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Weighted Inequalities for the Fractional Maximal Operator and the Fractional Integral Operator

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Abstract. A sufficient condition is given on weight functions u and v on $\mathbb{R}^n$ for which the fractional maximal operator M_s $(0 \leq s < n)$ defined by $(M_s f)(x) = \sup_{Q \ni x} |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\,dy$ or the fractional integral operator I_s $(0 < s < n)$ defined by $(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,dy$ is bounded from $L^p(\mathbb{R}^n, vdx)$ into $L^q(\mathbb{R}^n, udx)$ for $0 < q < p$ with $p > 1$, where Q is a cube and n a non-negative integer.

Keywords: *Weigthed inequalities, fractional maximal operators, fractional integral operators*
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1. Introduction

The fractional maximal operator M_s of order s $(0 \leq s < n)$ is defined by

$$(M_s f)(x) = \sup \left\{ |Q|^{\frac{s}{n}-1} \int_Q |f(y)|\,dy \,\middle|\, Q \text{ a cube with } Q \ni x \right\}.$$

Here n is a non-negative integer, and throughout this paper Q will denote a cube with sides parallel to the co-ordinate axes. The fractional integral operator I_s $(0 < s < n)$ is defined by

$$(I_s f)(x) = \int_{\mathbb{R}^n} |x-y|^{s-n} f(y)\,dy.$$

Our purpose is to derive a sufficient condition on weight functions u and v on $\mathbb{R}^n$, i.e. non-negative locally integrable functions, for which $T = M_s$ or $T = I_s$ is bounded from $L^p_v = L^p(\mathbb{R}^n, vdx)$ into $L^q(\mathbb{R}^n, udx)$ when $q < p$ and $1 < p, q < +\infty$. Precisely, we give a condition which ensures the existence of a constant $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} (Tf)^q(x) u(x)\,dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} f^p(x) v(x)\,dx \right)^{\frac{1}{p}} \tag{1.1}$$

for all $f \geq 0$. For simplicity, inequality (1.1) will be often denoted by $T : L^p_v \to L^q_u$.

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Inequality (1.1) is fundamental in analysis since many classical operators can be controlled by T. It is well-known [4] that (1.1) implies $\sigma = v^{-\frac{1}{p-1}} \in L^1_{loc}(\mathbb{R}^n, dx)$ and

$$|Q|^{\frac{s}{n}-1}\left(\int_Q u(y)\,dy\right)^{\frac{1}{q}}\left(\int_Q v^{-\frac{1}{p-1}}(y)\,dy\right)^{1-\frac{1}{p}} \leq A \tag{1.2}$$

for all cubes Q. So by the Lebesgue differentiation theorem, a necessary condition to (1.1) implicitely assumed is $\frac{1}{p} - \frac{1}{q} \leq \frac{s}{n}$. Consequently, for $M = M_0$ (i.e. the classical Hardy-Littlewood maximal operator) the embedding $M : L^p_v \to L^q_u$ has a non-trivial sense only for $q \leq p$.

For $1 < p \leq q < +\infty$, Sawyer [4] proved that $M_s : L^p_v \to L^q_u$ if and only if there is a constant $S > 0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[\left(M_s\chi_Q v^{-\frac{1}{p-1}}\right)(x)\chi_Q(x)\right]^q u(x)\,dx\right)^{\frac{1}{q}}& \\ \leq S\left(\int_{\mathbb{R}^n}\left[\chi_Q v^{-\frac{1}{p-1}}\right]^p(x)v(x)\,dx\right)^{\frac{1}{p}}&\end{aligned} \tag{1.3}$$

for all cubes Q with χ_Q being the characteristic function of Q. For $1 < q < p < +\infty$, the author proved in [3] that $M_s : L^p_v \to L^q_u$ if and only if

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\left[\left(M_s v^{-\frac{1}{p-1}}\sum_k \lambda_k\chi_{Q_k}\right)(x)\chi_{\cup Q_k}(x)\right]^q u(x)\,dx\right)^{\frac{1}{q}}& \\ \leq S\left(\int_{\mathbb{R}^n}\left[v^{-\frac{1}{p-1}}\sum_k \lambda_k\chi_{Q_k}\right]^p(x)v(x)\,dx\right)^{\frac{1}{p}}&\end{aligned} \tag{1.4}$$

for all cubes Q_k and all $\lambda_k > 0$, with S being independent on λ_k and Q_k. The main point here is the integration on the left restricted to $\cup_k Q_k$ and which implies that (1.4) is not a trivial condition for (1.1). Other characterizations for this embedding were found by Verbitsky [6] and D. Gu [1].

Although these characterizations of the embedding $M_s : L^p_v \to L^q_u$ are available, it is not easy in general to decide whether the test condition (1.3) or (1.4) holds since these conditions are expressed in term of M_s. The necessary and sufficient condition found in [6] or [1] is also too difficult for any practical use. Such situations lead us to investigate a sufficient condition for $T : L^p_v \to L^q_u$, when $q < p$, not too far to a suitable necessary condition and not expressed in term of the operator T. For the range $p \leq q$ a solution to this problem is known and due to Pérez [2]. One of the contribution of this paper is to bring a similar result when $q < p$.

The condition $(u, v) \in \widetilde{A}(s, p, q, t_1, t_2)$ $(t_1, t_2 \geq 1)$ found, is inspired from the necessary condition introduced by Verbitsky [6], and can be viewed as a substitute of the Fefferman-Phong condition $(u, v) \in A(s, p, q, t_1, t_2)$ which means

$$|Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}(1-\frac{1}{p})} \leq A$$

for all cubes Q. For the range $1 < p \le q < +\infty$, Pérez [2] got $M_s : L^p_v \to L^q_u$ whenever $(u,v) \in A(s,p,q,t_1,1)$ for some $t_1 > 1$. This condition is almost necessary in the sense that $A(s,p,q,1,1)$ (i.e. (1.2)) is a necessary condition for this embedding and $(u,v) \in A(s,p,q,t_1,1)$ becomes equivalent to $(u,v) \in A(s,p,q,1,1)$ when $\sigma = v^{-\frac{1}{p-1}} \in A_\infty$, i.e. $(v,v) \in A(0,t,t,1,1)$ for some $t > 1$. For $q < p$, we will see (in Theorem 2.1) that $M_s : L^p_v \to L^q_u$ whenever $(u,v) \in \widetilde{A}(s,p,q,t_1,1)$ for some $t_1 > 1$, with also the fact that $\widetilde{A}(s,p,q,1,1)$ is a necessary condition for the embedding. Moreover $(u,v) \in \widetilde{A}(s,p,q,t_1,1)$ becomes equivalent to $(u,v) \in \widetilde{A}(s,p,q,1,1)$ when $\sigma = v^{-\frac{1}{p-1}} \in A_\infty$.

The weight functions u and v for which $I_s : L^p_v \to L^q_u$ with $1 < p \le q < +\infty$, have been characterized by Sawyer [5]. However, like in the case of M_s, the corresponding necessary and sufficient condition is expressed in terms of I_s itself. Thus Pérez [2] proved that $I_s : L^p_v \to L^q_u$ whenever $(u,v) \in A(s,p,q,t_1,t_2)$ for some $t_1, t_2 > 1$. Similarly for $q < p$ we will get this last embedding from $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$.

For a large class of weight functions u and v it is more and less easy to decide whether the Fefferman-Phong condition $(u,v) \in A(s,p,q,t_1,t_2)$ holds (see Section 3). However the suitable Verbitski condition $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$ we use, appears to be more difficult to be checked than $(u,v) \in A(s,p,q,t_1,t_2)$. Thus our second purpose is to precise some cases of equivalence of these two test conditions (see Theorem 4.1 and Proposition 4.2 in Section 4). Such a study can be useful for people doing applications and explicit computations.

2. The first result

Let $t_1, t_2 \ge 1$ and let u, v be weight functions such that

$$0 < \int_Q v^{-\frac{t_1}{p-1}}(y)\,dy < +\infty \qquad \text{and} \qquad 0 < \int_Q u^{t_2}(y)\,dy < +\infty$$

for all cubes Q. Define the function $\Phi = \Phi_{s,n,p,u,v,t_1,t_2}$ by

$$\Phi(x) = \sup_{Q \ni x} \left\{ |Q|^{\frac{s}{n}} \left(\frac{1}{|Q|} \int_Q v^{-\frac{t_1}{p-1}}(y)\,dy \right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)} \left(\frac{1}{|Q|} \int_Q u^{t_2}(y)\,dy \right)^{\frac{1}{t_2 p}} \right\}.$$

For $q < p$ we write

$$(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$$

if

$$\int_{\mathbb{R}^n} \Phi^r(x) u(x)\,dx < +\infty \qquad \text{with } r = \frac{qp}{p-q}.$$

Really this condition is a variation on a condition introduced by Verbitsky [6] and which corresponds to $(u,v) \in \widetilde{A}(s,p,q,1,1)$. Since $\Phi(x) \ge \{|Q|^{\frac{s}{n}} \left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}$ $\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 p}}\}$, for each cube Q and $x \in Q$, then clearly $(u,v) \in \widetilde{A}(s,p,q,t_1,1)$ implies the usual Fefferman-Phong condition $(u,v) \in A(s,p,q,t_1,1)$. As we will see below, the converse is in general false.

Our first main result yields to a sufficient (resp. necessary) condition which ensures the embedding $T : L^p_v \to L^q_u$ when $q < p$, with $T = M_s$ or $T = I_s$.

Theorem 2.1. *Let $1 < p < +\infty$, $0 < q < p$ and $0 \le s < n$ $(0 < s$ in the case of $I_s)$:*

(A) *Suppose $M_s : L^p_v \to L^q_u$ or $I_s : L^p_v \to L^q_u$. Then $(u,v) \in \widetilde{A}(s,p,q,1,1)$.*

(B) *Conversely, suppose $(u,v) \in \widetilde{A}(s,p,q,t_1,1)$ for some $t_1 > 1$. Then $M_s : L^p_v \to L^q_u$. Similarly there is $I_s : L^p_v \to L^q_u$ whenever $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$ for some $t_1, t_2 > 1$.*

Part A was proved by Verbitski [6]. When $\sigma = v^{-\frac{1}{p-1}} \in A_\infty$ and $q < p$, then $(u,v) \in \widetilde{A}(s,p,q,1,1)$ is a necessary and sufficient condition for $M_s : L^p_v \to L^q_u$ $(0 \le s < n)$. Indeed, for such σ, by the reverse Hölder inequality, the condition $(u,v) \in \widetilde{A}(s,p,q,1,1)$ implies $(u,v) \in \widetilde{A}(s,p,q,t_1,1)$ for some $t_1 > 1$. Similarly, when $\sigma, u \in A_\infty$ and $q < p$, then $I_s : L^p_v \to L^q_u$ $(0 < s < n)$ if and only if $(u,v) \in \widetilde{A}(s,p,q,1,1)$.

Although for many weight functions it is easy to check the condition $(u,v) \in A(s,p,q,t_1,t_2)$ (see Proposition 3.1), it is not trivial to decide whether $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$. This problem will be studied in Section 4, for the moment consider the case of power weight functions.

Let $w(x) = |x|^{\alpha - n}$ and $v(x) = |x|^{\beta - n}$ with $\alpha > 0$ and $0 < \beta < np$. For $1 < p \le q < +\infty$, it is known (see also Section 3) that $M_s : L^p_v \to L^q_w$ as well as $I_s : L^p_v \to L^q_w$ if and only if $\frac{\beta}{p} = s + \frac{\alpha}{q}$. Also observe that $w, \sigma = v^{-\frac{1}{p-1}} \in A_\infty$ and $(w,v) \in A(s,p,q,1,1)$.

For $q < p$ we have the following negative result.

Proposition 2.2. *Let $1 < p < +\infty$, $0 < q < p$ and let α, β, v, w defined as above. Then M_s and I_s are not bounded from L^p_v into L^q_w.*

Therefore $(w,v) \in \widetilde{A}(s,p,q,1,1)$ is not equivalent to $(w,v) \in A(s,p,q,t_1,t_2)$.

Although for power weights the embedding $M_s : L^p_v \to L^q_u$ with $q < p$ is false, we can modify these weights to get a positive and explicit example.

Proposition 2.3. *Let $1 < p < +\infty, 0 < q < p, 0 < \gamma \le \alpha < +\infty$ and $0 < \beta < np$. Define $v(x) = |x|^{\beta - n}$ and $u(x) = |x|^{\alpha - n}\chi_{\{|x| \le 1\}}(x) + |x|^{\gamma - n}\chi_{\{|x|>1\}}(x)$ and suppose $\frac{\beta}{p} - \frac{\alpha}{p} \le s < \frac{\beta}{p} - \frac{\gamma}{q}$. Then $M_s : L^p_v \to L^q_u$ and $I_s : L^p_v \to L^q_u$.*

For instance take $p = 2$, $q = 1$, $n = 3$, $\alpha = 3$, $\beta = 4$, $\gamma = 1$ and v, u defined as in Proposition 2.3. Then $M_s : L^p_v \to L^q_u$ as well as $I_s : L^p_v \to L^q_u$ for all s with $\frac{1}{2} \le s < 1$.

3. The Fefferman-Phong condition $(u,v) \in A(s,p,q,t_1,t_2)$

The results in this section are not new (more and less known), but we write them for convenience and completness.

The problem of finding explicit examples of weights u and v for which $(u,v) \in A(s,p,p,1,1)$ was considered in [2]. For instance, let $0 \le s < \frac{n}{p}$ and $1 < p < +\infty$. If $u = w \in L^1_{loc}$ and $v = (M_{sp}w)$, then $(u,v) \in A(s,p,p,1,1)$. In order to describe more weight functions satisfying $(u,v) \in A(s,p,q,t_1,t_2)$ we introduce the growth weight condition $(\mathcal{C})$, for which $w \in \mathcal{C}$ means there are $c, C > 0$ such that

$$\sup_{\frac{1}{2}R<|x|<2R} w(x) \le \frac{C}{R^n} \int_{|y|<cR} w(y)\,dy$$

for all $R > 0$. Condition $(\mathcal{C})$ is very general since the case of radial non-increasing or non-decreasing weight functions are included. Also, if w is essentially constant on annuli, i.e. $w(y) \le cw(x)$ for $\frac{|y|}{2} \le |x| \le 2|y|$, then $w \in \mathcal{C}$.

For $u, v^{-\frac{1}{p-1}} \in \mathcal{C}$ then to obtain $(u,v) \in A(s,p,q,t_1,t_2)$ it is sufficient to check the similar condition for balls $B(0,R) = \{y \in \mathbb{R}^n : |y| < R\}$ centered at the origin.

Proposition 3.1. *Let* $1 \le t_1, t_2 < +\infty$, *and let* $0 \le s < n$, $1 < p < +\infty$ *and* $0 < q < +\infty$ *with* $\frac{1}{p} - \frac{1}{q} \le \frac{s}{n}$. *Let* u *and* v *be weight functions with* $u^{t_2}, v^{-\frac{t_1}{p-1}} \in \mathcal{C}$. *Then*

$$(u,v) \in A(s,p,q,t_1,t_2) \iff (u,v) \in A_0(s,p,q,t_1,t_2).$$

This last condition means there is $A > 0$ *such that*

$$R^{s+n[\frac{1}{q}-\frac{1}{p}]}\left(\frac{1}{R^n}\int_{|x|<R} v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}(1-\frac{1}{p})}\left(\frac{1}{R^n}\int_{|x|<R} u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}} \le A$$

for all $R > 0$.

We emphasize that, in applications, the condition $(u,v) \in A_0(s,p,q,t_1,t_2)$ is more interesting than $(u,v) \in A(s,p,q,t_1,t_2)$, since we avoid here the brake due to the integrations on arbitrary cubes non-centered at the origin.

A first consequence of this result is

Corollary 3.2. *Let* s, p, q *and* t_1, t_2 *be as in Proposition* 3.1. *Let* $u(x) = |x|^{\alpha-n}$ *and* $v(x) = |x|^{\beta-n}$, *with* $0 < \alpha < +\infty$ *and* $0 < \beta < np$. *Assume* $t_2 < \frac{n}{n-\alpha}$ *for* $\alpha < n$ *and* $t_1 < \frac{n(p-1)}{\beta-n}$ *for* $n < \beta$. *Then* $(u,v) \in A(s,p,q,t_1,t_2)$ *if and only if* $\frac{\beta}{p} = s + \frac{\alpha}{q}$.

Note that here $u, \sigma = v^{\frac{-1}{p-1}} \in A_\infty$. By Theorem 2.1 and Remark 2.4 in [2] we obtain

Corollary 3.3. *Let* $1 < p \le q < +\infty$ *with* $\frac{1}{p} - \frac{1}{q} \le \frac{s}{n}$, *and let* α, β *and* u, v *as in Corollary* 3.2. *Then* $M_s : L^p_v \to L^q_u$ *as well as* $I_s : L^p_v \to L^q_u$ *if and only if* $\frac{\beta}{p} = s + \frac{\alpha}{q}$.

4. The condition $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$

Now our purpose is to explicit a mean for checking the condition $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$. Let t_1, t_2 and u, v be as in the beginning of Section 2. Define the two cube functions $\mathcal{V} = \mathcal{V}_{s,n,p,u,v,t_1,t_2}$ and $\mathcal{A} = \mathcal{A}_{s,n,p,q,u,v,t_1,t_2}$ by

$$\mathcal{V}(Q) = |Q|^{\frac{s}{n}} \left(\frac{1}{|Q|}\int_Q \sigma^{t_1}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)} \left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 p}}$$

and

$$\mathcal{A}(Q) = |Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q \sigma^{t_1}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)} \left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}}.$$

The function Φ defined at the beginning of Section 2 is given by

$$\Phi(x) = \sup\left\{\mathcal{V}(Q) : Q \ni x\right\}. \tag{4.1}$$

For $q < p$, let $r > 1$ with $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. Then

$$\mathcal{A}(Q) = |Q|^{\frac{1}{r}}\, \mathcal{V}(Q) \left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 r}}. \tag{4.2}$$

Clearly $(u,v) \in A(s,p,q,t_1,t_2)$ if and only if there is $A > 0$ such that

$$\mathcal{A}(Q) < A \qquad \text{for all cubes } Q. \tag{4.3}$$

In other words, $(u,v) \in A(s,p,q,t_1,t_2)$ if $\mathcal{A}(Q)$ is uniformly bounded. This fact is not sufficient to get $(u,v) \in \widetilde{A}(s,n,p,q,t_1,t_2)$, for which more growth conditions on $\mathcal{A}$ and $\mathcal{V}$ are needed, as we will describe now.

To be explicit, we assume the existence of a cube $Q_0 = Q_0[0,R_0]$, centered at the origin and with sidelenght $R_0 > 0$, and for which the following hypotheses are satisfied:

(H$_1$) $\mathcal{V}(Q) \le C_1 \mathcal{V}(aQ_0)$ for all cubes $Q \subset 6Q_0$ with $|Q|^{\frac{1}{n}} \le R_0$, where $a \ge 3$.

(H$_2$) $\mathcal{V}(Q[0,R_2]) \le C_2 \mathcal{V}(Q[0,dR_1])$ for all R_1, R_2 with $R_0 \le R_1 \le R_2$, where $d \ge 1$.

(H$_3$) $\int_{Q[x_1,R_0]} u^{t_2}(y)\,dy \le C_3\left(\frac{R_0}{|x_1|}\right)^{n\varepsilon} \int_{Q[0,c_1|x_1|]} u^{t_2}(y)\,dy$ for all x_1 with $|x_1| > 2R_0$ (so $Q[x_1,R_0] \subset (3Q_0)^c$).

(H$_4$) $\mathcal{V}(Q[x,t]) \le C_4 \mathcal{V}(Q[0,c_1|x|])$ for all $x \in \mathbb{R}^n$ and $t > 0$ with $R_0 < |x|$ and $2t < |x|$.

(H$_5$) $\mathcal{A}(Q[0,R]) \le \frac{A}{R^{nr}}$ for all $R \ge R_0$.

Here

C_1, C_2, C_4, A, τ are non-negative constants depending on s; n, p, q, t_1, t_2, u, v

C_3, ε are non-negative constants depending on u, t_2, n

$c_1 = c_1(n)$ is a constant depending on n such that $c_1(n) \geq 1$.

Hypothesis (H_1) is some kind of control of $\mathcal{V}(Q)$ for each cube Q the centre of which is near the origin and with a small size. Condition (H_2) means that $R \to \mathcal{V}(Q[0, R])$ is an almost decreasing function for $R \geq R_0$. When $u^{t_2} \in \mathcal{C}$, then (H_3) is satisfied with $\varepsilon = 1$ since $|x| \approx |x_{Q_1}|$ for $x \in Q_1 = Q[x_1, R_0]$. The estimate of $\mathcal{V}(Q)$, for each cube Q with a centre far from the origin and small size, is described by (H_4). A control of $\mathcal{A}(Q[0, R])$ for large R is given by (H_5).

The second main result of this paper is as follows.

Theorem 4.1. *Let* $0 \leq s < n$, $1 < p < +\infty$, $0 < q < p$ *and* $1 \leq t_1, t_2 < +\infty$. *Assume hypotheses* ($\mathrm{H}_1$) - ($\mathrm{H}_5$) *are satisfied with* $1 < \varepsilon + r\tau t_2$ $(r = \frac{qp}{p-q})$. *Then the Fefferman-Phong condition* $(u, v) \in A(s, p, q, t_1, t_2)$ *implies* $(u, v) \in \widetilde{A}(s, p, q, t_1, t_2)$.

Therefore with the hypotheses of this result, $(u, v) \in \widetilde{A}(s, p, q, t_1, 1)$ is equivalent to $(u, v) \in A(s, p, q, t_1, 1)$ for $q < p$. And by Theorem 2.1, $M_s : L^p_v \to L^q_u$ whenever $(u, v) \in A(s, p, q, t_1, 1)$ for some $t_1 > 1$, and $I_s : L^p_v \to L^q_u$ whenever $(u, v) \in A(s, p, q, t_1, t_2)$ for some $t_1, t_2 > 1$.

For weight functions satisfying the growth condition ($\mathcal{C}$), the above hypotheses (H_1) - (H_5) can be simplified when there is $R_0 > 0$ for which the following conditions are fulfilled ($B(0, R)$ denotes the ball centered at the origin and with radius $R > 0$):

(H'_1) $\mathcal{V}(B(0, R)) \leq C_1 \mathcal{V}(B(0, aR_0))$ for all $R \leq R_0$, where $a \geq 3$.

(H'_2) $\mathcal{V}(B(0, R_2)) \leq C_2 \mathcal{V}(B(0, R_1))$ for all R_1 and R_2 with $R_0 \leq R_1 \leq R_2$.

(H'_5) $\mathcal{A}(B(0, R)) \leq \frac{A}{R^{n\tau}}$ for all $R \geq R_0$ where $\tau > 0$ and $A > 0$ are fixed.

Proposition 3.1 and Theorem 4.1 lead to

Proposition 4.2. *Let* s, p, q *and* t_1, t_2 *as in Theorem* 4.1. *Suppose* $u^{t_2}, \sigma^{t_1} = v^{-\frac{t_1}{p-1}} \in (\mathcal{C})$ *and assume hypotheses* (H'_1), (H'_2) and (H'_5) *are satisfied. Then* $(u, v) \in A_0(s, p, q, t_1, t_2)$ *implies* $(u, v) \in \widetilde{A}(s, p, q, t_1, t_2)$. *Thus* $(u, v) \in A_0(s, p, q, t_1, 1)$ *becomes equivalent to* $(u, v) \in \widetilde{A}_0(s, p, q, t_1, 1)$.

Now we give the proofs of our results. Theorem 2.1, Proposition 3.1 and Corollary 3.2 will be proved in Section 5. The next, Section 6, is devoted to the proof of Proposition 2.2. The proofs of Theorem 4.1 and Proposition 4.2 will be presented in Section 7. The last, Section 8, will contain the proof of Proposition 2.3.

5. Proofs of Theorem 2.1, Proposition 3.1 and Corollary 3.2

We start by giving the

Proof of Theorem 2.1. The fact that $M_s : L^p_v \to L^q_u$ implies $(u,v) \in \widetilde{A}(s,p,q,1,1)$ was proved by Verbitsky [6] via a theorem of Pisier on factorization trough $L^{p\infty}$. Since $(M_s f) \leq C(I_s f)$, then $(u,v) \in \widetilde{A}(s,p,q,1,1)$ is also a necessary condition for $I_s : L^p_v \to L^q_u$.

To prove Part B, suppose $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$ for some $t_1 > 1$ and $t_2 \geq 1$. Precisely, we take $t_2 = 1$ in the case of M_s and $t_2 > 1$ in the case of I_s. With $q < p$ and $r = \frac{qp}{p-q}$, the Hölder inequality yields

$$\left(\int_{\mathbb{R}^n} (Tf)^q(x)u(x)\,dx\right)^{\frac{1}{q}} \leq \left(\int_{\mathbb{R}^n} (Tf)^p(x)\widetilde{u}(x)\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \Phi^r(x)u(x)\,dx\right)^{\frac{1}{r}} \tag{5.1}$$

where $\widetilde{u} = u\Phi^{-p}$. Then $\widetilde{u}^{t_2} \in L^1_{loc}(\mathbb{R}^n, dx)$ since for all cubes Q

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \widetilde{u}^{t_2}(y)\,dy \\
&\leq \left[|Q|^{\frac{s}{n}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1 p'}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 p}}\right]^{-t_2 p} \\
&\quad\times\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right) \\
&= \left[|Q|^{\frac{s}{n}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1 p'}}\right]^{-t_2 p}
\end{aligned}$$

where $p' = \frac{p}{p-1}$, and so $|Q|^{\frac{s}{n}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1 p'}}\left(\frac{1}{|Q|}\int_Q \widetilde{u}^{t_2}(y)dy\right)^{\frac{1}{t_2 p}} \leq 1$, i.e. $(\widetilde{u}, v) \in A(s,p,p,t_1,t_2)$. Owing to the Pérez theorems [2: Theorems 2.1 and 2.11] this last condition implies $T : L^p_v \to L^p_{\widetilde{u}}$, i.e. there is $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} (Tf)^p(x)\widetilde{u}(x)\,dx\right)^{\frac{1}{p}} \leq C\left(\int_{\mathbb{R}^n} f^p(x)v(x)\,dx\right)^{\frac{1}{p}} \tag{5.2}$$

for all $f \geq 0$. By (5.1), (5.2) and condition $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$ we get $T : L^p_v \to L^q_u$ ∎

Proof of Proposition 3.1. Clearly $(u,v) \in A(s,p,q,t_1,t_2)$ implies $(u,v) \in A_0(s,p,q,t_1,t_2)$. Indeed, for each ball $B = B(0,R)$ it is sufficient to take the smallest cube Q containing B for which $B \subset Q \subset B(0,cR)$ with $c = c(n)$ depending only on n.

Conversely, suppose $(u,v) \in A_0(s,p,q,t_1,t_2)$ for some $A > 0$ and let $Q = Q[x_0, R]$ be a cube centered at x_0 and with sidelength $R > 0$. If $|x_0| \leq 2R$, then $Q \subset B(0,cR)$

for a fixed $c = c(n) > 0$, and with $\sigma = v^{-\frac{1}{p-1}}$ we have

$$\begin{aligned}\mathcal{A}(Q) &= |Q|^{\frac{s}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}}\left(\frac{1}{|Q|}\int_Q \sigma^{t_1}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}\\ &\le C(cR)^{s+n\left[\frac{1}{q}-\frac{1}{p}\right]}\\ &\quad\times\left(\frac{1}{(cR^n)}\int_{B(0,cR)} u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}}\left(\frac{1}{(cR)^n}\int_{B(0,cR)} \sigma^{t_1}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}\end{aligned}$$

where $C = C(s,n,p,q,t_1,t_2) > 0$. If $2R < |x_0|$, then $|x| \approx |x_0|$ for all $x \in Q$. So using $u^{t_2}, \sigma^{t_1} \in \mathcal{C}$, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q \sigma^{t_1}(y)\,dy &\le \frac{C}{(c|x_0|)^n}\int_{B(0,c|x_0|)} \sigma^{t_1}(y)\,dy\\ \frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy &\le \frac{C}{(c|x_0|)^n}\int_{B(0,c|x_0|)} u^{t_2}(y)\,dy\end{aligned}$$

with $c, C > 0$ not depending on $R > 0$ and $|x_0|$. Since $\frac{1}{p} - \frac{1}{q} \le \frac{s}{n}$, then

$$\begin{aligned}\mathcal{A}(Q) &\le C'(c|x_0|)^{s+n\left[\frac{1}{q}-\frac{1}{p}\right]}\\ &\quad\times\left(\frac{1}{(c|x_0|)^n}\int_{|y|<c|x_0|} v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}\left(\frac{1}{(c|x_0|)^n}\int_{|y|<c|x_0|} u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 q}}.\end{aligned}$$

Thus $\mathcal{A}(Q) \le C\sup_{R>0}\mathcal{A}(B(0,R)) \le CA$ for all cubes Q ∎

Proof of Corollary 3.2. Note that $[(\alpha-n)t_2+n] > 0$ since $t_2 < \frac{n}{n-\alpha}$ for $\alpha < n$. Then for all $R > 0$

$$\int_{|y|<R} u^{t_2}(y)\,dy = \int_{|y|<R} |y|^{[(\alpha-n)t_2+n]-n}\,dy \approx R^{[(\alpha-n)t_2+n]}. \tag{5.3}$$

The condition $0 < \beta < pn$ implies $\sigma = v^{-\frac{1}{p-1}} \in L^1_{loc}$ and

$$\int_{|y|<R} \sigma(y)\,dy = \int_{|y|<R} |y|^{\left[\frac{n-\beta}{p-1}+n\right]-n}\,dy \approx R^{\left[\frac{n-\beta}{p-1}+n\right]}. \tag{5.4}$$

Similarly $[n + t_1\frac{n-\beta}{p-1}] > 0$, since $t_1 < n\frac{p-1}{\beta-n}$ for $n < \beta$, and so

$$\int_{|y|<R} \sigma^{t_1}(y)\,dy = \int_{|y|<R} |y|^{\left[t_1\frac{n-\beta}{p-1}+n\right]-n}\,dy \approx R^{\left[t_1\frac{n-\beta}{p-1}+n\right]}.$$

By these computations

$$\mathcal{A}(B(0,R)) = \mathcal{A}_{s,n,p,q,u,v,t_1,t_2}(B(0,R)) \approx R^\lambda$$

where $\lambda = s + n(\frac{1}{q} - \frac{1}{p}) + \frac{1}{p}[n-\beta] + \frac{1}{q}(\alpha - n) = s - \frac{\beta}{p} + \frac{\alpha}{q} = 0$, and then $(u,v) \in A_0(s,p,q,t_1,t_2)$ ∎

6. Proof of Proposition 2.2

The proof of this result is based on the following two lemmas.

Lemma 6.1. *Let $1 < p < +\infty$, $t_1, t_2 \geq 1$, and u,v weight functions. Then*

$$\Phi(x) = \Phi_{s,n,p,u,v,t_1,t_2}(x) = \sup\left\{\mathcal{V}(Q) \middle| \, Q \text{ is a cube with } Q \ni x\right\}$$
$$\approx \Psi(x) = \sup_{R>0}\left\{\mathcal{V}(Q[x,R])\right\}.$$

Here and in the sequel of the paper $Q[x, R]$ denotes the cube centered at x and with sidelenght $R > 0$, and $\Phi \approx \Psi$ means $c_1\Phi \leq \Psi \leq c_2\Phi$ for fixed constants $c_1, c_2 > 0$.

Lemma 6.2. *Let $q < p$ and $p > 1$, $w(x) = |x|^{\alpha-n}$ and $v(x) = |x|^{\beta-n}$ with $\alpha > 0$, $0 < \beta < np$ and $\frac{\beta}{p} = s + \frac{\alpha}{q}$. Let $\Phi = \Phi_{s,n,p,w,v,1,1}$ be defined as in Section 2. Then*

$$\Phi(x) \approx |x|^{-\varepsilon} \quad \textit{with} \quad -\varepsilon = (s-n) + \frac{\alpha}{p} + \left[n + \frac{n-\beta}{p-1}\right]\left(1 - \frac{1}{p}\right) < 0.$$

By Theorem 2.1/(A), to prove that M_s does not send L^p_v into L^q_w (and consequently I_s does not map L^p_v into L^q_u), it is sufficient to see that $(w,v) \notin \widetilde{A}(s,p,q,1,1)$. And this is the case since

$$\int_{\mathbb{R}^n} \Phi^r(x) w(x)\,dx = \int_{\mathbb{R}^n} |x|^{\alpha - \varepsilon r - n} dx = +\infty$$

with $r = \frac{qp}{p-q}$.

Prove of Lemma 6.1. Obviously $\Psi \leq C\Phi$. Conversely, let Q be a cube and $x \in Q$. There is $c = c(n) \geq 1$ such that $Q \subset Q[x, cR]$ with $R = |Q|^{\frac{1}{n}}$. Then

$$\mathcal{V}(Q) = |Q|^{\frac{s}{n}}\left(\frac{1}{|Q|}\int_Q v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}\left(\frac{1}{|Q|}\int_Q u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 p}}$$
$$\leq C(cR)^s\left(\frac{1}{(cR)^n}\int_{Q[x,cR]} v^{-\frac{t_1}{p-1}}(y)\,dy\right)^{\frac{1}{t_1}\left(1-\frac{1}{p}\right)}\left(\frac{1}{(cR)^n}\int_{Q[x,cR]} u^{t_2}(y)\,dy\right)^{\frac{1}{t_2 p}}$$

for a constant $C > 0$ which does not depend on the cube Q. Therefore $\Phi \leq C\Psi$ and consequently $\Phi \approx \Psi$ ∎

Proof of Lemma 6.2. We first observe that ε is non-negative. Indeed, the condition $\frac{\beta}{p} = s + \frac{\alpha}{q}$ and $q < p$ imply

$$-\varepsilon = (s-n) + \frac{\alpha}{p} + \left[n + \frac{n-\beta}{p-1}\right]\left(1 - \frac{1}{p}\right) = s + \frac{\alpha}{p} - \frac{\beta}{p}$$
$$= \left(s + \frac{\alpha}{q} - \frac{\beta}{p}\right) + \alpha\left(\frac{1}{p} - \frac{1}{q}\right) = \alpha\left(\frac{1}{p} - \frac{1}{q}\right) < 0.$$

Since $\Phi \approx \Psi$ (by Lemma 6.1), it is sufficient to estimate

$$\Psi_1(x) = \sup_{0<|x|\leq 2R}\left[R^{s-n}\left(\int_{Q[x,R]} w(y)\,dy\right)^{\frac{1}{p}}\left(\int_{Q[x,R]} \sigma(y)dy\right)^{1-\frac{1}{p}}\right]$$
$$\Psi_2(x) = \sup_{0<2R<|x|}\left[R^{s-n}\left(\int_{Q[x,R]} w(y)\,dy\right)^{\frac{1}{p}}\left(\int_{Q[x,R]} \sigma(y)\,dy\right)^{1-\frac{1}{p}}\right]$$

because $\Psi \approx \Psi_1 + \Psi_2$. Note that w and $\sigma = v^{-\frac{1}{p-1}}$ are doubling weight functions (since $w, \sigma \in A_\infty$) Now let $R > 0$ and consider $|x| \le 2R$. By (5.3) and (5.4)

$$\int_{Q[x,R]} w(y)\,dy \approx \int_{|y|<cR} w(y)\,dy \approx R^\alpha$$
$$\int_{Q[x,R]} \sigma(y)\,dy \approx \int_{|y|<cR} \sigma(y)\,dy \approx R^{n+\frac{n-\beta}{p-1}}$$

where $c = c(n) \ge 1$. For $2R < |x|$ we use the fact that $|y| \approx |x|$ for all $y \in Q[x,R]$ to get

$$\int_{Q[x,R]} w(y)\,dy \approx \left(\frac{R}{|x|}\right)^n |x|^\alpha$$
$$\int_{Q[x,R]} \sigma(y)\,dy \approx \left(\frac{R}{|x|}\right)^n |x|^{n+\frac{n-\beta}{p-1}}.$$

Finally, by these last equivalences and $-\varepsilon = s + \frac{\alpha}{p} - \frac{\beta}{p} < 0$ then

$$\Psi_1(x) \approx \sup_{0<|x|\le 2R} \left[R^{s+\frac{\alpha}{p}-\frac{\beta}{p}}\right] \approx |x|^{-\varepsilon}$$
$$\Psi_2(x) \approx \sup_{0<2R\le|x|} \left[\left(\frac{R}{|x|}\right)^s |x|^{s+\frac{\alpha}{p}-\frac{\beta}{p}}\right] \approx |x|^{-\varepsilon}$$

as we claimed ∎

7. Proofs of Theorem 4.1 and Proposition 4.2

In this section, first we do some preliminaries, then we state a basic lemma. By this last we deduce the proof of Theorem 4.1, and finally we give the proof of this lemma.

Prelimiraries. Let $Q_0 = Q[0, R_0]$ be the cube centered at the origin and with sidelenght $R_0 = |Q_0|^{\frac{1}{n}}$. We decompose the space $\mathbb{R}^n$ as a union of cubes Q_{kl} whose interiors are pairwise disjoints and which have a common size $R = R_0$. More precisely, we write

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} E_k \tag{7.0}$$

with

$$E_0 = Q_0$$
$$E_1 = 3Q_0 \setminus Q_0 = \bigcup_{l\in\mathcal{I}_1} Q_{1l}$$
$$\vdots$$
$$E_k = (2k+1)Q_0 \setminus (2k-1)Q_0 = \bigcup_{l\in\mathcal{I}_k} Q_{kl} \qquad (k \ge 2)$$
$$\vdots$$

Moreover, there is a constant $N = N(n) \geq 1$ such that the cardinality $\#\mathcal{I}_k$ of $\mathcal{I}_k$ satisfies

$$\#\mathcal{I}_k \leq N\, k^{n-1}. \tag{7.1}$$

For each cube $Q_{kl} = Q[x_{kl}, R_0]$ (centered at x_{kl} and with sidelenght $R_0 > 0$) then

$$B(x_{kl}, \tfrac{1}{2}R_0) \subset Q_{kl} \subset B(x_{kl}, c_2\tfrac{1}{2}R_0) \subset 2Q_{kl} = Q[x_{kl}, 2R_0] \tag{7.2}$$

for a constant $c_2 = c_2(n) \geq 1$. Here $B(x,t) = \{y \in \mathbb{R}^n : |x-y| < t\}$ denotes the ball centered at x and with radius $t > 0$. On the other hand

$$Q_{1l} \subset 3Q_0 \qquad (l \in \mathcal{I}_1). \tag{7.3}$$

Note also that

$$|x| \approx |x_{kl}| \approx kR_0 \qquad (2 \leq k \in \mathbb{N}), \tag{7.4}$$

moreover $(2R_0) < |x_{kl}|$ and $\frac{3}{2}R_0 \leq |x|$.

We assume the Fefferman-Phong condition $(u,v) \in A(s,p,q,t_1,t_2)$ holds for a constant $A > 0$. With the assumptions (H_1) - (H_5) our purpose is to get $(u,v) \in \widetilde{A}(s,p,q,t_1,t_2)$, or

$$\int_{\mathbb{R}^n} \Phi^r(x)u(x)\,dx = \sum_{k\in\mathbb{N}, l\in\mathcal{I}_k} \int_{Q_{kl}} \Phi^r(x)u(x)\,dx < +\infty \tag{7.5}$$

$\left(r = \frac{qp}{p-q}\right)$ where $\Phi = \Phi_{s,n,p,u,v,t_1,t_2}$ is defined as in (4.1). Consequently it is sufficient to estimate each quantity $\int_{Q_{kl}} \Phi^r(x)u(x)\,dx$ by using the following

Basic Lemma. *Assume the hypotheses of Theorem* 4.1 *are satisfied. Then there is a constant* $C > 0$ *such that*

$$\int_{Q_0} \Phi^r(x)u(x)\,dx \leq CA^r \tag{7.6}$$

$$\int_{Q_{1l}} \Phi^r(x)u(x)\,dx \leq CA^r \quad (l \in \mathcal{I}_1) \tag{7.7}$$

$$\int_{Q_{kl}} \Phi^r(x)u(x)\,dx \leq CA^r \frac{1}{R_0^{nr\tau}} k^{-n\left[\frac{\varepsilon}{t_2} + \left(1 - \frac{1}{t_2}\right) + r\tau\right]} \qquad (2 \leq k \in \mathbb{N}) \tag{7.8}$$

where $\varepsilon > 0$ *and* $\tau > 0$ *are the constants in hypotheses* (H_3) *and* (H_5).

Now we prove inequality (7.5). Indeed, since $1 < \varepsilon + t_2 r\tau$ or $0 < \frac{1}{t_2}(\varepsilon - 1) + r\tau$, then, by this Basic Lemma and property (7.1), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi^r(x)u(x)\,dx &= \sum_{k\in\mathbb{N},\, l\in\mathcal{I}_k} \int_{Q_{kl}} \Phi^r(x)u(x)\,dx \\
&\leq (1+N)CA^r + CA^r \frac{1}{R_0^{rn\tau}} \sum_{k=2,\, l\in\mathcal{I}_k}^{+\infty} k^{-n\left[\frac{\varepsilon}{t_2} + \left(1-\frac{1}{t_2}\right) + r\tau\right]} \\
&\leq (1+N)CA^r + CA^r N \frac{1}{R_0^{rn\tau}} \sum_{k=2}^{+\infty} k^{-\left\{1 + n\left[\frac{\varepsilon}{t_2} + \left(1-\frac{1}{t_2}\right) + r\tau - 1\right]\right\}} \\
&\leq (1+N)CA^r \left[1 + \frac{1}{R_0^{rn\tau}} \sum_{k=2}^{+\infty} k^{-\left\{1 + n\left[\frac{1}{t_2}(\varepsilon-1) + r\tau\right]\right\}}\right] \\
&< +\infty.
\end{aligned}$$

Thus inequality (7.5) is proved.

Proof of the Basic Lemma. Estimate (7.6): The proof is reduced to get

$$\mathcal{V}(Q) \le CA|Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}} \tag{7.9}$$

for all cubes Q with $Q \cap Q_0 \neq \emptyset$ where $C > 0$ is a fixed constant. Indeed (7.9) yields

$$\Phi(x) \le CA|Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}$$

for each $x \in Q_0$ and consequently, by the Hölder inequality,

$$\begin{aligned}\int_{Q_0} \Phi^r(x)u(x)\,dx &\le (CA)^r|Q_0|^{-\left(1-\frac{1}{t_2}\right)} \left(\int_{Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{t_2}} \int_{Q_0} u(x)\,dx \\ &\le (CA)^r.\end{aligned}$$

To obtain (7.9) take an arbitrary cube Q with $Q \cap Q_0 \neq \emptyset$. If $|Q| \le |Q_0|$, then $Q \subset 3Q_0$, so by (H_1) and the Fefferman-Phong condition

$$\begin{aligned}\mathcal{V}(Q) &\le C_1\mathcal{V}(aQ_0) \le C_1A|aQ_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{aQ_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}} \\ &\quad \text{(remind that } a \ge 3) \\ &\le C_1A|Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}.\end{aligned}$$

If $|Q_0| \le |Q|$, then $Q_0 \subset 3Q$, so by the Fefferman-Phong condition

$$\begin{aligned}\mathcal{V}(Q) &\le c\mathcal{V}(3Q) \le cA|3Q|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{3Q} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}} \\ &\le cA|Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}.\end{aligned}$$

Estimate (7.7): As above, the proof is reduced to

$$\mathcal{V}(Q) \le CA|3Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{3Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}} \tag{7.10}$$

for all cubes Q with $Q \cap Q_{1l} \neq \emptyset$. Clearly (7.10) implies

$$\Phi(x) \le CA|3Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)} \left(\int_{3Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{t_2 r}}$$

for each $x \in Q_{1l}$ and since $Q_{1l} \subset 3Q_0$ (see (7.3)), then by this last inequality

$$\int_{Q_{1l}} \Phi^r(x)u(x)dx \le (CA)^r|3Q_0|^{-(1-\frac{1}{t_2})}\left(\int_{3Q_0} u^{t_2}(y)dy\right)^{-\frac{1}{t_2}}\left(\int_{Q_{1l}} u(x)dx\right)$$
$$\le (CA)^r.$$

To prove (7.10) take a cube Q with $Q \cap Q_{1l} \neq \emptyset$. For $5|Q|^{\frac{1}{n}} \le |Q_0|^{\frac{1}{n}}$ then $Q \subset 6Q_0$, so by (H_1) and the Fefferman-Phong condition

$$\mathcal{V}(Q) \le C_1\mathcal{V}(6Q_0) \le C_1A|6Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)}\left(\int_{(6Q_0)} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}$$
$$\le C_1A|3Q_0|^{\frac{1}{r}\left(1-\frac{1}{t_2}\right)}\left(\int_{3Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{r}}.$$

Next consider $|Q_0|^{\frac{1}{n}} \le 5|Q|^{\frac{1}{n}}$. Since $3Q_0 \cap 16Q \neq \emptyset$ and $|3Q_0|^{\frac{1}{n}} \le |16Q|^{\frac{1}{n}}$, there is $c = c(n) > 3$ such that $3Q_0 \subset cQ$. The Fefferman-Phong condition yields the conclusion since

$$\mathcal{V}(Q) \le c'\mathcal{V}(cQ) \le c'A|cQ|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)}\left(\int_{cQ} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}$$
$$\le c'A|3Q_0|^{-\frac{1}{r}\left(1-\frac{1}{t_2}\right)}\left(\int_{3Q_0} u^{t_2}(y)\,dy\right)^{-\frac{1}{rt_2}}.$$

Estimate (7.8): Since $\Phi \approx \Psi = \sup_{t>0}\mathcal{V}(Q[\cdot,t])$ (see Lemma 6.1) then to get (7.8) it is sufficient to obtain

$$\int_{Q_{kl}} \Psi_1^r(x)u(x)\,dx \le CA^r\frac{1}{R_0^{nr\tau}}\,k^{-n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)+r\tau\right]} \tag{7.11}$$

and

$$\int_{Q_{kl}} \Psi_2^r(x)u(x)\,dx \le CA^r\frac{1}{R_0^{nr\tau}}\,k^{-n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)+r\tau\right]} \tag{7.12}$$

with

$$\Psi_1(x) = \sup_{0<2t<|x|}\mathcal{V}(Q[x,t]) \qquad \text{and} \qquad \Psi_2(x) = \sup_{0<|x|\le 2t}\mathcal{V}(Q[x,t]).$$

Here $C > 0$ is a fixed constant, $k \in \mathbb{N}\backslash\{0,1\}$ and $l \in \mathcal{I}_k$.

The key for proving (7.11) is

$$\mathcal{V}(Q[x,t]) \le C_5\mathcal{V}(Q[0,c_1'|x_{kl}|]) \quad \text{for } 0 < 2t < |x| \text{ and } x \in Q_{kl} \tag{7.13}$$

where $c_1' = c_1'(n) > 1$, $C_5 = C_5(s,n,p,u,v,t_1,t_2) > 0$, and $k \ge 2$. To get inequality (7.13) observe that $R_0 < \frac{3}{2}R_0 \le |x|$ for all $x \in Q_{kl} = Q[x_{kl}, R_0]$ with $0 < 2t < |x|$. Since $|x| \approx |x_{kl}|$ (see (7.4)), we can choose a constant $c_1' = c_1'(n) \ge c_1$ such that $|x| \le c_1'|x_{kl}|$. Then by (H_4) we have $\mathcal{V}(Q[x,t]) \le C_4\mathcal{V}(Q[0,c_1|x|]) \le C_5\mathcal{V}(Q[0,c_1'|x_{kl}|])$.

To deduce inequality (7.11) from (7.13), first clearly $\Psi_1(x) \le C_5\mathcal{V}(Q[0, c_1'|x_{kl}|])$ for all $x \in Q_{kl}$. Next we obtain

$$
\begin{aligned}
\int_{Q_{kl}} \Psi_1^r(x)u(x)\,dx &\le C_5^r\Big[\mathcal{V}(Q[0,c_1'|x_{kl}|])\Big]^r \int_{Q_{kl}} u(x)\,dx \\
&\quad \textit{(by the last inequality)} \\
&\le C_5^r R_0^{n\left(1-\frac{1}{t_2}\right)}\Big[\mathcal{V}(Q[0,c_1'|x_{kl}|])\Big]^r \left(\int_{Q_{kl}} u^{t_2}(x)\,dx\right)^{\frac{1}{t_2}} \\
&\quad \textit{(by the Hölder inequality if } t_2>1) \\
&\le C_3^{\frac{1}{t_2}} C_5^r R_0^{n\left(1-\frac{1}{t_2}\right)}\left(\frac{R_0}{|x_{kl}|}\right)^{\frac{n\epsilon}{t_2}} \\
&\quad \times \Big[\mathcal{V}(Q[0,c_1'|x_{kl}|])\Big]^r \left(\int_{Q[0,c_1'|x_{kl}|]} u^{t_2}(x)\,dx\right)^{\frac{1}{t_2}} \\
&\quad \textit{(by } (H_3)) \\
&= c_3' C_3^{\frac{1}{t_2}} C_5^r \left(\frac{R_0}{|x_{kl}|}\right)^{n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)\right]} \\
&\quad \times \left[(c_1'|x_{kl}|)^{\frac{n}{r}\left(1-\frac{1}{t_2}\right)}\mathcal{V}(Q[0,c_1'|x_{kl}|])\left(\int_{Q[0,c_1'|x_{kl}|]} u^{t_2}\right)^{\frac{1}{rt_2}}\right]^r \\
&= c_3' C_3^{\frac{1}{t_2}} C_5^r \left(\frac{R_0}{|x_{kl}|}\right)^{n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)\right]} \left[\mathcal{A}\left(Q[0,c_1'|x_{kl}|]\right)\right]^r \\
&\quad \textit{(by the definition of } \mathcal{A}=\mathcal{A}_{s,n,p,q,u,v,t_1,t_2}) \\
&\le c_3' C_3^{\frac{1}{t_2}} C_5^r \left(\frac{R_0}{|x_{kl}|}\right)^{n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)\right]} \left[\frac{A}{(c_1'|x_{kl}|)^{n\tau}}\right]^r \\
&\quad \textit{(by } (H_5)) \\
&= (c_3' C_3^{\frac{1}{t_2}} C_5^r) A^r \frac{1}{R_0^{n r\tau}} \left(\frac{R_0}{|x_{kl}|}\right)^{n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)+\tau r\right]} \\
&\le \left(c_5 c_3' C_3^{\frac{1}{t_2}} C_5^r\right) A^r \frac{1}{R_0^{n r\tau}} k^{-n\left[\frac{\epsilon}{t_2}+\left(1-\frac{1}{t_2}\right)+\tau r\right]} \\
&\quad \textit{(by (7.4))}
\end{aligned}
$$

where $c_5 = c_5(n)$ and $c_3' = c_3'(s, n, r, t_1, t_2)$.

For the inequality (7.12), it will be sufficient to prove that an analogy of (7.13) remains true for all x and t with $|x| < 2t$ and $x \in Q_{kl}$ ($k \ge 2$). The keys are

(i) $R_0 \le c_1(n)|x_{kl}| \le c_4(n)t$

(ii) $Q[x,t] \subset Q[0, c_4(n)t]$ with $c_4 = c_4(n) \ge 1$.

Indeed $\mathcal{V}(Q[x,t]) \le c_5\mathcal{V}(Q[0, c_4(n)t])$ for a constant $c_5 = c_5(s, n, c_4)$ because $Q[x,t] \subset Q[0, c_4t]$. Further $\mathcal{V}(Q[x,t]) \le c_5C_2\mathcal{V}(Q[0, (dc_1)|x_{kl}|])$ by (H_2) and since $R_0 \le c_1|x_{kl}| \le$

$c_4 t$. So $\Psi_2(x) \leq c_5 C_2 \mathcal{V}(Q[0,(dc_1)|x_{kl}|])$ for all $x \in Q_{kl}$ and consequently the sequel for the estimate (7.12) will become as (7.13).

Assertion (i) is true since $c_1 = c_1(n) \geq 1$ and

$$R_0 < \frac{1}{2}|x_{kl}| \leq c_1|x_{kl}| \leq c_3|x| \leq 2c_3 t \leq \max\{2c_3, 4+c_2\}t = c_4 t.$$

Remind that $|x| \approx |x_{kl}|$ by (7.4) and here $c_2 = c_2(n) \geq 1$ is the constant described in (7.2) and $c_3 = c_3(n) > 1$. For Assertion (ii), take an $y \in Q[x,t]$. Then

$$|y| \leq |y-x| + |x| \leq c_2\frac{t}{2} + |x| \leq [4+c_2]\frac{t}{2} \leq \max\{2c_3, 4+c_2\}\frac{t}{2} = c_4\frac{t}{2}.$$

Thus, again by (7.2), we get $Q[x,t] \subset B(0, c_4\frac{t}{2}) \subset Q[0, c_4 t]$ ∎

Proof of Proposition 4.2. The proof is essentially reduced to get the assumptions in Theorem 4.1, from hypotheses (H'_1), (H'_2) and (H'_5) and from the growth condition $u^{t_2}, \sigma^{t_1} = v^{-\frac{t_1}{p-1}} \in \mathcal{C}$.

Hypothesis (H'_5) implies (H_5) since

$$\mathcal{A}(Q[0,R]) \leq a_1 \mathcal{A}(B(0, c_2\tfrac{R}{2})) \leq a_2 \mathcal{A}(B(0, c_2 R)) \leq \frac{a_2}{(c_2)^{n\tau}} \frac{A}{R^{n\tau}}$$

for all $R \geq R_0$. Here $c_2 = c_2(n) > 1$ is the constant defined by (7.2), and a_1, a_2 are constants depending on s, n, p, q, t_1 and t_2. Observe that $|x| \approx |x_1|$ for $|x_1| > 2R_0$ and $x \in Q[x_1, R_0]$. This equivalence, with $u^{t_2} \in \mathcal{C}$, implies (H_3) with $\varepsilon = 1$. Next let x, t with $R_0 < |x|$ and $0 < 2t < |x|$. Since $\sigma^{t_1}, u^{t_2} \in \mathcal{C}$, then

$$\mathcal{V}(Q[x,t]) \leq a_3 \left(\frac{t}{|x|}\right)^s \mathcal{V}(Q[0, a_4|x_Q|]) \leq a_3 \mathcal{V}(Q[0, a_4|x|])$$

and hence (H_4) holds. Hypothesis (H'_2) implies (H_2) since for all R_1, R_2 with $R_0 \leq R_1 \leq R_2$:

$$\begin{aligned}\mathcal{V}(Q[0,R_2]) &\leq a_4 \mathcal{V}(B(0, c_2\tfrac{R_2}{2})) \leq a_5 \mathcal{V}(B(0, c_2 R_2)) \\ &\leq a_5 C_1 \mathcal{V}(B(0,R_1)) \leq a_6 C_1 \mathcal{V}(Q[0, 2R_1]).\end{aligned}$$

Finally to get (H_1), take a cube $Q = Q[x_Q, R] \subset 6Q_0$ with $R \leq R_0$. Then $Q \in B(x_Q, c_2(n)\frac{R}{2})$. Note that $|x_Q| \leq 3c_2 R_0$. For $|x_Q| \leq 2c_2\frac{R}{2}$ then $B(x_Q, c_2\frac{R}{2}) \subset B(0, 3c_2\frac{R}{2})$ and hence

(i) $\mathcal{V}(Q) \leq a_7 \mathcal{V}(B(0, 3c_2\frac{R}{2}))$.

For $|x_Q| > 2c_2\frac{R}{2}$ then $|y| \approx |x_Q|$ for all $y \in Q$. Since $u^{t_2}, \sigma^{t_1} \in \mathcal{C}$ then

(ii) $\mathcal{V}(Q) \leq a_8 \mathcal{V}(B(0, a_9|x_Q|))$.

If $3c_2\frac{R}{2} \leq R_0$, then by (H'_1), $\mathcal{V}(B(0, 3c_2\frac{R}{2})) \leq C_1 \mathcal{V}(B(0, aR_0))$. In the case $R_0 \leq 3c_2\frac{R}{2}$ we use hypothesis (H'_2) to obtain $\mathcal{V}(B(0, 3c_2\frac{R}{2})) \leq C_1 \mathcal{V}(B(0, aR_0))$. Estimate (ii) can be obtained in the same manner by using (H'_1) or (H'_2). Therefore the hypothesis (H_1) holds ∎

8. Proof of Proposition 2.3

We will prove this result in four steps:

(8.1) The weight functions u and $\sigma = v^{-\frac{1}{p-1}}$ satisfy the growth condition $(\mathcal{C})$

(8.2) They also belong to the Muckenhoupt class A_∞

(8.3) $(u,v) \in A(s,p,q,1,1)$

(8.4) Hypotheses (H_1'), (H_2') and (H_5') are satisfied .

By Theorem 4.1, then (8.3) and (8.4) imply $(u,v) \in \widetilde{A}(s,n,p,q,1,1)$. By (8.2) and the reverse Hölder inequality, the condition $(u,v) \in \widetilde{A}(s,n,p,q,t_1,t_2)$ is satisfied for some $t_1 > 1$ and $t_2 > 1$. Consequently we get $M_s : L^p_v \to L^q_u$ and $I_s : L^p_v \to L^q_u$ in virtue of Theorem 2.1. Therefore the proof of Proposition 2.3 will be achieved with the help of the following two lemmas.

Lemma 8.1. *Suppose $0 < \gamma \le \alpha < +\infty$ and $R_0 \ge 1$. Let $u(x) = |x|^{\alpha-n}\chi_{\{|x|\le R_0\}} + |x|^{\gamma-n}\chi_{\{|x|>R_0\}}$. Then the following assertions are true:*

(A) $\int_{|x|<R} u(x)\,dx \approx \begin{cases} R^\alpha & \textit{if } 0 < R \le R_0 \\ R^\gamma & \textit{if } R > R_0. \end{cases}$

(B) *The weight function u satisfies the growth condition $(\mathcal{C})$.*

(C) *The weight function u satisfies the Muckenhoupt condition A_t for all $t > 1$ with $0 < \gamma \le \alpha < nt$.*

Lemma 8.2. *Let $0 < \gamma < \alpha$, $0 < \lambda$, $0 \le s < n$, $1 < p < +\infty$ and $0 < q < p$. Let $\sigma(x) = |x|^{\lambda-n}$ and $u(x) = |x|^{\alpha-n}\chi_{\{|x|\le R_0\}} + |x|^{\gamma-n}\chi_{\{|x|>R_0\}}$ with $R_0 \ge 1$. Define the function $\mathcal{V} = \mathcal{V}_{s,p,u,v,1,1}$ as in Section 2 and suppose*

$$\lambda\left(1-\frac{1}{p}\right)+\frac{\gamma}{q} < (n-s) \le \lambda\left(1-\frac{1}{p}\right)+\frac{\alpha}{p}.$$

Then

$$\mathcal{V}(R) \approx R^{\rho+\frac{\alpha}{p}} \quad \textit{and} \quad \mathcal{A}(R) \approx R^{\rho+\frac{\alpha}{q}} \quad \textit{if } R \le R_0$$

$$\mathcal{V}(R) \approx R^{\rho+\frac{\gamma}{p}} \quad \textit{and} \quad \mathcal{A}(R) \approx R^{\rho+\frac{\gamma}{q}} \quad \textit{if } R > R_0$$

since $\rho = \lambda(1-\frac{1}{p}) + (s-n)$. Consequently

$$\mathcal{A}(R) \le c\max\left\{R_0^{\rho+\frac{\alpha}{q}}, R_0^{\rho+\frac{\gamma}{q}}\right\}$$

with $\rho + \frac{\gamma}{p} < \rho + \frac{\gamma}{q} < 0 \le \rho + \frac{\alpha}{p} < \rho + \frac{\alpha}{q}$.

Lemma 8.1 yields immediately (8.1) and (8.2); and Lemma 8.2 ensures (8.3) and (8.4). Indeed, we are in the case of $\lambda = n + \frac{n-\beta}{p-1}$, and the main condition in Lemma 8.2 becomes $\frac{\beta}{p} - \frac{\alpha}{p} \le s < \frac{\beta}{p} - \frac{\gamma}{q}$ where $\rho = s - \frac{\beta}{p}$, and so $0 \le s - \frac{\beta}{p} + \frac{\alpha}{p}$ and $s - \frac{\beta}{p} + \frac{\gamma}{q} < 0$.

Proof of Lemma 8.1. Part (A): If $0 < R \le R_0$, then

$$\int_{|y|<R} u(y)\,dy = \int_{|y|<R} |y|^{\alpha-n}\,dy \approx R^{\alpha}.$$

and, for $R \ge R_0$,

$$\begin{aligned}\int_{|y|<R} u(y)\,dy &= \int_{|y|\le R_0} |y|^{\alpha-n}dy + \int_{R_0<|y|<R} |y|^{\gamma-n}dy\\ &\approx R_0^{\alpha} + R^{\gamma} - R_0^{\gamma} = R^{\gamma} + (R_0^{\alpha} - R_0^{\gamma}) = R^{\gamma}\Big(1 + \frac{R_0^{\alpha} - R_0^{\gamma}}{R^{\gamma}}\Big)\\ &\approx R^{\gamma}.\end{aligned}$$

Part (B): If $R_0 < \frac{1}{2}R < 2R$, then by part (A)

$$\sup_{\frac{1}{2}R<|x|<2R} u(x) = \sup_{\frac{1}{2}R<|x|<2R} |x|^{\gamma-n} \approx R^{\gamma-n} \le c\frac{1}{R^n}\int_{|x|<R} u(y)\,dy.$$

And for $\frac{1}{2}R < 2R \le R_0$

$$\sup_{\frac{1}{2}R<|x|<2R} u(x) = \sup_{\frac{1}{2}R<|x|<2R} |x|^{\alpha-n} \approx R^{\alpha-n} \approx \frac{1}{R^n}\int_{|y|<R} u(y)\,dy.$$

For $\frac{1}{2}R \le R_0 \le 2R$

$$\begin{aligned}u(x)\chi_{\{\frac{1}{2}R<|x|<2R\}} &= |x|^{\alpha-n}\chi_{\{\frac{1}{2}R<|x|<R_0\}}(x) + |x|^{\gamma-n}\chi_{\{R_0<|x|<2R\}}(x)\\ &\le c(\alpha,\gamma,n)\Big[A(R)\chi_{\{\frac{1}{2}R<|x|<R_0\}} + B(R)\chi_{\{R_0<|x|<2R\}}\Big]\end{aligned}$$

where

$$A(R) = \begin{cases} R^{\alpha-n} & \text{if } \alpha \le n\\ R_0^{\alpha-n} & \text{if } n \le \alpha\end{cases} \qquad \text{and} \qquad B(R) = \begin{cases} R_0^{\gamma-n} & \text{if } \gamma \le n\\ R^{\gamma-n} & \text{if } n \le \gamma.\end{cases}$$

Remember that $\frac{1}{2}R_0 \le R \le 2R_0$ or $R \approx R_0$. We estimate $A(R)$ and $B(R)$ by using part (A). For $\frac{1}{2}R_0 \le R \le R$ then

$$\begin{cases} R_0^{\alpha-n} \approx R^{\alpha-n} \approx \dfrac{1}{R^n}\displaystyle\int_{|y|<R} |y|^{\alpha-n}\,dy \approx \dfrac{1}{R^n}\displaystyle\int_{|x|<R} u(y)\,dy\\ R_0^{\gamma-n} \approx R^{\gamma-n} \approx R_0^{\gamma-\alpha}\dfrac{1}{R^n}\displaystyle\int_{|y|<R} |y|^{\alpha-n}\,dy \approx \dfrac{1}{R^n}\displaystyle\int_{|x|<R} u(y)\,dy\end{cases}$$

and for $R_0 \le R \le 2R_0$

$$\begin{cases} R_0^{\alpha-n} \approx R^{\alpha-n} \approx R_0^{\alpha-\gamma}\dfrac{1}{R^n}\displaystyle\int_{|y|<R} u(y)\,dy\\ R_0^{\gamma-n} \approx R^{\gamma-n} = \dfrac{1}{R^n}\displaystyle\int_{|y|<R} u(y)\,dy.\end{cases}$$

Part (C): Take $1 < t < +\infty$ with $0 < \gamma \le \alpha < nt$, and set

$$\mathcal{M}(Q) = |Q|^{-n}\left(\int_Q u(y)\,dy\right)^{\frac{1}{t}}\left(\int_Q u^{-\frac{1}{t-1}}(y)\,dy\right)^{1-\frac{1}{t}}$$

for all cubes Q. Since $u, u^{-\frac{1}{t-1}} \in C$ (see Part (B)), then as in Proposition 3.1 the estimate of $\mathcal{M}(Q)$ is reduced to that of

$$\mathcal{M}_0(R) = R^{-n}\left(\int_{|y|<R} u(y)\,dy\right)^{\frac{1}{t}}\left(\int_{|y|<R} u^{-\frac{1}{t-1}}(y)\,dy\right)^{1-\frac{1}{t}}$$

for all $R > 0$. If $R \le R_0$, then

$$\mathcal{M}_0(R) \approx R^{-n}R^{\frac{\alpha}{t}}R^{[n-\frac{\alpha-n}{t-1}][1-\frac{1}{t}]} = R^{-n+\frac{\alpha}{t}+n-\frac{n}{t}-\frac{\alpha}{t}+\frac{n}{t}} = 1.$$

And for $R > R_0$ we have

$$\mathcal{M}_0(R) \approx R^{-n}R^{\frac{\gamma}{t}}R^{[n-\frac{\gamma-n}{t-1}][1-\frac{1}{t}]} = 1.$$

Therefore $\mathcal{M}(Q) \le C$ for all cubes Q and a fixed constant $C > 0$ ∎

Proof of Lemma 8.2. By the main hypothesis (with $\rho = \lambda(1-\frac{1}{p}) + (s-n)$) we have

$$0 \le \rho + \frac{\alpha}{p} = (s-n) + \lambda\left(1-\frac{1}{p}\right) + \frac{\alpha}{p}$$

$$0 > \rho + \frac{\gamma}{q} = (s-n) + \lambda\left(1-\frac{1}{p}\right) + \frac{\gamma}{q}.$$

If $0 < R \le R_0$ then, by Lemma 8.1,

$$\mathcal{V}(R) \approx R^{(s-n)+\lambda\left(1-\frac{1}{p}\right)+\frac{\alpha}{p}} = R^{\rho+\frac{\alpha}{p}}$$

and

$$\mathcal{A}(R) = \mathcal{V}(R)\left(\int_{|y|<R} u(y)\,dy\right)^{\frac{1}{q}-\frac{1}{p}} \approx R^{\rho+\frac{\alpha}{p}+\alpha\left[\frac{1}{q}-\frac{1}{p}\right]} = R^{\rho+\frac{\alpha}{q}}.$$

Similarly for $R > R_0$ we get

$$\mathcal{V}(R) \approx R^{(s-n)+\lambda\left(1-\frac{1}{p}\right)+\frac{\gamma}{p}} = R^{\rho+\frac{\gamma}{p}}$$

and

$$\mathcal{A}(R) = \mathcal{V}(R)\left(\int_{|x|<R} u(y)\,dy\right)^{\frac{1}{q}-\frac{1}{p}} \approx R^{\rho+\frac{\gamma}{p}+\gamma\left[\frac{1}{q}-\frac{1}{p}\right]} = R^{\rho+\frac{\gamma}{q}}.$$

Since

$$0 \le \rho + \frac{\alpha}{p} < \rho + \frac{\alpha}{q} \quad \text{and} \quad \rho + \frac{\gamma}{p} < \rho + \frac{\gamma}{q} < 0,$$

it follows that $\mathcal{A}(R) \le c(s,n,p,q)\max\left\{R_0^{\rho+\frac{\alpha}{q}}, R_0^{\rho+\frac{\gamma}{q}}\right\}$ ∎

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