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A Note on the Bonnet-Myers Theorem

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Abstract. The aim of this note is to derive a compactness result for complete manifolds whose Ricci curvature is bounded from below. The classical result, usually stated as Bonnet-Myers theorem, provides an estimation of the diameter of a manifold whose Ricci curvature is greater than a strictly positive constant. Weaker assumptions that the Ricci curvature function tends slowly to zero (when the distance from a fixed point goes to infinity) were already considered in [2, 3]. We shall improve here their results.

Keywords: *Ricci curvature, Jacobi equation*

AMS subject classification: 53 C 20, 53 C 21

We will be concerned with the following analytic

Problem. Given a function $a : [r_0, +\infty) \to (0, +\infty)$ we consider positive solutions $y = y(r)$ of the differential equation

$$y'' + ay = 0$$

satisfying $y(r_0) = 0$. Obviously, y has to be concave. We have to determine the functions $a = a(r)$ for which y has a further zero $r_1 > r_0$ which may be bounded from above.

It is clear that there is such a bound in case a is a positive constant, but this bound tends to infinity as $a(r) \to 0$. The above problem seems to be interesting for functions a satisfying $\lim_{r\to+\infty} a(r) = 0$. It turns out that the right asymptotic is $a(r) \sim cr^{-2}$, with critical value $c = \frac{1}{4}$. In fact, for $c = \frac{1}{4} + v^2$ one gets the solution $y(r) = r^{\frac{1}{2}} \sin v\big(\log \frac{r}{r_0}\big)$, and hence there is a second zero. In this paper we show that in fact the constant v^2 may be replaced by a function which tends as weakly as an iterated logarithm to zero, which enters in our definition of some function $A_{k,v} = A_{k,v}(r)$.

Let us first make some notations. For each natural number k we set

$$Log_0(r) = r$$
$$L_k(r) = \underbrace{\log \ldots \log r}_{k}$$

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whenever is defined, and

$$A_{k,v}(r) = \frac{1}{4r^2}\left(1 + \frac{1}{L_1(r)^2} + \ldots + \frac{1}{L_1(r)^2 L_2(r)^2 \cdots L_{k-1}(r)^2} + \frac{1+4v^2}{L_1(r)^2 L_2(r)^2 \cdots L_k(r)^2}\right).$$

For a Riemannian manifold M, we denote by $Ric_x(Y)$ the Ricci curvature in the direction $Y \in T_x(M)$, for a point $x \in M$ and $T_x(M)$ being the tangent space to M in x. The space M is said to have an *almost positive asymptotic Ricci curvature* (abbreviated to be an AP-Riemannian space) if there exist $k, v, r_0 > 0$ and $p \in M$ such that

$$Ric_x(Y) \geq (n-1)\, A_{k,v}(r)|Y|^2 \tag{1}$$

holds for all $x \in M$ whose distance from a fixed point p is $r = dist(p,x) \geq r_0$ and for all vectors $Y \in T_x(M)$. Also $|\cdot|$ stands for the norm in the tangent space induced by the metric, and n is the dimension of M.

Our result can be stated now as follows.

Theorem 1. *A complete AP-Riemannian manifold is compact, and its diameter $d(M)$ is bounded by*

$$d(M) \leq e_{k-1}\left(L_{k-1}\left(\exp\frac{\pi}{v}\max\{r_0, e_k(0)\}\right)\right)$$

where $e_0(x) = x$ and $e_{m+1}(x) = \exp e_m(x)$ for $m > 0$.

Notice that the case $k = 0$ is discussed in [2] and the case $k = 1$ is covered by [3]. Also, Dekster and Kupka [3] proved that the constant $\frac{1}{4}$ is sharp, i.e. for any function $A = A(r)$ using in the place of $A_{k,r}$ so that Theorem 1 holds we must have

$$\lim_{r\to+\infty} A(r)r^2 \geq \frac{1}{4} \qquad \text{and} \qquad \lim_{r\to+\infty}\left(A(r)r^2 - \frac{1}{4}\right)(\log r)^2 \geq 1.$$

So our result identifies the higher order terms which might be added in spite to preserve the boundedness of the manifold. We think that the function $A_{k,v}$ is sharp.

Proof of Theorem 1. We write the Jacobi equation associated to the sectional curvature function $A_{k,v}$, namely

$$y'' + A_{k,v}(r)y = 0. \tag{2}$$

We claim that this equation admits the basic solutions

$$\Psi_0 = \Phi_k(r)\cos vL_k(r) \qquad \text{and} \qquad \Psi_1 = \Phi_k(r)\sin vL_k(r)$$

where

$$\Phi_k(r) = r^{\frac{1}{2}} L_1(r)^{\frac{1}{2}} \cdots L_{k-1}(r)^{\frac{1}{2}}.$$

For $k = 1$ it is easy to see that $r^{\frac{1}{2}}\cos(v\log r)$ and $r^{\frac{1}{2}}\sin(v\log r)$ are solutions for equation (2). By recurrence we prove first that the following relations are fulfilled (for $k = 1$ they are simply to check):

$$\Phi_k'' + A_{k,0}\Phi_k = 0 \qquad \text{and} \qquad 2\Phi_k'L_k' + \Phi_k L_k'' = 0.$$

In fact we have

$$\Phi_{k+1} = \Phi_k L_k^{\frac{1}{2}} \qquad \text{and} \qquad L_{k+1} = \log L_k,$$

hence

$$2\Phi_{k+1}'L_{k+1}' + \Phi_{k+1}kL_{k+1}'' = \left(2\Phi_k'L_k' + \Phi_k L_k'\right)L_k^{-\frac{1}{2}} = 0.$$

On the other hand

$$\frac{\Phi_{k+1}''}{\Phi_{k+1}} = -A_{k,0} + (L_k')^2 L_k^{-2} = -A_{k+1,0}$$

holds and the two relations stated above are proved.

Furthermore we verify that

$$\Psi_0'' = \Phi_k''\cos(vL_k) - v\left(2\Phi_k'L_k' + \Phi_k L_k'\right)\sin(vL_k) - v^2\Phi_k L_k'^2\cos(vL_k).$$

The two relations stated above and the obvious identity

$$L_k' = L_0^{-1}L_1^{-1} \cdots L_{k-1}^{-1}$$

complete the proof of our claim for Ψ_0 (the case of Ψ_1 is similar).

Both Ψ_0 and Ψ_1 are defined on the interval $[e_k(0), +\infty)$. Set $r_1 = \max\{r_0, e_k(0)\}$. Therefore, for each $\lambda \geq e_k(0)$ the linear combination

$$Y_{k,v,\lambda}(r) = -\sin(vL_k(\lambda))\,\Psi_0(r) + \cos(vL_k(\lambda))\,\Psi_1(r) \tag{3}$$

is a solution for equation (2), which satisfies also $Y_{k,v,\lambda}(\lambda) = 0$. Also, we may write

$$Y_{k,v,\lambda}(r) = \sin(v(L_k(r) - L_k(\lambda))\,\Phi_k(r)\,L_k(r)$$

so that $Y_{k,v,\lambda}$ is positive on the interval $(\lambda, \beta(\lambda))$ where $\beta(\lambda) = e_{k-1}\left(L_{k-1}(\lambda)\exp\left(\frac{\pi}{v}\right)\right)$ and vanishes again in $\beta(\lambda)$. This is a consequence of the straightforward formula

$$L_k(\beta(\lambda)) - L_k(\lambda) = \frac{\pi}{v}.$$

A standard argument (see, for instance, [1]) proves that the diameter of the manifold M is less than $\beta(r_1)$. Since M is complete from the Hopf-Rinow theorem it follows that M is in fact compact and this ends the proof of the theorem ∎

Remark 2. The form of the function $A_{k,v}$ is in some sense sharp. In fact, for $v = 0$ the analog result is false: We may choose on $M = \mathbb{R}^n - K$, with K being a sufficiently large compact, the metric with radial symmetry $dr + P_k(r)d\theta$ (in polar coordinates) where $d\theta$ is the metric form on the standard sphere S^{n-1} and

$$P_k(r) = r\left(\sum_{i=1}^{k} L_i(r)^{-2}\right)^{-\frac{1}{2}}.$$

Then a straightforward computation shows that $Ric_x(Y) = A_{k,0}(r)|Y|^2$ for all points x outside the compact K and all tangent vectors Y.

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