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On a Spatial Generalization of the Complex Π-Operator

K. Gürlebeck and U. Kähler

Abstract. The Π-operator plays a mayor role in complex analysis, especially in the theory of generalized analytic functions in the sense of Vekua. The present paper deals with a hypercomplex generalization of the complex Π-operator which turns out to have most of the useful properties of its complex origin such as mapping properties and invertibility. At the end an application of the generalized Π-operator to the solution of a hypercomplex Beltrami equation will be studied.

Keywords: *Clifford analysis, integral transforms, Π-operator, hypercomplex Beltrami equation*
AMS subject classification: Primary 30 G 35, secondary 30 G 20, 35 A 22, 35 A 35

1. Introduction

As it is well-known, there are close connections between Complex Analysis and the theory of partial differential equations. Using functional analytical methods many results of classical function theory can be applied for solving partial differential equations. Often the transformation of partial differential equations into integral equations is the starting point for such methods. The most important integral operators in this part of complex analysis are the so-called T- and Π-operator introduced by Vekua [15].

In multidimensional complex analysis many of these methods and efforts are lost. One of the most important reasons is the fact that in the theory of several complex variables there are no "good enough" analogies of the complex differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. That means these analogies have not the property to factorize the Laplacian. But, "good" elliptic analogies of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ exist in another generalization of one-dimensional complex analysis, namely, in the Clifford analysis. The hypercomplex Cauchy-Riemann operator has most of the properties of its complex origin.

For spatial generalization of Vekua's theory we have to generalize the integral operators used by Vekua. Till now, only one of these two operators, the T-operator, was systematically applied in generalized form [7, 13, 14]. It is not so well-known that Sprößig studied not only a generalization of the T-operator but also of the Π-operator [13].

K. Gürlebeck: Techn. Univ. Chemnitz – Zwickau, Dep. Math., D – 09107 Chemnitz
U. Kähler: Techn. Univ. Chemnitz – Zwickau, Dep. Math., D – 09107 Chemnitz

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This paper deals with this hypercomplex Π-operator in the following generalized Π-operator called. Starting from the results of [13] we study the Π-operator in Sobolev spaces. We prove an integral representation formula, continuity, invertibility, norm estimations, and some algebraic properties. An essential result for applications to the solution of boundary value problems is the description of the interaction between Π and the Bergman projection onto the L_2-space of monogenic functions. The application of this generalized Π-operator to the solution of a hypercomplex Beltrami equation shall only be sketched because this area of Clifford Analysis is still mainly undiscovered. But first steps have demonstrated that it seems to be possible to generalize not only the complex theory of generalized analytic functions by Vekua but also the complex methods for solving nonlinear systems of partial differential operators by Tutschke.

2. Preliminaries

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be an orthonormal basis of $\mathbb{R}^n$. The Clifford algebra $C\ell_{0,n}$ is the free algebra over $\mathbb{R}^n$ generated modulo the relation

$$x^2 = -|x|^2 \mathbf{e}_0$$

where $\mathbf{e}_0$ is the identity of $C\ell_{0,n}$. These algebras were introduced by Clifford in 1878 [2]. We remark that in the case of $n = 1$ the algebra $C\ell_{0,1}$ is isomorphic to $\mathbb{C}$. For the algebra $C\ell_{0,n}$ we have the multiplication rule

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij} \mathbf{e}_0 \qquad (i, j = 1, \ldots, n)$$

where δ_{ij} is the Kronecker symbol. Taking this rule we get a basis of this algebra in the form

$$\{\mathbf{e}_A\}_{A \subseteq \{1,\ldots,n\}}$$

with $\mathbf{e}_A = \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k}$, $\mathbf{e}_{\{i\}} = \mathbf{e}_i$ $(i = 1, \ldots, n)$ and $\mathbf{e}_\emptyset = \mathbf{e}_0$. Each element of the algebra $C\ell_{0,n}$ can be represented in the form

$$x = \sum_{A \subseteq \{1,\ldots,n\}} x_A \mathbf{e}_A$$

where x_A are real numbers. This algebra has the dimension 2^n. The elements of the Clifford algebra are called Clifford numbers. If the set A contains k elements, then we call $\mathbf{e}_A$ a *k-vector*. Likewise, we call each linear combination of k vectors a k-vector. The vector space of all k-vectors is denoted by $\bigwedge^k \mathbb{R}^n$. Obviously, $C\ell_{0,n}$ is the direct sum of all $\bigwedge^k \mathbb{R}^n$ for $k \leq n$. By

$$\bar{x} = \sum_{A \subseteq \{1,\ldots,n\}} x_A \bar{\mathbf{e}}_A$$

where $\bar{\mathbf{e}}_A = \bar{\mathbf{e}}_{i_k} \cdots \bar{\mathbf{e}}_{i_1}$ and $\bar{\mathbf{e}}_j = -\mathbf{e}_j$ $(j = 1, \ldots, n)$ we define a conjugate element.

In the following we identify the Euclidian space $\mathbb{R}^{n+1}$ with the direct sum $\bigwedge^0 \mathbb{R}^n \oplus \bigwedge^1 \mathbb{R}^n$. For all what follows let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain with a sufficiently

smooth boundary $\Gamma = \partial\Omega$. Then functions f defined in Ω with values in $C\ell_{0,n}$ are considered. These functions may be written as

$$f(z) = \sum_{k=0}^{n} \mathbf{e}_k f_k(z) \qquad (z \in \Omega).$$

Properties such as continuity, differentiability, integrability, and so on, which are ascribed to f have to be possessed by all components $f_k(z)$ $(k = 0, \ldots, n)$. In this way the usual Banach spaces of these functions are denoted by $\mathcal{C}^\alpha$, $\mathcal{L}_p$ and $\mathcal{W}_p^k$. In the case of $p = 2$ we introduce in $\mathcal{L}_2(\Omega)$ the $C\ell_{0,n}$-valued inner product

$$(u, v) = \int_\Omega \bar{u}(\xi) v(\xi)\, d\Omega_\xi. \tag{1}$$

We now define the generalized Cauchy-Riemann operator by

$$D = \sum_{k=0}^{n} \mathbf{e}_k \frac{\partial}{\partial x_k}.$$

For this operator we have that

$$D\overline{D} = \overline{D}D = \Delta \tag{2}$$

where Δ is the Laplacian and $\overline{D} = \sum_{k=0}^{n} \bar{\mathbf{e}}_k \frac{\partial}{\partial x_k}$ is the conjugate Cauchy-Riemann operator. A sufficiently smooth function $f : \Omega \mapsto C\ell_{0,n}$ is said to be *left-monogenic* if it satisfies the equation $(Df)(\xi) = 0$ for each $\xi \in \Omega$. The Cauchy-Riemann operator has a right inverse in the form

$$(Tf)(z) = -\frac{1}{\omega} \int_\Omega \frac{\overline{(\xi - z)}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi \qquad (z \in \Omega)$$

where $E(\xi, z) = -\frac{1}{\omega} \frac{\overline{(\xi - z)}}{|\xi - z|^{n+1}}$ is the generalized Cauchy kernel and ω stands for the surface area of the unit sphere in $\mathbb{R}^{n+1}$. This operator acts from $\mathcal{W}_p^k(\Omega)$ to $\mathcal{W}_p^{k+1}(\Omega)$ with $1 < p < \infty$ and $k \in \cup\{0\}$ (see [7]). We remark that for $n = 1$ we get from this definition the complex T-operator

$$T_\Omega h(z) = -\frac{1}{\pi} \iint_\Omega \frac{h(\zeta)}{\zeta - z}\, d\xi d\eta \qquad (z \in \Omega)$$

up to the factor two.

Introducing the boundary integral operator

$$(F_\Gamma f)(z) = \frac{1}{\omega} \int_\Gamma \frac{\overline{(\xi - z)}}{|\xi - z|^{n+1}} \alpha(\xi) f(\xi)\, d\Gamma_\xi \qquad (z o t in \Gamma)$$

where $\alpha(\xi)$ is the outward pointing normal unit vector to Γ at the point ξ, we get the well-known Borel-Pompeiu formula

$$(F_\Gamma f)(z) + (TDf)(z) = f(z) \qquad (z \in \Omega).$$

Obviously, $DF_\Gamma f = 0$ and $F_\Gamma T f = 0$ hold. In [7] it is proved that F_Γ acts from $\mathcal{W}_p^{k-\frac{1}{p}}(\Gamma)$ into $\mathcal{W}_p^k(\Omega)$ $(1 < p < \infty;\ k \in)$.

Notice that above all our integral operators will be defined in spaces of Hölder-continuous functions. It is possible to extend these operators to Sobolev spaces in the classical way be approximation (with Hoelder-continuous functions). We omit the detailed discussion here. We remark that then all the referred formulas have to be understood in the generalized sense.

Taking the traces of $F_\Gamma f$ we introduce the projections

$$(P_\Gamma f)(z) = \lim_{\substack{\xi \to z \\ \xi \in \Omega,\ z \in \Gamma}} (F_\Gamma f)(\xi)$$

$$(Q_\Gamma f)(z) = \lim_{\substack{\xi \to z \\ \xi \in \mathbb{R}^{n+1} \setminus \overline{\Omega},\ z \in \Gamma}} (-F_\Gamma f)(\xi).$$

P_Γ is the projection onto the space of all Clifford-valued functions which may be left-monogenic extended into the domain Ω. Q_Γ is the projection onto the space of all Clifford-valued functions which may be left-monogenic extended into the domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. For these projections $Q_\Gamma P_\Gamma f = P_\Gamma Q_\Gamma f = 0$ holds for all f (for further details see [7]). For an integral operator

$$(Af)(z) = af(z) + \int_\Omega k(\xi - z)\, f(\xi)\, d\Omega_\xi \qquad (z \in \Omega)$$

where $a \in C\ell_{0,n}$ we define a conjugate operator $\overline{A}$ by

$$(\overline{A}f)(z) = af(z) + \int_\Omega \overline{k(\xi - z)}\, f(\xi)\, d\Omega_\xi \qquad (z \in \Omega).$$

In the same way we define for a boundary integral operator

$$(A_\Gamma f)(z) = af(z) + \int_\Gamma k(\xi - z)\, \alpha(\xi)\, f(\xi)\, d\Gamma_\xi$$

a conjugate operator by

$$(\overline{A}_\Gamma f)(z) = af(z) + \int_\Gamma \overline{k(\xi - z)}\, \overline{\alpha(\xi)}\, f(\xi)\, d\Gamma_\xi.$$

It will be clear from the context whether $z \in \Omega$ or $z \in \Gamma$.

For this paper we also need the following decompositions of the space $\mathcal{L}_2(\Omega)$.

Theorem 1. *The right Hilbert modul $\mathcal{L}_2(\Omega)$ allows the orthogonal decompositions*

$$\mathcal{L}_2(\Omega) = \ker D(\Omega) \cap \mathcal{L}_2(\Omega) \oplus \overline{D}(\overset{\circ}{\mathcal{W}}{}_2^1(\Omega)) \tag{3}$$

$$\mathcal{L}_2(\Omega) = \ker \overline{D}(\Omega) \cap \mathcal{L}_2(\Omega) \oplus D(\overset{\circ}{\mathcal{W}}{}_2^1(\Omega)) \tag{4}$$

with respect to the inner product (1).

For the idea of the proof we refer to a proof of a similar theorem in [7: Theorem 3.1] made in the case of the Dirac operator. Besides we will only prove the decomposition (3), because the proof of (4) is analogous to that of (3).

Proof of Theorem 1. The right linear sets

$$\mathcal{X}_1 = \mathcal{L}_2(\Omega) \cap \ker D(\Omega) \qquad \text{and} \qquad \mathcal{X}_2 = \mathcal{L}_2(\Omega) \ominus \mathcal{X}_1$$

are subspaces of $\mathcal{L}_2(\Omega)$. For any $u \in \mathcal{L}_2(\Omega)$ we have $\overline{T}u \in \mathcal{W}_2^1(\Omega)$. From this it follows that there exists a function $v \in \mathcal{W}_2^1(\Omega)$ with $u = \overline{D}v$. Let $u \in \mathcal{X}_2$. Then, we have for all $g \in \mathcal{X}_1$

$$\int_\Omega \overline{\overline{D}v}\, g\, d\Omega = 0$$

and, in particular, for any $l \in$

$$\int_\Omega \overline{\overline{D}v}\, g_l\, d\Omega = 0 \tag{5}$$

with

$$g_l(x) = \frac{\overline{(x - y_l)}}{|x - y_l|^{n+1}} \qquad (l \in,\ y_l \in \mathbb{R}^{n+1} \setminus \overline{\Omega}).$$

Obviously, $g_l \in \ker D(\Omega) \cap \mathcal{L}_2(\Omega)$. We assume that the set $\{y_l\}_{l\in}$ is dense in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Then we get for any y_l

$$\begin{aligned}
\int_\Omega \overline{\overline{D}v}\, g_l\, d\Omega_x &= \sum_{i,j=0}^{n} \int_\Omega \overline{\bar{\mathbf{e}}_i \frac{\partial}{\partial x_i} v_j \mathbf{e}_j}\, g_l\, d\Omega_x \\
&= -\sum_{i,j=0}^{n} \int_\Omega \bar{\mathbf{e}}_j \mathbf{e}_i v_j \frac{\partial}{\partial x_i} g_l\, d\Omega_x + \sum_{i,j=0}^{n} \int_\Gamma \bar{\mathbf{e}}_j\, \mathbf{e}_i\, v_j\, \alpha_i\, g_l\, d\Gamma_x \\
&= -\int_\Omega \bar{v}\, D g_l\, d\Omega_x + \int_\Gamma \bar{v}\, \alpha g_l\, d\Gamma_x \\
&= \overline{\int_\Gamma \bar{g}_l\, \bar{\alpha}\, v\, d\Gamma_x} \\
&= \overline{\int_\Gamma \frac{(x - y_l)}{|x - y_l|^{n+1}}\, \bar{\alpha}\, v\, d\Gamma_x} \\
&= \omega(\overline{F}_\Gamma(\operatorname{tr} v))(y_l)
\end{aligned}$$

where $\operatorname{tr} v$ denotes the trace of v. Using equation (5) we get $\overline{F}_\Gamma(\operatorname{tr} v) = 0$ in $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ and it follows that $\operatorname{tr} v \in \operatorname{im}\overline{P}_\Gamma \cap \mathcal{W}_2^{\frac{1}{2}}(\Gamma)$. Consequently, there exists a function $h \in \mathcal{W}_2^1(\Omega) \cap \ker \overline{D}(\Omega)$ with the property that $\operatorname{tr} h = \operatorname{tr} v$. Taking the function $w = v - h \in \mathring{\mathcal{W}}_2^1(\Omega)$ we get that $\overline{D}w \in \overline{D}(\mathring{\mathcal{W}}_2^1(\Omega))$. The result now follows from $u = \overline{D}\, v = \overline{D}\, w$ ∎

Remark 1. The decompositions of the space $\mathcal{L}_2(\Omega)$ define the complementary orthoprojections

$$\mathbf{P} : \mathcal{L}_2(\Omega) \mapsto \ker D(\Omega) \cap \mathcal{L}_2(\Omega)$$

$$\mathbf{Q} : \mathcal{L}_2(\Omega) \mapsto \overline{D}(\mathring{\mathcal{W}}_2^1(\Omega))$$

and

$$\overline{\mathbf{P}} : \mathcal{L}_2(\Omega) \mapsto \ker \overline{D}(\Omega) \cap \mathcal{L}_2(\Omega)$$

$$\overline{\mathbf{Q}} : \mathcal{L}_2(\Omega) \mapsto D(\mathring{\mathcal{W}}_2^1(\Omega)).$$

It is easy to see that $\mathbf{Q} = I - \mathbf{P}$ and $\overline{\mathbf{Q}} = I - \overline{\mathbf{P}}$. Furthermore, the projection $\mathbf{P}$ is the Bergman operator. We have to remark here that there are other approaches to the Bergman operator which do not use the decompositions (3) and (4) for the definition and representation of $\mathbf{P}$. We refer to [1, 3] where the authors proved integral representation formulae of $\mathbf{P}$ in some special cases with explicitly known kernel functions. In the paper [12] integral representations of the Bergman operator are contained, based on derivatives of the Green function with respect to the Dirichlet problem for the Laplacian. This line of consideration is devoted to the study of more general properties of the kernel function and of the Bergman projection. Using the representations from [12] results concerning the singularities of the kernel function and its behaviour near the boundary are possible as well as a first description of the operator algebra generated by $\mathbf{P}$.

For more information about these topics and general Clifford analysis see [1, 4 – 7].

3. Definition and properties of the generalized Π-operator

Starting with his generalization of the complex T-operator in [13] Sprößig proposed the following definition of a hypercomplex analogy of the complex Π-operator.

Definition 1. The operator Π defined by

$$\Pi f = \overline{D} T f$$

is called *generalized Π-operator*.

This operator acts from $C^\alpha(\Omega)$ into $C^\alpha(\Omega)$ $(0 < \alpha \leq 1)$ [13]. Applying the definition of the T- and the $\overline{D}$-operator we get an integral representation formula for the Π-operator.

Theorem 2. *Assume that* $f \in C^\alpha(\Omega)$ $(0 < \alpha \leq 1)$. *Then*

$$(\Pi f)(z) = -\frac{1}{\omega} \int_\Omega \frac{(n-1) + (n+1)\frac{\overline{(\xi - z)}^2}{|\xi - z|^2}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi + \frac{1-n}{1+n} f(z) \tag{6}$$

holds.

Proof. Using [10: Chapter IX/§7] we get for $f \in C^\alpha(\Omega)$ $(0 < \alpha \leq 1;\ k = 0, \ldots, n)$ the equation

$$\frac{\partial}{\partial z_k} \int_\Omega \frac{\overline{(\xi - z)}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi$$

$$= \int_\Omega \frac{-\bar{e}_k + (n+1)(\xi_k - z_k)\frac{\overline{(\xi - z)}}{|\xi - z|^2}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi - \omega \frac{\bar{e}_k}{n+1} f(z)$$

because

$$\frac{\partial}{\partial z_k} \left[\frac{\overline{(\xi - z)}}{|\xi - z|^{n+1}} \right] = \frac{-\bar{e}_k + (n+1)(\xi_k - z_k)\frac{\overline{(\xi - z)}}{|\xi - z|^2}}{|\xi - z|^{n+1}}$$

and

$$\int_S \frac{\overline{(\xi - z)}}{|\xi - z|} \cos(r, z_k)\, dS = \omega \frac{\bar{e}_k}{n+1}.$$

From this we obtain our representation formula by summation over k ∎

Remark 2. Obviously we get from this representation formula that Π is a strongly singular operator of Calderon-Zygmund type.

Remark 3. In the case of $n = 1$ this hypercomplex Π-operator coincides with the usual complex Π-operator

$$\Pi_\Omega h(z) = \frac{\partial}{\partial z} T_\Omega h(z) = -\frac{1}{\pi} \iint_\Omega \frac{h(\zeta)}{(\zeta - z)^2} d\xi d\eta$$

up to the factor two.

Theorem 3. *Assume that* $f \in C^\alpha(\Omega)$ $(0 < \alpha \leq 0)$. *Then for the conjugate operator* $\overline{\Pi}$ *of* Π

$$\overline{\Pi} f = D\overline{T} f$$

holds.

Proof. Using our representation formula of the Π-operator and the definition of the conjugate operator we get

$$(\overline{\Pi} f)(z) = -\frac{1}{\omega} \int_\Omega \frac{(n-1) + (n+1)\frac{(\xi - z)^2}{|\xi - z|^2}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi + \frac{1-n}{1+n} f(z)$$

with $f \in C^\alpha(\Omega)$ $(0 < \alpha \leq 1)$. Calculating $D\overline{T} f$ we have

$$\begin{aligned} D\overline{T} f &= \sum_{k=0}^{n} \mathbf{e}_k \left(-\frac{1}{\omega}\right) \int_\Omega \frac{-\mathbf{e}_k + (n+1)(\xi_k - z_k)\frac{(\xi - z)}{|\xi - z|^2}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi + \sum_{k=0}^{n} \frac{\mathbf{e}_k^2}{n+1} f(z) \\ &= -\frac{1}{\omega} \int_\Omega \frac{(n-1) + (n+1)\frac{(\xi - z)^2}{|\xi - z|^2}}{|\xi - z|^{n+1}} f(\xi)\, d\Omega_\xi + \frac{1-n}{1+n} f(z). \end{aligned}$$

From this we conclude $\overline{\Pi} f = D\overline{T} f$ ∎

Remark 4. In general the relation

$$\overline{\Pi} f = \overline{T} D f$$

is not true.

If we look for applications of the generalized Π-operator, then we need its mapping properties within Sobolev spaces. Because we have that the Π-operator is an operator of Calderon-Zygmund type we can apply the theory of Calderon and Zygmund. This means that first we have to look for the symbol of the Π-operator. In the theory of singular integral operators the symbol is the main tool for investigating these operators. For the sake of brevity we use the same notations as in [10] and we refer to this textbook for more information and the details. In our case the symbol of a singular integral operator

$$(Af)(z) = af(z) + \int_\Omega k(\xi - z)\, f(\xi)\, d\Omega_\xi$$

is defined by

$$\Phi(\theta) = a + \mathcal{F}(k)(\theta)$$

where $\mathcal{F}(k)(\theta)$ is the Fourier transform of $k(\zeta) = k(\xi - z)$ $\left(\theta = \frac{z}{|z|}\right)$. Using the representation of the symbol by the characteristic $\kappa(\theta') = k(\zeta)|\zeta|^{n+1}$ with $\theta' = \frac{\zeta}{|\zeta|}$ we get for the generalized Π-operator

$$\begin{aligned}\Phi(\theta) &= \frac{1-n}{1+n} + \tilde{\Phi}(\theta)\\ \tilde{\Phi}(\theta) &= \int_S \kappa(\theta') \left[\ln \frac{1}{|\cos\gamma|} + \frac{i\pi}{2}\operatorname{sign}\cos\gamma\right] dS_{\theta'}\end{aligned}$$

where $\tilde{\Phi}(\theta)$ is the symbol of the singular integral and S is the unit sphere in $\mathbb{R}^{n+1}$. We denote by γ the angle between θ and θ' (in the Fourier transform). For the characteristic $\kappa(\theta)$ we have

$$\kappa(\theta) = \frac{1}{\omega}\Big((n-1) + (n+1)\overline{\theta}^2\Big).$$

Applying the theorem of Calderon and Zygmund [10: Chapter XI 9.1] we obtain the following

Theorem 4. *Suppose that* $1 < p < \infty$ *and* $k \in \cup\{0\}$. *Then*

$$\Pi : \mathcal{W}_p^k(\Omega) \mapsto \mathcal{W}_p^k(\Omega)$$

holds.

By the help of the theory of Calderon and Zygmund we have that the norm of the Π-operator is equal to the essential supremum of the absolute value of the symbol. That means we obtain

$$\|\Pi\|_{[\mathcal{L}_2(\Omega),\mathcal{L}_2(\Omega)]} = \operatorname{supess}|\Phi(\theta)|.$$

Or with other words if we use the representation of the symbol $\Phi(\theta) = \frac{1-n}{1+n} + \tilde{\Phi}(\theta)$ by the characteristic, then we get the estimate

$$\begin{aligned}|\tilde{\Phi}(\theta)|^2 &= \left|\int_S \kappa(\theta)\left[\ln\frac{1}{|\cos\gamma|} + \frac{i\pi}{2}\operatorname{sign}\cos\gamma\right] dS\right|^2\\ &\le \int_S |\kappa(\theta)|^2 \left|\ln\frac{1}{|\cos\gamma|} + \frac{i\pi}{2}\operatorname{sign}\cos\gamma\right|^2 dS\cdot\omega\\ &\le \left(\int_S |\kappa(\theta)|^4\, dS\right)^{\frac{1}{2}} \left(\int_S \left|\ln\frac{1}{|\cos\gamma|} + \frac{i\pi}{2}\operatorname{sign}\cos\gamma\right|^4 dS\right)^{\frac{1}{2}}\cdot\omega\\ &\le \left(\int_S |\kappa(\theta)|^4 dS\right)^{\frac{1}{2}}\cdot c_4\cdot\omega\end{aligned}$$

with some constant c_4 (see below), where

$$\begin{aligned}\left(\int_S |\kappa(\theta)|^4\, dS\right)^{\frac{1}{2}} &\leq \left(\int_S \left|\frac{(n-1)+(n+1)\overline{\theta}^2}{\omega}\right|^4 dS\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\omega^4}\int_S (n-1+n+1)^4\, dS\right)^{\frac{1}{2}} \\ &\leq \frac{4n^2}{\omega^{\frac{3}{2}}}.\end{aligned}$$

From this we obtain an estimate of the norm of the generalized Π-operator in the form

$$\|\Pi\|_{[\mathcal{L}_2(\Omega),\mathcal{L}_2(\Omega)]} \leq \frac{n-1}{1+n} + \frac{2n\sqrt{c_4}}{\omega^{\frac{1}{4}}} \tag{7}$$

where

$$c_4 = \left(\int_S \left|\ln\frac{1}{|\cos\gamma|} + \frac{i\pi}{2}\operatorname{sign}\cos\gamma\right|^4 dS_{\theta'}\right)^{\frac{1}{2}}$$

is a constant. We remark that in the case of $C\ell_{0,2} \cong \mathbb{H}$ we can calculate the integral $\left(\int_S |\kappa(\theta)|^4\, dS\right)^{1/2}$ exactly. So we get

$$\begin{aligned}\left(\int_S |f(\theta)|^4\, dS\right)^{\frac{1}{2}} &= \left(\int_0^{\pi}\int_0^{\pi}\int_0^{2\pi} |f(\theta)|^4 \sin^2\varphi_1 \sin\varphi_2\, d\varphi_3\, d\varphi_2\, d\varphi_1\right)^{\frac{1}{2}} \\ &= \left(\int_0^{\pi}\int_0^{\pi} \frac{\left(6\sin\varphi_1 + 4\sin(3\varphi_1)\right)^2 \sin\varphi_2}{\pi^7}\, d\varphi_2\, d\varphi_1\right)^{\frac{1}{2}} \\ &= \frac{\sqrt{26}}{\pi^3}\end{aligned}$$

or for our norm estimate

$$|\Phi(\theta)| \leq \frac{1}{2} + \sqrt{\frac{2\sqrt{26}}{\pi}\cdot c_4}.$$

For further details see [8]. Notice that Theorem 4 remains true in the case of $\Omega = \mathbb{R}^{n+1}$.

Theorem 5. *Suppose that $f \in \mathcal{W}_p^1(\Omega)$ $(1 < p < \infty)$. Then*

$$D\Pi f = \overline{D} f \tag{8}$$

$$\Pi D f = \overline{D} f - \overline{D} F_\Gamma f \tag{9}$$

$$F_\Gamma \Pi f = (\overline{D} T - T\overline{D}) f \tag{10}$$

$$(D\Pi - \Pi D) f = \overline{D} F_\Gamma f \tag{11}$$

$$F_\Gamma \Pi \overline{F}_\Gamma f = \Pi \overline{F}_\Gamma f \tag{12}$$

hold.

Proof. For the proof of (8) - (12) we have always in one line

$$\begin{aligned} D\Pi f &= D\overline{D}Tf = \overline{D}DTf = \overline{D}f \\ \Pi Df &= \overline{D}TDf = \overline{D}(I - F_\Gamma)f = \overline{D}f - \overline{D}F_\Gamma f \\ F_\Gamma \Pi f &= (I - TD)\overline{D}Tf = \overline{D}Tf - T\overline{D}DTf = (\overline{D}T - T\overline{D})f \\ (D\Pi - \Pi D)f &= (\overline{D} - \overline{D} + \overline{D}F_\Gamma)f = \overline{D}F_\Gamma f \\ F_\Gamma \Pi \overline{F}_\Gamma f &= (\overline{D}T - T\overline{D})(I - \overline{T}\,\overline{D})f = \overline{D}T(I - \overline{T}\,\overline{D})f = \Pi \overline{F}_\Gamma f \end{aligned}$$

and the proof is finished ∎

From equation (8) we obtain the following important mapping property of the generalized Π-operator.

Proposition 1. *The relation*

$$\Pi : \ker \overline{D}(\Omega) \cap \mathcal{L}_2(\Omega) \mapsto \ker D(\Omega) \cap \mathcal{L}_2(\Omega)$$

is true.

In terms of the orthoprojections we obtained $\Pi : \operatorname{im} \overline{\mathbf{P}} \mapsto \operatorname{im} \mathbf{P}$. From equation (9) and $F_\Gamma u = 0$ for any function $u \in \overset{\circ}{\mathcal{W}}{}_2^1(\Omega)$ we get the following

Proposition 2. *The relation*

$$\Pi : D(\overset{\circ}{\mathcal{W}}{}_2^1(\Omega)) \mapsto \overline{D}(\overset{\circ}{\mathcal{W}}{}_2^1(\Omega))$$

is true.

Proposition 2 means that

$$\Pi : \operatorname{im} \overline{\mathbf{Q}} \mapsto \operatorname{im} \mathbf{Q}.$$

Therefore, the Bergman operator preserves the orthogonal decomposition of $\mathcal{L}_2(\Omega)$ in a certain sense. More exactly, decompositions generated by D are transformed into those generated by $\overline{D}$. In [11] the authors study the $\bar{\partial}$-problem for quaternionic-valued functions. They prove the existence and a representation formula of the solution using the subspaces $\operatorname{im} \mathbf{P}$ and $\ker \mathbf{P}$, respectively. In [7] the orthoprojection $\mathbf{P}$ is used to solve second order boundary value problems of Dirichlet's type. Therefore, from the present point of view it seems to be advantageous to preserve the mentioned invariance properties also in the class of problems connected with applications of Π.

Using the same ideas we can obtain similar results for $\overline{\Pi}$, namely we have the following

Remark 5. Again investigating the above mapping properties for the conjugate operator $\overline{\Pi}$ we get

$$\begin{aligned} \overline{\Pi} &: \operatorname{im} \mathbf{P} \mapsto \operatorname{im} \overline{\mathbf{P}} \\ \overline{\Pi} &: \operatorname{im} \mathbf{Q} \mapsto \operatorname{im} \overline{\mathbf{Q}}. \end{aligned}$$

The complex Π-operator is a unitary operator for special domains (see, e.g., [15]). Investigating the hypercomplex Π-operator we get the following connection between the Π-operator and its conjugate operator $\overline{\Pi}$.

Theorem 6. *Suppose* $f \in \mathcal{W}_p^k(\Omega)$ $(1 < p < \infty;\ k \in)$. *Then we have*

$$\overline{\Pi}\Pi f = D\overline{T}\ \overline{D}Tf = D(I - \overline{F}_\Gamma)Tf = f - D\overline{F}_\Gamma Tf$$
$$\Pi\overline{\Pi} f = \overline{D}T D\overline{T}f = \overline{D}(I - F_\Gamma)\overline{T}f = f - \overline{D}F_\Gamma\overline{T}f.$$

The first part of this theorem was proved by Sprößig in the case of Hölder-continuous functions.

Corollary 1. *The relations*

$$\overline{\Pi}\Pi f = f \qquad \textit{for all } f \in \textit{im}\ \overline{\mathbf{Q}}$$
$$\Pi\overline{\Pi} f = f \qquad \textit{for all } f \in \textit{im}\ \mathbf{Q}$$

hold.

Proof. From $f \in \operatorname{im}\overline{\mathbf{Q}}$ we get $tr\ Tf = 0$ and $f \in \operatorname{im}\mathbf{Q}$ implies $tr\ \overline{T}f = 0$. ∎

The following formula for functions $f \in \mathcal{W}_p^k(\Omega)$ $(k \in,\ 1 < p < \infty)$ is also of some interest for the invertibility of Π in case of bounded domains.

Corollary 2. *Suppose* $f \in \mathcal{W}_p^k(\Omega)$ $(k \in, 1 < p < \infty)$. *Then*

$$\overline{T}\,\overline{D}\,\overline{\Pi}\Pi f = \overline{T}\,\overline{D}f$$

holds.

Proof. Let $f \in \mathcal{W}_p^k(\Omega)$ $(k \in;\ 1 < p < \infty)$. Then we obtain $\overline{T}\,\overline{D}D\overline{T}\,\overline{D}Tf = \overline{T}\,\overline{D}f$ by the help of equation (2) ∎

Now let Ω be the whole $\mathbb{R}^{n+1}$. Then the following theorem was proved in [13].

Theorem 7. *Let* $f \in C_0^\infty(\mathbb{R}^{n+1})$. *Then*

$$\overline{\Pi}\Pi u = \Pi\overline{\Pi}u = u$$

holds.

Using this theorem and (7) we get a result for the invertibility of Π defined on a class of functions containing essentially more than in the above theorem.

Proposition 3. *Suppose* $u \in \mathcal{L}_2(\mathbb{R}^{n+1})$. *Then*

$$\overline{\Pi}\Pi u = \Pi\overline{\Pi}u = u$$

holds.

Proof. The proof follows from the fact that $C_0^\infty(\mathbb{R}^{n+1})$ is dense in $\mathcal{L}_2(\mathbb{R}^{n+1})$ and from the boundedness of Π as an operator from $\mathcal{L}_2(\mathbb{R}^{n+1})$ into $\mathcal{L}_2(\mathbb{R}^{n+1})$ ∎

Now, let us discuss the case of Sobolev spaces $\mathring{\mathcal{W}}_2^k(\Omega)$ $(k \in)$. Moreover, for a given operator A we denote by A^* the $\mathcal{L}_2$-adjoint operator with respect to our scalar product (1).

Theorem 8. *Assume* $u, w \in \overset{\circ}{\mathcal{W}}{}_2^k(\Omega)$ $(k \in)$. *Then*

$$(\Pi u, v) = (u, \overline{T} D v)$$

holds.

For the proof we make use of $T^* = -\overline{T}$ and $D^* = -\overline{D}$. Taking this theorem we can put a representation of the $\mathcal{L}_2$-adjoint operator Π^* in a special case.

Proposition 4. *Let* $f \in \overset{\circ}{\mathcal{W}}{}_2^k(\Omega)$ $(k \in)$. *Then*

$$\Pi^* f = \overline{T} D f.$$

holds.

Applying this result we obtain that in this special case Π^* is a left inverse of the generalized Π-operator.

Proposition 5. *We have*

$$\Pi^* \Pi u = u$$

for all $u \in \overset{\circ}{\mathcal{W}}{}_2^k(\Omega)$ *and* $k \in$.

Proof. Let $u, v \in \overset{\circ}{\mathcal{W}}{}_2^k(\Omega)$ $(k \in)$. Then

$$(\Pi^* \Pi u, v) = (\overline{T}\, \overline{D} u, v) = (u, DTv) = (u, v)$$

for all $v \in \overset{\circ}{\mathcal{W}}{}_2^k(\Omega)$ from which $\Pi^* \Pi u = u$ follows ∎

Considering the whole space $\mathbb{R}^{n+1}$ and using the fact that the spaces $\overset{\circ}{\mathcal{W}}{}_2^k(\mathbb{R}^{n+1})$ $(k \in)$ are dense in $\mathcal{L}_2(\mathbb{R}^{n+1})$ we can derive from Proposition 3 an expression of the $\mathcal{L}_2$-adjoint operator in the case of $\mathcal{L}_2(\mathbb{R}^{n+1})$.

Proposition 6. *Suppose* $u \in \mathcal{L}_2(\mathbb{R}^{n+1})$. *Then*

$$\Pi^* u = \overline{\Pi} u$$

holds.

We see from Propositions 3 and 6 that $\Pi : \mathcal{L}_2(\mathbb{R}^{n+1}) \longrightarrow \mathcal{L}_2(\mathbb{R}^{n+1})$ is a unitary operator. Therefore, we get from this proposition a property important for applications of the hypercomplex Π-operator.

Proposition 7. *Let* $u, v \in \mathcal{L}_2(\mathbb{R}^{n+1})$. *Then*

$$(\Pi u, \Pi v) = (u, v) \qquad \textit{and} \qquad \|\Pi u\|_{\mathcal{L}_2(\mathbb{R}^{n+1})} = \|u\|_{\mathcal{L}_2(\mathbb{R}^{n+1})}$$

hold.

4. Application of the Π-operator to the solution of a hypercomplex Beltrami equation

oindent In complex function theory the Beltrami equation plays an important role for solving systems of partial differential equations. In hypercomplex function theory the Beltrami equation has not yet this importance, but nevertheless, it is a basic condition for the transfer of complex methods and efforts for solving partial differential equations to the hypercomplex case. In this section we will see how this equation can be solved using our generalized Π-operator.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, simply connected domain with a sufficiently smooth boundary, and $q : \Omega \mapsto C\ell_{0,n}$ a measurable function. Moreover, let $w : \Omega \mapsto C\ell_{0,n}$ be a sufficiently smooth function. Then we call the equation

$$Dw = q\,\overline{D}w \tag{13}$$

generalized Beltrami equation. Applying the ansatz

$$w = \phi + Th \tag{14}$$

where ϕ is an arbitrary left-monogenic function we transform the equation (13) into the integral equation

$$h = q\,(\overline{D}\phi + \Pi h). \tag{15}$$

Obviously, w is a solution of equation (13) if h is a solution of equation (15). On the other hand each solution of equation (13) can be represented by (14). Investigating the norm of the operator $q\Pi$ we have that in the case of

$$||q|| \leq \frac{1}{||\Pi||}$$

this operator is contractive. That means we can get a solution of equation (15) using the Banach fixed-point theorem (the other conditions of the Banach fixed-point theorem can easily be verified). Applying our norm estimate (7) for the generalized Π-operator we get the condition

$$||q||_{[\mathcal{L}_2(\Omega),\mathcal{L}_2(\Omega)]} \leq \frac{1}{\frac{n-1}{n+1} + \frac{2n\sqrt{c_4}}{\omega^{1/4}}}$$

being sufficient for the existence of a solution of equation (15). We only remark that in the case of the complex Beltrami equation we have the condition $||q|| \leq ||q_0|| < 1$.

In [9] we can find another hypercomplex generalization of the complex Beltrami equation. For this we have to introduce the real linear mappings $J_j : C\ell_{0,n} \mapsto C\ell_{0,n}$, by

$$J_j(\mathbf{e}_j) = \bar{\mathbf{e}}_j \quad \text{and} \quad J_j(\mathbf{e}_k) = \mathbf{e}_k \quad (k, j = 1, \ldots, n;\ k \neq j).$$

Especially define the mapping

$$J_0 : \sum_{k=0}^{n} a_k \mathbf{e}_k \mapsto \sum_{k=0}^{n} a_k \bar{\mathbf{e}}_k.$$

Let $A = \{\alpha_1, \ldots, \alpha_i\}$ with $0 \leq \alpha_1 < \ldots < \alpha_i \leq n$, $J_A = J_{\alpha_1} \cdots J_{\alpha_i}$ a composition of mappings J_j and $J_\emptyset$ the identical mapping. Using these mappings we can represent each real linear mapping $\mathbf{L} : C\ell_{0,n} \mapsto C\ell_{0,n}$ in the form

$$\mathbf{L}(x) = \sum_A c_A J_A(x)$$

with suitably chosen hypercomplex coefficients $c_A \in C\ell_{0,n}$ where $A \subseteq \{0, \ldots, n\}$ [6]. Let $\vec{Q}_A = (Q_{1A}, \ldots, Q_{nA})$ be arbitrary vector functions mapping Ω into the space $C\ell^n_{0,n} = C\ell_{0,n} \times \cdots \times C\ell_{0,n}$. Then the equation

$$Dw = \sum_A \prec \vec{Q}_A; \nabla_{\vec{x}} J_A w \succ$$

with $\nabla_{\vec{x}} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$ defined as a formal vector and $\prec \cdot;\cdot \succ$ standing for the Euclidian scalar product is a generalization of the complex Beltrami equation. This generalization is not a direct generalization, i.e. unlike the generalized Beltrami equation (13) we do not get the usual Beltrami equation in the case of $C\ell_{0,1} = \mathbb{C}$, because the differentiation with respect to x_0 is not in $\nabla_{\vec{x}}$. On the first view we can think that there is not a great difference, but to solve this equation we have to introduce a new generalization of the Π-operator, a generalization which acts between spaces of vector functions where the components are Clifford-valued functions [9].

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