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Regularity for a Variational Inequality with a Pseudodifferential Operator of Negative Order

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Abstract. We prove that the solution of a variational inequality on a submanifold in $\mathbb{R}^n$ involving a pseudodifferential operator of order -1 is bounded.

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1. Introduction

Consider the variational inequality to find $u \in K$ such that $(v-u, Au) \geq (v-u, b)$ for all $v \in K$, where $b \in W^{\frac{1}{2},2}(S)$ is given, K denotes the positive cone of the Hilbert space $W^{-\frac{1}{2},2}(S)$ and A is an elliptic pseudodifferential operator of the negative order -1 on a closed manifold $S \subset \mathbb{R}^n$.

Variational inequalities are nonlinear problems even if the operator A is linear because K fails to be a linear subspace of $W^{-\frac{1}{2},2}(S)$. The usual setting is that A maps a Banach or Hilbert space X into its dual X^*. In many applications X is a Sobolev space and A denotes a linear elliptic differential operator of order m. By energetic considerations, for example, it is often easy to prove the (weak) solvability of the variational inequality. Concerning the regularity of weak solutions we find two different situations: For elliptic *equations* $Au = b$ the inclusion $b \in W^{k,2}$ implies, in general, the inclusion $u \in W^{k+m,2}$. In contrast to this case, problems for variational *inequalities* have limited regularity, i.e. even if b is smooth, their solutions u cannot overcome a certain threshold of smoothness. For instance, Shamir [14] gave an example where $u \notin W^{3,2}(\Omega) \cup W^{2,4}(\Omega)$ for $A = -\Delta + I$, $b \in W^{1,p}$ for all $p > 1$ and $K = \{u \in W^{1,2}(\Omega) : u \geq 0 \text{ on } \Gamma \subset \partial\Omega\}$ (cf. Lions [9: Section 8.2] and Rodrigues [12: p. 279]). For variational inequalities with elliptic differential operators the regularity of solutions was investigated, e.g., by Kinderlehrer [6], Kinderlehrer and Stampacchia [8], and Uralzeva [2, 17]. The case of systems of variational inequalities with one-sided obstacles was treated in the papers of Kinderlehrer [7] (systems in $\mathbb{R}^2$) and Schumann [13] (Lamé's system of elasticity in $\mathbb{R}^N$ $(N \geq 2)$.

It seems however that problems concerning regularity of solutions of variational inequalities have not been considered if the operator A is a pseudodifferential operator

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of negative order. This case can also be motivated by a physical example (see [10]). A-priori the solution u of the variational inequality only belongs to the Sobolev space $W^{-\frac{1}{2},2}(S)$ of negative order $-\frac{1}{2}$. Thus we are interested to prove more regularity for the solution. In Section 5 we shall prove the following result.

Theorem. *Suppose $b \in W^{\gamma,r}(S)$ for some $\gamma \in (1,2)$ and $r > \frac{N}{\gamma-1}$. Then the solution $u \in K$ of the variational inequality* (1) *below is essentially bounded, i.e. $u \in L_\infty(S)$.*

We use the following notation. The norm in the Lebesgue space $L_p(U)$ where $U \subset \mathbb{R}^n$ denotes an open set is

$$\|u\|_p = \|u\|_{p,U} = \left(\int_U |u(x)|^p dx\right)^{1/p},$$

and

$$\|u\|_{\gamma,p} = \left(\|u\|_p^p + |u|_{\gamma,p}^p\right)^{1/p}$$

denotes the norm in the Sobolev space $W^{\gamma,p}(U)$ with $\gamma \in (0,1)$ where the seminorm $|u|_{\gamma,p}$ is defined by

$$|u|_{\gamma,p} = \left(\iint_{U\times U} |x-y|^{-N-\gamma p}\,|u(x)-u(y)|^p dxdy\right)^{1/p}.$$

The set of pseudodifferential operators of order m acting on U is denoted by $\Psi^m(U)$.

2. Problem and approximation (I)

We suppose that S is a smooth compact N-dimensional manifold ($N \geq 2$) without boundary ($\partial S = \emptyset$). Consider the following variational inequality:

Find $u \in K$ such that

$$(v-u, Au) \geq (v-u, b) \qquad \textit{for all } v \in K \tag{1}$$

where $b \in W^{\frac{1}{2},2}(S)$ is given and K is the positive cone of the Hilbert space $W^{-\frac{1}{2},2}(S)$, i.e.

$$K = \left\{v \in W^{-\frac{1}{2},2}(S) : (v,\varphi) \geq 0 \textit{ for all } \varphi \in \mathcal{D}(S) \textit{ such that } \varphi \geq 0 \textit{ on } S\right\}. \tag{2}$$

Clearly K is a closed cone of the Sobolev space $X = W^{-\frac{1}{2},2}(S)$. We denote the norm in X by $\|\cdot\|_{-\frac{1}{2},2}$ and make the following hypotheses on the linear continuous operator $A : W^{-\frac{1}{2},2}(S) \to W^{\frac{1}{2},2}(S)$:

(H1) There exists a constant $c > 0$ such that $(v, Av) \geq c\|v\|^2_{-\frac{1}{2},2}$ for all $v \in X$.

(H2) For sake of technical simplicity, we assume that a part Γ of S lies in the hyperplane $\mathbb{R}^N \subset \mathbb{R}^n$ $(n = N+1)$. Furthermore we suppose that the principal symbol of the pseudodifferential operator $A \in \Psi^{-1}(S)$ on Γ is given by

$$\sigma_{-1}(A)(x', \xi') = |\xi'|^{-1} \qquad \text{for } (x', 0) \in \Gamma \tag{3}$$

where $x' = (x_1, \ldots, x_N)$ and $\xi' = (\xi_1, \ldots, \xi_N)$ (the general case can be handled after a coordinate transform).

It follows from hypothesis (H1) that the variational inequality (1) has a unique solution $u \in K$ (for a proof cf. Lions [9: Chapter 2.8.2/Theorem 8.1]). Hypothesis (H2) will be used in Sections 4 and 5 to prove regularity of the solution.

To prove regularity we first approximate the solution u of variational inequality (1) by solutions u^δ $(\delta > 0)$ of the following family of variational inequalities:

Find $u^\delta \in K_1$ such that

$$\delta(v - u^\delta \mid u^\delta) + (v - u^\delta, Au^\delta) \geq (v - u^\delta, b^\delta) \qquad \textit{for all } v \in K_1 \tag{4}$$

where

$$K_1 = K \cap L_2(S) = \Big\{ v \in L_2(S) : v(x) \geq 0 \ \ a.e. \ on \ S \Big\},$$

$b^\delta \in W^{\frac{1}{2},2}$ and $(\cdot \mid \cdot)$ denotes the inner product in $L_2(S)$.

We will show that the family $(u^\delta)_{\delta>0}$ of solutions of variational inequalities (4) approximates the solution u of variational inequality (1).

Proposition 1. *Let $b, b^\delta \in W^{\frac{1}{2},2}(S)$. Then the following assertions are true.*

1. *For any $\delta > 0$, there exists a unique solution $u^\delta \in K_1$ of inequality (4).*

2. *If $\sup_\delta \|b^\delta\|_{\frac{1}{2},2} < +\infty$, then $\sup_\delta \|u^\delta\|_{-\frac{1}{2},2} < +\infty$.*

3. *If $b^\delta \to b$ in $W^{\frac{1}{2},2}(S)$ as $\delta \to +0$, then $u^\delta \to u$ in $X = W^{-\frac{1}{2},2}(S)$ where u is the unique solution of inequality (1).*

Proof. Assertion 1: K_1 is a closed, convex cone of $L_2(S)$. The linear continuous operator $\mathcal{A}$ defined by

$$(v, \mathcal{A}u) = \delta(v \mid u) + (v, Au) \qquad \text{for all } u, v \in X \tag{5}$$

is strongly coercive on $L_2(S)$ since $(u, \mathcal{A}u) \geq \delta\|u\|_2^2 + c\|u\|_{-\frac{1}{2},2}^2$ for all $u \in L_2(S)$ (cf. (2)). Thus existence and uniqueness of the solution u^δ of variational inequality (4) follow immediately.

Assertion 2: We set $v = 0$ in (4) and get

$$\delta\|u^\delta\|_2^2 + (u^\delta, Au^\delta) \leq \|b^\delta\|_{\frac{1}{2},2}\|u^\delta\|_{-\frac{1}{2},2}.$$

Thus, by (2) and Young's inequality

$$\delta\|u^\delta\|_2^2 + \frac{c}{2}\|u^\delta\|_{-\frac{1}{2},2}^2 \le c_1\|b^\delta\|_{\frac{1}{2},2}^2.$$

This means that there exists a constant $C > 0$ such that

$$\sup_\delta \|u^\delta\|_{-\frac{1}{2},2} \le C \qquad \text{and} \qquad \sup_\delta \sqrt{\delta}\|u^\delta\|_2 \le C. \tag{6}$$

Assertion 3: Now, we suppose that $b^\delta \to b$ in $W^{\frac{1}{2},2}(S)$ and that (δ_n) is a sequence converging to zero. For simplicity we write only δ instead of δ_n in what follows. Then we may conclude that, at least for a subsequence, $u^\delta \rightharpoonup u_1 \in K$ in X and $\sqrt{\delta}u^\delta \rightharpoonup w$ in $L_2(S)$. By compact embedding, $\sqrt{\delta}u^\delta \to w$ in X. Since (u^δ) is bounded in X it follows that $\sqrt{\delta}u^\delta \to 0$ in X as $\delta \to +0$. Therefore $w = 0$ and $\sqrt{\delta}u^\delta \rightharpoonup 0$ in $L_2(S)$.

(a) To prove $u = u_1$ we want to show that u_1 satisfies the inequality

$$(v - u_1, Au_1) \ge (v - u_1, b) \qquad \text{for all } v \in K_1. \tag{7}$$

Then a density argument proves that u_1 is a solution of inequality (1) and the uniqueness of the solution gives $u = u_1$. Indeed, from (4) we get

$$\delta(u^\delta \mid u^\delta) + (u^\delta, Au^\delta) \le (u^\delta - v, b^\delta) + \delta(v \mid u^\delta) + (v, Au^\delta). \tag{8}$$

Since the positive bilinear form $v \mapsto (Av, v)$ is weakly sequentially lower semicontinuous (cf. Zeidler [19: Vol. 3, p. 156]) it follows from $\delta \to +0$ that

$$\begin{aligned}(u_1, Au_1) &\le \liminf(u^\delta, Au^\delta)\\ &\le \liminf\left((u^\delta, Au^\delta) + \delta\|u^\delta\|^2\right)\\ &\le (u_1 - v, b) + (v, Au_1)\end{aligned} \tag{9}$$

for all $v \in K_1$. Thus (7) is proved and we have $u = u_1$. A well-known argument concerning subsequences (cf. Zeidler [19: Vol. 1, p. 480]) shows that the whole sequence (u^{δ_n}) is weakly convergent to u.

(b) We prove the *strong* convergence $u^\delta \to u$ in X. Let us use (8) with $v = u$ to get

$$\begin{aligned}(u, Au) &\le \liminf(u^\delta, Au^\delta)\\ &\le \limsup(u^\delta, Au^\delta)\\ &\le \limsup\left((u^\delta, Au^\delta) + \delta\|u^\delta\|^2\right)\\ &\le \limsup\left((u^\delta - u, b^\delta) + \delta(u \mid u^\delta) + (u, Au^\delta)\right)\\ &= (u, Au)\end{aligned}$$

and therefore $(u^\delta, Au^\delta) \to (u, Au)$ as $\delta \to +0$. Then (2) implies

$$c\|u^\delta - u\|_{-\frac{1}{2},2}^2 \le (u^\delta - u, Au^\delta - Au) \longrightarrow 0$$

and Assertion 3 is proved ∎

3. Approximation (II)

In Section 2 we replaced the variational inequality (1) acting in $X = W^{-\frac{1}{2},2}(S)$ by a family of approximate variational inequalities depending on $\delta > 0$ with cone $K_1 \subseteq L_2(S)$ (see (4)). Now we suppose that $\delta > 0$ is fixed and introduce a penalization of the negative part of the functions of $L_2(S)$. The aim is to get a variational inequality over the *whole of* $L_2(S)$. This variational inequality has a unique solution $u_\varepsilon = u_\varepsilon^\delta$ where $\varepsilon > 0$ is the penalization parameter. (Since δ is fixed in this section we shall omit the supercript δ.) Later, in Sections 4 and 5 we are going to derive bounds on the solutions depending neither on ε nor on δ in order to get regularity results for the solution u of variational inequality (1).

Suppose $\varepsilon > 0$. We construct the following approximation of the variational inequality (4):

Find $u_\varepsilon \in L_2(S)$ *such that*

$$\delta(v - u_\varepsilon \mid u_\varepsilon) + (v - u_\varepsilon, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) \geq (v - u_\varepsilon, b_\varepsilon) \tag{10}$$

for all $v \in L_2(S)$, *where* $b_\varepsilon \in W^{\frac{1}{2},2}(S)$ *and the penalization functional* F_ε *is defined by*

$$F_\varepsilon(v) = \frac{1}{2\varepsilon}\int_S |v^-|^2 dS$$

for $v \in L_2(S)$, *denoting for any real function* φ *by* $\varphi^\pm$ *the positive and negative parts of* φ, *respectively, i.e.* $\varphi = \varphi^+ + \varphi^-$.

Parallel with (10) we consider the following variational inequality:

Find $u^\delta \in L_2(S)$ *such that*

$$\delta(v - u^\delta \mid u^\delta) + (v - u^\delta, Au^\delta) + F(v) - F(u^\delta) \geq (v - u^\delta, b^\delta) \tag{11}$$

for all $v \in L_2(S)$, *where* F *is the indicatrix of the convex set* K_1, *i.e. for* $v \in L_2(S)$ *we have* $F(v) = 0$ *if* $v \in K_1$ *and* $F(v) = +\infty$ *otherwise.*

We get now the following statement.

Proposition 2. *Let* $\delta > 0$ *be fixed and* $b^\delta, b^\varepsilon \in W^{\frac{1}{2},2}(S)$. *Then the following assertions are true.*

1. *For any* $\varepsilon > 0$ *the variational inequality* (10) *has exactly one solution* $u_\varepsilon \in L_2(S)$.
2. *The variational inequality* (11) *has exactly one solution* $u^\delta \in L_2(S)$.
3. *If* $M_0 = \sup_\varepsilon \|b_\varepsilon\|_{\frac{1}{2},2} < +\infty$, *then there exists a constant* $M > 0$ *independent of* δ *such that* $M = \sup_\varepsilon \left(\|u_\varepsilon\|^2_{-\frac{1}{2},2} + \delta\|u_\varepsilon\|_2^2 + F_\varepsilon(u_\varepsilon)\right) < +\infty$.
4. $b_\varepsilon \to b^\delta$ *in* $W^{\frac{1}{2},2}(S)$ *as* $\varepsilon \to +0$ *implies* $u_\varepsilon \to u^\delta$ *in* $L_2(S)$ *and in* $W^{-\frac{1}{2},2}(S)$.

Proof. Assertion 1 follows from the coercivity of the operator $\mathcal{A}$ defined by (5) and the fact that $F_\varepsilon(v) \geq 0$ for all $v \in L_2(S)$ (cf. Lions [9: Chapter 2.8.5/Theorem 8.5]). Since (11) and (4) are equivalent Assertion 2 is obvious. To prove Assertion 3 we set $v = 0$ in (10). As $F_\varepsilon(0) = 0$ we get

$$\delta\|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon) + F_\varepsilon(u_\varepsilon) \leq \|b_\varepsilon\|_{\frac{1}{2},2}\|u_\varepsilon\|_{-\frac{1}{2},2}.$$

Therefore

$$\delta\|u_\varepsilon\|_2^2 + \frac{c}{2}\|u_\varepsilon\|_{-\frac{1}{2},2}^2 + F_\varepsilon(u_\varepsilon) \leq c_1\|b_\varepsilon\|_{\frac{1}{2},2}^2 \tag{12}$$

which gives Assertion 3.

To prove Assertion 4 suppose $\varepsilon = \varepsilon_n \to +0$. If $\|b_\varepsilon - b^\delta\|_{\frac{1}{2},2} \to 0$ we get from estimate (12) that at least for a subsequence $u_\varepsilon \rightharpoonup u_1$ in $L_2(S)$. Thus $u_\varepsilon \to u_1$ in X. We need to prove that $u_1 = u^\delta$. From the variational inequality (10) it follows that

$$\delta\|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon) \leq \delta(v \mid u_\varepsilon) + (v, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) + (u_\varepsilon - v, b_\varepsilon) \tag{13}$$

for all $v \in L_2(S)$. By virtue of Barbu and Precupanu [3: Theorem 2.3/p. 107] we have

$$F_\varepsilon(\varphi) = \frac{1}{2\varepsilon}\|\varphi - J_\varepsilon\varphi\|_2^2 + F(J_\varepsilon\varphi) \tag{14}$$

where $J_\varepsilon = (I + \varepsilon\partial F)^{-1}$ denotes the resolvent of ∂F. Then $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$ implies $\|u_\varepsilon - J_\varepsilon u_\varepsilon\| \to 0$ if $\varepsilon \to +0$. Therefore we have $J_\varepsilon u_\varepsilon \rightharpoonup u_1$ in $L_2(S)$ and, since the convex funtion F is weakly sequentially lower semincontinuous (see [3: p. 102]),

$$\begin{aligned} F(u_1) &\leq \liminf F(J_\varepsilon u_\varepsilon) \\ &\leq \liminf \Big(- \frac{1}{2\varepsilon}\|u_\varepsilon - J_\varepsilon u_\varepsilon\|^2 + F_\varepsilon(u_\varepsilon)\Big) \\ &\leq \liminf F_\varepsilon(u_\varepsilon). \end{aligned} \tag{15}$$

Since $u_\varepsilon \rightharpoonup u_1$ in $L_2(S)$ and $u_\varepsilon \to u_1$ in X we get from (13)

$$\begin{aligned} &\delta\|u_1\|^2 + (u_1, Au_1) \\ &\quad\leq \liminf \big(\delta\|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon)\big) \\ &\quad\leq \limsup \big(\delta\|u_\varepsilon\|_2^2 + (u_\varepsilon, Au_\varepsilon)\big) \\ &\quad\leq \limsup \big\{\delta(v \mid u_\varepsilon) + (v, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) + (u_\varepsilon - v, b_\varepsilon)\big\} \\ &\quad\leq F(v) - \liminf F_\varepsilon(u_\varepsilon) + \delta(v \mid u_1) + (v, Au_1) + (u_1 - v, b) \\ &\quad\leq F(v) - F(u_1) + \delta(v \mid u_1) + (v, Au_1) + (u_1 - v, b) \end{aligned} \tag{16}$$

for all $v \in L_2(S)$, i.e. u_1 is a solution of variational inequality (11). Observe that $F_\varepsilon(v) \to F(v)$ for all $v \in L_2(S)$ (see Barbu and Precupanu [3: p. 107]). Uniqueness implies $u_1 = u^\delta$ ∎

4. Regularity

In this section we derive L_p-bounds for the solution $u_\varepsilon = u_\varepsilon^\delta$ of the variational inequality (10) that are *independent of* ε *and* δ. (Here again, we shall omit the supercript δ.) We are going to consider u_ε on the hyperplane part Γ of S defined in hypothesis (H2). The solution $u_\varepsilon \in L_2(S)$ satifies the inequality

$$\delta(v - u_\varepsilon \mid u_\varepsilon) + (v - u_\varepsilon, Au_\varepsilon) + F_\varepsilon(v) - F_\varepsilon(u_\varepsilon) \geq (v - u_\varepsilon, b_\varepsilon) \tag{17}$$

for all $v \in L_2(S)$. We multiply inequality (17) by the test function $v = u_\varepsilon + t\eta$, where $0 \neq t \in \mathbb{R}$ and $\eta \in C_0^\infty(S)$ satisfies the condition $\operatorname{supp}\eta \subset\subset \Gamma$. Thus

$$\delta(\eta \mid u_\varepsilon) + (\eta, Au_\varepsilon) + \frac{1}{t}\big(F_\varepsilon(u_\varepsilon + t\eta) - F_\varepsilon(u_\varepsilon)\big) \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} (\eta, b_\varepsilon) \quad \text{for } t \begin{Bmatrix} > \\ < \end{Bmatrix} 0.$$

From

$$\lim_{t\to 0} \frac{1}{t}\big(F_\varepsilon(u_\varepsilon + t\eta) - F_\varepsilon(u_\varepsilon)\big) = \varepsilon^{-1} \int_\Gamma \eta u_\varepsilon^- dS$$

it follows that

$$\delta \int_S \eta\, u_\varepsilon\, dS + \int_S \eta\, Au_\varepsilon\, dS + \varepsilon^{-1} \int_S \eta\, u_\varepsilon^-\, dS = \int_S \eta\, b_\varepsilon\, dS \tag{18}$$

for all $\eta \in C_0^\infty(S)$ and by approximation for all $\eta \in L_2(S)$ with $\operatorname{supp}\eta \subset\subset \Gamma$. Since η can be chosen arbitrarily we get

$$\delta u_\varepsilon + Au_\varepsilon + \varepsilon^{-1} u_\varepsilon^- = b_\varepsilon \qquad \text{in } L_2^{loc}(\Gamma). \tag{19}$$

4.1 (Localization and preliminary regularity). In the following we are going to use local properties of pseudodifferential operators. We choose an open subset $U \subset\subset \Gamma$ and an arbitrary but fixed test function $\varphi \in C_0^\infty(U)$ with $\varphi \geq 0$. Setting $g_\varepsilon = \varphi u_\varepsilon$, relation (19) gives

$$\delta g_\varepsilon + \varphi Au_\varepsilon + \varepsilon^{-1} g_\varepsilon^- = \varphi b_\varepsilon =: \widetilde{b}_\varepsilon. \tag{20}$$

Remark that $\operatorname{supp}\widetilde{b}_\varepsilon \subset U$. Furthermore we choose a function $\mu \in C_0^\infty(U)$ such that $\mu \equiv 1$ on an open set $W \subset\subset U$ with $K_\varphi = \operatorname{supp}\varphi \subset W$. Then relation (20) may be written in the form

$$\delta g_\varepsilon + (\varphi A\mu)u_\varepsilon + \varepsilon^{-1} g_\varepsilon^- = \widetilde{b}_\varepsilon - \varphi A(1-\mu)u_\varepsilon = \widetilde{b}_\varepsilon + R_1 u_\varepsilon = \widetilde{b}_\varepsilon + \mu R_1 u_\varepsilon \tag{21}$$

where $R_1 = -\varphi A(1-\mu)$ is a so-called regularizing ψdo: $R_1 \in \Psi^{-\infty}(S)$ (see Dieudonné [4: Vol. 7, Prop. 23.26.11/p. 212]). Therefore $R_1 : W^{-\frac{1}{2},2}(S) \longrightarrow W^{m,2}(U) \subset W^{m,2}(S)$ is a continuous operator for all $m \in \mathbb{N}$.

Next we make use of the principal symbol $\sigma_{-1}(A)$ defined in hypothesis (H2). Let us agree to write $x \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^N$ in the following instead of x' and ξ', respectively. Since the principal symbols of both $\varphi A\mu$ and $\mu A\varphi$ are the same: $\sigma_{-1}(\varphi A\mu)(x,\xi) = \sigma_{-1}(\mu A\varphi)(x,\xi) = \varphi(x)|\xi|^{-1}$, we only get a perturbation of order -2 exchanging φ and

μ in the term $(\varphi A\mu)$ of (21): $\varphi A\mu = \mu A\varphi + P_{-2}$ where $P_{-2} \in \Psi^{-2}(U)$ is a proper ψdo of order -2. Thus

$$\delta g_\varepsilon + (\mu A\varphi)u_\varepsilon + \varepsilon^{-1}g_\varepsilon^- = \widetilde{b}_\varepsilon + \mu R_1 u_\varepsilon + P_{-2}g_\varepsilon =: f_\varepsilon. \tag{22}$$

By the mapping properties of proper ψdo's, we see that $P_{-2} : W^{-\frac{1}{2},2}(U) \longrightarrow W^{\frac{3}{2},2}(U)$ is a continuous linear mapping. Introducing a third cut-off function μ_1 such that $\mu_1 \equiv 1$ on $\operatorname{supp}\mu$ we can re-write (22) as

$$\delta g_\varepsilon + (\mu A\mu_1)g_\varepsilon + \varepsilon^{-1}g_\varepsilon^- = f_\varepsilon. \tag{23}$$

The principal symbol of $\mu A\mu_1$ on Γ is $\sigma_{-1}(\mu A\mu_1) = \mu(x)|\xi|^{-1}$.

Let us fix $\varepsilon > 0$ and study the individual function g_ε for a moment.

Lemma 1. *Let us assume $b_\varepsilon \in W^{1,p}_{loc}(\Gamma)$ for all $p < +\infty$. Then $g_\varepsilon = \varphi u^\delta_\varepsilon \in W^{1,p}(U)$ for all $\varepsilon, \delta > 0$ and $p < +\infty$.*

Proof. The solution u_ε of inequality (17) belongs to $L_2(S)$. Therefore $f_\varepsilon \in W^{1,2}(U)$. From Treves [15: Theorem 2.1/p. 16] we get $(\mu A\mu_1)g_\varepsilon \in W^{1,2}(U)$ and relation (23) gives the inclusion

$$\delta g_\varepsilon + \varepsilon^{-1}g_\varepsilon^- \in W^{1,2}(U) \tag{24}$$

Therefore δg_ε^+ and $(\delta + \varepsilon^{-1})g_\varepsilon^-$ both belong to $W^{1,2}(U)$, and $g_\varepsilon \in W^{1,2}(U)$ for each fixed pair $\delta, \varepsilon > 0$. From the embedding theorem it follows that $g_\varepsilon \in L_{p_1}(U)$ with $p_1 = \frac{2N}{N-2}$ for $N \geq 3$ and $p_1 < +\infty$ arbitrary for $N = 2$. From the same argument we derive the inclusion $f_\varepsilon, (\mu A\mu_1)g_\varepsilon \in W^{1,p_1}(U)$ and finally $g_\varepsilon \in W^{1,p_1}(S) \subset L_{p_2}(U)$ with $p_2 = \frac{2N}{N-4}$ for $N \geq 5$ and $p_2 < +\infty$ arbitrary for $N \leq 4$. Repeating the argument we conclude that for each $\varepsilon, \delta > 0$

$$g_\varepsilon = \varphi u^\delta_\varepsilon \in W^{1,p}(U) \qquad \text{for all } p < +\infty. \tag{25}$$

Then it follows from the embedding theorem that $g_\varepsilon \in C^\beta(U)$ for all $\beta \in (0,1)$ ∎

4.2 (L_p-regularity). We intend first to apply a ψdo P with principal symbol $|\xi|$ to equality (23). Then we multiply it by the test function $\langle g_\varepsilon\rangle^{p-1} = |g_\varepsilon|^{p-2}g_\varepsilon$. In order to avoid additional regularizing terms containing $\varepsilon^{-1}g_\varepsilon^-$ we need some preparation. For this define

$$(Pv)(x) = \int_{\mathbb{R}^N} e^{ix\xi}\chi(\xi)|\xi|\hat{v}(\xi)\frac{d\xi}{(2\pi)^N}$$

for $v \in C_0^\infty(\mathbb{R}^N)$, where $\chi \in C^\infty(\mathbb{R}^N)$ is a cut-off function characterized, e.g., by

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi| < 1 \\ 1 & \text{if } |\xi| \geq 2. \end{cases}$$

Now we put $\int Pv \cdot w\,dx$ into a form adapted for considerations of the positive and negative part of the functions involved. Taking real functions $v, w \in C_0^\infty(\mathbb{R}^N)$ the theorem of Fubini gives

$$\begin{aligned}(Pv,w) &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} e^{ix\xi}\chi(\xi)|\xi|\hat{v}(\xi)\frac{d\xi}{(2\pi)^N}\right)w(x)\,dx \\ &= \int_{\mathbb{R}^N}\chi(\xi)|\xi|\hat{v}(\xi)\,\overline{\hat{w}(\xi)}\,\frac{d\xi}{(2\pi)^N} \\ &= \int_{\mathbb{R}^N}|\xi|\hat{v}(\xi)\,\overline{\hat{w}(\xi)}\,\frac{d\xi}{(2\pi)^N} + \int_{\mathbb{R}^N}(\chi(\xi)-1)|\xi|\hat{v}(\xi)\,\overline{\hat{w}(\xi)}\,\frac{d\xi}{(2\pi)^N} \\ &=: I_1 + I_2.\end{aligned} \tag{26}$$

The operator R_2 defined by

$$(R_2v)(x) = \int_{\mathbb{R}^N} e^{ix\xi}(\chi(\xi)-1)|\xi|\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N} \qquad \text{for } v \in C_0^\infty(\mathbb{R}^N)$$

is regularizing: $R_2 \in \Psi^{-\infty}(\mathbb{R}^N)$, since the amplitude $\chi(\xi)-1$ vanishes outside the ball $B_2(0)$ (cf. Dieudonné [4: Remark 23.19.5(iii)/p.149]). Applying Parseval's equality to I_1 we get

$$I_1 = a\iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^{-N-1}\big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\,dxdy \tag{27}$$

where $a = a(N) > 0$ is a constant (see Wloka [18: p. 97] and Hörmander [5: Vol. 1/p. 241]). We stress that both integrals I_1 and I_2 depend on v and w. We have $(R_2v,w) = I_2$ and define an operator J_1 by

$$(J_1v,w) = I_1 = a\iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^{-N-1}\big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\,dxdy$$

for all $v,w \in C_0^\infty(\mathbb{R}^N)$ to get

$$(J_1v,w) = (Pv,w) - (R_2v,w). \tag{28}$$

We now prove L_p-regularity of the solution u of the variational inequality (1).

Theorem 1. *Let $2 \le p < +\infty$ and $b \in W^{\frac12,2}(S)\cap W^{1,p}_{loc}(\Gamma)$. Then $u \in L_p^{loc}(\Gamma)$.*

Remark 1. For $2 \le p < +\infty$, the inclusion $b \in W^{1,p}(S)$ implies the inclusion $u \in L_p(S)$ if after a coordinate transform the operator A has the principal symbol (3) in each coordinate patch of a partition of unity on S.

Proof of Theorem 1. To prove the theorem we consider the approximate problems and derive uniform bounds for the solutions $u_\epsilon = u_\epsilon^\delta$ of inequality (10) and u^δ of inequality (4).

(a) For simplicity we set $b^\delta = b \in W^{\frac12,2}(S)\cap W^{1,p}_{loc}(\Gamma)$. By approximation, we may assume that the family (b_ϵ) belongs to $W^{\frac12,2}(S)\cap W^{1,q}_{loc}(\Gamma)$ for all $q < +\infty$ and, furthermore, $b_\varepsilon \to b^\delta$ in $W^{\frac{1}{2},2}(S)$ and in $W^{1,p}_{loc}(\Gamma)$ as $\varepsilon \to +0$. In particular, for any open set $O \subset\subset \Gamma$

$$\sup_{\varepsilon} \|b_\varepsilon\|_{1,p,O} \le M = M(O) < +\infty. \tag{29}$$

It follows from Lemma 1 that $g_\varepsilon \in W^{1,q}(U)$ for all $q < +\infty$. Therefore also $g_\varepsilon^- \in W^{1,q}(U)$ for all $q < +\infty$. Suppose $q > N$, arbitrary. Then $W^{1,q}(U)$ is a Banach algebra (see Adams [1: p. 115]) and it follows that $\langle g_\varepsilon \rangle^{p-1} = |g_\varepsilon|^{p-2} g_\varepsilon \in W^{1,q}(U)$ for each $q \ge 2$. It is our goal to show that (29) implies

$$\sup_{\varepsilon} \|g_\varepsilon\|_p \le M_1 < +\infty \tag{30}$$

where the constant M_1 is independent of δ. This gives the local boundedness of $u_\varepsilon \in L_p(\Gamma)$. In fact, we may choose φ such that $\varphi \equiv 1$ on any open set $V \subset\subset U$ and estimation (30) implies

$$\sup_{\varepsilon} \|u_\varepsilon\|_{p,V} \le M_1 < +\infty. \tag{31}$$

(b) We apply operator J_1 to equality (23) and multiply it by $h_\varepsilon = \langle g_\varepsilon \rangle^{p-1}$ to get

$$\delta(J_1 g_\varepsilon, h_\varepsilon) + \big(J_1(\mu A \mu_1) g_\varepsilon, h_\varepsilon\big) + \varepsilon^{-1}(J_1 g_\varepsilon^-, h_\varepsilon) = (J_1 f_\varepsilon, h_\varepsilon),$$

that is

$$\begin{aligned} L_1 + L_2 + L_3 := \delta a \iint |x-y|^{-N-1}\big(g_\varepsilon(x) - g_\varepsilon(y)\big)\big(h_\varepsilon(x) - h_\varepsilon(y)\big)\,dx dy \\ + \big(P(\mu A \mu_1) g_\varepsilon, h_\varepsilon\big) \\ + \varepsilon^{-1} a \iint |x-y|^{-N-1}\big(g_\varepsilon^-(x) - g_\varepsilon^-(y)\big)\big(h_\varepsilon(x) - h_\varepsilon(y)\big)\,dx dy \\ = \big((P - R_2) f_\varepsilon, h_\varepsilon\big) + \big(R_2(\mu A \mu_1) g_\varepsilon, h_\varepsilon\big). \end{aligned} \tag{32}$$

Now we have to consider the terms L_1, L_2 and L_3 of (32) separately. The function $t \mapsto |t|^{p-2} t$ is uniformly monotone for $p \ge 2$:

$$\big(|s|^{p-2}s - |t|^{p-2}t\big)(s-t) \ge c|s-t|^p \qquad \text{for all } s, t \in \mathbb{R} \tag{33}$$

where $c > 0$ is a constant (cf. Zeidler [19: Vol. 2/p. 503]). Then

$$\begin{aligned} L_1 &= \delta a \iint |x-y|^{-N-1}\big(g_\varepsilon(x) - g_\varepsilon(y)\big)\big(|g_\varepsilon(x)|^{p-2} g_\varepsilon(x) - |g_\varepsilon(y)|^{p-2} g_\varepsilon(y)\big)\,dx dy \\ &\ge \delta c a \iint |x-y|^{-N-1} |g_\varepsilon(x) - g_\varepsilon(y)|^p\,dx dy \\ &= \delta c a |g_\varepsilon|^p_{\frac{1}{p},p}. \end{aligned}$$

The third term L_3 in (32) is the penalization term. Oberserving that

$$\big(|s|^{p-2}s - |t|^{p-2}t\big)(s^- - t^-) \ge \big(|s^-|^{p-2}s^- - |t^-|^{p-2}t^-\big)(s^- - t^-)$$

it follows from (33) that

$$\begin{aligned}L_3 &= \varepsilon^{-1}a\iint |x-y|^{-N-1}\big(g_\varepsilon^-(x)-g_\varepsilon^-(y)\big)\big(h_\varepsilon(x)-h_\varepsilon(y)\big)\,dx\,dy\\ &\geq \varepsilon^{-1}ca\iint |x-y|^{-N-1}|g_\varepsilon^-(x)-g_\varepsilon^-(y)|^p\,dx\,dy\\ &= \varepsilon^{-1}ca|g_\varepsilon^-|^p_{\frac{1}{p},p}.\end{aligned}$$

The second term of $L_2 = \big(P(\mu A\mu_1)g_\varepsilon, h_\varepsilon\big)$ of (32) contains the composition of $P \in \Psi^1(U)$ and the proper ψdo $\mu A\mu_1 \in \Psi^{-1}(U)$. The principal symbol of $P(\mu A\mu_1) \in \Psi^0(U)$ is $\sigma_0\big(P(\mu A\mu_1)\big)(x,\xi) = \chi(\xi)\mu(x)$. Thus there exists a ψdo $P_{-1} \in \Psi^{-1}(U)$ such that

$$\begin{aligned}&\int (P\mu A\mu_1)(v)\cdot w\,dx\\ &= \int\left\{\widetilde{\iint} e^{i(x-y)\xi}\chi(\xi)\mu(y)v(y)\,dy\frac{d\xi}{(2\pi)^N}\right\}w(x)\,dx + \int P_{-1}v\cdot w\,dx\\ &= \int\left(\int e^{ix\xi}\chi(\xi)\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N}\right)w(x)\,dx + (P_{-1}v,w)\\ &= \int\left(\int e^{ix\xi}\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N}\right)w(x)\,dx\\ &\quad + \int\left(\int_{\mathbb{R}^N} e^{ix\xi}(\chi(\xi)-1)\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N}\right)w(x)\,dx + (P_{-1}v,w)\\ &= \int v\,w\,dx + (R_3v,w) + (P_{-1}v,w)\end{aligned} \tag{34}$$

for all $v, w \in C_0^\infty(W)$ where $\widetilde{\iint}$ denotes an oscillatory integral and R_3 is regularizing by the argument already used for R_2. Then, by approximation,

$$L_2 = \int_\Gamma |g_\varepsilon|^p dx + (R_3g_\varepsilon, h_\varepsilon) + (P_{-1}g_\varepsilon, h_\varepsilon).$$

By Hölder's inequality, equations (32) and (34) together give

$$\begin{aligned}&\delta ca|g_\varepsilon|^p_{\frac{1}{p},p} + \|g_\varepsilon\|_p^p + \varepsilon^{-1}ca|g_\varepsilon^-|^p_{\frac{1}{p},p}\\ &\quad\leq \Big(\|(P-R_2)f_\varepsilon\|_p + \|R_2(\mu A\mu_1)g_\varepsilon\|_p + \|R_3g_\varepsilon\|_p + \|P_{-1}g_\varepsilon\|_p\Big)\|g_\varepsilon\|_p^{p-1}\\ &\quad\leq C\Big(\|\varphi b_\varepsilon\|_{1,p,W} + \|(P-R_2)R_1u_\varepsilon\|_{p,W} + \|P_{-2}g_\varepsilon\|_{1,p,W}\\ &\qquad + \|R_2(\mu A\mu_1)g_\varepsilon\|_{p,W} + \|R_3g_\varepsilon\|_{p,W} + \|P_{-1}g_\varepsilon\|_{p,W}\Big)\|g_\varepsilon\|_p^{p-1}\end{aligned}$$

since $K_\varphi = \operatorname{supp}\varphi \subset W \subset\subset U$. Young's inequality and Proposition 2 imply

$$\begin{aligned}&\delta|g_\varepsilon|^p_{\frac{1}{p},p} + \|g_\varepsilon\|_p^p + \varepsilon^{-1}|g_\varepsilon^-|^p_{\frac{1}{p},p}\\ &\quad\leq C\Big(\|b_\varepsilon\|^p_{1,p,W} + \|u_\varepsilon\|^p_{-\frac{1}{2},2,S} + \|P_{-2}g_\varepsilon\|^p_{1,p,W} + \|g_\varepsilon\|^p_{-\frac{1}{2},2} + \|P_{-1}g_\varepsilon\|^p_{p,W}\Big)\\ &\quad\leq C\Big(1 + \|P_{-2}g_\varepsilon\|^p_{1,p,W} + \|P_{-1}g_\varepsilon\|_{p,W}\Big)\end{aligned} \tag{35}$$

since R_1 and R_2 are regularizing.

(c) We are going to apply a bootstrap argument. Using the embedding theorem and the fact that $P_{-1} : W^{-\frac{1}{2},2}_{comp}(U) \longrightarrow W^{\frac{1}{2},2}_{loc}(U)$ and $P_{-2} : W^{-\frac{1}{2},2}_{comp}(U) \longrightarrow W^{\frac{3}{2},2}_{loc}(U)$ are continuous linear mappings we get

$$\|P_{-1}g_\varepsilon\|_{q_1,W} \le c_1\|P_{-1}g_\varepsilon\|_{\frac{1}{2},2,W} \le c_2\|g_\varepsilon\|_{-\frac{1}{2},2} \tag{36}$$

$$\|P_{-2}g_\varepsilon\|_{1,q_1,W} \le c_1\|P_{-2}g_\varepsilon\|_{\frac{3}{2},2,W} \le c_2\|g_\varepsilon\|_{-\frac{1}{2},2} \tag{37}$$

for some constants $c_1 > 0$ and $c_2 > 0$, where $q_1 = \frac{2N}{N-1}$. We stress that these constants depend upon W and K_φ, but neither on ε nor on δ. It follows from (35) with $p = q_1$ that

$$\sup_\varepsilon \left(\delta|g_\varepsilon|^{q_1}_{\frac{1}{q_1},q_1} + \|g_\varepsilon\|^{q_1}_{q_1} + \varepsilon^{-1}|g^-_\varepsilon|^{q_1}_{\frac{1}{q_1},q_1}\right) < +\infty. \tag{38}$$

This implies $\sup_\varepsilon \|g_\varepsilon\|_{q_1} < +\infty$. As in the first step we get

$$\sup_\varepsilon \{\|P_{-2}g_\varepsilon\|_{2,q_1,W} + \|P_{-1}g_\varepsilon\|_{1,q_1,W}\} < +\infty.$$

With $q_2 = \frac{2N}{N-3}$ the embedding theorem implies

$$\|P_{-2}g_\varepsilon\|_{1,q_2,W} \le c_3\|P_{-2}g_\varepsilon\|_{2,q_1,U} \quad \text{and} \quad \|P_{-1}g_\varepsilon\|_{q_2,W} \le c_3\|P_{-1}g_\varepsilon\|_{1,q_1,W}$$

and we get from (35) with $p = q_2$

$$\sup_{\varepsilon,\delta} \left(\delta|g_\varepsilon|^{q_2}_{\frac{1}{q_2},q_2} + \|g_\varepsilon\|^{q_2}_{q_2} + \varepsilon^{-1}|g^-_\varepsilon|^{q_2}_{\frac{1}{q_2},q_2}\right) < +\infty.$$

We can repeat this procedure as far as $q_j \le p$. In the last step we get

$$\sup_\varepsilon \left(\delta|g_\varepsilon|^{p}_{\frac{1}{p},p} + \|g_\varepsilon\|^{p}_{p} + \varepsilon^{-1}|g^-_\varepsilon|^{p}_{\frac{1}{p},p}\right) \le M_1 < +\infty \tag{39}$$

where the estimates used above show that the constant M_1 is independent of $\delta > 0$. This proves estimations (30) and (31).

(d) Let $\varepsilon_n \to +0$ for fixed $\delta > 0$. Since $\sup_n \|g_{\varepsilon_n}\|_p \le M_1$ we can extract a subsequence with $\varphi u_\varepsilon \rightharpoonup g^\delta$ in $L_p(U)$. As $u_\varepsilon \to u^\delta$ in $L_2(S)$ (Proposition 2) we conclude that $g^\delta = \varphi u^\delta \in L_p(S)$, i.e. $u^\delta \in L^{loc}_p(\Gamma)$. Let $\varphi \equiv 1$ on V. The weak sequential lower semicontinuity of the norm gives $\|u^\delta\|_{p,V} \le \|\varphi u^\delta\|_p \le M_1$ for $V \subset\subset U$.

(e) If $\delta_n \to +0$, there exists a subsequence such that $\varphi u^\delta \rightharpoonup u_0$ in $L_p(S)$ and $\varphi u^\delta \to u_0$ in $W^{-\frac{1}{2},2}(U)$. Proposition 1 gives $\varphi u^\delta \to \varphi u$ in $W^{-\frac{1}{2},2}(S)$. Consequently $u_0 = \varphi u \in L_p(U)$, and it follows that $u \in L^{loc}_p(\Gamma)$ with $\|u\|_{p,V} \le \|\varphi u\|_p \le M_1$ for $V \subset\subset U$ ∎

5. $\mathbf{L_\infty}$-regularity

5.1. To prove L_∞-regularity for the solutions u_ε of equation (19) we apply a method from the classical theory of differential equations due to Stampacchia. It depends on estimates for the size of level sets. As in Subsection 4.2 we begin with a kind of differentiation of equation (23). Here we are going to use the operator

$$(P^\gamma v)(x) := \int_{\mathbb{R}^N} e^{ix\xi}\chi(\xi)|\xi|^\gamma \hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N} \tag{40}$$

for $v \in C_0^\infty(\mathbb{R}^N)$ where $1 < \gamma < 2$ and $\chi \in C^\infty(\mathbb{R}^N)$ is the same function as in Subsection 4.2. For $g_\varepsilon = \varphi u_\varepsilon$ we have the following estimate.

Lemma 2. *Suppose $b_\varepsilon \in W^{\gamma,2}(U)$ for some $\gamma \in (1,2)$. Then there exist appropriate ψdo's Q_γ and $Q_{\gamma-2}$ from $\Psi^\gamma(U)$ and $\Psi^{\gamma-2}(U)$, repectively, such that*

$$\begin{aligned}\delta a\big|[g_\varepsilon(x)-k]^+\big|^2_{\frac{\gamma}{2},2} &+ a\big|[g_\varepsilon(x)-k]^+\big|^2_{\frac{\gamma-1}{2},2}\\ &\le \int_U \big(|Q_\gamma f_\varepsilon| + |Q_{\gamma-2}g_\varepsilon|\big)[g_\varepsilon(x)-k]^+\,dx.\end{aligned} \tag{41}$$

Proof. (a) For $v, w \in C_0^\infty(\mathbb{R}^N)$ we get

$$\begin{aligned}(P^\gamma v, w) &= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} e^{ix\xi}\chi(\xi)|\xi|^\gamma\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N}\right)w(x)\,dx\\ &= \int_{\mathbb{R}^N}|\xi|^\gamma\hat{v}(\xi)\,\overline{\hat{w}(\xi)}\,\frac{d\xi}{(2\pi)^N} + \int_{\mathbb{R}^N}(\chi(\xi)-1)|\xi|^\gamma\hat{v}(\xi)\,\overline{\hat{w}(\xi)}\,\frac{d\xi}{(2\pi)^N}\\ &=: I_1^\gamma + I_2^\gamma.\end{aligned} \tag{42}$$

Concerning the integral I_2^γ we observe that the operator R_2^γ defined by

$$(R_2^\gamma v)(x) = \int_{\mathbb{R}^N} e^{ix\xi}(\chi(\xi)-1)|\xi|^\gamma\hat{v}(\xi)\,\frac{d\xi}{(2\pi)^N}$$

for $v \in C_0^\infty(\mathbb{R}^N)$ is regularizing, whereas Parseval's inequality implies

$$I_1^\gamma = a\iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^{-N-\gamma}\big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\,dx\,dy \tag{43}$$

with $a = a(\gamma, N) > 0$. Defining

$$\begin{aligned}(J_\gamma v, w) &= I_1^\gamma\\ &= a\iint_{\mathbb{R}^N\times\mathbb{R}^N}|x-y|^{-N-\gamma}\big(v(x)-v(y)\big)\big(w(x)-w(y)\big)\,dx\,dy\end{aligned}$$

for all $v, w \in C_0^\infty(\mathbb{R}^N)$ we get

$$(J_\gamma v, w) = (P^\gamma v, w) - (R_2^\gamma v, w). \tag{44}$$

(b) The application of the operator J_γ to equality (23) and scalar multiplication by a test function h_ε gives

$$\begin{aligned} L_1 + L_2 + L_3 := \delta a \iint |x-y|^{-N-\gamma}\big(g_\varepsilon(x) - g_\varepsilon(y)\big)\big(h_\varepsilon(x) - h_\varepsilon(y)\big)dxdy \\ + \big(P^\gamma(\mu A \mu_1) g_\varepsilon, h_\varepsilon\big) \\ + \varepsilon^{-1} a \iint |x-y|^{-N-\gamma}\big(g_\varepsilon^-(x) - g_\varepsilon^-(y)\big)\big(h_\varepsilon(x) - h_\varepsilon(y)\big)dxdy \\ = \big((P^\gamma - R_2^\gamma)f_\varepsilon, h_\varepsilon\big) + \big(R_2^\gamma(\mu A \mu_1)g_\varepsilon, h_\varepsilon\big). \end{aligned} \tag{45}$$

For $k \geq 0$, choose $h_\varepsilon = [g_\varepsilon - k]^+ \in W^{\frac{\gamma}{2},2}(U)$ in (45). It follows that $\operatorname{supp}[g_\varepsilon(x) - k]^+ \subseteq \operatorname{supp}\varphi$ for $k \geq 0$. We first get

$$\begin{aligned} L_1 &= \delta a \iint |x-y|^{-N-\gamma}\big([g_\varepsilon(x) - k] - [g_\varepsilon(y) - k]\big) \cdot \\ &\quad \times \big([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+\big)dxdy \\ &\geq \delta a \iint |x-y|^{-N-\gamma}\Big|[g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+\Big|^2 dxdy \\ &= \delta a |[g_\varepsilon - k]^+|^2_{\frac{\gamma}{2},2}. \end{aligned}$$

Observing that

$$(s^- - t^-)\big([s-k]^+ - [t-k]^+\big) \geq 0 \qquad \text{for all } s,t \in \mathbb{R}$$

we see that

$$\begin{aligned} L_3 &= \varepsilon^{-1} a \iint |x-y|^{-N-\gamma}\big(g_\varepsilon^-(x) - g_\varepsilon^-(y)\big) \\ &\quad \times \big([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+\big)dxdy \\ &\geq 0. \end{aligned}$$

In the second term L_2 of (45), the principal symbol of the composition $P^\gamma(\mu A \mu_1) \in \Psi^{\gamma-1}(U)$ is $\sigma_{\gamma-1}\big(P^\gamma(\mu A \mu_1)\big)(x,\xi) = \mu(x)|\xi|^{\gamma-1}\chi(\xi)$. It follows that there exists a ψdo $P_{\gamma-2} \in \Psi^{\gamma-2}(U)$ such that $P^\gamma(\mu A \mu_1) = P^{\gamma-1}\mu + P_{\gamma-2}$ where $P^{\gamma-1} \in \Psi^{\gamma-1}(U)$ is defined by (40) with γ replaced by $\gamma - 1$. Thus (44) with $\gamma - 1$ instead of γ gives

$$\begin{aligned} L_2 &= (P^{\gamma-1} g_\varepsilon, h_\varepsilon) + (P_{\gamma-2} g_\varepsilon, h_\varepsilon) \\ &= a \iint |x-y|^{-N-\gamma+1}\big([g_\varepsilon(x) - k] - [g_\varepsilon(y) - k]\big)\big([g_\varepsilon(x) - k]^+ - [g_\varepsilon(y) - k]^+\big) \\ &\quad + (R_3 g_\varepsilon, h_\varepsilon) + (P_{\gamma-2} g_\varepsilon, h_\varepsilon) \\ &\geq a |[g_\varepsilon - k]^+|_{\frac{\gamma-1}{2},2} + (R_3 g_\varepsilon, h_\varepsilon) + (P_{\gamma-2} g_\varepsilon, h_\varepsilon). \end{aligned}$$

The regularizing operator $R_3 = R_2^{\gamma-1}$ arises from (44). Observe that $\mu \equiv 1$ on $K_\varphi = \operatorname{supp}\varphi$. Summarizing we get

$$\begin{aligned} &\delta a |[g_\varepsilon - k]^+|^2_{\frac{\gamma}{2},2} + a |[g_\varepsilon - k]^+|^2_{\frac{\gamma-1}{2},2} \\ &\quad \leq \int_U \Big\{\big((P^\gamma - R_2^\gamma)f_\varepsilon\big) + \big(R_2^\gamma(\mu A \mu_1)g_\varepsilon - P_{\gamma-2} g_\varepsilon - R_3 g_\varepsilon\big)\Big\}[g_\varepsilon(x) - k]^+ dx \\ &\quad = \int_U (Q_\gamma f_\varepsilon + Q_{\gamma-2} g_\varepsilon)[g_\varepsilon(x) - k]^+ dx \end{aligned} \tag{46}$$

where we have introduced $Q_\gamma = P^\gamma - R_2^\gamma$ and $Q_{\gamma-2} = R_2^\gamma(\mu A\mu_1) - P_{\gamma-2} - R_3$ to keep the notation short. This proves the lemma ∎

5.2 We prove an embedding theorem which is needed later in this section.

Lemma 3. *Suppose $\Omega \subset \mathbb{R}^N$ is a domain and $s \in (0,1)$ is given. We set $\frac{1}{q} = \frac{1}{2} - \frac{s}{N}$, i.e. $q = \frac{2N}{N-2s} > 2$. Then the following assertions are true.*

1. *We have the continuous embedding $W^{s,2}(\Omega) \subset L_q(\Omega)$, such that*

$$\|u\|_q \leq c\|u\|_{s,2} \qquad \text{for all } \ u \in W^{s,2}(\Omega).$$

2. *If $\Omega_1 \subset\subset \Omega$ is an open set, then there exists a constant $C = C(\Omega, \Omega_1) > 0$ such that*

$$\|u\|_q \leq C|u|_{s,2} \qquad \text{for all } \ u \in W^{s,2}(\Omega) \text{ with } \operatorname{supp} u \subseteq \Omega_1. \tag{47}$$

Proof. For Assertion 1 cf. Triebel [16: p. 196]. For Assertion 2 we prove that

$$u \longmapsto \|u\|_a = \left\{|u|_{s,2}^2 + \int_{\Omega\setminus\Omega_1} |u|^2\right\}^{1/2} \tag{48}$$

is an equivalent norm on $W^{s,2}(\Omega)$, i.e. there exist constants $c_1, c_2 > 0$ such that

$$c_1\left\{|u|_{s,2}^2 + \int_{\Omega\setminus\Omega_1} |u|^2 dx\right\} \leq \left\{|u|_{s,2}^2 + \int_{\Omega} |u|^2 dx\right\} \leq c_2\left\{|u|_{s,2}^2 + \int_{\Omega\setminus\Omega_1} |u|^2 dx\right\} \tag{49}$$

for all $u \in W^{s,2}(\Omega)$. The first inequality in (49) is obvious. To prove the second one we suppose the contrary. Then there exists an sequence $(u_n)_{n\in\mathbb{N}}$ such that $\|u_n\|_{s,2} \geq n\|u_n\|_a$ $(n \in \mathbb{N})$. We define $v_n = \frac{u_n}{\|u_n\|_{s,2}}$. Thus $\|v_n\|_{s,2} = 1$ and $\|v_n\|_a \to 0$, and we can select a subsequence, again denoted by (v_n) such that $v_n \rightharpoonup v$ in $W^{s,2}(\Omega)$, $v_n \to v$ in $L_2(\Omega)$ and $v_n(x) \to v(x)$ a.e. in Ω. From $\|v_n\|_a \to 0$ it follows that

$$v_n(x) \to 0 \qquad \text{a.e. in } \ \Omega \setminus \Omega_1 \tag{50}$$

and

$$|v_n|_{s,2}^2 = \iint_{\Omega\times\Omega} |x-y|^{-N-2s}|v_n(x) - v_n(y)|^2 dx dy \longrightarrow 0.$$

Therefore $|v_n(x) - v_n(y)| \to 0$ a.e. in $\Omega \times \Omega$ and (50) implies $v_n(x) \to 0$ a.e. in Ω. This gives $v_n \to 0$ in $L_2(\Omega)$ and because of $|v_n|_{s,2} \to 0$ we see that $\|v_n\|_{s,2} \to 0$, which contradicts $\|v_n\|_{s,2} = 1$. Thus (49) is proved, and (47) follows immediately ∎

5.3 Now we define sets $A_\varepsilon(k)$ where $g_\varepsilon = \varphi u_\varepsilon^\delta$ superceeds a level k:

$$A_\varepsilon(k) = \left\{x \in \Gamma : g_\varepsilon \geq k\right\}.$$

We age going to estimate the size of $A_\varepsilon(k)$. Remember that $1 < \gamma < 2$.

Lemma 4. *We suppose $b \in W^{\gamma,r}_{loc}(\Gamma)$ for some $\gamma \in (1,2)$ and $r > \frac{N}{\gamma-1}$. Set $b_\varepsilon := b^\delta := b$. Then there exist constants $C > 0$ and $\beta > 1$, independent from ε and δ, such that*

$$|A_\varepsilon(h)| \le \frac{C}{(h-k)^q}\,|A_\varepsilon(k)|^\beta \qquad \textit{for all } \; h > k \ge 0 \tag{51}$$

where $q = \frac{2N}{N+1-\gamma}$.

Proof. Set $s = \frac{\gamma-1}{2}$, $q = \frac{2N}{N-2s} = \frac{2N}{N+1-\gamma} > 2$ and $\frac{1}{q} + \frac{1}{q'} = 1$. It follows from Lemma 2, Lemma 3 and the inclusion $\operatorname{supp}[g_\varepsilon(x) - k]^+ \subseteq \operatorname{supp}\varphi \subset\subset U$ that

$$\begin{aligned}&\left\{\int_{A_\varepsilon(k)} |[g_\varepsilon(x) - k]^+|^q dx\right\}^{2/q} \\ &\quad\le c\left\{\int_{A_\varepsilon(k)} \left(|Q_\gamma f_\varepsilon| + |Q_{\gamma-2} g_\varepsilon|\right)^{q'} dx\right\}^{1/q'} \left\{\int_{A_\varepsilon(k)} |[g_\varepsilon(x) - k]^+|^q dx\right\}^{1/q}\end{aligned} \tag{52}$$

for $k \ge 0$. Young's inequality gives

$$\left\{\int_{A_\varepsilon(k)} |[g_\varepsilon(x) - k]^+|^q dx\right\}^{2/q} \le c\left\{\int_{A_\varepsilon(k)} \left(|Q_\gamma f_\varepsilon| + |Q_{\gamma-2} g_\varepsilon|\right)^{q'} dx\right\}^{2/q'}.$$

Therefore, for $h > k \ge 0$,

$$|A_\varepsilon(h)|(h-k)^q \le c\left\{\int_{A_\varepsilon(k)} \left(|Q_\gamma f_\varepsilon| + |Q_{\gamma-2} g_\varepsilon|\right)^{q'} dx\right\}^{q/q'}$$

and, by Hölder's inequality with $r > \frac{q}{q-2} = \frac{N}{\gamma-1}$ and $r > q'$,

$$|A_\varepsilon(h)|(h-k)^q \le c\Big(\|Q_\gamma f_\varepsilon\|_{r,U} + \|Q_{\gamma-2} g_\varepsilon\|_{r,U}\Big)^q |A_\varepsilon(k)|^{q-1-\frac{q}{r}}. \tag{53}$$

We see that $\beta = q - 1 - \frac{q}{r} > 1$. It follows from (22) and (30) in the proof of Theorem 1 that $\sup\left(\|Q^\gamma f_\varepsilon\|_{r,U} + \|Q_{\gamma-2} g_\varepsilon\|_{r,U}\right) < +\infty$. This gives (51) ∎

Now we are in the position to prove the uniform boundedness of the family $(u_\varepsilon) = (u_\varepsilon^\delta)$. We are going to use the following result of Stampacchia.

Lemma 5 (see Kinderlehrer and Stampacchia [8: p. 63]). *Let $\phi : [k_0, +\infty) \to \mathbb{R}$ be a non-negative and non-increasing function such that*

$$\phi(h) \le \frac{C}{(h-k)^\alpha}[\phi(k)]^\beta \qquad \textit{for } \; h > k > k_0 \tag{54}$$

where C, α and β are positive constants with $\beta > 1$. Then

$$\phi(k_0 + M) = 0$$

where

$$M = 2^{\frac{\beta}{\beta-1}} C^{\frac{1}{\alpha}} [\phi(k_0)]^{\frac{\beta-1}{\alpha}}. \tag{55}$$

Theorem 2. *Suppose $b \in W^{\gamma,r}(U)$ for some $\gamma \in (1,2)$ and $r > \frac{N}{\gamma-1}$. Then the solution u of the variational inequality (1) is locally bounded on Γ : $u \in L_\infty^{loc}(\Gamma)$, i.e. for all $V \subset\subset \Gamma$ there exists a constant $M > 0$ such that $0 \le u(x) \le M$ a.e. on V.*

Remark 2. Under the hypotheses of Remark 1 one may prove the inclusion $u \in L_\infty(S)$.

Proof of Theorem 2. We shall prove the theorem in three steps.

(a) First we define $b_\varepsilon^\delta = b^\delta := b$ for all $\varepsilon, \delta > 0$. We are going to apply Lemma 5 and suppress the superscript δ again. Set $\phi_\varepsilon(k) = |A_\varepsilon(k)|$ and $k_0 = 0$. Then $\phi_\varepsilon(k_0) = |\{x \in \Gamma : g_\varepsilon \ge 0\}| \le |U|$ and it follows from (51) that there exists a bound $M > 0$ *independent* of ε and δ such that

$$\varphi(x) u_\varepsilon^\delta(x) = g_\varepsilon(x) \le M := \sup_\varepsilon 2^{\frac{\beta}{\beta-1}} C^{\frac{1}{q}} [\phi_\varepsilon(0)]^{\frac{\beta-1}{q}} \le c_1 |U|^{\frac{\beta-1}{q}} \tag{56}$$

a.e. on U.

(b) Next, we keep $\delta > 0$ fixed and let $\varepsilon := \varepsilon_n \to +0$. For simplicity, we omit the subscipt n. From Proposition 2 we know that $u_\varepsilon \to u^\delta$ in $L_2(S)$, $g_\varepsilon \to g^\delta = \varphi u^\delta$ in $L_2(U)$ and along a subsequence $g_\varepsilon(x) \to g^\delta(x)$ a.e. in U. Since $u^\delta \in K_1$ (56) gives

$$0 \le \varphi(x) u^\delta(x) = g^\delta(x) \le M$$

a.e. in U.

(c) Finally, let $\delta := \delta_n \to +0$. As in the proof of Theorem 1 we have $\varphi u^\delta \to \varphi u$ in $L_2(U)$, and $\varphi u^\delta \rightharpoonup \varphi u$ in $W^{-\frac{1}{2},2}(S)$. Along a subsequence, a theorem of Banach and Saks (see Riesz and Sz.-Nagy [11: p.72]) implies the strong L_2-convergence of the sequence of arithmetic means, i.e. $v_n = \frac{1}{n}(\varphi u^{\delta_1} + \varphi u^{\delta_2} + \ldots + \varphi u^{\delta_n}) \to \varphi u$ in $L_2(U)$. Again, passing to a subsequence if necessary, $v_n(x) \to \varphi(x)u(x)$ a.e. in U. Since for the means $0 \le v_n(x) \le M$ we have also $0 \le \varphi(x)u(x) \le M$ a.e. in U. As we may choose φ in Subsection 4.1 such that $\varphi \equiv 1$ on an arbitrary open set $V \subset\subset U$ the assertion follows ∎

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