Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 2, 345–356

On x-Analytic Solutions to the Cauchy Problem for Partial Differential Equations with Retarded Variables

A. Augustynowicz and H. Leszczyński

Abstract. We prove some existence results for solutions analytic with respect to the spatial variables to first-order equations with a delay and some deviations not only at the function, but also at its derivative. We construct a natural Banach space and a norm which make an adequate integral operator contractive. Due to a useful relation of partial order in this space the main problem is also placed in the theory of monotone iterative techniques.

Keywords: *Analyticity, iterative methods, Banach contraction principle, Bielecki's norm*

AMS subject classification: Primary 35 A 10, secondary 35 C 10, 35 E 15

0. Introduction

We consider the Cauchy problem

$$\begin{aligned} D_t u(t,x) &= g\Big(t,x,u\big(\alpha(t),\beta(t,x)\big),D_x u\big(\gamma(t),\delta(t,x)\big)\Big) \\ u(0,x) &= 0. \end{aligned} \tag{0}$$

The present paper is aimed at providing us with some sufficient conditions for the existence of solutions, analytic at least with respect to the second variable. In [3] we can find a classical version of the Cauchy-Kovalevsky theorem for partial differential equations. That result is extended in [5] onto some evolution equations with functional dependence. The proof is based on the Banach contraction principle. The author assumes in particular that the right-hand side of the equation is a Lipschitz continuous function with a sufficiently small Lipschitz constant, and that there exist solutions analytic with respect to x, although the derivatives lessen the regularity of functions, because an integral operator generated by the differential equation turns to restore it and maps a Banach space into itself.

In [1] we consider the equation

$$D_t u(t,x) = f\Big(t,x,u\big(\alpha(t),x\big),D_x u\big(\beta(t),x\big)\Big) \tag{0$'$}$$

A. Augustynowicz: Univ. Gdańsk, Dep. Math., ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
H. Leszczyński: Univ. Gdańsk, Dep. Math., ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

ISSN 0232-2064 / $ 2.50

where $0 \le \alpha(t) \le t$ and $0 \le \beta(t) \le t$. We introduce there a relation $\overset{s}{\le}$ between analytic functions. We say that $z(t,x) \overset{s}{\le} u(t,x)$ and $f(t,x,p,q) \overset{s}{\le} g(t,x,p,q)$ if u and g are analytic majorants of z and f, respectively. The essential result of [1] can be expressed in the following sample

Theorem 0. *Assume that the functions*

$$\alpha, \beta, \bar{\alpha}, \bar{\beta} : [0,T] \to \mathbb{R}$$
$$f, g : [0,T] \times (-c_1, c_1) \times \ldots \times (-c_n, c_n) \to \mathbb{R}$$

are analytic and

$$f \overset{s}{\le} g, \qquad \alpha \overset{s}{\le} \bar{\alpha}, \qquad \beta \overset{s}{\le} \bar{\beta}.$$

Suppose that

$$\bar{u} : [0,T] \times (-c_1, c_1) \times \ldots \times (-c_n, c_n) \to \mathbb{R}$$

is an analytic solution to the Cauchy problem

$$\left.\begin{aligned} D_t u(t,x) &= g\Big(t, x, u\big(\bar{\alpha}(t), x\big), D_x u\big(\bar{\beta}(t), x\big)\Big) \\ u(0,x) &= 0. \end{aligned}\right\} \qquad (0)''$$

Then there exists a unique analytic solution to problem $(0)'$.

In view of Theorem 0 and the classical Cauchy-Kovalevsky theorem we obtain analytic solutions for deviations $\alpha(t) = d_0 t$ and $\beta(t) = d_1 t$ with $d_0, d_1 \in [0,1]$ in a two-dimensional case.

In the present paper we introduce a relation which partially orders the space of functions analytic with respect to the second variable in a similar way as the relation $\overset{s}{\le}$ did in [1]. We take at least threefold advantage of this relation. First of all, we find easy to define an appropriate Banach space and force an integral operator to be a contraction mapping that space into itself. Secondly, we bring the theory of differential equations with deviations closer to monotone iterative techniques (cf. [4]), and the main difference is that we replace the ordinary inequality $\le$ between functions by the above mentioned relation $\overset{x}{\le}$, which means the same as the usual inequality between the respective coefficients of each of the formal series associated with these functions. Finally, we present an existence result which basically concerns the same class of problems as that in [5], it is shown, however, from a different point of view, and it is obtained under some assumptions which are different from those in [5].

Our result differs from the results of [1] and [5] also in that we allow of variety of non-linear deviations, especially with respect to the second variable, whereas they would cause enormous technical problems if we were to generalize any theorems from [1] onto such equations, and there is no indication that the model of equation analysed in [5], though apparently more general compared with ours, and the existence theorem can be easily adapted to the ground of the equations with complicated deviations at the derivatives. It should be admitted, however, that the conditions on the right-hand side assumed in the present paper are also quite restrictive, because even in a linear case we cannot avoid multiplying the derivative of the solution z by x^2 and this function itself by x.

1. Basic notations and assumptions

We say that

$$z(t,x)=\sum_{j=0}^{\infty} z_j(t)x^j \qquad \text{is } x\text{-analytic in a domain } \Omega \in \mathbb{R}^2$$

if z is continuous on Ω and for every t the series which defines $z(t,\cdot)$ converges almost uniformly in $\{x|\,(t,x)\in\Omega\}$. We define a relation $\overset{x}{\leq}$ in the space of x-analytic functions. Given two x-analytic functions

$$z(t,x)=\sum_{j=0}^{\infty} z_j(t)x^j \qquad \text{and} \qquad y(t,x)=\sum_{j=0}^{\infty} y_j(t)x^j,$$

we set

$$y(t,x)\overset{x}{\leq} z(t,x) \ \text{ or simply } \ y\overset{x}{\leq} z \qquad \text{if} \qquad y_j(t)\leq z_j(t) \text{ for all } j\in\mathbb{N}_0.$$

This relation can be in a natural way extended onto the case either z or y is a formal series with the radius of convergence equal to 0. In a similar way we define the relation $\overset{x}{\geq}$.

We will say that a function h of variables $(t,r_1,\dots,r_k)$ is $(r_1,\dots,r_k)$-analytic if there are continuous real functions $h_{j_1,\dots,j_k}$ such that

$$h(t,r_1,\dots,r_k)=\sum_{j_1,\dots,j_k=0}^{\infty} h_{j_1,\dots,j_k}(t)\,r_1^{j_1}\cdots r_k^{j_k},$$

where the above series converges almost uniformly on each section of a region in $\mathbb{R}^{1+k}$, and we denote

$$|h|_*(t,r_1,\dots,r_k)=\sum_{j_1,\dots,j_k=0}^{\infty} |h_{j_1,\dots,j_k}(t)|\,r_1^{j_1}\cdots r_k^{j_k}.$$

The operation $|\cdot|_*$ is valid for h being a formal series as well. Observe that the notion of x-analyticity with a clear reference to functions of variables (t,x) we can extend onto all functions h of variables $(t,x_1,\dots,x_k)$ calling them to be $(x_1,\dots,x_k)$-analytic.

We consider the equation

$$u(t,x)=\int_0^t g\Big(s,x,u\big(\alpha(s),\beta(s,x)\big),D_xu\big(\gamma(s),\delta(s,x)\big)\Big)\,ds \qquad (s\in[0,T]) \qquad (1)$$

where $T,c>0$ and

$$g:[0,T]\times(-c,c)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$$
$$\alpha,\gamma:[0,T]\to[0,T]$$
$$\beta,\delta:[0,T]\times(-c,c)\to(-c,c).$$

Note that the above integral equation is a weak formulation of problem (0).

We introduce the following assumptions.

(H$_0$) 1^0 Given any $x \in (-c,c)$ and $p,q \in \mathbb{R}$, the functions $\alpha, \gamma, \beta(\cdot,x), \delta(\cdot,x)$ and $g(\cdot,x,p,q)$ are continuous on $[0,T]$.

2^0 The functions β, δ are x-analytic, g is (x,p,q)-analytic.

3^0 $\alpha(s) \leq s$ and $\gamma(s) \leq s$ on $[0,T]$.

4^0 $|\beta(s,x)|, |\delta(s,x)| \leq |x|$ for $(s,x) \in [0,T] \times (-c,c)$.

(H$_1$) There exists an x-analytic function $\omega : [0,T] \times (-c,c) \to \mathbb{R}$ with $\omega(t,x) \overset{x}{\geq} 0$ and

$$\omega(t,x) \overset{x}{\geq} \int_0^t |g|_*\Big(s,x,\omega\big(\alpha(s),|\beta|_*(s,x)\big), D_x\omega\big(\gamma(s),|\delta|_*(s,x)\big)\Big)ds \tag{2}$$

for all $s \in [0,T]$.

(H_2) There exist a $\theta \in (0,1)$ and an x-analytic function $\lambda : [0,T] \times (-c,c) \to \mathbb{R}$ such that $\lambda(t,x) \overset{x}{\geq} \omega(t,x)$ and additionally

$$\begin{aligned}\lambda(t,x) \overset{x}{\geq} \frac{1}{\theta}\int_0^t \Big(&|D_p g|_*\big(P(s,x;\omega)\big)\, \lambda\big(\alpha(s),|\beta|_*(s,x)\big) \\ &+ |D_q g|_*\big(P(s,x;\omega)\big)\, D_x\lambda\big(\gamma(s),|\delta|_*(s,x)\big)\Big)ds\end{aligned} \tag{3}$$

where $P(s,x;\omega) = \big(s,x,\omega\big(\alpha(s),|\beta|_*(s,x)\big),\omega\big(\gamma(s),|\delta|_*(s,x)\big)\big)$.

2. The existence result via the monotone iterative method

Iterative techniques have been widely applied in the existence theory for differential and differential-functional equations (see [4]). Due to our relation $\overset{x}{\leq}$ altogether with an appropriate definition of a Banach space we are able to prove the following existence result by means of the iterative method.

Theorem 1. *If assumptions* (H$_0$) - (H$_2$) *are satisfied, then equation* (1) *has an x-analytic solution.*

Proof. Define the set

$$W = \Big\{ z : \ |z|_*(t,x) \overset{x}{\leq} \omega(t,x) \ \text{ on } \ [0,T] \times (-c,c) \Big\}$$

and the operator F acting on it by

$$(Fu)(t,x) = \int_0^t g\Big(s,x,u\big(\alpha(s),\beta(s,x)\big), D_x u\big(\gamma(s),\delta(s,x)\big)\Big)ds.$$

If $u \in W$, it is clear that F satisfies the inequality

$$|(Fu)|_*(t,x) \overset{x}{\leq} \int_0^t |g|_*\Big(s,x,\omega\big(\alpha(s),|\beta|_*(s,x)\big),D_x\omega\big(\gamma(s),|\delta|_*(s,x)\big)\Big)ds$$
$$\overset{x}{\leq} \omega(t,x)$$

and consequently $F: W \to W$. Let us introduce the norm in W by

$$\|u\| = \sup_{j\in\mathbb{N}_0}\ \sup_{t\in[0,T]} \frac{|u_j(t)|}{\lambda_j(t)} \qquad \text{for } u \in W.$$

This norm is well defined because $\lambda \overset{x}{\geq} \omega$. Next, for $\phi,\psi \in W$ we obtain

$$\begin{aligned}(F\phi - F\psi)(t,x) &\\ = \int_0^t\int_0^1 \Big\{ & D_p g\big(\bar P(s,x;\tau,\phi,\psi)\big)\Big(\phi\big(\alpha(s),\beta(s,x)\big) - \psi\big(\alpha(s),\beta(s,x)\big)\Big)\\ &+ D_q g\big(\bar P(s,x;\tau,\phi,\psi)\big)\Big(D_x\phi\big(\alpha(s),\beta(s,x)\big) - D_x\psi\big(\alpha(s),\beta(s,x)\big)\Big)\Big\}\,d\tau\, ds\end{aligned}$$

where

$$\begin{aligned}\bar P(s,x;\tau,\phi,\psi) &= \tau Q(s,x;\phi) + (1-\tau)Q(s,x;\psi)\\ Q(s,x;\phi) &= \Big(s,x,\phi\big(\alpha(s),\beta(s,x)\big)D_x\phi\big(\gamma(s),\delta(s,x)\big)\Big)\\ Q(s,x;\psi) &= \Big(s,x,\psi\big(\alpha(s),\beta(s,x)\big)D_x\psi\big(\gamma(s),\delta(s,x)\big)\Big).\end{aligned}$$

It follows that

$$\begin{aligned}|F\phi - F\psi|_*(t,x) &\overset{x}{\leq} \int_0^t \Big\{|D_pg|_*\big(P(s,x;\omega)\big)|\phi-\psi|_*\big(\alpha(s),|\beta|_*(s,x)\big)\\ &\qquad + |D_qg|_*\big(P(s,x;\omega)\big)|D_x\phi - D_x\psi|_*\big(\gamma(s),|\delta|_*(s,x)\big)\Big\}\,ds\\ &\overset{x}{\leq} \int_0^t \Big\{|D_pg|_*\big(P(s,x;\omega)\big)\|\phi-\psi\|\lambda\big(\alpha(s),|\beta|_*(s,x)\big)\\ &\qquad + |D_qg|_*\big(P(s,x;\omega)\big)\|\phi-\psi\|D_x\lambda\big(\gamma(s),|\delta|_*(s,x)\big)\Big\}\,ds\\ &\overset{x}{\leq} \theta\lambda(t,x)\|\phi-\psi\|\end{aligned}$$

hence

$$\|F\phi - F\psi\| \leq \theta\|\phi-\psi\|.$$

Since the set W is a complete metric space with respect to the norm $\|\cdot\|$, the above inequality and the Banach contraction principle finish the proof ∎

Using the monotone operators method we can prove that if there is a function ω which fulfills assumption (H_1), then there exists a solution to the equation

$$u(t,x) = \int_0^t |g|_*\Big(s, x, u\big(\alpha(s), |\beta|_*(s,x)\big), D_x u\big(\gamma(s), |\delta|_*(s,x)\big)\Big)\,ds \tag{2$'$}$$

for all $s \in [0,T]$, which corresponds to inequality (2). We draw the Reader's attention to the fact that this result has the following structure:

If there is an upper solution, then there is a solution. This solution is obtained as a limit of the monotone sequence of recursively defined functions starting from the upper solution.

Proposition 1. *Suppose that assumption* (H_1) *and conditions* $1^0, 2^0$ *of assumption* (H_0) *are satisfied. Then there exists a unique x-analytic solution* $\bar{w} : [0,T] \times (-c,c) \to \mathbb{R}$ *to equation* $(2)'$ *such that* $0 \overset{x}{\leq} \bar{w}(t,x) \overset{x}{\leq} \omega(t,x)$.

Proof. Define the set

$$W = \Big\{ z :\ 0 \overset{x}{\leq} z(t,x) \overset{x}{\leq} \omega(t,x) \Big\}$$

and the operator $\mathcal{T}$ acting on it as

$$(\mathcal{T}z)(t,x) = \int_0^t |g|_*\Big(s, x, u\big(\alpha(s), |\beta|_*(s,x)\big), D_x u\big(\gamma(s), |\delta|_*(s,x)\big)\Big)\,ds.$$

It is an obvious fact that $\mathcal{T}(W) \subset W$. The operator $\mathcal{T}$ is monotone with respect to the relation $\overset{x}{\leq}$. Thus the sequence $\{u^n\}_{n\in\mathbb{N}_0}$ defined by $u^0 = \omega$ and $u^{n+1} = \mathcal{T}u^n$ ($n \in \mathbb{N}_0$) is non-increasing and we have

$$\bar{w}_j(t) = \lim_{n\to\infty} u_j^n(t), \qquad \text{where } u^n(t,x) = \sum_{j=0}^{\infty} u_j^n(t) x^j.$$

Define

$$\bar{w}(t,x) = \sum_{j=0}^{\infty} \bar{w}_j(t) x^j.$$

This clearly forces

$$\begin{aligned}\lim_{n\to\infty} \frac{d}{dt}(\mathcal{T}u^n)(t,x) &= |g|_*\Big(t, x, \bar{w}\big(\alpha(t), |\beta|_*(t,x)\big), D_x \bar{w}\big(\gamma(t), |\delta|_*(t,x)\big)\Big)\\ &\overset{x}{\leq} \frac{d}{dt}(\mathcal{T}\omega)(t,x).\end{aligned}$$

It follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}\bar{w}(t,x) &= \lim_{n\to\infty} u^{n+1}(t,x)\\ &= \int_0^t \lim_{n\to\infty}\Big(\frac{d}{ds}(\mathcal{T}u^n)(s,x)\Big)\,ds\\ &= \int_0^t |g|_*\Big(s,x,\bar{w}\big(\alpha(s),|\beta|_*(s,x)\big),D_x\bar{w}\big(\gamma(s),|\delta|_*(s,x)\big)\Big)\,ds.\end{aligned}$$

The fact that $\bar{w}$ is x-analytic and the inequalities $0 \overset{x}{\leq} \bar{w}(t,x) \overset{x}{\leq} \omega(t,x)$ are obvious ∎

We can obtain a similar result for inequality (3). From the above proposition we easily get

Corollary 1. *If assumption* (H_1) *and conditions* 1^0, 2^0 *of assumption* (H_0) *are satisfied and each of the power series for the functions* $g(t,\cdot,\cdot,\cdot)$, $\beta(t,\cdot)$ *and* $\delta(t,\cdot)$ *have non-negative coefficients, then there is a solution to equation* (1).

Remark 1. We can generalize Theorem 1, Proposition 1 and Corollary 1 onto the multi-dimensional case, which refers to both: the dimension of space and the replacement of a single equation by a system of many equations, i.e. $x = (x_1,\dots,x_k)$ and $g = (g_1,\dots,g_l)$. Although the proofs are nearly the same, they demand more complicated notations and more careful treatment, so we omit them. On the other hand, we can generalize these results onto equations with many delays, which take the form

$$u(t,x) = \int_0^t g\Big(s,x,u_1(s,x),\dots,u_n(s,x),v_1(s,x),\dots,v_k(s,x)\Big)\,ds$$

where

$$\begin{aligned}u_1(s,x) &= u\big(\alpha_1(s),\beta_1(s,x)\big), \;\dots\;, u_n(s,x) = u\big(\alpha_n(s),\beta_n(s,x)\big)\\ v_1(s,x) &= D_x u\big(\gamma_1(s),\delta_1(s,x)\big), \;\dots\;, v_k(s,x) = D_x u\big(\gamma_k(s),\delta_k(s,x)\big).\end{aligned}$$

3. Further assumptions

Using the technique of Bielecki's norms (see [2]) we obtain (as it will be stated below) an existence result in a quite wide class of linear equations of the form

$$\begin{aligned}u(t,x) = &\int_0^t xa(s,x)u\big(\alpha(s),\beta(s,x)\big)\,ds\\ &+\int_0^t x^2 b(s,x)D_x u\big(\gamma(s),\delta(s,x)\big)\,ds + F(t,x).\end{aligned}\tag{4}$$

We need the following assumptions.

(H$_3$) Suppose that the functions a, b and β, δ are x-analytic, and there are $\bar{a}, \bar{b}, R \in (0, \infty)$ such that

$$a(t,x) = \sum_{j=0}^{\infty} a_j(t)x^j \qquad b(t,x) = \sum_{j=0}^{\infty} b_j(t)x^j$$

and

$$\beta(t,x) = \sum_{j=0}^{\infty} \beta_j(t)x^j \qquad \delta(t,x) = \sum_{j=0}^{\infty} \delta_j(t)x^j$$

for $(t,x) \in [0,T] \times (-c,c)$ and

$$|a_j(t)| \le \bar{a}R^{-j} \qquad \text{and} \qquad |b_j(t)| \le \bar{b}R^{-j}$$

for $t \in [0,T]$ and $j \in \mathbb{N}_0$. Moreover, assume that there are $\bar{\beta}, \bar{\delta} \in (0, \infty)$ such that for all $s \in [0,T]$ and $j, k \in \mathbb{N}$ we have

$$\beta_0(s) = 0 \qquad \text{and} \qquad \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} |\beta_{p_l}(s)| \le \bar{\beta}, \tag{5}$$

$$\delta_0(s) = 0 \qquad \text{and} \qquad \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} |\delta_{p_l}(s)| \le \bar{\delta}. \tag{5$'$}$$

(H$_4$) $F(t,x) = \sum_{j=0}^{\infty} F_j(t)x^j$ is x-analytic and there are $\bar{F}, r, m > 0$ such that $R > r$ and $1 > r$, and

$$m > \frac{R}{R-r}\,\frac{1}{1-r}(\bar{a}\bar{\beta}r + \bar{b}\bar{\delta}) \qquad \text{and} \qquad |F_j(t)| \le \bar{F}r^{-j}e^{jmt}$$

for $t \in [0,T]$ and $j \in \mathbb{N}$.

Remark 2. If conditions (5) and (5)$'$ are satisfied, then $|\beta_j(s)| \le 1$ and $|\delta_j(s)| \le 1$ for all $s \in [0,T]$ and $j \in \mathbb{N}$. These assumptions are satisfied, in particular, when

$$\sum_{j=1}^{\infty} |\beta_j(s)| \le 1 \qquad \text{and} \qquad \sum_{j=1}^{\infty} |\delta_j(s)| \le 1$$

for $s \in [0,T]$ and $j \in \mathbb{N}$. Note also that there are finite numbers of non-zero coefficients in the series that appears in (5) and (5)$'$. This is due to the condition $p_1 + \ldots + p_j = k$ and $p_l \ge 0$ for $j \in \mathbb{N}$. Moreover, some coefficients vanish for we have $\beta_0(s) = \delta_0(s) = 0$.

4. The existence theorem for the linear equation

Any deviations at spatial derivatives on the right-hand sides of first-order equations significantly affect all methods concerning their qualitative theory. Even the existence and uniqueness of their solutions may become not obvious, especially because neither characteristics nor bi-characteristics are any longer present when non-trivial deviations appear. There does not exist any suitable substitute. Nevertheless, we give some effective conditions for the existence of x-analytic solutions to linear equations. Our result shows that the differential problems with deviations should be regarded as quite difficult.

Theorem 2. *Suppose that assumptions* (H_0), (H_3) *and* (H_4) *are fulfilled. Then there exists an x-analytic solution to equation* (4) *defined on* $[0,T]\times(-\bar{c},\bar{c})$, *where* $\bar{c}=re^{-mT}$.

Proof. Let us define the spaces

$$V=\left\{u\in C([0,T]\times(-\bar{c},\bar{c}),\mathbb{R})\;\middle|\; \begin{array}{l} u(t,x)=\sum_{j=0}^{\infty}u_j(t)x^j \\ |u_j(t)|\le Mr^{-j}e^{jmt} \text{ for some } M>0 \end{array}\right\}$$

$$V_0=\left\{u\in V:\ u_0(t)=u_1(t)=0\right\},$$

the function

$$(\mathcal{G}u)(t,x)=\int_0^t xa(s,x)u\big(\alpha(s),\beta(s,x)\big)\,ds+\int_0^t x^2b(s,x)D_xu\big(\gamma(s),\delta(s,x)\big)\,ds$$

for $u\in V$ and the norms

$$\|u\|=\sup_{t\in[0,T]}\sup_{j\in\mathbb{N}_0}|u_j(t)|r^je^{-jmt}\ (u\in V)\qquad\text{and}\qquad \|\mathcal{G}\|=\sup_{u\in V,\,\|u\|\le1}\|\mathcal{G}u\|.$$

We prove that $\mathcal{G}:V_0\to V_0$ and $\|\mathcal{G}\|<1$.

For $u\in V_0$ we obtain

$$\begin{aligned}(\mathcal{G}u)(t,x)&=\int_0^t\left(x\sum_{i=0}^{\infty}a_i(s)x^i\sum_{j=0}^{\infty}u_j(\alpha(s))\left(\sum_{k=0}^{\infty}\beta_k(s)x^k\right)^j\right)ds\\&\quad+\int_0^t\left(x^2\sum_{i=0}^{\infty}b_i(s)x^i\sum_{j=0}^{\infty}(j+1)u_{j+1}(\gamma(s))\left(\sum_{k=0}^{\infty}\delta_k(s)x^k\right)^j\right)ds\\&=\int_0^t\left(\sum_{i=0}^{\infty}\sum_{j=2}^{\infty}\sum_{k=0}^{\infty}x^{1+i+k}a_i(s)u_j(\alpha(s))\sum_{\substack{p_1,\dots,p_j=0\\p_1+\dots+p_j=k}}^{\infty}\prod_{l=1}^{j}\beta_{p_l}(s)\right)ds\\&\quad+\int_0^t\left(\sum_{i=0}^{\infty}\sum_{j=1}^{\infty}\sum_{k=0}^{\infty}x^{2+i+k}b_i(s)(j+1)u_{j+1}(\gamma(s))\sum_{\substack{p_1,\dots,p_j=0\\p_1+\dots+p_j=k}}^{\infty}\prod_{l=1}^{j}\delta_{p_l}(s)\right)ds\end{aligned}$$

$$= \int_0^t \Bigg(\sum_{\nu=3}^{\infty} x^\nu \sum_{\substack{i=0,\, k\geq 2 \\ i+k=\nu-1}}^{\infty} \sum_{j=2}^{k} a_i(s) u_j\big(\alpha(s)\big) \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} \beta_{p_l}(s) \Bigg) ds$$

$$+ \int_0^t \Bigg(\sum_{\nu=3}^{\infty} x^\nu \sum_{\substack{i=0,\, k\geq 1 \\ i+k=\nu-2}}^{\infty} \sum_{j=1}^{k} b_i(s)(j+1) u_{j+1}\big(\gamma(s)\big) \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} \delta_{p_l}(s) \Bigg) ds.$$

Now, it is obvious that $(\mathcal{G}u)_0(t) = (\mathcal{G}u)_1(t) = 0$ on $[0,T]$.

Take $\theta \in \mathbb{R}_+$ defined by

$$\theta = \frac{1}{m} \frac{R}{R-r} \frac{1}{1-r} \big(\bar{a}\bar{\beta} r + \bar{b}\bar{\delta}\big).$$

It follows from assumption (H_4) that $\theta < 1$. Given $u \in V_0$ we will show that $\|\mathcal{G}u\| \leq \theta \|u\|$. From the above calculations we get

$$\begin{aligned}
\|\mathcal{G}u\| \leq & \sup_{t\in[0,T],\, \nu\geq 3} r^\nu e^{-\nu m t} \\
& \times \Bigg\{ \int_0^t \Bigg(\sum_{\substack{i=0,\, k\geq 2 \\ i+k=\nu-1}}^{\infty} \sum_{j=2}^{k} |a_i(s)|\, \big|u_j\big(\alpha(s)\big)\big| \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} |\beta_{p_l}(s)| \Bigg) ds \\
& + \int_0^t \Bigg(\sum_{\substack{i=0,\, k\geq 1 \\ i+k=\nu-2}}^{\infty} \sum_{j=1}^{k} |b_i(s)|\,(j+1) \big|u_{j+1}\big(\gamma(s)\big)\big| \sum_{\substack{p_1,\dots,p_j=0 \\ p_1+\dots+p_j=k}}^{\infty} \prod_{l=1}^{j} |\delta_{p_l}(s)| \Bigg) ds \Bigg\} \\
\leq & \sup_{\nu\geq 3} r^\nu e^{-\nu m t} \|u\| \\
& \times \Bigg(\bar{a}\bar{\beta} \sum_{\substack{i=0,\, k\geq 2 \\ i+k=\nu-1}}^{\infty} \sum_{j=2}^{k} R^{-i} e^{jmt} \frac{r^{-j}}{mj} + \bar{b}\bar{\delta} \sum_{\substack{i=0,\, k\geq 1 \\ i+k=\nu-2}}^{\infty} \sum_{j=1}^{k} R^{-i} e^{(j+1)mt} \frac{r^{-j-1}}{m} \Bigg) \\
\leq & \frac{\|u\|}{m} \sup_{\nu\geq 3} r^\nu \Bigg(\bar{a}\bar{\beta} \sum_{\substack{i=0,\, k\geq 2 \\ i+k=\nu-1}}^{\infty} \sum_{j=2}^{k} R^{-i} \frac{r^{-j}}{j} + \bar{b}\bar{\delta} \sum_{\substack{i=0,\, k\geq 1 \\ i+k=\nu-2}}^{\infty} \sum_{j=1}^{k} R^{-i} r^{-j-1} \Bigg) \\
\leq & \|u\| \frac{1}{m(1-r)} \big(\bar{a}\bar{\beta} r + \bar{b}\bar{\delta}\big) \frac{1}{1-\frac{r}{R}} \\
= & \theta \|u\|.
\end{aligned}$$

Since $u \in V_0$, these estimates show that $\mathcal{G}u \in V_0$ and $\mathcal{G}$ is a contraction on V_0.

Define

$$\begin{aligned}
\rho(t,x) &= F_0(t) + \Big(F_1(t) + \int_0^t a_0 F_0\big(\alpha(s)\big)\, ds \Big) x \\
V' &= \Big\{ u \in V \,:\, u - \rho \in V_0 \Big\} \\
\mathcal{F}u &= \mathcal{G}u + F \quad \text{for } u \in V'.
\end{aligned}$$

It is easy to verify that $\mathcal{F} : V' \to V'$ and, in addition, $\mathcal{F}u - \mathcal{F}v \in V_0$ for $u, v \in V'$. Moreover, we get

$$\|\mathcal{F}u - \mathcal{F}v\| = \|\mathcal{G}(u - v)\| \leq \theta \|u - v\|$$

for $u, v \in V'$. We can conclude from this that there exists a unique fixed point $\bar{u} \in V'$ of the operator $\mathcal{F}$ ∎

The above theorem is only a local existence result with respect to x. Consequently, we have to decrease significantly the radius of convergence of solutions compared with the radius of convergence of given functions. If we assume more about given functions $\beta(t, x)$ and $\delta(t, x)$, we obtain a global existence result.

Theorem 3. *Suppose that the assumptions* (H_0), (H_3) *and* (H_4) *are satisfied,*

$$|\beta(t, x)| < |x| \quad \textit{and} \quad |\delta(t, x)| < |x| \qquad \textit{for} \quad x \neq 0.$$

Then there exists a solution of class C^∞ to equation (4) *defined on* $[0, T] \times (-c, c)$.

Proof. It follows from Theorem 2 that there exists an x-analytic solution to equation (4) defined on $(-\bar{c}, \bar{c})$, where $\bar{c} \in (0, c)$. Let $\tilde{u}$ be an extention of this solution of class C^∞ onto $[0, T] \times (-d, d)$ for a positive constant $d \geq \bar{c}$. Suppose that $d < c$ and there is no smooth extension of this solution onto $[0, T] \times (-\bar{d}, \bar{d})$ for $\bar{d} \in (d, c)$. Choose $\varepsilon > 0$ such that $d + \varepsilon \leq c$ and $|\beta(t, x)|, |\delta(t, x)| < d$ for $|x| < d + \varepsilon$. The existence of such a constant follows from the continuity of functions β and δ and from the assumptions of Theorem 3. Now, it is seen that the right-hand side of equation (4) is of class C^∞ also for $d \leq |x| < d + \varepsilon$, and it clearly defines a solution to equation (4) in this region. This contradicts the pre-supposed fact that the solution $\tilde{u}$ cannot be extended onto the above region ∎

We cannot expect that the global solution obtained in Theorem 3 turns to be expressed in the form

$$u(t, x) = \sum_{j=0}^{\infty} u_j(t) x^j,$$

where the series is convergent for $x \in (-c, c)$. For this reason, we slightly modify the meaning of x-analyticity: namely, a function $u \in C([0, T] \times (-c, c), \mathbb{R})$ will be said to be x-analytic if for every $x_0 \in (-c, c)$ the function $u(t, x + x_0)$ of variables (t, x) is x-analytic in a neighbourhood of $[0, T] \times \{0\}$.

Theorem 4. *Suppose that the assumptions* (H_0), (H_3) *and* (H_4) *are satisfied,*

$$|\beta(t, x)| < |x| \quad \textit{and} \quad |\delta(t, x)| < |x| \qquad \textit{for} \quad x \neq 0.$$

Then there exists an x-analytic solution to equation (4) *defined on* $[0, T] \times (-c, c)$.

Proof. We omit the proof as it is similar to that of Theorem 3 ∎

References

[1] Augustynowicz, A. and H. Leszczyński: *On the existence of analytic solutions to the Cauchy problem for first-order partial differential equations with retarded variables.* Unpublished paper.

[2] Bielecki, A.: *Une remarque sur la methode de Banach-Cacciopoli-Tikhonov dans la theorie des equations differentielles ordinaires.* Bull. Acad. Polon. Sci. Sér. Sci. Tech. 4 (1956), 261 - 264.

[3] John, F.: *Partial Differential Equations.* New York-Heidelberg-Berlin: Springer – Verlag 1975.

[4] Ladde, G. S., Lakshmikantham, V. and A. S. Vatsala: *Monotone Iterative Techniques for Nonlinear Differential Equations* (Pitman Advanced Publishing Program). Boston-London-Melbourne: Pitman 1985.

[5] Walter, W.: *Functional differential equations of the Cauchy-Kovalevsky type.* Aeq. Math. 28 (1985), 102 - 113.

Received 20.04.1995; in revised form 04.12.1995