

On the Application of the Newton-Kantorovich Method to Nonlinear Partial Integral Equations

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Abstract. We discuss the applicability of the Newton-Kantorovich method to a nonlinear equation which contains partial integrals with Uryson type kernels. A basic ingredient of this method consists in verifying a local Lipschitz condition for the Fréchet derivatives of the nonlinear partial integral operators generated by such kernels. The abstract results are illustrated in the space C of continuous functions and the Lebesgue space L_p for $1 \leq p \leq \infty$. In particular, it is shown that a local Lipschitz condition for the derivative in the space L_p for $p < \infty$ leads to a degeneracy of the corresponding kernels. For ordinary integral operators, such a degeneracy occurs for $p \leq 2$ only.

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0. Introduction

Let S and T be subsets of Euclidean space with finite Lebesgue measure, and let

$$\begin{aligned}l &: T \times S \times T \times \mathbb{R} \rightarrow \mathbb{R} \\m &: T \times S \times S \times \mathbb{R} \rightarrow \mathbb{R} \\n &: T \times S \times T \times S \times \mathbb{R} \rightarrow \mathbb{R}\end{aligned}$$

be given Carathéodory functions (i.e. functions which are continuous in the last variable and measurable in the other variables). The purpose of this paper is to investigate the nonlinear partial integral equation of Uryson type

$$\begin{aligned}x(t, s) &= \int_T l(t, s, \tau, x(\tau, s)) d\tau \\&+ \int_S m(t, s, \sigma, x(t, \sigma)) d\sigma + \int_T \int_S n(t, s, \tau, \sigma, x(\tau, \sigma)) d\sigma d\tau\end{aligned}\tag{1}$$

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by means of the Newton-Kantorovich method.

Equations of this type arise in both the theory and applications of boundary value problems for partial differential equations [2, 13, 14], as well as in various nonlinear transport problems. They describe, for example, the propagation of radiation through the atmosphere of planets and stars [12, 21], or the transfer of neutrons through thin plates and membranes in nuclear reactors [22].

The first two integrals occurring in the right-hand side of equation (1) are usually called "partial integrals", inasmuch as the integration is carried out only with respect to some variables, while the other variables are "frozen". As a matter of fact, most of the classical tools of nonlinear analysis (degree theory, fixed point principles, variational methods based on degree arguments, etc.) do not apply to operators involving such integrals. One reason for this is that, in contrast to ordinary integral operators, partial integral operators are not compact. The Newton-Kantorovich method is one of the few, though important, tools which may be used to prove the solvability of nonlinear equations involving partial integral operators. Moreover, this method makes it also possible to "calculate" approximate solutions with prescribed accuracy.

For the general theory of partial integral operators in so-called Lebesgue spaces with mixed norm we refer the reader to the papers [3, 4, 7, 15 - 19], where various boundedness and continuity conditions are studied in detail. The results of the present paper are in a certain sense parallel to those of the paper [1], where existence results are given for ordinary integral equations of Uryson type. However, there are important differences in both the methods and results which we will point out in the corresponding places.

1. The Newton-Kantorovich method

The Newton-Kantorovich method is one of the basic tools for finding approximate solutions of the operator equation

$$F(x) = 0, \quad (2)$$

where F is some nonlinear operator in a Banach space X . In the corresponding iterative scheme

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \quad (n = 0, 1, 2, \dots) \quad (3)$$

one has to require, in particular, that the Fréchet derivative F' of F at all points x_n exists and is invertible in the algebra $\mathcal{L}(X)$ of all bounded linear operators in X . The direct verification of this requirement may cause essential difficulties in practice. However, in the last years several new ideas have been developed to overcome these difficulties. For the reader's ease, let us briefly sketch some of these ideas related to the Newton-Kantorovich method [28, 29].

Suppose that F is defined on the closure $\overline{B}_R(X)$ of some ball $B_R(X) = \{x \in X : \|x\| < R\}$ and admits a Fréchet derivative $F'(x)$ at each point of $B_R(X)$ such that F' satisfies a Lipschitz condition

$$\|F'(x_1) - F'(x_2)\| \leq k(r) \|x_1 - x_2\| \quad (x_1, x_2 \in B_r(X), 0 \leq r \leq R) \quad (4)$$

as a map from $B_R(X)$ into $\mathcal{L}(X)$. We also assume that the Fréchet derivative at zero is invertible and put

$$a = \|F'(0)^{-1}F(0)\| \tag{5}$$

$$b = \|F'(0)^{-1}\|. \tag{6}$$

Since the (minimal) Lipschitz constant $k = k(r)$ in (4) is non-negative, the function $\varphi : [0, R] \rightarrow [0, \infty)$ defined by

$$\varphi(r) = a + b \int_0^r (r-t)k(t) dt \quad (0 \leq r \leq R) \tag{7}$$

is convex. The scalar equation

$$r = \varphi(r) \tag{8}$$

may have no solution, a unique solution, or many solutions in $[0, R]$. Let us suppose that (8) has a unique solution $r_* \in [0, R]$ and $\varphi(R) \leq R$. In this case equation (2) has a solution $x_* \in \overline{B}_{r_*}(X)$, and this solution is unique in the ball $B_R(X)$ (see [28]). Moreover, the iterations (3) are defined for every n and converge to the solution x_* .

The usefulness of the Newton-Kantorovich method does not only consist in reducing the operator equation (2) in a Banach space to the scalar equation (8) on a real interval. It is also possible to give estimates for the convergence speed. In fact, let

$$U(r) = \frac{\varphi(r) - r}{1 - \varphi'(r)}, \quad V(r) = U(r + U^{-1}(r)), \quad W(r) = \sum_{k=0}^{\infty} V^k(r)$$

where by V^k we mean the k -th iterate of the operator V . Then the estimates

$$\|x_{n+1} - x_n\| \leq V^n(a) \quad \text{and} \quad \|x_* - x_n\| \leq W(V^n(a))$$

hold [28]. So, in order to study equation (2) from the viewpoint of existence, uniqueness, and approximation it suffices in many cases

- to calculate (or estimate) the constants a and b
- to calculate (or estimate) the scalar function $k = k(r)$.

As already mentioned, the simplest case is when the scalar equation (8) has a unique solution r_* in $[0, R]$ and $\varphi(R) \leq R$. Other cases may be reduced to this case. For example, if (8) has another solution $r^* \in [0, R]$ with $r^* > r_*$, say, we simply take $R < r^*$. Likewise, if (8) has a whole continuum of solutions $[r_*, r^*] \subset (0, R]$, we can choose $R = r_*$.

2. Lipschitz conditions for derivatives and norm estimates

Throughout we denote the (partial) integral operator generated by some kernel function by the corresponding capital letter, i. e.

$$L(x)(t, s) = \int_T l(t, s, \tau, x(\tau, s)) d\tau \tag{9}$$

$$M(x)(t, s) = \int_S m(t, s, \sigma, x(t, \sigma)) d\sigma \tag{10}$$

$$N(x)(t, s) = \int_T \int_S n(t, s, \tau, \sigma, x(\tau, \sigma)) d\sigma d\tau. \tag{11}$$

The nonlinear partial integral equation (1) may then be rewritten as operator equation (2) if we put

$$F(x) = x - L(x) - M(x) - N(x) \quad (x \in B_R(X)), \tag{12}$$

where X is some Banach space of real functions over $T \times S$.

Suppose now that the three kernel functions in (1) have partial derivatives in the last argument

$$\begin{aligned} l_1(t, s, \tau, u) &= \frac{\partial l}{\partial u}(t, s, \tau, u) \\ m_1(t, s, \sigma, u) &= \frac{\partial m}{\partial u}(t, s, \sigma, u) \\ n_1(t, s, \tau, \sigma, u) &= \frac{\partial n}{\partial u}(t, s, \tau, \sigma, u). \end{aligned}$$

Consider the operators L^* , M^* and N^* defined by

$$L^*(x)(t, s, \tau) = l_1(t, s, \tau, x(\tau, s)) \tag{13}$$

$$M^*(x)(t, s, \sigma) = m_1(t, s, \sigma, x(t, \sigma)) \tag{14}$$

$$N^*(x)(t, s, \tau, \sigma) = n_1(t, s, \tau, \sigma, x(\tau, \sigma)). \tag{15}$$

These operators are not superposition operators in the usual sense, since they map functions of the two variables (t, s) into functions of three (or even four) variables; we call such operators "generalized superposition operators" in the sequel. The operators (13) - (15) may be considered between the space X and the kernel spaces $\mathfrak{R}_T(X, X)$, $\mathfrak{R}_S(X, X)$, and $\mathfrak{R}_{T \times S}(X, X)$ defined by the norms

$$\|p\|_{\mathfrak{R}_T(X, X)} = \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T |p(t, s, \tau) h(\tau, s)| d\tau \right\|_X \tag{16}$$

$$\|q\|_{\mathfrak{R}_S(X, X)} = \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_S |q(t, s, \sigma) h(t, \sigma)| d\sigma \right\|_X \tag{17}$$

$$\|r\|_{\mathfrak{R}_{T \times S}(X, X)} = \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T \int_S |r(t, s, \tau, \sigma) h(\tau, \sigma)| d\sigma d\tau \right\|_X, \tag{18}$$

respectively, where we have written $\|(t, s) \mapsto f(t, s)\|$ instead of $\|f\|$ to point out the variables of the function involved. Of course, the functionals (16) - (18) are nothing else than the operator norms of the moduli [8 - 11, 24 - 27] of the corresponding (regular) linear partial integral operators

$$Ph(t, s) = \int_T p(t, s, \tau) h(\tau, s) d\tau \tag{19}$$

$$Qh(t, s) = \int_S q(t, s, \sigma) h(t, \sigma) d\sigma \tag{20}$$

$$Rh(t, s) = \int_T \int_S r(t, s, \tau, \sigma) h(\tau, \sigma) d\sigma d\tau \tag{21}$$

in the algebra $\mathfrak{L}(X)$ of bounded linear operators on X . We set

$$L_1(x)h(t, s) = \int_T l_1(t, s, \tau, x(\tau, s)) h(\tau, s) d\tau \tag{22}$$

$$M_1(x)h(t, s) = \int_S m_1(t, s, \sigma, x(t, \sigma)) h(t, \sigma) d\sigma \tag{23}$$

$$N_1(x)h(t, s) = \int_T \int_S n(t, s, \tau, \sigma, x(\tau, \sigma)) h(\tau, \sigma) d\sigma d\tau. \tag{24}$$

Lemma 1. *Suppose that the generalized superposition operators L^*, M^* and N^* defined by (13) - (15) act from $B_R(X)$ into $\mathfrak{R}_T(X, X)$, $\mathfrak{R}_S(X, X)$ and $\mathfrak{R}_{T \times S}(X, X)$, respectively, and satisfy a Lipschitz condition. Then the operators L, M and N defined by (9) - (11) are Fréchet differentiable with $L' = L_1, M' = M_1$ and $N' = N_1$. Consequently,*

$$F'(x) = I - L_1(x) - M_1(x) - N_1(x) \quad (x \in B_R(X)). \tag{25}$$

Proof. The assertion has been proved in [1] for integral operators N of the form (11), so we have to prove it only for partial integral operators L and M of the form (9) and (10), respectively. For $x \in B_R(X)$ and $h \in X$ we have

$$\begin{aligned} & [L(x + h) - L(x) - L_1(x)h](t, s) \\ &= \int_T [l(t, s, \tau, x(\tau, s) + h(\tau, s)) - l(t, s, \tau, x(\tau, s)) - l_1(t, s, \tau, x(\tau, s)) h(\tau, s)] d\tau \\ &= \int_T \int_0^1 [l_1(t, s, \tau, x(\tau, s) + \lambda h(\tau, s)) - l_1(t, s, \tau, x(\tau, s))] h(\tau, s) d\lambda d\tau \\ &= \left\{ \int_0^1 [L_1(x + \lambda h) - L_1(x)] h d\lambda \right\} (t, s). \end{aligned}$$

Since the operator $L_1 : B_R(X) \rightarrow \mathfrak{L}(X)$ satisfies a Lipschitz condition, by assumption, we conclude that

$$\|L(x + h) - L(x) - L_1(x)h\| \leq \int_0^1 \|L_1(x + \lambda h) - L_1(x)\| \|h\| d\lambda = o(\|h\|),$$

which means that $L'(x) = L_1(x)$. The equality $M'(x) = M_1(x)$ is proved similarly ■

Applying Lemma 1 allows us to “find” the constants a and b for the equation (2), where F is given by (12). In fact, the function $h = F'(0)^{-1}F(0)$ satisfies the linear partial integral equation

$$\begin{aligned}
 h(t, s) &= \int_T l_1(t, s, \tau, 0) h(\tau, s) d\tau \\
 &\quad - \int_S m_1(t, s, \sigma, 0) h(t, \sigma) d\sigma - \int_T \int_S n_1(t, s, \tau, \sigma, 0) h(\tau, \sigma) d\sigma d\tau \quad (26) \\
 &= g(t, s)
 \end{aligned}$$

where

$$g(t, s) = - \int_T l(t, s, \tau, 0) d\tau - \int_S m(t, s, \sigma, 0) d\sigma - \int_T \int_S n(t, s, \tau, \sigma, 0) d\sigma d\tau. \quad (27)$$

Suppose that the (unique!) solution of equation (26) may be written in the form

$$\begin{aligned}
 h(t, s) &= g(t, s) + \int_T r_l(t, s, \tau) g(\tau, s) d\tau \\
 &\quad + \int_S r_m(t, s, \sigma) g(t, \sigma) d\sigma + \int_T \int_S r_n(t, s, \tau, \sigma) g(\tau, \sigma) d\sigma d\tau \quad (28)
 \end{aligned}$$

with some resolvent kernels r_l , r_m and r_n defined through the kernels l_1 , m_1 and n_1 , respectively. Then the constant a in (5) is of course nothing else than the norm $\|h\|_X$ of the function (28) in the space X . Since the explicit form of these resolvent kernels is in general hard to find, one usually looks for a representation of the form $F'(0) = T - E$, where T is boundedly invertible and the norm of E in $\mathcal{L}(X)$ is small. The elementary equality $T - E = T(I - T^{-1}E)$ implies then that, under the hypotheses of Lemma 1, the estimates

$$\begin{aligned}
 a &= \|F'(0)^{-1}F(0)\| \leq \frac{\|T^{-1}F(0)\|}{1 - \|E\|\|T^{-1}\|} \\
 b &= \|F'(0)^{-1}\| \leq \|T^{-1}\| \|(I - T^{-1}E)^{-1}\| \leq \|T^{-1}\| \sum_{k=0}^{\infty} \|T^{-1}E\|^k \leq \frac{\|T^{-1}\|}{1 - \|E\|\|T^{-1}\|}
 \end{aligned}$$

are true. In this way, we have proved the following

Lemma 2. *Suppose that the generalized superposition operators L^* , M^* and N^* given by (13) - (15) act from $\mathfrak{R}_T(X, X)$, $\mathfrak{R}_S(X, X)$ and $\mathfrak{R}_{T \times S}(X, X)$, respectively, satisfy a Lipschitz condition, and $F'(0) = T - E$, where T is invertible in $\mathcal{L}(X)$ and $\|E\| \leq \varepsilon$. Then the estimates*

$$a \leq \frac{\|T^{-1}F(0)\|}{1 - \varepsilon\|T^{-1}\|} \quad \text{and} \quad b \leq \frac{\|T^{-1}\|}{1 - \varepsilon\|T^{-1}\|}$$

hold.

Under the conditions of Lemma 2 it is natural to write the linear operators T and E also as sums of (partial) integral operators like (9) - (11). Various conditions for the "smallness" of the norm of E may then be found in [4 - 7]. On the other hand, the invertibility of T is often not easy to verify, except for particular cases. Assume, for instance, that T has the special form

$$Th(t, s) = h(t, s) - \int_T \phi(t, \tau) h(\tau, s) d\tau - \int_S \psi(s, \sigma) h(t, \sigma) d\sigma - c \int_T \int_S \phi(t, \tau) \psi(s, \sigma) h(\tau, \sigma) d\sigma d\tau,$$

where $\phi : T \times T \rightarrow \mathbb{R}$, $\psi : S \times S \rightarrow \mathbb{R}$ and $c \in \mathbb{R}$ are given. Then T is invertible if and only if

$$1 \notin \sigma(\Phi) + \sigma(\Psi) + c\sigma(\Phi)\sigma(\Psi), \tag{29}$$

where σ denotes the spectrum, Φ and Ψ are the integral operators generated by the kernels ϕ and ψ , respectively.¹⁾ Moreover, in some cases the operator T^{-1} may be expressed explicitly through the operators Φ and Ψ . For example, in case $c = 0$ the formula

$$T^{-1} = -\frac{1}{4\pi^2} \int_{\Gamma_\Psi} \int_{\Gamma_\Phi} \frac{(\Phi - \xi I)^{-1} \otimes (\Psi - \eta I)^{-1}}{1 - \xi - \eta} d\xi d\eta$$

holds, where Γ_Φ and Γ_Ψ are simple closed contours around $\sigma(\Phi)$ and $\sigma(\Psi)$, respectively. If the kernels ϕ and ψ are symmetric or degenerate, the relation (29) may be verified by standard schemes.

3. Lipschitz conditions for partial Uryson operators

The crucial assumption in Lemma 1 is of course the Lipschitz condition for the operators L^* , M^* and N^* defined by (13) - (15), respectively. In this section we take a closer look to this condition from a general point of view. More information in some specific function spaces which arise frequently in applications will be given in subsequent sections.

In what follows, we study all operators in "ideal spaces" of functions over $T \times S$. Recall that a Banach space X of measurable or continuous functions $x : T \times S \rightarrow \mathbb{R}$ is called *ideal space* if the norm on X is monotone, i.e. from $|u(t, s)| \leq |v(t, s)|$ a.e. on $T \times S$ and $u, v \in X$ it follows that $\|u\| \leq \|v\|$.

Suppose that the three kernel functions l, m and n in (1) also have second partial derivatives in the last argument

$$\begin{aligned} l_2(t, s, \tau, u) &= \frac{\partial^2 l}{\partial u^2}(t, s, \tau, u) \\ m_2(t, s, \sigma, u) &= \frac{\partial^2 m}{\partial u^2}(t, s, \sigma, u) \\ n_2(t, s, \tau, \sigma, u) &= \frac{\partial^2 n}{\partial u^2}(t, s, \tau, \sigma, u) \end{aligned}$$

¹⁾ A detailed proof of formula (29) can be found, e.c., in the paper of T. Ichinose: *Spectral properties of tensor products of linear operators*. Trans. Amer. Math. Soc. 235 (1978), 75 - 113.

and that

$$|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)| \leq \tilde{l}_2(t, s, \tau, w) |u_1 - u_2| \tag{30}$$

$$|m_1(t, s, \sigma, u_1) - m_1(t, s, \sigma, u_2)| \leq \tilde{m}_2(t, s, \sigma, w) |u_1 - u_2| \tag{31}$$

$$|n_1(t, s, \tau, \sigma, u_1) - n_1(t, s, \tau, \sigma, u_2)| \leq \tilde{n}_2(t, s, \tau, \sigma, w) |u_1 - u_2| \tag{32}$$

for $|u_1|, |u_2| \leq w$. Here

$$\begin{aligned} \tilde{l}_2(t, s, \tau, w) &= \sup_{|u| \leq w} |l_2(t, s, \tau, u)| \\ \tilde{m}_2(t, s, \sigma, w) &= \sup_{|u| \leq w} |l_2(t, s, \sigma, u)| \\ \tilde{n}_2(t, s, \tau, \sigma, w) &= \sup_{|u| \leq w} |l_2(t, s, \tau, \sigma, u)|, \end{aligned}$$

respectively. It is then natural to state the Lipschitz condition (4) in terms of the generalized superposition operators

$$L^{**}(x)(t, s, \tau) = l_2(t, s, \tau, x(\tau, s)) \tag{33}$$

$$M^{**}(x)(t, s, \sigma) = m_2(t, s, \sigma, x(t, \sigma)) \tag{34}$$

$$N^{**}(x)(t, s, \tau, \sigma) = n_2(t, s, \tau, \sigma, x(\tau, \sigma)) \tag{35}$$

which are second order analogues to the operators L^*, M^* and N^* defined by (13) - (15), respectively. As a matter of fact, we want to replace the generalized superposition operators L^{**}, M^{**} and N^{**} given by (33) - (35), respectively, by the usual superposition operators

$$\tilde{L}^{**}(x)(t, s, \tau) = l_2(t, s, \tau, x(t, s, \tau)) \tag{36}$$

$$\tilde{M}^{**}(x)(t, s, \sigma) = m_2(t, s, \sigma, x(t, s, \sigma)) \tag{37}$$

$$\tilde{N}^{**}(x)(t, s, \tau, \sigma) = n_2(t, s, \tau, \sigma, x(t, s, \tau, \sigma)) \tag{38}$$

defined on functions over $T \times S \times T$, $T \times S \times S$ and $T \times S \times T \times S$, respectively. To make this precise, we need some special definitions. First, given an ideal space X of real functions over $T \times S$, we denote by

$$X_T, \quad X_S, \quad X_{T \times S}$$

the spaces of functions

$$p: T \times S \times T \rightarrow \mathbb{R}, \quad q: T \times S \times S \rightarrow \mathbb{R}, \quad r: T \times S \times T \times S \rightarrow \mathbb{R}$$

defined by the norms

$$\begin{aligned} \|p\|_{X_T} &= \inf \left\{ \|\hat{p}\|_X : |p(t, s, \tau)| \leq \hat{p}(\tau, s) \text{ for } \hat{p} \in X \right\} \\ \|q\|_{X_S} &= \inf \left\{ \|\hat{q}\|_X : |q(t, s, \sigma)| \leq \hat{q}(t, \sigma) \text{ for } \hat{q} \in X \right\} \\ \|r\|_{X_{T \times S}} &= \inf \left\{ \|\hat{r}\|_X : |r(t, s, \tau, \sigma)| \leq \hat{r}(\tau, \sigma) \text{ for } \hat{r} \in X \right\}, \end{aligned}$$

respectively, where the inequalities in these definitions are considered everywhere on $T \times S \times T$, $T \times S \times S$ and $T \times S \times T \times S$, respectively. Further, we define spaces

$$\tilde{\mathfrak{R}}_T(X, X), \quad \tilde{\mathfrak{R}}_S(X, X), \quad \tilde{\mathfrak{R}}_{T \times S}(X, X)$$

of measurable functions

$$p : T \times S \times T \rightarrow \mathbb{R}, \quad q : T \times S \times S \rightarrow \mathbb{R}, \quad r : T \times S \times T \times S \rightarrow \mathbb{R}$$

by the norms

$$\|p\|_{\tilde{\mathfrak{R}}_T(X, X)} = \sup_{\|z\|_X, \|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T |p(t, s, \tau)x(\tau, s)h(\tau, s)| d\tau \right\|_X \quad (39)$$

$$\|q\|_{\tilde{\mathfrak{R}}_S(X, X)} = \sup_{\|z\|_X, \|h\|_X \leq 1} \left\| (t, s) \mapsto \int_S |q(t, s, \sigma)x(t, \sigma)h(t, \sigma)| d\sigma \right\|_X \quad (40)$$

$$\|r\|_{\tilde{\mathfrak{R}}_{T \times S}(X, X)} = \sup_{\|z\|_X, \|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T \int_S |r(t, s, \tau, \sigma)x(\tau, \sigma)h(\tau, \sigma)| d\sigma d\tau \right\|_X \quad (41)$$

respectively.

The construction described above is not as trivial as it seems to be. For example, if X is a Lebesgue space L_p with $p \geq 2$, one can easily see that the spaces $\tilde{\mathfrak{R}}_T(X, X)$, $\tilde{\mathfrak{R}}_S(X, X)$ and $\tilde{\mathfrak{R}}_{T \times S}(X, X)$ consist of kernels of linear integral operators (19) - (21) acting from $L_{\frac{p}{2}}$ into L_p ; in the case $1 \leq p < 2$ these spaces strongly degenerate (i.e. contain only the zero function). More generally, if X is an Orlicz space L_M (see, e.g., [20]), the spaces $\tilde{\mathfrak{R}}_T(X, X)$, $\tilde{\mathfrak{R}}_S(X, X)$ and $\tilde{\mathfrak{R}}_{T \times S}(X, X)$ contain kernels of linear integral operators (19) - (21) acting from L_N into L_M , where the Orlicz space L_N is generated by any Young function N satisfying for some $c > 0$ the asymptotic condition

$$\limsup_{u \rightarrow \infty} \frac{N(cu^2)}{M(u)} < \infty.$$

Below we use the usual abbreviation $(u \vee v)(t, s) = \sup \{u(t, s), v(t, s)\}$ and put

$$\gamma(r) = \sup \{ \|u \vee v\| : \|u\|, \|v\| \leq r \} \quad (r > 0).$$

Moreover, given a bounded nonlinear operator F between two normed spaces, by

$$\mu(F; \rho) = \sup \{ \|Fx\| : \|x\| \leq \rho \} \quad (\rho > 0)$$

we denote the *growth function* of F .

Lemma 3. *Let X be a Banach space of real functions over $T \times S$. Then the following holds:*

(a) *If the superposition operator \tilde{L}^{**} in (36) is bounded from X_T into $\tilde{\mathfrak{R}}_T(X, X)$, then the generalized superposition operator L^* given by (13) satisfies the Lipschitz condition*

$$\|L^*(x_1) - L^*(x_2)\|_{\mathfrak{R}_T(X, X)} \leq \mu(\tilde{L}^{**}; \gamma(r)) \|x_1 - x_2\|_X \tag{42}$$

where $\|x_1\|_X, \|x_2\|_X \leq r$.

(b) *If the superposition operator \tilde{M}^{**} in (37) is bounded from X_S into $\tilde{\mathfrak{R}}_S(X, X)$, then the generalized superposition operator M^* given by (14) satisfies the Lipschitz condition*

$$\|M^*(x_1) - M^*(x_2)\|_{\mathfrak{R}_S(X, X)} \leq \mu(\tilde{M}^{**}; \gamma(r)) \|x_1 - x_2\|_X \tag{43}$$

where $\|x_1\|_X, \|x_2\|_X \leq r$.

(c) *If the superposition operator \tilde{N}^{**} in (38) is bounded from $X_{T \times S}$ into $\tilde{\mathfrak{R}}_{T \times S}(X, X)$, then the generalized superposition operator N^* given by (15) satisfies the Lipschitz condition*

$$\|N^*(x_1) - N^*(x_2)\|_{\mathfrak{R}_{T \times S}(X, X)} \leq \mu(\tilde{N}^{**}; \gamma(r)) \|x_1 - x_2\|_X \tag{44}$$

where $\|x_1\|_X, \|x_2\|_X \leq r$.

Proof. Let us prove (42), the estimates (43) and (44) are proved in the same way. For any $x_1, x_2 \in X$ with $\|x_1\|_X, \|x_2\|_X \leq r$ we have, by (30),

$$\begin{aligned} & \|L^*(x_1) - L^*(x_2)\|_{\mathfrak{R}_T(X, X)} \\ &= \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T |l_1(t, s, \tau, x_1(\tau, s)) - l_1(t, s, \tau, x_2(\tau, s))| |h(\tau, s)| d\tau \right\|_X \\ &\leq \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T |\tilde{l}_2(t, s, \tau, |x_1(\tau, s)| \vee |x_2(\tau, s)|) \right. \\ &\quad \left. \times |x_1(\tau, s) - x_2(\tau, s)| |h(\tau, s)| d\tau \right\|_X. \end{aligned}$$

By the Krasnosel'skij-Ladyzhenskij lemma (see, e.g., [10]) there exists a function $w : T \times S \times T \rightarrow \mathbb{R}$ such that $|w(t, s, \tau)| \leq |x_1(\tau, s)| \vee |x_2(\tau, s)|$ and

$$|l_2(t, s, \tau, w(t, s, \tau))| = \tilde{l}_2(t, s, \tau, |x_1(\tau, s)| \vee |x_2(\tau, s)|).$$

Thus,

$$\begin{aligned} & \|L^*(x_1) - L^*(x_2)\|_{\mathfrak{R}_T(X, X)} \\ &\leq \sup_{\|h\|_X \leq 1} \left\| (t, s) \mapsto \int_T |l_2(t, s, \tau, w(t, s, \tau))| |x_1(\tau, s) - x_2(\tau, s)| |h(\tau, s)| d\tau \right\|_X. \end{aligned}$$

This implies (42), by the definition of $\gamma(r)$ and $\mu(\tilde{L}^{**}; \gamma(r))$ ■

4. The case $X = C(T \times S)$

In many cases, Lemma 3 is sufficient to find the constants a and b given by (5) and (6), respectively, and the function $k = k(r)$ from (4) in the space C explicitly. Consider the scalar functions

$$k_l(r) = \sup_{(t,s) \in T \times S} \int_T \sup_{|u_1|, |u_2| \leq r} \frac{|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)|}{|u_1 - u_2|} d\tau \tag{45}$$

$$k_m(r) = \sup_{(t,s) \in T \times S} \int_S \sup_{|u_1|, |u_2| \leq r} \frac{|m_1(t, s, \sigma, u_1) - m_1(t, s, \sigma, u_2)|}{|u_1 - u_2|} d\sigma \tag{46}$$

$$k_n(r) = \sup_{(t,s) \in T \times S} \int_T \int_S \sup_{|u_1|, |u_2| \leq r} \frac{|n_1(t, s, \tau, \sigma, u_1) - n_1(t, s, \tau, \sigma, u_2)|}{|u_1 - u_2|} d\sigma d\tau. \tag{47}$$

These functions are finite for $r \leq R$ if and only if the operators L^*, M^* and N^* given by (13) - (15) satisfy a local Lipschitz condition in C . Moreover, the numbers given in (45) - (47) are then the minimal Lipschitz constants for the corresponding operators on $\bar{B}_r(C)$. This may be stated in a more precise and convenient way:

Theorem 1. *The operators L_1, M_1 and N_1 given by (22) - (24) satisfy a Lipschitz condition on $\bar{B}_R(C)$ if and only if the three kernel functions l, m and n in (1) have second partial derivatives in the last argument*

$$\begin{aligned} l_2(t, s, \tau, u) &= \frac{\partial^2 l(t, s, \tau, u)}{\partial u^2} \\ m_2(t, s, \sigma, u) &= \frac{\partial^2 m(t, s, \sigma, u)}{\partial u^2} \\ n_2(t, s, \tau, \sigma, u) &= \frac{\partial^2 n(t, s, \tau, \sigma, u)}{\partial u^2} \end{aligned}$$

for all $(t, s) \in T \times S$ and almost all $(\tau, u) \in T \times \mathbb{R}$, $(\sigma, u) \in S \times \mathbb{R}$ and $(\tau, \sigma, u) \in T \times S \times \mathbb{R}$, respectively, and the three functions

$$\tilde{k}_l(r) = \sup_{(t,s) \in T \times S} \int_T \sup_{|u| \leq r} |l_2(t, s, \tau, u)| d\tau \tag{48}$$

$$\tilde{k}_m(r) = \sup_{(t,s) \in T \times S} \int_S \sup_{|u| \leq r} |m_2(t, s, \sigma, u)| d\sigma \tag{49}$$

$$\tilde{k}_n(r) = \sup_{(t,s) \in T \times S} \int_T \int_S \sup_{|u| \leq r} |n_2(t, s, \tau, \sigma, u)| d\sigma d\tau \tag{50}$$

are finite for $r \leq R$. Moreover, the numbers $\tilde{k}_l(r)$, $\tilde{k}_m(r)$ and $\tilde{k}_n(r)$ are then the minimal Lipschitz constants for the operators L_1, M_1 and N_1 , respectively, on $\bar{B}_r(C)$. Finally, the minimal Lipschitz constant $k(r)$ for the operator F' by (25) on $\bar{B}_r(C)$ satisfies the two-sided estimate

$$\max \{ \tilde{k}_l(r), \tilde{k}_m(r), \tilde{k}_n(r) \} \leq k(r) \leq \tilde{k}_l(r) + \tilde{k}_m(r) + \tilde{k}_n(r). \tag{51}$$

Proof. The proof for the integral operator N_1 given by (24) is contained in [1]. Let us prove the assertion for the partial integral operator L_1 given by (22), the proof for the operator M_1 given by (23) is similar. By what has been observed before, for this it is necessary and sufficient to show that the function $k_l(r)$ given by (45) is finite for $r \leq R$.

Suppose first that $k_l(r)$ is finite for all $r \leq R$. This means that

$$\int_T \sup_{|u_1|, |u_2| \leq r} \frac{|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)|}{|u_1 - u_2|} d\tau \leq k_l(r) < \infty$$

for all $(t, s) \in T \times S$, and hence the function $\lambda_{t,s}$ given by

$$\lambda_{t,s}(\tau) = \sup_{|u_1|, |u_2| \leq r} \frac{|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)|}{|u_1 - u_2|}$$

is finite a.e. on T . Since

$$|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)| \leq \lambda_{t,s}(\tau) |u_1 - u_2|$$

for $|u_1|, |u_2| \leq r$, the map $u \mapsto l_1(t, s, \tau, u)$ is absolutely continuous. Consequently, the partial derivative $l_2 = \frac{\partial l_1}{\partial u}$ exists for almost all u and satisfies $\sup_{|u| \leq r} |l_2(t, s, \tau, u)| \leq \lambda_{t,s}(\tau)$. But this implies

$$\tilde{k}_l(r) \leq \sup_{(t,s) \in T \times S} \int_T \lambda_{t,s}(\tau) d\tau < \infty.$$

Conversely, suppose that the function $\tilde{k}_l(r)$ given by (48) is finite for $r \leq R$. This implies that the function $\tilde{\lambda}_{t,s}$ given by $\tilde{\lambda}_{t,s}(\tau) = \sup_{|u| \leq r} |l_2(t, s, \tau, u)|$ is finite a.e. on T , for all $(t, s) \in T \times S$. Consequently, for $|u_1|, |u_2| \leq r$ we have

$$|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)| = \left| \int_{u_2}^{u_1} l_2(t, s, \tau, u) du \right| \leq \tilde{\lambda}_{t,s}(\tau) |u_1 - u_2|.$$

We conclude that

$$k_l(r) \leq \sup_{(t,s) \in T \times S} \int_T \tilde{\lambda}_{t,s}(\tau) d\tau < \infty.$$

Of course, the proof shows also that $\tilde{k}_l(r) = k_l(r)$ for all $r \leq R$ ■

Theorem 1 implies, in particular, that the estimate $k(r) \leq \tilde{k}_l(r) + \tilde{k}_m(r) + \tilde{k}_n(r)$ holds for the Lipschitz constant $k(r)$ in (4) in case $X = C$. The problem of calculating the numbers a and b given by (5) and (6), respectively, is quite easy. In fact, suppose that the partial integral equation (26) has a unique solution (28) in the space $X = C$. From the definition of the norm in the space C we obtain then the equality

$$a = \sup_{(t,s) \in T \times S} \left| g(t, s) + \int_T r_l(t, s, \tau) g(\tau, s) d\tau + \int_S r_m(t, s, \sigma) g(t, \sigma) d\sigma + \int_T \int_S r_n(t, s, \tau, \sigma) g(\tau, \sigma) d\sigma d\tau \right|, \tag{52}$$

and from explicit formulas for the norm of a linear partial integral operator in the space C (see [5, 6]) the equality

$$b = 1 + \sup_{(t,s) \in T \times S} \left[\int_T |r_l(t, s, \tau)| d\tau + \int_S |r_m(t, s, \sigma)| d\sigma + \int_T \int_S |r_n(t, s, \tau, \sigma)| d\sigma d\tau \right] \tag{53}$$

where g is defined by (27), and r_l, r_m and r_n are the resolvent kernels corresponding to l_1, m_1 and n_1 , respectively.

The resolvent kernels r_l, r_m and r_n are in general difficult to compute explicitly. An exceptional case is that of degenerate kernels. We illustrate this by means of the following very elementary example.

Example 1. Let $S = T = [0, 1]$ and

$$l(t, s, \tau, u) = \lambda(u), \quad m(t, s, \sigma, u) = \mu(u), \quad n(t, s, \tau, \sigma, u) \equiv 0$$

where λ and μ are real C^2 -functions with $\lambda'(0) \neq 1, \mu'(0) \neq 1$ and $\lambda'(0) + \mu'(0) \neq 1$. For any $g \in C([0, 1] \times [0, 1])$ equation (26) has then the unique solution

$$h(t, s) = g(t, s) + \frac{\lambda'(0)}{1 - \lambda'(0)} \int_0^1 g(\tau, s) d\tau + \frac{\mu'(0)}{1 - \mu'(0)} \int_0^1 g(t, \sigma) d\sigma + \frac{\lambda'(0)\mu'(0)(2 - \lambda'(0) - \mu'(0))}{(1 - \lambda'(0))(1 - \mu'(0))(1 - \lambda'(0) - \mu'(0))} \int_0^1 \int_0^1 g(\tau, \sigma) d\sigma d\tau.$$

In particular, since $g(t, s) \equiv -[\lambda(0) + \mu(0)]$ in this case, we get here the constant solution $h(t, s) \equiv \frac{g(t,s)}{1 - \lambda'(0) - \mu'(0)}$. Putting this into (52) and (53) yields

$$a = \left| \frac{\lambda(0) + \mu(0)}{1 - \lambda'(0) - \mu'(0)} \right|$$

$$b = 1 + \left| \frac{\lambda'(0)}{1 - \lambda'(0)} \right| + \left| \frac{\mu'(0)}{1 - \mu'(0)} \right| + \left| \frac{\lambda'(0)\mu'(0)(2 - \lambda'(0) - \mu'(0))}{(1 - \lambda'(0))(1 - \mu'(0))(1 - \lambda'(0) - \mu'(0))} \right|.$$

The Lipschitz constant $k = k(r)$ in (4) may in turn be estimated by

$$k(r) \leq \sup_{|u| \leq r} |\lambda''(u)| + \sup_{|u| \leq r} |\mu''(u)|.$$

This gives a sufficiently effective “recipe” for finding the scalar function φ given by (7), and hence for applying the Newton-Kantorovich method to equation (1) in this special case. To be more specific, suppose that the functions λ and μ are quadratic polynomials

$$\lambda(u) = \lambda_2 u^2 + \lambda_1 u + \lambda_0 \quad \text{and} \quad \mu(u) = \mu_2 u^2 + \mu_1 u + \mu_0 \tag{54}$$

which is the simplest nonlinear case. A trivial calculation shows that then

$$a = \left| \frac{\lambda_0 + \mu_0}{1 - \lambda_1 - \mu_1} \right|$$

$$b = 1 + \left| \frac{\lambda_1}{1 - \lambda_1} \right| + \left| \frac{\mu_1}{1 - \mu_1} \right| + \left| \frac{\lambda_1 \mu_1 (2 - \lambda_1 - \mu_1)}{(1 - \lambda_1)(1 - \mu_1)(1 - \lambda_1 - \mu_1)} \right|,$$

and $k(r) = 2(|\lambda_2| + |\mu_2|)$, hence $\varphi(r) = a + cr^2$, where $c = b(|\lambda_2| + |\mu_2|)$. Consequently, the number of real solutions of the fixed point equation (8) depends on the sign of the discriminant $D = 1 - 4ac$.

We remark that this effective calculation also applies to the more general case

$$l(t, s, \tau, u) = a(t)b(s)c(\tau)\lambda(u) \quad \text{and} \quad m(t, s, \sigma, u) = d(t)e(s)f(\sigma)\mu(u)$$

and also to the case of degenerate kernels. Here one may make use of an algorithm proposed by Vitova [23] for solving partial integral equations with degenerate kernels.

5. The case $X = L_\infty(T \times S)$

In rather the same way as in $X = C(T \times S)$, the Lipschitz conditions for the operator F' given by (25) and the operators L_1, M_1 and N_1 given by (22) - (24), respectively, are also equivalent in the space $X = L_\infty(T \times S)$. This may again be analyzed by imposing appropriate conditions on the kernels l_1, m_1 and n_1 and the corresponding operators L^*, M^* and N^* given by (13) - (15), respectively.

For $r > 0$ and $\delta > 0$, let

$$k_l(r, \delta) = \text{ess sup}_{(t,s) \in T \times S} \int_T \sup_{\substack{|\mu_1|, |\mu_2| \leq r \\ |\mu_1 - \mu_2| \leq \delta}} |l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)| d\tau \tag{55}$$

$$k_m(r, \delta) = \text{ess sup}_{(t,s) \in T \times S} \int_S \sup_{\substack{|\mu_1|, |\mu_2| \leq r \\ |\mu_1 - \mu_2| \leq \delta}} |m_1(t, s, \sigma, u_1) - m_1(t, s, \sigma, u_2)| d\sigma \tag{56}$$

$$k_n(r, \delta) = \text{ess sup}_{(t,s) \in T \times S} \int_T \int_S \sup_{\substack{|\mu_1|, |\mu_2| \leq r \\ |\mu_1 - \mu_2| \leq \delta}} |n_1(t, s, \tau, \sigma, u_1) - n_1(t, s, \tau, \sigma, u_2)| d\sigma d\tau. \tag{57}$$

Lemma 4. *The following three conditions are equivalent:*

(a) *The limits*

$$k_l(r) = \lim_{\delta \rightarrow 0} \frac{k_l(r, \delta)}{\delta}, \quad k_m(r) = \lim_{\delta \rightarrow 0} \frac{k_m(r, \delta)}{\delta}, \quad k_n(r) = \lim_{\delta \rightarrow 0} \frac{k_n(r, \delta)}{\delta} \tag{58}$$

are finite for $r \leq R$.

(b) *The operators L_1, M_1 and N_1 given by (22) - (24) satisfy a Lipschitz condition from $B_R(L_\infty)$ into $\mathcal{L}(L_\infty)$.*

(c) The operator F' given by (25) satisfies a Lipschitz condition from $B_R(L_\infty)$ into $\mathfrak{L}(L_\infty)$.

Proof. We prove this again for the function k_l given by (55) and the operator L_1 given by (22). Suppose that assertion (a) holds. For $\varepsilon > 0$ choose $\hat{\delta} > 0$ such that $k_l(\tau, \delta) \leq (k_l(\tau) + \varepsilon)\delta$ for $0 < \delta \leq \hat{\delta}$. By the definition (55) of k_l we get

$$\|L_1(x_1)h - L_1(x_2)h\|_{L_\infty} \leq k_l(\tau, \delta) \|x_1 - x_2\|_{L_\infty}$$

for $\|h\| \leq 1$, $\|x_1\|, \|x_2\| \leq r$ and $\|x_1 - x_2\| \leq \delta$. In fact, for $\|x_1 - x_2\| < \delta' \leq \delta$ and fixed $(t, s) \in T \times S$ we have

$$\begin{aligned} & \left| \int_T \left[l_1(t, s, \tau, x_1(\tau, s)) - l_1(t, s, \tau, x_2(\tau, s)) \right] h(\tau, s) d\tau \right| \\ & \leq \delta' \int_{T(x_1, x_2)} \frac{|l_1(t, s, \tau, x_1(\tau, s)) - l_1(t, s, \tau, x_2(\tau, s))|}{|x_1(\tau, s) - x_2(\tau, s)|} d\tau \\ & \leq \delta' \int_T \sup_{\substack{|u_1|, |u_2| \leq r \\ |u_1 - u_2| \leq \delta}} \frac{|l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)|}{|u_1 - u_2|} d\tau \end{aligned}$$

where we have put

$$T(x_1, x_2) = \{\tau : x_1(\tau, s) \neq x_2(\tau, s)\}.$$

Since $\delta' > \|x_1 - x_2\|$ is arbitrary, this implies that

$$\|L_1(x_1) - L_1(x_2)\|_{\mathfrak{L}(L_\infty)} \leq k_l(\tau, \delta) \|x_1 - x_2\|_{L_\infty}$$

for $\|x_1 - x_2\| \leq \delta$. Now, for arbitrary $x_1, x_2 \in L_\infty$, fix $m \in \mathbb{N}$ such that $\|x_1 - x_2\| \leq m\delta$. Then

$$\begin{aligned} & \|L_1(x_1) - L_1(x_2)\|_{\mathfrak{L}(L_\infty)} \\ & \leq \sum_{j=1}^m \left\| L_1 \left[\left(1 - \frac{j}{m}\right)x_1 + \frac{j}{m}x_2 \right] - L_1 \left[\left(1 - \frac{j-1}{m}\right)x_1 + \frac{j-1}{m}x_2 \right] \right\|_{\mathfrak{L}(L_\infty)} \\ & \leq m(k_l(\tau) + \varepsilon) \frac{\|x_1 - x_2\|}{m} \\ & = (k_l(\tau) + \varepsilon) \|x_1 - x_2\|, \end{aligned}$$

and hence assertion (b) is true. Conversely, suppose that assertion (b) holds. As was shown in [30] (see also [1]), the equality

$$\begin{aligned} & \sup_{\substack{\|x_1\|, \|x_2\| \leq r \\ \|x_1 - x_2\| \leq \delta}} \sup_{\|h\| \leq 1} |L_1(x_1)h(t, s) - L_1(x_2)h(t, s)| \\ & = \int_T \sup_{\substack{|u_1|, |u_2| \leq r \\ |u_1 - u_2| \leq \delta}} |l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)| d\tau \end{aligned}$$

holds in the space L_∞ , and assertion (a) follows by taking L_∞ -norms. The equivalence of assertions (b) and (c) is clear ■

The following statement is parallel to Theorem 1.

Theorem 2. *The operators L_1, M_1 and N_1 given by (22) - (24), respectively, satisfy a Lipschitz condition on $\overline{B}_R(L_\infty)$ if and only if the three kernel functions l, m and n in equation (1) have second partial derivatives in the last argument*

$$\begin{aligned}
 l_2(t, s, \tau, u) &= \frac{\partial^2 l(t, s, \tau, u)}{\partial u^2} \\
 m_2(t, s, \sigma, u) &= \frac{\partial^2 m(t, s, \sigma, u)}{\partial u^2} \\
 n_2(t, s, \tau, \sigma, u) &= \frac{\partial^2 n(t, s, \tau, \sigma, u)}{\partial u^2}
 \end{aligned}$$

for all $(t, s) \in T \times S$ and almost all $(\tau, u) \in T \times \mathbb{R}$, $(\sigma, u) \in S \times \mathbb{R}$, and $(\tau, \sigma, u) \in T \times S \times \mathbb{R}$, respectively, and the three functions

$$\tilde{k}_l(r) = \operatorname{ess\,sup}_{(t,s) \in T \times S} \int_T \sup_{|u| \leq r} |l_2(t, s, \tau, u)| \, d\tau \tag{59}$$

$$\tilde{k}_m(r) = \operatorname{ess\,sup}_{(t,s) \in T \times S} \int_S \sup_{|u| \leq r} |m_2(t, s, \sigma, u)| \, d\sigma \tag{60}$$

$$\tilde{k}_n(r) = \operatorname{ess\,sup}_{(t,s) \in T \times S} \int_T \int_S \sup_{|u| \leq r} |n_2(t, s, \tau, \sigma, u)| \, d\sigma \, d\tau \tag{61}$$

are finite for $r \leq R$. Moreover, the numbers $\tilde{k}_l(r)$, $\tilde{k}_m(r)$ and $\tilde{k}_n(r)$ are then the minimal Lipschitz constants for the operators L_1, M_1 and N_1 given by (22) - (24), respectively, on $\overline{B}_r(L_\infty)$. Finally, the minimal Lipschitz constant $k(r)$ for the operator F' given by (25) on $\overline{B}_r(L_\infty)$ satisfies the two-sided estimate (51).

The example of the operator F given by

$$Fx(t, s) = x(t, s) - \int_0^1 tsx^2(\tau, s) \, d\tau - \int_0^1 (1-t)(1-s)x^2(t, \sigma) \, d\sigma$$

shows that, in general, the equality

$$k(r) = \tilde{k}_l(r) + \tilde{k}_m(r) + \tilde{k}_n(r)$$

is not true.

Theorem 2 gives an effective algorithm for estimating the Lipschitz constant $k(r)$ in (4) in the space $X = L_\infty$. Analogously to what we have done in the preceding section for $X = C$, we may calculate the numbers a and b given by (5) and (6), respectively, in the space $X = L_\infty$. However, the proof is somewhat more technical, so we state this separately as

Theorem 3. Suppose that equation (26) has a unique solution (28) in the space $X = L_\infty$. Then the numbers a and b given by (5) and (6), respectively, may be calculated in L_∞ by means of the formulas

$$a = \operatorname{ess\,sup}_{(t,s) \in T \times S} \left| g(t,s) + \int_T r_l(t,s,\tau)g(\tau,s) d\tau + \int_S r_m(t,s,\sigma)g(t,\sigma) d\sigma + \int_T \int_S r_n(t,s,\tau,\sigma)g(\tau,\sigma) d\sigma d\tau \right| \tag{62}$$

$$b = 1 + \operatorname{ess\,sup}_{(t,s) \in T \times S} \left[\int_T |r_l(t,s,\tau)| d\tau + \int_S |r_m(t,s,\sigma)| d\sigma + \int_T \int_S |r_n(t,s,\tau,\sigma)| d\sigma d\tau \right] \tag{63}$$

where the function g is defined by (27), and r_l, r_m and r_n are the resolvent kernels corresponding to l_1, m_1 and n_1 , respectively.

Proof. Obviously, it suffices to prove equality (63). Denote by $[A] = \sup\{|Ax(t,s)| : \|x\| \leq 1\}$ the abstract norm of a linear operator $A \in \mathfrak{L}(L_\infty)$. Then $[A] = \|[A]\|$, where $|A|$ is the modulus of A , i.e. the minimal positive majorant of A (see, e.g., [8 - 11, 24 - 27]). Consequently, we have $\|[A]\| = \|[A]\| = \|[A]\|$, where all norms are taken in $\mathfrak{L}(L_\infty)$. Now, in [7] is was shown that the operator A given by the right-hand side of (28) has the modulus

$$|A|x(t,s) = x(t,s) + \int_T |r_l(t,s,\tau)|x(\tau,s) d\tau + \int_S |r_m(t,s,\sigma)|x(t,\sigma) d\sigma + \int_T \int_S |r_n(t,s,\tau,\sigma)|x(\tau,\sigma) d\sigma d\tau. \tag{64}$$

Putting $x(t,s) \equiv 1$ in (64), we conclude that $b = \|[A]\|$ is just given by (63) ■

We illustrate the results of this section again by means of Example 1. The constants a and b given by (5) and (6), respectively, may be calculated precisely as in the space $X = C$. The functions \tilde{k}_l, \tilde{k}_m and \tilde{k}_n given by (59) - (61), respectively, have the form

$$\tilde{k}_l(r) = \sup\{|\lambda''(u)| : |u| \leq r\}, \quad \tilde{k}_m(r) = \sup\{|\mu''(u)| : |u| \leq r\}, \quad \tilde{k}_n(r) \equiv 0.$$

For the polynomials λ and μ given by (54) this gives, in particular, $\tilde{k}_l(r) \equiv 2\lambda_2$ and $\tilde{k}_m(r) \equiv 2\mu_2$.

6. The case $X = L_p(T \times S)$ ($1 \leq p < \infty$)

The analysis of the preceding two sections becomes more difficult when passing to the case of the Lebesgue space L_p with $1 \leq p < \infty$. One reason for this is that the unit ball in L_p contains lots of unbounded functions, and therefore one "cannot get rid of the function h " under the integrals in the right-hand sides of the norms (16) - (18). But this is not just a technical problem: in fact, imposing a Lipschitz condition like (4) in L_p may lead to a strong degeneracy of the kernel functions involved! For the integral operator N given by (11), for example, it was shown in [1] that the derivative N' satisfies a Lipschitz condition in L_2 only if the corresponding kernel n_1 satisfies a Lipschitz condition in u , and in L_p for $1 \leq p < 2$ only if n_1 does not depend on u , i.e. the kernel n is linear in u .

We shall show now that the situation is even worse for the partial integral operators L and M given by (9) and (10), respectively: a Lipschitz condition for the derivatives L' and M' necessarily leads to linear kernels for all values of p !

Theorem 4. *The derivatives of the operators L and M given by (9) and (10), respectively, satisfy a Lipschitz condition in $X = L_p(T \times S)$ ($1 \leq p < \infty$) if and only if the corresponding kernel functions l and m are linear in the last argument.*

Proof. We prove the assertion for the operator L' or, what is equivalent by Lemma 1, for the operator L_1 given by (22). Of course, if the kernel l of L is linear in u , the kernel l_1 of L_1 is independent of u , and there is nothing to prove. Suppose that the operator L_1 satisfies a Lipschitz condition in L_p , i.e.

$$\int_S \int_T \left| \int_T [l_1(t, s, \tau, x_1(\tau, s)) - l_1(t, s, \tau, x_2(\tau, s))] h(\tau, s) d\tau \right|^p dt ds \leq k_l^p(r) \|x_1 - x_2\|^p \|h\|^p \quad (\|x_1\|, \|x_2\| \leq r) \tag{65}$$

where all norms are taken in $L_p(T \times S)$. Choosing, in particular, $x_i(t, s) = u_i \chi_D(t) \chi_E(s)$ and $h(t, s) = \chi_D(t) \chi_E(s)$, where $D \subset T$ and $E \subset S$ satisfy $u_i \text{mes} D \text{mes} E \leq r^p$ ($i = 1, 2$) and putting this into (65) yields

$$\frac{1}{\text{mes} E} \int_E \int_T \left| \int_D [l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)] d\tau \right|^p dt ds \leq k_l^p(r) |u_1 - u_2|^p (\text{mes} D)^2 (\text{mes} E). \tag{66}$$

Letting $\text{mes} E$ in (66) tend to zero, we get

$$\int_T \left| \int_D [l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)] d\tau \right|^p dt = 0 \tag{67}$$

for almost all $s \in E$. From (67) it follows in turn that

$$\int_D [l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2)] d\tau = 0$$

for almost all $(t, s) \in T \times E$. Since D is an arbitrary measurable set, we conclude that $l_1(t, s, \tau, u_1) - l_1(t, s, \tau, u_2) = 0$ for almost all $(t, s, \tau) \in T \times S \times T$, and the assertion follows ■

Theorem 4 is, of course, rather disappointing: the Newton-Kantorovich method applies to equation (1) in L_p ($1 \leq p < \infty$) only if the kernels l and m are linear in u . Only for the kernel n we have a larger choice in L_p , provided that $p \geq 2$. Taking into account this degeneracy, we close with another example in $X = L_2$.

Example 2. Let $S = T = [0, 1]$ and $p = 2$. By what has been observed before, this choice of p forces us to choose the kernels l and m linear in u . For example, let

$$\begin{aligned} l(t, s, \tau, u) &= \lambda_1(t)\lambda_2(s)u + \lambda_0(t, s, \tau) \\ m(t, s, \sigma, u) &= \mu_1(t)\mu_2(s)u + \mu_0(t, s, \sigma) \\ n(t, s, \tau, \sigma, u) &\equiv 0. \end{aligned}$$

The function g in (27) is here

$$g(t, s) = - \int_0^1 \lambda_0(t, s, \tau) d\tau - \int_0^1 \mu_0(t, s, \sigma) d\sigma$$

and equation (26) for h takes the form

$$h(t, s) = \lambda_1(t)\lambda_2(s)\phi(s) + \mu_1(t)\mu_2(s)\psi(t) + g(t, s) \tag{68}$$

where we have put

$$\phi(s) = \int_0^1 h(\tau, s) d\tau \quad \text{and} \quad \psi(t) = \int_0^1 h(t, \sigma) d\sigma. \tag{69}$$

Inserting (68) into (69) we arrive at the system

$$\left. \begin{aligned} \phi(s) &= \int_0^1 \lambda_1(\tau)\lambda_2(s)\phi(s) d\tau + \int_0^1 \mu_1(\tau)\mu_2(s)\psi(\tau) d\tau + \int_0^1 g(\tau, s) d\tau \\ \psi(t) &= \int_0^1 \lambda_1(t)\lambda_2(\sigma)\phi(\sigma) d\sigma + \int_0^1 \mu_1(t)\mu_2(\sigma)\psi(t) d\sigma + \int_0^1 g(t, \sigma) d\sigma \end{aligned} \right\}$$

for the unknown functions ϕ and ψ . If we suppose that

$$\alpha(s) = 1 - \lambda_2(s) \int_0^1 \lambda_1(\tau) d\tau \neq 0 \quad \text{and} \quad \beta(t) = 1 - \mu_1(t) \int_0^1 \mu_2(\sigma) d\sigma \neq 0$$

and put

$$\gamma(s) = \int_0^1 g(\tau, s) d\tau \quad \text{and} \quad \delta(t) = \int_0^1 g(t, \sigma) d\sigma,$$

we end up at a system of two scalar equations

$$\left. \begin{aligned} - \left(\int_0^1 \frac{\lambda_1(\tau)\mu_1(\tau)}{\beta(\tau)} d\tau \right) \xi + \eta &= \int_0^1 \frac{\lambda_1(\tau)\delta(\tau)}{\beta(\tau)} d\tau \\ \xi - \left(\int_0^1 \frac{\lambda_2(\sigma)\mu_2(\sigma)}{\alpha(\sigma)} d\sigma \right) \eta &= \int_0^1 \frac{\lambda_2(\sigma)\gamma(\sigma)}{\alpha(\sigma)} d\sigma \end{aligned} \right\}$$

for the unknown real numbers

$$\xi = \int_0^1 \lambda_2(\sigma)\phi(\sigma) d\sigma \quad \text{and} \quad \eta = \int_0^1 \mu_1(\tau)\psi(\tau) d\tau.$$

The last system has a unique solution $(\xi, \eta) \in \mathbb{R}^2$ if and only if

$$\left(\int_0^1 \frac{\lambda_1(\tau)\mu_1(\tau)}{\beta(\tau)} d\tau \right) \left(\int_0^1 \frac{\lambda_2(\sigma)\mu_2(\sigma)}{\alpha(\sigma)} d\sigma \right) \neq 1 \quad (70)$$

and this solution may be used to find (ϕ, ψ) and, consequently, the solution h of equation (26).

In this way, we may find the numbers a and b given by (5) and (6), respectively, by means of well-known upper estimates for the L_2 -norm of a linear integral operator. The Lipschitz constant $k(r)$ in (4) is very easy to compute in this case, since

$$l_1(t, s, \tau, u) = \lambda_1(t)\lambda_2(s) \quad \text{and} \quad m_1(t, s, \sigma, u) = \mu_1(t)\mu_2(s)$$

do not depend on u .

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