1}(-1)| = 4U_{2n-3}(3)$, if only this is not less than $\max_{j=0,1,2} |\psi^*_j(-1)| = \frac{3}{2}$, and this is true for all $n \geq 2$.

The proof of equations (29) and (30) goes along the same lines ∎

The comparison of (28) with the result of W. Gautschi [3: Inequality 5.9] is very interesting. He investigates the map G_n instead of K_n and obtains the estimate

$$\mathrm{cond}_\infty(G_n, m) > \frac{(17 + 6\sqrt{8})^n}{64n^2}$$

where cond_∞ is defined via (3) with the norm $\|\cdot\| = \|\cdot\|_\infty$. From our equation (28) together with (33) we have the asymptotics

$$\|K'_n\|_\infty \sim 4\,\frac{1}{2\sqrt{8}}\,(3 + \sqrt{8})^{2n-2} = \frac{2(17 - 6\sqrt{8})}{\sqrt{8}}\,(17 + 6\sqrt{8})^n.$$

6. Bernstein polynomials

The set $\mathcal{B}_N = (B_{N,j})_{j=0}^N$ of well-known Bernstein polynomials defined by

$$B_{N,j}(x) = \binom{N}{j} x^j (1-x)^{N-j}$$

form a basis of $\mathcal{P}_N$. Since $\mathcal{B}_N$ is not a subset of $\mathcal{B}_M$ for $M \geq N$, we have to indicate the dependence of coordinate functionals and norms on the degree N explicitly:

$$q(x) = \sum_{j=0}^{N} c_{N,j}(q) B_{N,j}(x) \quad (q \in \mathcal{P}_N) \qquad \text{and} \qquad a_{N,\infty}(q) = \sum_{j=0}^{N} |c_{N,j}(q)|$$

(we consider here the supremum norm only).

Lemma 3. *Let $q \in \mathcal{P}_N \subset \mathcal{P}_M$ (i.e. $N \leq M$). Then we have the inequality*

$$a_{M,\infty}(q) \leq \frac{M+1}{N+1}\,a_{N,\infty}(q). \tag{36}$$

Proof. Obviously, we have to prove our proposition only for $M = N+1$, the general case follows via induction. From the identity

$$B_{N,j}(x) = \frac{N+1-j}{N+1}\,B_{N+1,j}(x) + \frac{j+1}{N+1}\,B_{N+1,j+1}(x)$$

for $j = 0, \dots, N$ we obtain

$$c_{N+1,j}(q) = \frac{N+1-j}{N+1}\, c_{N,j}(q) + \frac{j}{N+1}\, c_{N,j-1}(q)$$

for $q \in \mathcal{P}_N$ and $j = 0, \dots, N+1$ (the undefined functionals $c_{N,N+1}$ and $c_{N,-1}$ have coefficient zero). This gives

$$\begin{aligned} a_{N+1,\infty}(q) &= \sum_{j=0}^{N+1} |c_{N+1,j}(q)| \\ &\le \sum_{j=0}^{N} \frac{N+1-j}{N+1} |c_{N,j}(q)| + \sum_{j=1}^{N+1} \frac{j}{N+1} |c_{N,j-1}(q)| \\ &= \sum_{j=0}^{N} \left(\frac{N+1-j}{N+1} + \frac{j+1}{N+1} \right) |c_{N,j}(q)| \\ &= \frac{N+2}{N+1} \sum_{j=0}^{N} |c_{N,j}(q)| \\ &= \frac{N+2}{N+1}\, a_{N,\infty}(q) \end{aligned}$$

and the assertion is proved ∎

The following result enables us to evaluate $a_{N,\infty}(q)$ explicitly in some special cases

Lemmea 4. *Let $q \in \mathcal{P}_N$, and let its coefficients $c_{n,j}(q)$ in the basis $\mathcal{B}_N$ have alternating sign. Then the equality*

$$a_{N,\infty}(q) = (N+1) \left| \int_0^1 (2u-1)^N q\left(\frac{u}{2u-1}\right) du \right| \tag{37}$$

holds. The coefficients have alternating sign, if all zeros of the polynomial q are real and contained in $[0,1]$.

Proof. Suppose first that all coefficients $c_{N,j}(q)$ are positive. From the well-known fact that $\int_0^1 B_{N,j}(u)\, du = \frac{1}{N+1}$ we obtain

$$a_{N,\infty}(q) = \sum_{j=0}^{N} c_{N,j}(q) = (N+1) \int_0^1 q(u)\, du.$$

The first proposition now follows easily from the identity

$$(2u-1)^N B_{N,j}\left(\frac{u}{2u-1}\right) = (-1)^{N-j} B_{N,j}(u).$$

If all zeros $x_1, \dots, x_N$ of q are real and contained in $[0,1]$, then we have

$$q(x) = c \prod_{i=1}^{N} (x - x_i) = c \prod_{i=1}^{N} \Big((1-x_i)x - x_i(1-x) \Big)$$

and this polynomial obviously has coefficients of alternating sign in the basis $\mathcal{B}_N$ ∎

We choose $N = 2n - 1$, and equation (17) takes the form

$$\|K_n'\|_\infty = \max_{0\le j\le N} a_{N,\infty}(\psi_j).$$

Of course, our equation (37) enables us to evaluate $a_{j,\infty}(\psi_j)$ only. But in some cases we will be able to show

$$\max_{0\le j\le N} a_{N,\infty}(\psi_j) = a_{N,\infty}(\psi_N)$$

using the inequality

$$a_{N,\infty}(\psi_j) \le \frac{N+1}{j+1}\, a_{j,\infty}(\psi_j).$$

Since the natural interval for Bernstein polynomials is $[0,1]$, we illustrate our results with shifted Chebyshev polynomials only.

Theorem 3. *Let the measure σ be defined as in Example 4.2* (shifted Chebyshev polynomials of first kind). Then the supremum norm of K_n' can be estimated as*

$$\|K_n'\|_\infty = \frac{n}{2}\left(2^{2n-2}\,\frac{\Gamma(\frac{1}{2})\Gamma(2n-\frac{3}{2})}{\Gamma(2n-1)} - 2^{2n}\,\frac{\Gamma(\frac{3}{2})\Gamma(2n-\frac{1}{2})}{\Gamma(2n+1)} + O(1)\right) \tag{38}$$

for $n \to \infty$. In the case of Example 4.2 (shifted Chebyshev polynomials of second kind) we have the estimate*

$$\|K_n'\|_\infty = \frac{n}{2}\left(2^{2n-1}\,\frac{\Gamma(\frac{1}{2})\Gamma(2n+\frac{1}{2})}{\Gamma(2n+1)} + O(1)\right) \tag{39}$$

for $n \to \infty$.

Proof. First, we consider shifted Chebyshev polynomials of first kind and define the polynomials $q_j = T_j^* - T_{j-2}^*$ for $j \ge 2$. ¿From (35) we have $q_j(x) = 8x\,(x-1)\,U_{j-2}^*(x)$. This polynomial has all its zeros in $[0,1]$ and thus fulfils the assumptions of Lemma 4. Using equation (33) we get

$$q_j(x) = 2\sqrt{x^2-x}\Big(\big(2x-1+2\sqrt{x^2-x}\big)^{j-1} - \big(2x-1-2\sqrt{x^2-x}\big)^{j-1}\Big)$$

and from this

$$(2u-1)^j q_j\left(\frac{u}{2u-1}\right) = 2\sqrt{u-u^2}\Big(\big(1+2\sqrt{u-u^2}\big)^{j-1} - \big(1-2\sqrt{u-u^2}\big)^{j-1}\Big).$$

Our Lemma 4 yields

$$\frac{1}{j+1}\, a_{j,\infty}(q_j) = \int_0^1 2\sqrt{u-u^2}\Big(\big(1+2\sqrt{u-u^2}\big)^{j-1} - \big(1-2\sqrt{u-u^2}\big)^{j-1}\Big)du.$$

Now we can see that the integrand is monotonically increasing in j, and consequently we obtain

$$\frac{1}{j+1}\, a_{j,\infty}(q_j) \le \frac{1}{N+1}\, a_{N,\infty}(q_N) \qquad \text{for } 2 \le j \le N.$$

Together with (36) this implies

$$a_{N,\infty}(q_j) \le \frac{N+1}{j+1}\, a_{j,\infty}(q_j) \le a_{N,\infty}(q_N) \qquad \text{for } 2 \le j \le N.$$

Since according to equations (21) and (22) we have $\psi_j = \frac{1}{16}\, q_j$ for even $j > 2$ and $\psi_j = \frac{1}{4}\, q_j$ for odd $j > 2$ and $N = 2n-1$ is odd, the inequality

$$a_{N,\infty}(\psi_j) \le \frac{1}{4}\, a_{N,\infty}(q_j) \le \frac{1}{4}\, a_{N,\infty}(q_N) = a_{N,\infty}(\psi_N) \qquad \text{for } 2 < j \le N$$

follows immediately. This yields the equation

$$\max_{2<j\le N} a_{N,\infty}(\psi_j) = a_{N,\infty}(\psi_N),$$

and since we will see that $a_{N,\infty}(\psi_N)$ tends to infinity as $N \to \infty$, we have

$$\|K_n'\|_\infty = \max_{0\le j\le N} a_{N,\infty}(\psi_j) = a_{N,\infty}(\psi_N) = \frac{1}{4}\, a_{N,\infty}(q_N)$$

for N large enough. This quantity can be estimated easily: Our Lemma 4 gives the expression

$$\frac{1}{N+1}\, a_{N,\infty}(q_N) = \int_0^1 2\sqrt{u-u^2}\Big(\big(1+2\sqrt{u-u^2}\big)^{N-1} - \big(1-2\sqrt{u-u^2}\big)^{N-1}\Big)\, du.$$

But $|1-2\sqrt{u-u^2}| \le 1$, and consequently the right-hand side equals

$$\int_0^1 2\sqrt{u-u^2}\Big(\big(1+2\sqrt{u-u^2}\big)^{N-1} + \big(1-2\sqrt{u-u^2}\big)^{N-1}\Big) du + O(1).$$

The last integral can be evaluated: First, in the integral

$$\int_0^1 2\sqrt{u-u^2}\big(1+2\sqrt{u-u^2}\big)^{N-1} du = 2\int_0^{1/2} 2\sqrt{u-u^2}\big(1+2\sqrt{u-u^2}\big)^{N-1} du$$

we substitute $v = \sqrt{u} + \sqrt{1-u}$ to obtain

$$2\int_1^{\sqrt{2}} \frac{(v^2-1)^2}{\sqrt{2-v^2}}\, v^{2N-2}\, dv.$$

Further, in the integral

$$\int_0^1 2\sqrt{u-u^2}\big(1-2\sqrt{u-u^2}\big)^{N-1} du = 2\int_0^{1/2} 2\sqrt{u-u^2}\big(1-2\sqrt{u-u^2}\big)^{N-1} du$$

we substitute $v = \sqrt{1-u} - \sqrt{u}$ to obtain

$$2\int_0^1 \frac{(v^2-1)^2}{\sqrt{2-v^2}}\, v^{2N-2}\, dv.$$

Putting these integrals together, we obtain the equation

$$\frac{1}{N+1}\, a_{N,\infty}(q_N) = 2\int_0^{\sqrt{2}} \frac{(v^2-1)^2}{\sqrt{2-v^2}}\, v^{2N-2}\, dv + O(1).$$

Using the simple identity

$$\frac{(v^2-1)^2}{\sqrt{2-v^2}}\, v^{2N-2} = \frac{1}{\sqrt{2-v^2}}\, v^{2N-2} - \sqrt{2-v^2}\, v^{2N},$$

the substitution $v = \sqrt{2t}$ and the well-known formulas for the Beta integrals (see [1: 6.2.1 and 6.2.2])

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

we finally arrive at

$$\frac{1}{N+1}\, a_{N,\infty}(q_N) = 2^{N-1}\, \frac{\Gamma(\frac{1}{2})\Gamma(N-\frac{1}{2})}{\Gamma(N)} - 2^{N+1}\, \frac{\Gamma(\frac{3}{2})\Gamma(N+\frac{1}{2})}{\Gamma(N+2)} + O(1).$$

Inserting this in equation (40) and observing $N = 2n-1$, we obtain our proposition (38). The proof of our second proposition (39) is almost identical to the above one and will be omitted here ∎

7. Lagrange interpolation polynomials

Suppose we are given $N+1$ (different) nodes $x_0, \ldots, x_N$ of interpolation. If we denote

$$\omega_{N+1}(x) = (x-x_0)\cdots(x-x_N),$$

then the basic polynomials of interpolation in Lagrange form will be

$$l_{N,j}(x) = \frac{\omega_{N+1}(x)}{\omega'_{N+1}(x_j)(x-x_j)}$$

and we have the identity

$$q(x) = \sum_{j=0}^{N} q(x_j)\, l_{N,j}(x) \qquad \text{for } \; q \in \mathcal{P}_N.$$

Thus, our coordinate functionals c_j have the simple form $c_j(q) = q(x_j)$ for all $q \in \mathcal{P}_N$, and clearly the following result holds.

Lemma 5. *Let $q \in \mathcal{P}_N$ and $[a,b]$ be any interval containing given nodes x_j $(j = 0,\dots,N)$. Then we have the estimate*

$$a_\infty(q) \le (N+1)\,\|q\|_\infty \tag{41}$$

where $\|q\|_\infty$ is the usual sup-norm: $\|q\|_\infty = \sup_{x\in[a,b]} |q(x)|$.

In many practically important cases we have bounds for the modulus of orthonormal polynomials on the support of the measure.

Lemma 6. *Let the measure σ have support $S(\sigma) \subset [a,b]$ and assume*

$$\sigma([x,y]) \ge C\,(y-x)^\gamma \qquad \text{for } x,y \in [a,b]$$

with some constant $C > 0$. Then we have the bound

$$\|\pi_n\|_\infty \le \frac{2^{(\gamma+1)}}{\sqrt{C(b-a)^\gamma}}\, n^\gamma \qquad \text{for } n \ge 1. \tag{42}$$

Proof. We denote for brevity $\|\pi_n\|_\infty$ by M_n. Then clearly $M_n = \pi_n(x^*)$ for some $x^* \in [a,b]$, and from the classical Markov theorem we know

$$|\pi_n'(y)| \le \frac{2n^2}{b-a} M_n \qquad \text{for } y \in [a,b].$$

With the notation

$$B = \left\{ x \in [a,b] : |x - x^*| \le \frac{b-a}{4n^2} \right\}$$

this implies

$$|\pi_n(x)| = \left| \pi_n(x^*) + \int_{x^*}^{x} \pi_n'(y)\,dy \right| \ge |\pi_n(x^*)| - |x - x^*|\,\frac{2n^2}{b-a} M_n \ge \frac{1}{2} M_n$$

for $x \in B$, and we obtain the inequality

$$1 = \int \pi_n^2(x)\,d\sigma(x) \ge \int_B \pi_n^2(x)\,d\sigma(x) \ge \frac{1}{4} M_n^2 \cdot \sigma(B) \ge \frac{1}{4} M_n^2 \cdot C \left(\frac{b-a}{4n^2} \right)^\gamma.$$

Solving this inequality for M_n, we arrive at our proposition (42) (the idea of this proof is well known, compare the proof of Theorem 1.10 in [7: p. 145]) ∎

Now we are able to give simple estimates for the norm $\|K_n'\|_\infty$ in the case of these measures.

Theorem 4. *Let the measure σ fulfill the assumptions of Lemma 6 and let all nodes $x_0,\dots,x_N$, where $N = 2n-1$, be in the interval of orthogonality $[a,b]$. Then we can estimate*

$$\|K_n'\|_\infty = O(n^{2\gamma+1}). \tag{43}$$

Proof. Since the support $S(\sigma)$ is compact, the coefficients β_n are bounded, and from inequality (42) and the definitions (15) and (16) we deduce $\|\psi_j\|_\infty = O(j^{2\gamma})$. Our proposition now follows from the inequality

$$\|K_n'\|_\infty = \max_{0\le j\le N} a_\infty(\psi_j) \le (N+1) \max_{0\le j\le N} \|\psi_j\|_\infty$$

(cf. Lemma 5) ∎

Of course, the bound (42) is very rough. We can obtain sharper bounds for our illustrative examples.

Theorem 5. *Let the measure σ be defined as in example* 4.1 (*Chebyshev polynomials of first kind*), *and let all nodes $x_0, \ldots, x_N$, where $N = 2n - 1$, be in the interval of orthogonality* $[-1, 1]$. *Then the suppremum norm of K_n' can be estimated as*

$$\|K_n'\|_\infty \leq 2n. \tag{44}$$

In the case of example 4.2 (*Chebyshev polynomials of second kind*) *we have the estimate*

$$\|K_n'\|_\infty \leq 2n^2. \tag{45}$$

Proof. In the case of Chebyshev polynomials of first kind, we can see from our equations (19) and (20) that $\|\psi_j\|_\infty \leq 1$ for all $j \geq 0$. Consequently, we have the estimate

$$\|K_n'\|_\infty = \max_{0 \leq j \leq N} a_\infty(\psi_j) \leq (N+1) \max_{0 \leq j \leq N} \|\psi_j\|_\infty \leq N + 1 = 2n.$$

In the case of Chebyshev polynomials of second kind, our proposition follows easily from equations (23) and (24) and from $\|\psi_j\|_\infty \leq \frac{1}{2}\|U_j\|_\infty = \frac{1}{2}(j+1)$ for $j > 0$ ∎

Surprisingly, we have simple estimates for $\|K_n'\|_\infty$ of only polynomial growth in n. Moreover, the result does not depend on the distribution of the nodes inside the interval of orthogonality. However, this is somewhat misleading. The norm $\|K_n'\|_\infty$ is a measure for sensitivity to small *absolute* perturbations of the modified moments. But if the distribution of the nodes is bad, the basic polynomials may have large peaks, and consequently the modified moments may vary greatly in magnitude (compare the discussion in [4: Sections 2 and 6]). If the support of our measure σ is an interval, we could, of course, use well-known good nodes of interpolation (e.g. Chebyshev nodes), where the basic polynomials remain uniformly bounded. Unfortunately, this is only theoretically useful, too. It is hardly ever possible to compute the "Lagrange moments" practically even in the case of very simple measures σ.

8. Orthonormal systems

Let now $\{p_j\}_{n=0}^N$ be a system of orthonormal polynomials with respect to some measure s. Then the coordinate functional c_j will be $c_j(q) = \int q(x)\, p_j(x)\, ds(x)$. We shall concentrate in this case on the Frobenius norm. Clearly we have

$$a_2(q) = \sum_{j=0}^{N} c_j^2(q) = \int q^2(x)\, ds(x) \qquad \text{for } q \in \mathcal{P}_N.$$

This immediately yields, by (15), (16) and (18), the following result.

Theorem 6. *The Frobenius norm of K'_n is*

$$\|K'_n\|_F = \left\{ \int w_n(x)\, ds(x) \right\}^{1/2} \tag{46}$$

where

$$\begin{aligned} w_n(x) &= \sum_{j=0}^{2n-1} \psi_j^2(x) \\ &= \sum_{j=0}^{n-1} \left\{ \beta_j^2 \left(\pi_j^2(x) - \pi_{j-1}^2(x)\right)^2 + \left(\beta_{j+1}^{1/2} \pi_{j+1}(x) - \beta_j^{1/2} \pi_{j-1}(x)\right)^2 \pi_j^2(x) \right\}. \end{aligned}$$

For the class of measures mentioned in the preceding section we are able to give bounds for the norm $\|K'_n\|_F$ growing only polynomially in n.

Theorem 7. *Let the measure σ have support $S(\sigma) \subset [a,b]$ and assume $\sigma([x,y]) \geq C\,(y-x)^\gamma$ for $x, y \in [a,b]$ with some constant $C > 0$. Then we have the bound $\|K'_n\|_F = O(n^{2\gamma+\frac{1}{2}})$.*

Proof. Our assumptions imply $\|\psi_j\| = O(j^{2\gamma})$, and from equation (46) we deduce $\|K'_n\|_F^2 \leq s([a,b]) \sum_{j=0}^{2n-1} \|\psi_j\|^2$. The proposition follows immediately from $\sum_{j=0}^{2n-1} j^{4\gamma} = O(n^{4\gamma+1})$ ∎

Again, we illustrate our results with Examples 4.1 and 4.2.

Theorem 8. *Let the measure σ be defined as in example* (4.1) *(Chebyshev polynomials of first kind), and let s be any measure with support $\subset [-1,1]$. Then the polynomial w_n defined above can be bounded by*

$$w_n(x) \leq 2n \qquad \text{for } x \in [-1,1] \tag{47}$$

and consequently the Frobenius norm of K'_n can be estimated by

$$\|K'_n\|_F \leq \sqrt{2n\, s([-1,1])}. \tag{48}$$

In the case of Example (4.2) *(Chebyshev polynomials of second kind) we have the estimates*

$$w_n(x) \leq 1 + \frac{(2n-1)(4n^2+5n+3)}{12} \qquad \text{for } x \in [-1,1] \tag{49}$$

and

$$w_n(x) \leq 1 + \frac{1}{4}\frac{2n-1}{1-x^2} \qquad \text{for } x \in (-1,1) \tag{50}$$

and consequently for the special measure $ds(x) = dx$ on $[-1,1]$ we obtain the inequality

$$\|K'_n\|_F \leq \sqrt{2 + \frac{2n-1}{4} + \frac{2n-1}{4} \ln\left(\frac{16}{3}n^2 + \frac{20}{3}n + 3\right)}. \tag{51}$$

Proof. The proof of the inequality (47) is trivial, since for Chebyshev polynomials of first kind we saw $|\psi_j(x)| \leq 1$ in $[-1,1]$ for all $j \geq 0$. The estimate (48) is an immediate consequence of formula (46).

In the case of Chebyshev polynomials of second kind, we have $\psi_0 =$ and $\psi_j^2(x) \leq \frac{1}{4} U_j^2(x)$ for all $j \geq 1$ (see equations (23) and (24)). This gives the inequality

$$w_n(x) \leq 1 + \frac{1}{4} \sum_{j=1}^{2n-1} U_j^2(x). \tag{52}$$

Using the inequality $|U_j(x)| \leq j+1$ for $x \in [-1,1]$ (this is [1: 22.14.6]) and the elementary formula

$$\sum_{j=1}^{2n-1} (j+1)^2 = \sum_{j=1}^{2n} j^2 - 1 = \frac{(2n-1)(4n^2+5n+3)}{3}$$

we obtain inequality (49).

Recalling the trigonometric forms of the definitions of Chebyshev polynomials (compare equations [1: 22.3.15 and 22.3.16])

$$T_{j+1}(\cos\theta) = \cos(j+1)\theta \qquad \text{and} \qquad U_j(\cos\theta) = \frac{\sin(j+1)\theta}{\sin\theta}$$

we obtain (with $x = \cos\theta$) the identity $T_{j+1}^2(x) + (1-x^2)\,U_j^2(x) \equiv 1$. Consequently, the inequality $(1-x^2)\,U_j^2(x) \leq 1$ holds for all $j \geq 0$ and all real x. Our proposition (50) is an easy consequence of this inequality and (52).

The proof of the estimate (51) is now straightforward. From equation (46) we obtain

$$\|K_n'\|_F^2 = \int_{-1}^{1} w_n(x)\,dx = 2 + \int_{-1}^{1} \big(w_n(x) - 1\big)\,dx.$$

If we introduce the notation $\varepsilon_n = \frac{3}{8n^2+10n+6}$, we can estimate (using (49 and 50))

$$\int_{1-\varepsilon_n \leq |x| \leq 1} \big(w_n(x)-1\big)\,dx \leq \int_{1-\varepsilon_n \leq |x| \leq 1} \frac{2n-1}{8\varepsilon_n}\,dx = \frac{2n-1}{4}$$

and

$$\int_{-1+\varepsilon_n}^{1-\varepsilon_n} \big(w_n(x)-1\big)\,dx \leq \frac{2n-1}{4} \int_{-1+\varepsilon_n}^{1-\varepsilon_n} \frac{dx}{1-x^2} = \frac{2n-1}{4} \ln\left(\frac{2}{\varepsilon_n} - 1\right).$$

Putting these estimates together, we obtain our inequality (51) ∎

It is very interesting to compare our results with those of W. Gautschi [4]. He investigated the sensitivity of the map G_n instead (defined in our Section 2) and showed the equality

$$\|G_n'\|_F = \left(\int g_n(x)\, ds(x)\right)^{1/2}$$

where the polynomial g_n of degree $4n-2$ is defined via the weights and nodes of a quadrature formula for the measure σ (see the following section). For Chebyshev polynomials of first kind, he conjectured the polynomials g_n to be uniformly bounded on $[-1,1]$. This is true, as we will show in the following Appendix. Consequently, the norms $\|G_n'\|_F$ are bounded, too. In the case of Chebyshev polynomials of second kind, the polynomials g_n have large peaks near the ends of the interval $[-1,1]$, and the numerical values for $\|G_n'\|_F$ show moderate growth with n. This is in good agreement with our estimate (51) for the norms $\|K_n'\|_F$.

9. Appendix

In this section we estimate the polynomials g_n introduced by W. Gautschi [4] in the case of Chebyshev polynomials of first kind. We have

$$g_n(x) = \sum_{j=1}^{n} h_j^2(x) + \sum_{j=1}^{n} \sigma_j^{-2} k_j^2(x) \tag{53}$$

where

$$h_j(x) = l_j^2(x)\big(1 - 2l_j'(\tau_j)(x-\tau_j)\big) \tag{54}$$

and

$$k_j(x) = l_j^2(x)(x-\tau_j) \tag{55}$$

are the fundamental Hermite interpolation polynomials in terms of the fundamental Lagrange interpolation polynomials

$$l_j(x) = \frac{T_n(x)}{T_n'(\tau_j)(x-\tau_j)}. \tag{56}$$

The nodes τ_j are the zeros of the polynomial T_n (obviously, $|\tau_j| < 1$ for $j = 1, \ldots, n$). First, we have the identity

$$\sum_{j=1}^{n} h_j(x) = 1 \tag{57}$$

(this is just Hermite interpolation of the "polynomial" which is equal 1 everywhere). But in our case we obtain $l_j'(\tau_j) = \frac{1}{2}\tau_j(1-\tau_j^2)^{-1}$ (see [4: Equation 4.2]) and from (54) there follows $h_j(x) = l_j^2(x)\,\frac{1-x\tau_j}{1-\tau_j^2}$. Consequently, for $x \in [-1,1]$ we can estimate

$$h_j(x) \geq l_j^2(x)\,\frac{1-|\tau_j|}{1-\tau_j^2} = l_j^2(x)\,\frac{1}{1+|\tau_j|} \geq \frac{1}{2}\, l_j^2(x). \tag{58}$$

This gives immediately $h_j(x) \geq 0$, and from (57) we get $h_j(x) \leq 1$. Thus, the first sum on the right-hand side of (53) can be estimated as

$$\sum_{j=1}^{n} h_j^2(x) \leq \sum_{j=1}^{n} h_j(x) = 1.$$

To deal with the second sum, we note $\sigma_j = \frac{1}{n}$, and (55) and (56) yield

$$\sum_{j=1}^{n} \sigma_j^{-2} k_j^2(x) = n^2 T_n^2(x) \sum_{j=1}^{n} \frac{1}{T_n'^2(\tau_j)} l_j^2(x).$$

The derivative of T_n at the zeros τ_j can be easily calculated: ¿From the well-known identity $T_n(\cos\theta) = \cos n\theta$ we obtain $\sin\theta\, T_n'(\cos\theta) = n \sin n\theta$. Thus, $\sin^2\theta\, T_n'^2(\cos\theta) + n^2 \cos^2 n\theta = n^2$, and substituting $x = \cos\theta$ we obtain $(1 - x^2) T_n'^2(x) + n^2 T_n^2(x) = n^2$. From this identity the equation $T_n'^2(\tau_j) = \frac{n^2}{1-\tau_j^2}$ for the zeros of T_n follows immediately. Consequently, our sum can be written as

$$\sum_{j=1}^{n} \sigma_j^{-2} k_j^2(x) = T_n^2(x) \sum_{j=1}^{n} (1 - \tau_j^2)\, l_j^2(x).$$

The last sum can be estimated easily:

$$\sum_{j=1}^{n} (1 - \tau_j^2)\, l_j^2(x) \leq \sum_{j=1}^{n} l_j^2(x) \leq 2 \sum_{j=1}^{n} h_j(x) = 2$$

where we used the inequality $l_j^2(x) \leq 2h_j(x)$ following from (58). Finally, from (53) we obtain the interesting inequality

$$g_n(x) \leq 1 + 2 = 3 \tag{60}$$

uniformly in n. This result (via σ_j) depends on the normalization of the measure σ of orthogonality of the Chebyshev polynomials. W. Gautschi [4] used $\sigma([-1,1]) = \pi$, instead. This would imply $\sigma_j = \frac{\pi}{n}$ and

$$g_n(x) \leq 1 + \frac{2}{\pi^2} = 1.20264\ldots$$

which is not so far from the inequality $g_n(x) \leq 1$ conjectured by W. Gautschi.

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Received 08.06.1995