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Boundary-Blow-Up Problems in a Fractal Domain

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Abstract. Assume that Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, which satisfies a uniform interior and exterior cone condition. We determine uniform a priori lower and upper bounds for the growth of solutions and their gradients, of the problem $\Delta u(x) = f(u(x))$ $(x \in \Omega)$ with boundary blow-up, where $f(t) = e^t$ or $f(t) = t^p$ with $p \in (1, +\infty)$. The boundary estimates imply existence and uniqueness of a solution of the above problem. For $f(t) = t^p$ with $p \in (1, +\infty)$ the solution is positive. These results are used to construct a solution of the problem when $\Omega \subset \mathbb{R}^2$ is the von Koch snowflake domain.

Keywords: *Blow-up, snowflake, fractals, gradient estimates*

AMS subject classification: Primary 35 J 65, secondary 35 B 05

0. Introduction and notation

Let Ω be an open, connected subset (a domain) of $\mathbb{R}^N$ with $N \geq 2$. The problem to find positive solutions of

$$\left.\begin{aligned} &\Delta u(x) = f(u(x)) \quad \text{for } x \in \Omega \\ &u(x) \to \infty \quad \text{as } x \to \partial\Omega \end{aligned}\right\} \tag{0.1}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function satisfying some conditions, has a long history. L. Bieberbach [6] considered the problem for $f(t) = e^t$ in a planar domain with a smooth boundary, inspired by a problem in Riemannian geometry concerning scalar curvature. Since then several mathematicians have studied problems of type (0.1). A quick review of the history of the problem is given in C. Bandle and M. Essén [2], including references. Here we mention some of these results, which are particularly important to our work. In 1957 J. B. Keller [9] and R. Osserman [12] proved existence of solutions of problem (0.1) for rather general non-linearities f. In the paper [2] by C. Bandle and M. Essén from 1994, the authors presented (in particular) a complete characterization of the boundary behaviour of solutions of problem (0.1) in domains with C^2-boundary. Recently, C. Bandle and M. Marcus [5] published results analogous to those of C. Bandle and M. Essén, with the Laplacean replaced by a more general uniformly elliptic second order differential operator defined on a Riemannian manifold Ω with a sufficiently smooth boundary.

The problem (0.1) has applications in physics. In particular, if Ω is the interior of a hot metal sphere and if $f(t) = e^t$, then u represents the electromagnetic potential in

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Ω (see [7: p. 838]). Further connections with physics may be found in the references in [2].

The underlying idea of the technical calculations in this paper is the following. Assume that Ω has some kind of boundary regularity, e.g. that $\partial\Omega$ satisfies a uniform interior and exterior cone condition (the definition is given in Section 2). We find a solution u_k of the equation $\Delta u = f(u)$ with boundary value k ($k \in \mathbb{Z}_+$) and conclude, by the maximum principle, that the sequence $\{u_k\}_{k\in\mathbb{Z}_+}$ thus obtained is increasing. Next, we solve the problem (0.1) in an interior cone. Finally, a maximum principle argument shows that every function u_k is bounded from above (in the cone) by the interior cone solution and a compactness argument proves the existence of a solution u of the problem (0.1), since the interior cone condition is uniform in Ω.

Typically, upper bounds for the growth of the solution $u = u(x)$ of problem (0.1) as x approaches the boundary of Ω are obtained from the interior cone condition, while lower bounds are obtained from the exterior cone condition. However, there is a rescaling argument (Section 7) which transforms the problem of determining the boundary blow-up rate of ∇u to the corresponding problem in an infinite cone, which uses the interior cone condition only. It turns out that this rescaling argument provides both upper and lower bounds of u too.

The results proved by the above technique (Sections 1 - 7) are adapted to solve the problem (0.1) when Ω is the bounded domain whose boundary is the von Koch snowflake curve. A crucial geometric property in this context is that there exists a dense subset of $\partial\Omega$ which meets a semi-uniform interior and exterior cone condition (Section 8).

The solution of the problem (0.1) is unique when Ω meets a uniform interior and exterior cone condition, but our uniqueness proof fails when Ω is the von Koch snowflake domain. The proof technique is taken from [2]. Uniqueness results for the problem (0.1) in certain non-smooth domains have also been obtained by A. C. Lazer and P. J. McKenna [10], and by M. Marcus and L. Véron [11] who also proved blow-up estimates for $f(t) = t^p$ with $p \in (1, +\infty)$.

Throughout the paper we use a result from [5], which we state here. First we have to introduce some notation and assumptions. For functions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ we define the semilinear second order differential operator L by

$$Lu(x) = \sum_{i,j=1}^{N} a_{ij}(x)\, u_{ij}(x) + \sum_{k=1}^{N} b_k(x)\, u_k(x)$$

where u_i denotes differentiation with respect to the i-th component of $x \in \Omega$. It is assumed that L is uniformly elliptic with $a_{ij} = a_{ji}$ and $a_{ij}, b_k \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$. Consider the problem

$$\left.\begin{array}{ll} Lu(x) = g(x, u(x)) & \text{for } x \in \Omega \\ u(x) \to \infty & \text{as } x \to \partial\Omega \end{array}\right\} \qquad (0.2)$$

where $\partial\Omega$ satisfies a uniform interior and exterior sphere condition and g is continuous in its domain of definition. Suppose that there are two continuous functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \bar{\Omega} \to \mathbb{R}_+$ such that the following four conditions $(\mathrm{G})_1$ and $(\mathrm{F})_1$ - $(\mathrm{F})_3$ hold:

(G)$_1$ $\lim_{t\to+\infty} \frac{g(x,t)}{f(t)} = h(x)$ uniformly in $\bar{\Omega}$.

(F)$_1$ $f \in C^1(\mathbb{R}_+)$, $f(t) \to \infty$ as $t \to +\infty$, and $f(t), f'(t) \geq 0$ for $t \geq t' \geq 0$ for some t'.

Define

$$t_0 = \inf\left\{\tau > t' : f(t) \leq f(\tau) \text{ for every } t \leq \tau\right\}.$$

Condition (F)$_1$ implies that $t_0 < \infty$. Let F denote the primitive function of f with $F(t_0) = 0$. The third condition is the follwing one.

(F)$_2$ $\Psi(t) = \int_t^\infty [2F(s)]^{-1/2} ds$ exists for all $t \geq t_0$.

Finally we assume

(F)$_3$ $\lim_{t\to+\infty} \frac{\Psi(\beta t)}{\Psi(t)} > 1$ for all $\beta \in (0,1)$.

Then the following theorem holds.

Theorem 0.1. *Assume that conditions* (G)$_1$ *and* (F)$_1$ - (F)$_3$ *hold. Then there exists at least one solution of problem* (0.2). *Let* Φ *denote the inverse of* Ψ. *Then the boundary behaviour of any solution of problem* (0.2) *is given by*

$$\lim_{x\to\partial\Omega} \frac{u(x)}{\Phi\left(\sqrt{h(x)}\delta(x)\right)} = 1 \tag{0.3}$$

uniformly in Ω. *Here* $\delta(x)$ *denotes the distance from* $x \in \Omega$ *to the boundary* $\partial\Omega$, *in the metric* $ds^2 = b_{ij}(x)\,dx_i dx_j$, *where* $(b_{ij}(x))$ *is the inverse matrix of* $(a_{ij}(x))$.

1. Solutions of $\Delta u = u^p$ in an open cone

Let $N \geq 2$ and let Ω_{N-1} be a domain in S^{N-1} with a C^2-boundary, where S^{N-1} denotes the unit sphere in $\mathbb{R}^N$. For $x \in \mathbb{R}^N$ let (r, θ) denote the polar coordinates of x, i.e. $r = |x|$ and $\theta \in S^{N-1}$.

Definition 1.1 (see [3]). Let Ω_{N-1} be as above. We call a set C defined by

$$C = \left\{x \in \mathbb{R}^N : x \text{ has polar coordinates } (r,\theta) \text{ with } r > 0 \text{ and } \theta \in \Omega_{N-1}\right\}$$

an *open cone* in $\mathbb{R}^N$.

In this section we are interested in solutions of the problem

(P) $\Delta u(x) = u^p(x)$ for $x \in C$
$u(x) \to \infty$ as $x \to \partial C$

for $p \in (1, +\infty)$. In particular we prove the following

Theorem 1.2. *Let $1 < p < +\infty$ if $N = 2$ and let $1 < p < \frac{N}{N-2}$ if $N \geq 3$. Then there exists a positive solution $u(x) \equiv u(r,\theta) = r^{-\frac{2}{p-1}}\alpha(\theta)$ of problem* (P)*, such that $\alpha(\theta)$ solves the problem*

$$\left.\begin{aligned} &\Delta_\theta \alpha(\theta) = \alpha^p(\theta) + C_{N,p} \cdot \alpha(\theta) \quad \text{for } \theta \in \Omega_{N-1} \\ &\alpha(\theta) \to \infty \quad \text{as } \theta \to \partial\Omega_{N-1} \end{aligned}\right\} \tag{1.1}$$

where $C_{N,p} = -\frac{2}{p-1} \cdot (2 - N + \frac{2}{p-1})$ and Δ_θ denotes the Laplace-Beltrami operator on the unit sphere S^{N-1} of $\mathbb{R}^N$.

Proof. We have

$$\Delta u(r,\theta) = \frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{N-1}{r} \cdot \frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2} \cdot \Delta_\theta u(r,\theta).$$

Suppose $u(r,\theta) = r^q \cdot \alpha(\theta)$ solves our problem. Substitution in (P) gives

$$r^{q-2} \cdot \Big[q(q + N - 2) \cdot \alpha(\theta) + \Delta_\theta \alpha(\theta)\Big] = r^{pq} \cdot \alpha^p(\theta).$$

With the choice $q = -\frac{2}{p-1}$ the r-dependence vanishes and the remaining equation is exactly the equation of problem (1.1) ∎

Let us investigate some properties of problem (1.1). First, note that the Laplace-Beltrami operator Δ_θ is uniformly elliptic in Ω_{N-1}. Next, we see that the right-hand member of our equation may be written as $g(\alpha(\theta))$, where $g(t) = t^p + C_{N,p} \cdot t$ for all $t \geq 0$. The function g satisfies $\lim_{t\to\infty} \frac{g(t)}{t^p} = 1$. Hence our equation satisfies the conditions $(G)_1$ and $(F)_1$ - $(F)_3$ with $f(t) = t^p$ for all $t \geq 0$, and Theorem 0.1 allows us to conclude that there exists an $\alpha(\theta)$ with the desired properties.

Note that $g(t)$ is allowed to be negative for $t \leq T$, for some finite T, since in the proof of Theorem 0.1 the maximum principle is used close to the boundary of Ω, where u is large enough for $g(u) > 0$ to hold.

In view of an elementary calculation of the function Φ corresponding to f, we get from Theorem 1.2 and Theorem 0.1 the following

Corollary 1.3. *The solution $u(x) \equiv u(r,\theta) = r^{-\frac{2}{p-1}} \cdot \alpha(\theta)$ of problem* (P) *given by Theorem* 1.2 *satisfies for every $r > 0$*

$$\lim_{\theta \to \partial\Omega_{N-1}} u(r,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot r^{\frac{2}{p-1}} \cdot \delta^{\frac{2}{p-1}}(\theta) = 1 \tag{1.2}$$

where $\delta(\theta)$ denotes the distance from $\theta \in \Omega_{N-1}$ to $\partial\Omega_{N-1}$ in the metric of the unit sphere S^{N-1} of $\mathbb{R}^N$.

Of course, the idea to separate variables in a cone as in the proof of Theorem 1.2 is well-known. For example, C. Bandle and H. A. Levine [4] used this technique when solving a reaction-diffusion problem in a sectorial domain.

Notice that although we cannot construct a solution of problem (1.1) with separated variables when $p \geq \frac{N}{N-2}$, it is easy to find a supersolution of this form, with the boundary behaviour (1.2): Pick $\alpha(\theta)$ which solves $\Delta_\theta \alpha(\theta) = \alpha^p(\theta)$ with boundary blow-up. Theorem 0.1 says that such a function exists.

Next, we will prove a result (Proposition 1.8) which may be used to study problem (0.1) in more general domains than an open cone C. However, first we have to state another definition and prove some preliminary results.

Definition 1.4. Let $R > 0$ be given and let C be an open cone in $\mathbb{R}^N$. Define a cut-off open *cone* C_R in $\mathbb{R}^N$ by

$$C_R = \{x \in C : r = |x| < R\}.$$

Lemma 1.5. *Suppose that $\phi \in C^2(C_R)$ solves the problem*

$$\left.\begin{array}{ll}\Delta\phi(x) \leq \phi^p(x) & \text{for } x \in C_R \\ \phi(x) \to \infty & \text{as } x \to \partial C_R.\end{array}\right\} \tag{1.3}$$

If $u \in C^2(C_R) \cap C^0(\bar{C}_R)$ is any solution of the problem

$$\Delta u(x) = u^p(x) \quad \text{for } x \in C_R,$$

then we have $u(x) \leq \phi(x)$ for all $x \in C_R$.

Proof. Let $v = u - \phi$. Suppose there is a point $x_0 \in C_R$ such that $v(x_0) > 0$. Since $v(x) \to -\infty$ as $x \to \partial C_R$, this implies that there must be a point $y_0 \in C_R$ where v attains its positive maximum. Hence there is an open neighbourhood $B_\epsilon(y_0)$ of y_0 such that $v(y) > 0$ for all $y \in B_\epsilon(y_0)$. So we have

$$\Delta v(y) = \Delta u(y) - \Delta\phi(y) \geq u^p(y) - \phi^p(y) > 0$$

for every $y \in B_\epsilon(y_0)$, and v is subharmonic in $B_\epsilon(y_0)$. However, this contradicts the fact that v attains a maximum in y_0. Thus there is no $x_0 \in C_R$ such that $v(x_0) > 0$ and the lemma is proved ∎

Clearly, Lemma 1.5 still holds when C_R is replaced by any bounded domain Ω. An obvious modification of the proof allows us to treat the case $u \in C^2(\Omega)$ with boundary blow-up.

Lemma 1.6. *Assume that the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ satisfies an exterior cone condition. Let*

$$K = \left\{v \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega) : v|_{\partial\Omega} = k \text{ and } v \geq 0\right\}$$

where $k \in \mathbb{Z}_+$. Then there exists a unique solution $u \in K$ of the problem

$$\left.\begin{array}{ll}\Delta u(x) = u^p(x) & \text{for } x \in \Omega \\ u(x) = k & \text{for } x \in \partial\Omega\end{array}\right\} \tag{1.4}$$

where $p \in (1, +\infty)$.

Proof. The proof is based on a theorem of K. Akô [1]. Let $v \in K$. Consider the problem

$$\left.\begin{array}{ll} \Delta u_v(x) = v^p(x) & \text{for } x \in \Omega \\ u_v(x) = k & \text{for } x \in \partial\Omega. \end{array}\right\} \tag{1.5}$$

According to [8: Theorem 6.13/Problem 6.3] there exists a unique $u_v \in K$ which solves problem (1.5) (this result is called Lemma D in [1]; we have to refer to [8] since in [1] the domain is nicer than our domain Ω). Define $T : K \to K$ by $T(v) = u_v$. The functions ϕ_1 and ϕ_2 defined by $\phi_1(x) = 0$ for all $x \in \bar{\Omega}$ and $\phi_2(x) = k$ for all $x \in \bar{\Omega}$ are a sub- and a supersolution of problem (1.4), respectively. By [1: Main Theorem] there exists a fixed point $u \in K$ of T, i.e. u solves problem (1.4) ∎

Lemma 1.7. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which satisfies an exterior cone condition. Then there exists an increasing sequence $\{u_m\}_{m=1}^{\infty}$ of positive $C^2(\Omega)$-functions such that*

$$\left.\begin{array}{ll} \Delta u_m(x) = u_m^p(x) & \text{for } x \in \Omega \\ \lim\limits_{m \to +\infty} u_m(x) = \infty & \text{for } x \in \partial\Omega. \end{array}\right\} \tag{1.6}$$

Proof. Let $m \in \mathbb{Z}_+$. By Lemma 1.6 there exists a unique, positive solution $u_m \in C^2(\Omega) \cap C^0(\bar{\Omega})$ of problem (1.4). By the maximum principle (use the argument in the proof of Lemma 1.5) $u_m \leq u_{m+1}$ holds throughout Ω. Hence $\{u_m\}_{m=1}^{\infty}$ is a sequence as stated ∎

Proposition 1.8. *Let C_R be a cut-off open cone in $\mathbb{R}^N$ $(N \geq 2)$. There is exactly one positive solution u of the problem*

$$\left.\begin{array}{ll} \Delta u(x) = u^p(x) & \text{for } x \in C_R \\ u(x) \to \infty & \text{as } x \to \partial C_R. \end{array}\right\} \tag{1.7}$$

For $N = 2$ and for $N \geq 3$ with $1 < p < \frac{N}{N-2}$ the boundary behaviour of this solution is controlled by the formula

$$\bar{u}(r,\theta) \leq u(r,\theta) \leq \bar{u}(r,\theta) + v(r).$$

For $N \geq 3$ with $p \geq \frac{N}{N-2}$ the upper bound still holds. The boundary blow-up of the functions $\bar{u}$ and v satisfies

$$\lim_{(r,\theta) \to \partial C_R} \bar{u}(r,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot r^{\frac{2}{p-1}} \cdot \delta^{\frac{2}{p-1}}(\theta) = 1$$

$$\lim_{r \to R} v(r) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot (R-r)^{\frac{2}{p-1}} = 1$$

where (r,θ) are the polar coordinates of $x \in C_R$ and $\delta(\theta)$ denotes the distance from $\theta \in \Omega_{N-1}$ to $\partial\Omega_{N-1}$ in the metric of the unit sphere S^{N-1} of $\mathbb{R}^N$.

Proof. If $N = 2$ or $N \geq 3$ with $1 < p < \frac{N}{N-2}$ let $\bar{u}$ denote the restriction to C_R of the solution of problem (P) in the open cone C which is given in Theorem 1.2. If $N \geq 3$ and $p \geq \frac{N}{N-2}$, instead pick as $\bar{u}$ the supersolution with the same boundary behaviour. Let v denote the (radially symmetric) maximal solution of the problem

$$\left.\begin{aligned} &\Delta v(x) = v^p(x) \quad \text{for } x \in B_R(0) \\ &v(x) \to \infty \quad \text{as } x \to \partial B_R(0) \end{aligned}\right\}$$

as in [2]. Now define

$$U(x) = \bar{u}(x) + v(x) \qquad \text{for } x \in C_R \cap B_R(0) = C_R.$$

Then

$$\Delta U(x) = \Delta\bar{u}(x) + \Delta v(x) \leq \bar{u}^p(x) + v^p(x) \leq \big(\bar{u}(x) + v(x)\big)^p = U^p(x)$$

for every $x \in C_R$. Hence U is a solution of problem (1.3) in Lemma 1.5. By an argument given in [9: Theorem III] and applicable because of Lemma 1.5, Lemma 1.7 and the monotonicity of the right-hand member of our problem, we may conclude that there exists a solution u of our problem: Pick an increasing sequence $\{u_m\}_{m=1}^{\infty}$ of positive solutions of problem (1.6). By Lemma 1.7, such a sequence does exist. Lemma 1.5 says that this sequence is uniformly bounded from above by U. As in [9], this fact and the conditions $(F)_1$ and $(F)_2$ on $f(t) = t^p$ imply that there exists a solution $u \in C^2(\Omega)$ of problem (1.7). By construction, $u(x) \leq U(x)$ for every $x \in \Omega$. So, Corollary 1.3 and [2: Theorem 2.6] yield the desired upper growth control of $u(x)$ as $x \to \partial C_R$.

A. C. Lazer and P. J. McKenna proved in [10: Theorem 2] that problem (0.1) has at most one positive solution if Ω is a domain bounded star-shaped with respect to some point. Hence the above constructed solution is unique. Furthermore, when $N = 2$ or $N \geq 3$ with $1 < p < \frac{N}{N-2}$, the uniqueness proof of A. C. Lazer and P. J. McKenna gives the lower bound $\bar{u}(r,\theta)$ of $u(r,\theta)$, and the proof is complete ∎

Remark 1.9. Let $r > 0$ be fixed. As $\theta \in \Omega_{N-1}$ approaches $\partial\Omega_{N-1}$, $r \cdot \delta(\theta)$ tends to the N-dimensional Euclidean distance $\operatorname{dist}(x, \partial C_R)$, and the boundary behaviour of $\bar{u}$ coincides with that of a solution of problem (0.1) in a smooth domain, calculated in [2: Theorem 2.6].

2. Solutions of $\Delta u = u^p$ in a domain with a non-smooth boundary

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. The following definition is very important in the present paper.

Definition 2.1. Ω is said to meet a *uniform interior cone condition* if there exists a fixed number $R > 0$ such that for each $z \in \partial\Omega$ there exists a cut-off open cone $C_R \subset \Omega$ with $z \in \partial C_R \cap \partial\Omega$. If z is the vertex of C_R, then the cone is denoted by $C_R(z)$.

In particular, the uniform interior cone condition means that there is a fixed subset Ω_{N-1} of S^{N-1} such that for each $z \in \partial\Omega$ it is possible to construct a cone with the desired properties from R and a suitably rotated copy of Ω_{N-1}.

With obvious changes in Definition 2.1 we may define a uniform *exterior* cone condition.

Assume that Ω meets a uniform interior cone and a uniform exterior sphere condition. We determine an upper bound for the growth as $x \to \partial\Omega$ of positive solutions of the problem

(P)′ $\Delta u(x) = u^p(x)$ for $x \in \Omega$
$u(x) \to \infty$ as $x \to \partial\Omega$

for $p \in (1, +\infty)$. The technique developed by C. Bandle and M. Essén in [2] immediately yields a lower bound for the growth of $u(x)$ as x approaches $\partial\Omega$.

Let $C_{\gamma(\varepsilon)}(z)$ denote a cut-off open cone with cut-off radius $\gamma(\varepsilon) > 0$ and vertex at $z \in \partial\Omega$, such that $C_{\gamma(\varepsilon)}(z) \subset \Omega$. By our uniform interior cone condition, there exists such a cone for every $z \in \partial\Omega$ and for every cut-off radius $\gamma(\varepsilon) \in (0, R)$. With this notation we are ready to state and prove the following result.

Theorem 2.2. *Let u be a positive solution of problem* (P)′. *Then for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that for every $x \in \cup_{z\in\partial\Omega} C_{\gamma(\varepsilon)}(z) \subset \Omega$ the estimates*

$$u(x) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot |x-z|^{\frac{2}{p-1}} \geq 1-\epsilon \tag{2.1}$$

$$u(|x-z|,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot |x-z|^{\frac{2}{p-1}} \cdot \delta^{\frac{2}{p-1}}(\theta) \leq 1+\epsilon \tag{2.2}$$

hold.

Proof. The lower bound (2.1) of the growth of $u(x)$ as x approaches $\partial\Omega$ is derived by copying the proof using the uniform exterior sphere condition discovered by C. Bandle and M. Essén in [2].

The uniform interior cone condition gives the upper bound (2.2) of u. Let $z \in \partial\Omega$. Pick an $\varepsilon > 0$ and pick a cut-off open cone $C_{\gamma(\varepsilon)}(z) \subset \Omega$ with cut-off radius $\gamma(\varepsilon) \in (0, R)$ to be determined later, and vertex at $z \in \partial\Omega$. Let $\Omega_{N-1} \subset S^{N-1}$ denote (the possibly rotated copy of) the subset of the unit sphere which defines the interior cone $C_R(z)$ with vertex at z; this subset also defines $C_{\gamma(\varepsilon)}(z)$. By Proposition 1.8, there is a unique solution U of problem (P) in this cone. Lemma 1.5 implies that u is dominated by U throughout $C_{\gamma(\epsilon)}(z)$. Thus Proposition 1.8 gives the estimate (2.2) in the cone $C_{\gamma(\varepsilon)}(z)$ if $\gamma(\varepsilon)$ is chosen small enough. z was chosen arbitrarily on the boundary of Ω, and by Proposition 1.8 the same $\gamma(\varepsilon)$ works for all z, so the upper bound for u is proved ∎

3. A partial result in a more general domain

Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$, which satisfies a uniform interior cone condition. We establish an upper and a lower bound for the growth as $x \to \partial\Omega$ of positive solutions of the problem (P)' in Section 2, with Ω as above. For technical reasons, if $N \geq 3$, the lower bound holds for "small" values of p only.

With the notation of Section 2, we state the following result.

Theorem 3.1. *Suppose that the domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ satisfies a uniform interior cone condition. Let u be a positive solution of problem (P)' with $p \in (1, +\infty)$ if $N = 2$ and $p \in (1, \frac{N}{N-2})$ if $N \geq 3$. Then for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that for every $x \in \cup_{z\in\partial\Omega} C_{\gamma(\varepsilon)}(z) \subset \Omega$ the estimates*

$$u(x) \cdot \left(\frac{2}{p-1}\left(2 - N + \frac{2}{p-1}\right)\right)^{-\frac{1}{p-1}} \cdot |x-z|^{\frac{2}{p-1}} \geq 1 - \epsilon \tag{3.1}$$

$$u(|x-z|, \theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot |x-z|^{\frac{2}{p-1}} \cdot \delta^{\frac{2}{p-1}}(\theta) \leq 1 + \varepsilon \tag{3.2}$$

hold.

Proof. The upper bound (3.2) of u follows from the uniform interior cone condition: The existence of a $\gamma_1(\varepsilon) > 0$ such that (3.2) holds was proved in Theorem 2.2.

Normally, the lower bound (3.1) would follow from e.g. a uniform exterior sphere condition, but actually we use the interior condition again. However, the best lower bound for the less general domain in the next section is obtained from a uniform exterior cone condition.

To prove (3.1), pick an $\varepsilon > 0$. Let $z \in \partial\Omega$ and choose $R > 0$ such that the set $\partial B_R(z) \cap \Omega$ is non-empty, where $B_R(z)$ denotes a ball of radius R centered at z. Because of the uniform interior cone condition, R may be chosen independently of $z \in \partial\Omega$. For $r > 0$, define

$$v(r) = (-C_{N,p})^{\frac{1}{p-1}} \cdot r^{-\frac{2}{p-1}} \tag{3.3}$$

where $C_{N,p}$ was defined in Theorem 1.2. Note that $(-C_{N,p})^{-\frac{1}{p-1}}$ is the constant in (3.1). An elementary calculation gives

$$\Delta v(r) = v^p(r) \qquad (r > 0). \tag{3.4}$$

Here the restriction on p for $N \geq 3$ is needed, since $-C_{N,p}$ must be positive. Now put

$$\bar{v}(r) = v(r) - (-C_{N,p})^{\frac{1}{p-1}} \cdot R^{-\frac{2}{p-1}} \qquad \text{for } r \in (0, R].$$

Then $\bar{v} > 0$ in the ball $B_R(0)$ and $\bar{v}(R) = 0$. Finally define

$$v_1(x) = \bar{v}(|x-z|) \qquad \text{for } x \in \bar{B}_R(z) \cap \Omega =: \Omega'.$$

It is a consequence of (3.4) and Lemma 1.5 that $v_1(x) \leq u(x)$ for all $x \in \Omega'$. Since R does not depend on the choice of $z \in \partial\Omega$, there is a $\gamma_2(\varepsilon) > 0$ such that the lower bound (3.1) holds in $C_{\gamma_2(\varepsilon)}(z)$ for every $z \in \partial\Omega$. Choose $\gamma(\varepsilon) = \min(\gamma_1(\varepsilon), \gamma_2(\varepsilon))$ ∎

Remark 3.2. It is of course desirable to choose the constant in equation (3.3) as big as possible. If $(-C_{N,p})^{\frac{1}{p-1}}$ is exchanged for a larger number, the equality sign in equation (3.4) is transformed into an inequality sign. Unfortunately, the resulting inequality goes in the wrong direction for Lemma 1.5 to give the desired conclusion $v_1 \le u$ in Ω'.

Note the slightly annoying fact that the constant in the lower bound (3.1) is strictly less than the constant in the upper bound (3.2). There is no reason to believe that the lower bound is the best possible for every domain satisfying a uniform interior cone condition. In fact, if we impose on Ω a uniform outer cone condition, we are able to find a technique (Sections 4 and 7) which gives a better lower bound.

4. An improved lower bound

Assume that Ω meets both a uniform interior and exterior cone condition. Then we are able to construct a better comparison function than in the previous section. First we prove a proposition, in which we assume that $R > 0$ and $\Omega_{N-1} \subset S^{N-1}$ is a domain with C^2-boundary. Furthermore, let C_R denote the corresponding cut-off open cone.

Proposition 4.1. *If $p \in (1, +\infty)$ when $N = 2$ or if $p \in (1, \frac{N}{N-2})$ when $N \ge 3$, then there exists a positive solution v of the problem*

$$\left.\begin{array}{ll} \Delta v(x) \ge v^p(x) & \text{for } x \in C_R \\ v(x) \to \infty & \text{as } x \to \partial C_R \setminus \{x \in \bar{C}_R : |x| = R\} \\ v(x) = 0 & \text{for } x \in \{x \in \bar{C}_R : |x| = R\}. \end{array}\right\} \tag{4.1}$$

Furthermore, for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) \in (0, R)$ such that for every $x = (r, \theta) \in C_{\gamma(\varepsilon)}$ the estimates

$$(1-\varepsilon)\cdot\alpha(\theta) \le v(r,\theta)\cdot r^{\frac{2}{p-1}} \le \alpha(\theta) \tag{4.2}$$

hold where $\alpha(\theta) > 0$ denotes the solution of problem (1.1) in Ω_{N-1}.

Proof. First we construct v satisfying problem (4.1). Let $\bar{v}(r,\theta) = r^{-\frac{2}{p-1}} \cdot \alpha(\theta)$ be the restriction to $C_R \cup \{x \in \bar{C}_R : |x| = R\}$ of the solution of problem (P) in C, given in Theorem 1.2. For $(r,\theta) \in C_R$ define

$$v(r,\theta) = \bar{v}(r,\theta) - R^{-\frac{2}{p-1}} \cdot \alpha(\theta).$$

Hence $v(R,\theta) = 0$ for every $\theta \in \Omega_{N-1}$, $v(r,\theta) \ge 0$ for every $(r,\theta) \in C_R$ and $v(r,\theta) \to \infty$ as $(r,\theta) \to \partial C_R$ if $r < R$. It remains to check that $\Delta v \ge v^p$. Let $(r,\theta) \in C_R$. Then

$$\begin{aligned} \Delta v(r,\theta) &= r^{-2}\cdot\left(r^{-\frac{2}{p-1}} - R^{-\frac{2}{p-1}}\right)\cdot\alpha^p(\theta) - C_{N,p}\cdot R^{-\frac{2}{p-1}}\cdot r^{-2}\cdot\alpha(\theta) \\ &\ge r^{-2}\cdot\left(r^{-\frac{2}{p-1}} - R^{-\frac{2}{p-1}}\right)\cdot\alpha^p(\theta) \end{aligned}$$

since $-C_{N,p} \ge 0$ for N and p as in the proposition. Here the restriction on p for $N \ge 3$ is necessary. Furthermore,

$$r^{-2}\cdot\left(r^{-\frac{2}{p-1}} - R^{-\frac{2}{p-1}}\right) \ge \left(r^{-\frac{2}{p-1}} - R^{-\frac{2}{p-1}}\right)^p$$

for every $r \in (0, R]$. Hence, $\Delta v(r,\theta) \geq v^p(r,\theta)$ for $(r,\theta) \in C_R$ and $v(r,\theta)$ solves problem (4.1).

Next, we prove that our v meets estimate (4.2). The lower bound holds in $C_{\gamma_1(\varepsilon)}$ if we choose $\gamma_1(\varepsilon)$ such that $\gamma_1^{\frac{2}{p-1}}(\varepsilon) \cdot R^{-\frac{2}{p-1}} \leq \varepsilon$. The upper bound of $v(r,\theta)$ follows from our construction: $v(r,\theta)$ is dominated by $\bar{v}(r,\theta)$. But this means that for any $\gamma_2(\varepsilon) > 0$ the inequality $v(r,\theta) \cdot r^{\frac{2}{p-1}} \leq \alpha(\theta)$ holds for every $r \in (0, \gamma_2(\varepsilon))$, since $\bar{v}(r,\theta) \cdot r^{\frac{2}{p-1}} = \alpha(\theta)$. Define $\gamma(\varepsilon) = \min\left(\gamma_1(\varepsilon), \gamma_2(\varepsilon)\right)$. Then equation (4.2) holds in $C_{\gamma(\varepsilon)}$ ∎

An immediate consequence of Proposition 4.1 and Corollary 1.3 is the following

Corollary 4.2. *For every $\varepsilon > 0$ there is a cut-off radius $R'(\varepsilon) \in (0, R)$ and a $\gamma(\varepsilon) > 0$ such that*

$$1 - \varepsilon \leq v(r,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot r^{\frac{2}{p-1}} \cdot \delta^{\frac{2}{p-1}}(\theta) \leq 1 + \varepsilon \tag{4.3}$$

for every $(r,\theta) \in C_{R'(\varepsilon)}$ satisfying $\delta(\theta) \leq \gamma(\varepsilon)$.

The natural idea to subtract $R^{-\frac{2}{p-1}} \cdot \alpha(\theta)$ from $\bar{v}(r,\theta)$ to obtain $v(r,\theta)$ in the proof of Proposition 4.1 does not work if p is larger than $\frac{N}{N-2}$ (when $N \geq 3$), since in this case the inequality $\Delta v(r,\theta) \geq v^p(r,\theta)$ will not hold in C_R.

Using Proposition 4.1 it is not difficult to prove the next Theorem 4.4, providing boundary-blow-up control for solutions of our problem. Denote by Ω^i_{N-1} and Ω^e_{N-1} the subsets of S^{N-1} defining the interior and the exterior cones. Recall that by $C_\gamma(z)$ we understand the inner cone with vertex at $z \in \partial\Omega$, which is defined by a (possibly rotated) copy of Ω^i_{N-1} and cut at the distance $\gamma > 0$ from z.

Theorem 4.4. *Suppose that the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ satisfies a uniform interior and exterior cone condition. Let u be a positive solution of the problem* (P)$'$ *with $p \in (1, +\infty)$ if $N = 2$ and $p \in (1, \frac{N}{N-2})$ if $N \geq 3$. Then for any $\varepsilon > 0$ there exists a $\gamma(\varepsilon) > 0$ such that for every $x \in \cup_{z\in\partial\Omega} C_{\gamma(\varepsilon)}(z) \subset \Omega$ the estimates*

$$u(|x-z|,\theta) \cdot |x-z|^{\frac{2}{p-1}} \geq (1-\varepsilon) \cdot \beta(\theta) \tag{4.4}$$

$$u(|x-z|,\theta) \cdot |x-z|^{\frac{2}{p-1}} \leq (1+\varepsilon) \cdot \alpha(\theta) \tag{4.5}$$

hold where $\alpha(\theta)$ and $\beta(\theta)$ denote the solutions of problem (1.1) *in Ω^i_{N-1} and $S^{N-1} \setminus \bar{\Omega}^e_{N-1}$. If $N \geq 3$, estimate* (4.5) *holds for $p \geq \frac{N}{N-2}$ too, if $\alpha(\theta)$ denotes the supersolution of problem* (1.1) *as in Proposition* 1.8.

Proof. Pick an $\varepsilon > 0$. The existence of a $\gamma_1(\varepsilon) > 0$ such that the upper bound (4.5) holds was proved in Theorem 2.2: The upper bound was proved using the uniform interior cone condition.

To deduce the lower bound we use the uniform exterior cone condition. Define $\omega_{N-1} = S^{N-1} \setminus \bar{\Omega}^e_{N-1}$ and $\omega'_{N-1} = \Omega^i_{N-1}$. Pick a $z \in \partial\Omega$. In accordance with the uniform interior and exterior cone condition, we may choose an $R > 0$ which is independent of z, such that the cut-off open cone $C'_R(z)$ with vertex at z, defined by R and a (possibly rotated) copy of ω'_{N-1} is a subset of Ω. The cone $C_R(z)$ defined by R and ω_{N-1} contains $C'_R(z)$ and $\omega'_{N-1} \subset\subset \omega_{N-1}$. By Proposition 4.1 there exists a v which solves problem (4.1) in $C_R(z)$. Lemma 1.5 implies that $u \geq v$ in $C_R(z) \cap \Omega$, in particular this is true in $C'_R(z)$. Hence the lower bound for v, given in (4.2), is a lower bound for u too. Thus, Proposition 4.1 says that there exists a $\gamma_2(\varepsilon) > 0$ such that equation (4.4) holds in $C_{\gamma_2(\varepsilon)}(z)$, where $\beta(\theta)$ solves problem (1.1) in ω_{N-1}. This estimate is independent of $z \in \partial\Omega$, so the theorem follows if we choose $\gamma(\varepsilon) = \min(\gamma_1(\varepsilon), \gamma_2(\varepsilon))$ ∎

5. The equation $\Delta u = e^u$

Two problems of type (0.1) in domains with regular boundaries have been the subject of a lot of research since L. Bieberbach's survey [6] from 1916. These are the cases $f(t) = t^p$ $(p \in (1, +\infty))$ and $f(t) = e^t$. It turns out that the methods of Sections 1 - 4 may be applied also when $f(t) = e^t$. We prove results analogous to Theorem 1.2 and Proposition 1.8. Then, for a domain satisfying a uniform interior and exterior cone condition and condition (I), the proof of Theorem 4.4 may be copied to obtain the desired growth control for the solutions of our problem.

Let C be an open cone in $\mathbb{R}^N$ $(N \geq 2)$. Explicitly, the problem we are interested in for the moment is

(P)$_1$ $\Delta u(x) = e^{u(x)}$ for $x \in C$
$u(x) \to \infty$ as $x \to \partial C$.

Theorem 5.1. *There is a solution $u(x) \equiv u(r,\theta) = \alpha(\theta) - 2\log r$ of problem* (P)$_1$ *such that $\alpha(\theta)$ solves the problem*

$$\left.\begin{array}{ll} \Delta_\theta \alpha(\theta) = 2(N-2) + e^{\alpha(\theta)} & \text{for } \theta \in \Omega_{N-1} \\ \alpha(\theta) \to \infty & \text{as } \theta \to \partial\Omega_{N-1}. \end{array}\right\} \tag{5.1}$$

Proof. We have

$$\Delta u(r,\theta) = \frac{2}{r^2} + \frac{N-1}{r} \cdot \left(-\frac{2}{r}\right) + \frac{1}{r^2} \cdot \Delta_\theta \alpha(\theta) = \frac{1}{r^2} \cdot e^{\alpha(\theta)}$$

if and only if $\alpha(\theta)$ solves problem (5.1). The right-hand member of (5.1) may be written as $g(\alpha(\theta))$ where $g(t) = 2(N-2) + e^t$ $(t \geq 0)$. Hence $\lim_{t\to\infty} \frac{g(t)}{e^t} = 1$ and $g(t)$ satisfies the conditions (G)$_1$ and (F)$_1$ - (F)$_3$ with $f(t) = e^t$ $(t \geq 0)$ (Section 0). Theorem 0.1 implies that there exists an $\alpha(\theta)$ with the desired properties. This concludes the proof ∎

Corollary 5.2. *The solution $u(r,\theta) = \alpha(\theta) - 2\log r$ of problem* (P)$_1$ *given by Theorem* 5.1 *satisfies for every $r > 0$*

$$\lim_{\theta \to \partial\Omega_{N-1}} \left[u(r,\theta) + 2\log(r \cdot \delta(\theta))\right] = \log 2. \tag{5.2}$$

Proof. A computation of the function Φ corresponding to $f(t) = e^t$, Theorem 0.1 and Theorem 5.1 gives (5.2) ∎

Remark 5.3. The solution in Theorem 5.1 need not be positive, not even when r is small. Indeed, if C is a sector of opening angle $2s$ with $s \in (\frac{1}{\sqrt{2}}\pi, \pi)$, then $\alpha(\theta)$ is symmetric under reflection in $\theta = s$, i.e. for $\theta < s$ we have $\alpha(\theta) = \alpha(2s - \theta)$, and

$$\alpha(\theta) = \log\left(\frac{\pi^2}{2s^2}\right) - 2\log\sin\left(\frac{\pi\theta}{2s}\right)$$

for $\theta \in (0, s)$ (see [2]).

By using the technique of the proof of Lemma 1.5, we prove the following

Lemma 5.4. *Suppose that $\phi \in C^2(C_R)$ solves the problem*

$$\left.\begin{aligned} &\Delta\phi(x) \le e^{\phi(x)} \quad \text{for } x \in C_R \\ &\phi(x) \to \infty \quad \text{as } x \to \partial C_R. \end{aligned}\right\} \tag{5.3}$$

If $u \in C^2(C_R) \cap C^0(\bar{C}_R)$ is any solution of

$$\Delta u(x) = e^{u(x)} \qquad (x \in C_R),$$

then we have $u(x) \le \phi(x)$ for all $x \in C_R$.

Again, Lemma 5.4 remains valid when we replace C_R by any bounded domain Ω, and a slight modification of the proof allows us to consider $u \in C^2(\Omega)$ with boundary blow-up.

Before proving the following Proposition 5.7 which is the analogue of Proposition 1.8, we need two more lemmas.

Lemma 5.5. *Assume that the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ satisfies an exterior cone condition. Let*

$$K = \left\{v \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega) : v|_{\partial\Omega} = k\right\}$$

where $k \in \mathbb{Z}_+$. Then there exists a unique solution $u \in K$ of the problem

$$\left.\begin{aligned} &\Delta u(x) = e^{u(x)} \quad \text{for } x \in \Omega \\ &u(x) = k \quad \text{for } x \in \partial\Omega. \end{aligned}\right\} \tag{5.4}$$

Proof. Let $v \in K$. The proof is based on the before mentioned (see Section 1) theorem by K. Akô [1]. Consider the problem

$$\left.\begin{aligned} &\Delta u_v(x) = e^{v(x)} \quad \text{for } x \in \Omega \\ &u_v(x) = k \quad \text{for } x \in \partial\Omega. \end{aligned}\right\} \tag{5.5}$$

According to [8: Theorem 6.13/Problem 6.3] there exists a unique $u_v \in K$ which solves problem (5.5). This result is called Lemma D in [1]; we have to refer to [8] since in [1] the domain is nicer than our domain Ω. Define $T : K \to K$ by $T(v) = u_v$. In [2] it is shown that the maximal solution of the problem $\Delta U(x) = e^{U(x)}$ in the strip

$$H_s = \left\{(x_1, \ldots, x_N) = x \in \mathbb{R}^N : 0 < x_1 < 2s\right\}$$

depends on x_1 only. In fact, $U(x) = U(x_1)$ enjoys $U(x_1) = U(2s - x_1)$ and

$$U(x_1) = \log\left(\frac{\pi^2}{2s^2}\right) - 2\log\sin\left(\frac{\pi x_1}{2s}\right)$$

for $x_1 \in (0, s)$. Since Ω is bounded, we may pick a strip H_s containing Ω such that $U(x_1) < k$ throughout $\Omega \cap H_s$. Thus, $\phi_1(x) = U(x_1)$ $(x \in \bar{\Omega})$ and $\phi_2(x) = k$ $(x \in \bar{\Omega})$ are a sub- and a supersolution of problem (5.4). By [1: Main Theorem] there exists a fixed point $u \in K$ of T, i.e. u solves problem (5.4) ∎

As in the case $\Delta u = u^p$, Lemmas 5.4 and 5.5 prove the following result.

Lemma 5.6. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which satisfies an exterior cone condition. Then there exists an increasing sequence $\{u_m\}_{m=1}^{\infty}$ of $C^2(\Omega)$-functions such that*

$$\left.\begin{aligned} \Delta u_m(x) &= e^{u_m(x)} \quad \text{for } x \in \Omega \\ \lim_{m\to\infty} u_m(x) &= \infty \quad \text{for } x \in \partial\Omega. \end{aligned}\right\} \tag{5.6}$$

Proposition 5.7. *Let $R > 0$ and put $c(N,R) = \max\left\{0, \log\frac{2R^4}{N}\right\} + \log 2$. There exists a unique solution u of the problem*

$$\left.\begin{aligned} \Delta u(x) &= e^{u(x)} \quad \text{for } x \in C_R \\ u(x) &\to \infty \quad \text{as } x \to \partial C_R. \end{aligned}\right\} \tag{5.7}$$

The boundary behaviour of u is controlled by the formula

$$\bar{u}(r,\theta) \le u(r,\theta) \le \bar{u}(r,\theta) + v(r) + c(N,R)$$

where $v(r) = \log\frac{4NR^2}{(R^2-r^2)^2}$ $\big(r \in [0,R)\big)$ and the boundary blow-up of $\bar{u}$ satisfies

$$\lim_{(r,\theta)\to\partial C_R}\left[\bar{u}(r,\theta) + 2\log\left(r\cdot\delta(\theta)\right)\right] = \log 2.$$

Proof. Let $\bar{u}$ denote the restriction to C_R of the solution of problem $(\mathrm{P})_1$ in the open cone C which is given in Theorem 5.1. The lower bound is proved as in Proposition 1.8, using Lemma 5.4 instead of Lemma 1.5.

To prove that there exists a solution of problem (5.7) and to deduce the upper bound for this solution we suggest the following argument. Define $v(r) = \log\frac{4NR^2}{(R^2-r^2)^2}$ $\big(r \in [0,R)\big)$. A simple computation shows that v is a radially symmetric solution of the problem

$$\left.\begin{aligned} \Delta v(x) &\le e^{v(x)} \quad \text{for } x \in B_R(0) \\ v(x) &\to \infty \quad \text{as } x \to \partial B_R(0). \end{aligned}\right\} \tag{5.8}$$

As usual $B_R(0)$ denotes a ball of radius R, centered at 0. Define

$$U(x) \equiv U(r,\theta) := \bar{u}(r,\theta) + v(r) + c(N,R) \qquad ((r,\theta) \in C_R \cap B_R(0) = C_R).$$

Then, by (5.8),

$$\Delta U(r,\theta) = \Delta\bar{u}(r,\theta) + \Delta v(r) \le e^{\bar{u}(r,\theta)} + e^{v(r)} \le e^{\bar{u}(r,\theta)+c_1} + e^{v(r)+c_2} \tag{5.9}$$

where

$$c_1 = \log 2 + \max\{0, \log 2 + 2\log R\} \qquad \text{and} \qquad c_2 = \max\left\{0, \log 2 + \log\frac{R^2}{4N}\right\}$$

and thus $c_1 + c_2 = c(N,R)$. Furthermore, it is not difficult to prove that

$$v(r) + c_2 \ge v(0) + c_2 = \log\frac{4N}{R^2} + c_2 \ge \log 2$$
$$\bar{u}(r,\theta) + c_1 = \alpha(\theta) - 2\log r + c_1 \ge \log 2$$

for every $(r,\theta) \in C_R$. In view of these inequalities, inequality (5.9) implies

$$\Delta U(r,\theta) \le e^{\bar{u}(r,\theta)+v(r)+c_1+c_2} = e^{U(r,\theta)} \tag{5.10}$$

for every $(r,\theta) \in C_R$. By the argument in the proof of Proposition 1.8, replacing Lemma 1.5 and Lemma 1.7 by Lemma 5.4 and Lemma 5.6, inequality (5.10) is enough to prove that there exists a solution u of problem (5.7), which is bounded from above by $\bar{u} + v + c(N,R)$. The boundary behaviour of $\bar{u}$ is an immediate consequence of Corollary 5.2. The uniqueness of u follows from [10] where a uniqueness proof for star-shaped domains is given ∎

Proposition 5.7 is the main tool in the proof of Theorem 5.8, providing growth control for solutions of the problem $\Delta u = e^u$ in a domain satisfying a uniform interior and exterior cone condition. As in Section 4, denote by Ω^i_{N-1} and Ω^e_{N-1} the subsets of S^{N-1} defining the interior and the exterior cone. Now we introduce the following condition:

(I) If Ω is such that whenever $z \in \partial\Omega$ is a corner and $C^e(z)$ is the open cone which defines the exterior cone $C^e_R(z)$, then $C^e(z) \cap \Omega = \emptyset$.

Theorem 5.8. *Suppose that the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \ge 2)$ satisfies a uniform interior and exterior cone condition as well as condition* (I). *Let u be a solution of the problem*

(P)$'_1$ $\Delta u(x) = e^{u(x)}$ *for* $x \in \Omega$
$u(x) \to \infty$ *as* $x \to \partial\Omega$.

Then for every sufficiently small $\gamma > 0$ and for every $x \in \cup_{z\in\partial\Omega} C_\gamma(z) \subset \Omega$ the estimates

$$\beta(\theta) \le u(r,\theta) + 2\log r \le \alpha(\theta) + \log\frac{4NR^2}{(R^2 - r^2)^2} \tag{5.11}$$

hold where $\alpha(\theta)$ and $\beta(\theta)$ denote the solutions of problem (5.1) *in Ω^i_{N-1} and $S^{N-1} \setminus \bar{\Omega}^e_{N-1}$, $r = |x - z|$ and R is the (uniform) cut-off radius of the interior cones.*

Proof. Copy the proof of the upper bound in Theorem 4.4, but use Proposition 5.7 instead of Proposition 1.8. The lower bound is proved by choosing R so large that the cone $C_R(z)$ with vertex at $z \in \partial\Omega$, which is defined from $S^{N-1} \setminus \bar{\Omega}^e_{N-1}$, contains Ω. Proposition 5.7 and Lemma 5.4 give the desired lower bound ∎

Condition (I) may be omitted without changing the statement of the theorem. This will be proved in Section 7.

6. Existence and uniqueness results

In the present section we prove existence of unique solutions of our problems in a bounded domain satisfying a uniform interior and exterior cone condition. To obtain such results we need Lemmas 1.7 and 5.6.

Lemma 6.1. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which satisfies an exterior cone condition. Then there exists an increasing sequence $\{u_m\}_{m=1}^{\infty}$ of $C^2(\Omega)$-functions such that*

$$\left.\begin{aligned} &\Delta u_m(x) = f(u_m(x)) \quad \textit{for } x \in \Omega \\ &\lim_{m\to\infty} u_m(x) = \infty \quad \textit{for } x \in \partial\Omega \end{aligned}\right\} \tag{6.1}$$

where either $f(t) = t^p$ $(p > 1)$ or $f(t) = e^t$. The functions u_m are positive if $f(t) = t^p$.

The existence part of the following theorem is a simple consequence of Lemma 6.1.

Theorem 6.2. *Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain which satisfies a uniform interior and exterior cone condition and let either $f(t) = t^p$ $(p > 1)$ or $f(t) = e^t$. Then there exists a unique solution of the problem*

$$\left.\begin{aligned} &\Delta u(x) = f(u(x)) \quad \textit{for} \quad x \in \Omega \\ &u(x) \to \infty \quad \textit{as} \quad x \to \partial\Omega. \end{aligned}\right\} \tag{6.2}$$

In the case $f(t) = t^p$, u is positive.

Proof. Pick a sequence $\{u_m\}_{m=1}^{\infty}$ of functions as in Lemma 6.1. Define

$$u(x) = \lim_{m\to\infty} u_m(x) \qquad (x \in \Omega). \tag{6.3}$$

We claim that this is a well-defined function. Let $K \subset \Omega$ be a compact subset of Ω. The uniform interior cone condition makes it possible to cover Ω by interior cones. In each of these cones C_R, Proposition 1.8 (or Proposition 5.7, depending on the choice of f) provides a comparison function Φ which solves the problem

$$\left.\begin{aligned} &\Delta\Phi(x) = f(\Phi(x)) \quad \text{for} \quad x \in C_R \\ &\Phi(x) \to \infty \quad \text{as} \quad x \to \partial C_R. \end{aligned}\right\} \tag{6.4}$$

By Lemma 1.5 (or Lemma 5.4) every function in the increasing sequence $\{u_m\}_{m=1}^{\infty}$ is bounded by Φ in C_R. Hence, in K the sequence $\{u_m\}_{m=1}^{\infty}$ is uniformly convergent. This proves our claim. The function u belongs to $C^2(\Omega)$ and solves $\Delta u(x) = f(u(x))$ $(x \in \Omega)$ (see [7: pp.787]). By construction, $u(x)$ blows up at the boundary.

To prove uniqueness, we use the a priori bounds in Theorems 4.4 and 5.8. Note that by Corollaries 7.4 and 7.7 the extra conditions in these theorems may be omitted. We consider the case $f(t) = t^p$ only, since that of $f(t) = e^t$ is treated similarly. Assume that u_1 and u_2 are two solutions of problem (6.2). Let $\alpha \in (0,1)$. Then, by Theorem 4.4

$$\lim_{x \to \partial\Omega} \frac{u_1(x)}{u_2(x)} = 1. \tag{6.5}$$

Hence $\alpha u_1 < u_2$ close enough to the boundary of Ω. Furthermore

$$\Delta(\alpha u_1(x)) = \alpha u_1^p(x) \geq (\alpha u_1(x))^p \tag{6.6}$$

so αu_1 is a subsolution of problem (6.2). Lemma 1.5 implies

$$\alpha u_1(x) < u_2(x) \tag{6.7}$$

for every $x \in \Omega$. Letting $\alpha \to 1$ proves the inequality

$$u_1(x) \leq u_2(x). \tag{6.8}$$

Exchanging the roles of u_1 and u_2 in the above argument proves the reverse inequality of (6.8). Hence $u_1 = u_2$ and the proof is complete ∎

Remark 6.3. The technique used in the existence proof of Theorem 6.2 is due to J. B. Keller (see [9: pp. 504 – 505]), and the uniqueness part follows C. Bandle and M. Essén [2].

7. Boundary blow-up of the gradient

C. Bandle and M. Essén [2] found a rescaling argument which allowed them to determine the boundary behaviour of the gradient of a solution of our problem (0.1) when $f(t) = t^p$ or $f(t) = e^t$, in a domain Ω with C^2-boundary. C. Bandle and M. Marcus [5] were able to adapt this argument to problem (0.2) with the Laplacean replaced by a more general second order semilinear differential operator L.

It turns out that the rescaling argument applies to our situation too. Again, we perform a separation of variables and use the results in [2] (for $\mathbb{R}^2$) and in [5] (for $\mathbb{R}^N$ with $N \geq 3$). For the sake of convenience, we restrict ourselves to the planar case. It is obvious from the proof how the results in [5] can be used to obtain corresponding results, at least for $f(t) = t^p$, if $\Omega \subset \mathbb{R}^N$ with $N \geq 3$. Therefore we state the theorem in this case without proof.

We would like to mention that in [2], the rescaling argument is used in interior spheres instead of in interior cones as in our case. The authors of [2] stress the fact their rescaling argument flattens the boundary of Ω. We do not use flattening of the boundary, since this would remove the corners of Ω.

As a corollary of the main theorems of this section, we are able to improve the lower bounds for the growth of our solutions. The condition on p for $N \geq 3$ in Theorem 4.4 and the condition (I) in Theorem 5.8 may be omitted without changing the conclusions of these theorems.

Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain which satisfies a uniform interior and exterior cone condition. Let $z_0 \in \partial\Omega$ and let $C_R(z_0)$ denote an interior cone with vertex at z_0 and cut-off radius R. Let L_0 denote the symmetry axis of $C_R(z_0)$. Now, let $c \in L_0 \cap \Omega$ and pick an $R' \in (0, R)$ such that $\partial B_{R'}(c) \cap L_0 \subset \Omega$ (this is a two-point set if R' is small enough). Denote by a_0 the point in $\partial B_{R'}(c) \cap L_0$ which is closest to z_0. Finally, introduce the cone $C_{2R'}(a_0)$ with vertex at a_0 and which is congruent to $C_R(z_0)$. Figure 1 shows the above described construction.

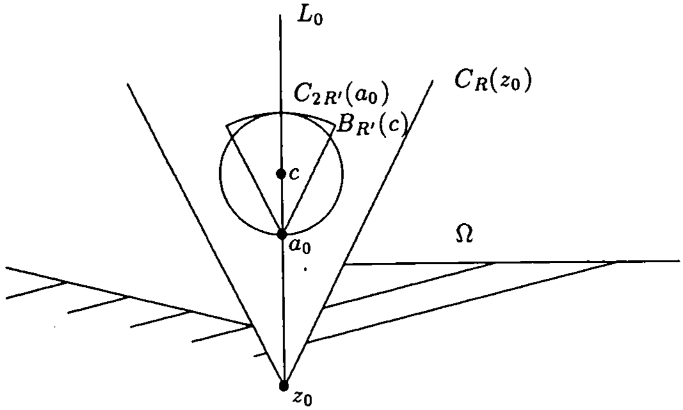

Figure 1

Take a_0 to be the origin of a new Cartesian coordinate system with the x_1-axis parallel to L_0 and directed towards c. Hence, the coordinates of c are $(R', 0, \dots, 0)$. Put

$$D(\beta) = C_{2R'}(a_0) \cap \left\{ x = (x_1, x') : 0 < x_1 < 2\beta^{1-\nu} \text{ and } |x'| < \beta^{1-\nu} \right\} \tag{7.1}$$

where $\beta > 0$ and $\nu \in (0, \frac{1}{2})$. Rescale the coordinates by

$$(\xi_1, \xi') = \frac{1}{\beta} \cdot (x_1, x')$$

and choose $\beta = |a_0 - z_0|$. Then we have the equivalence

$$(x_1, x') \in D(\beta) \iff (\xi_1, \xi') \in \tilde{D}(\beta) \tag{7.2}$$

where

$$\tilde{D}(\beta) = C_{2R'/\beta}(a_0) \cap \tilde{Z}(\beta) \tag{7.3}$$

with

$$\tilde{Z}(\beta) = \left\{ \xi = (\xi_1, \xi') : 0 < \xi_1 < 2\beta^{-\nu} \text{ and } |\xi'| < \beta^{-\nu} \right\}.$$

Define the two boundary sets

$$\Gamma(\beta) = \partial C_{2R'}(a_0) \cap \partial D(\beta) \qquad \text{and} \qquad \tilde{\Gamma}(\beta) = \partial C_{2R'/\beta}(a_0) \cap \partial \tilde{D}(\beta).$$

For $x \in \Omega$, let (r, θ) denote the polar coordinates of x in a coordinate system with origin in z_0. The following proposition is an immediate consequence of the definitions made above (cf. [2: Proposition 3.1]).

Proposition 7.1. *Let $C(z_0)$ denote the open cone with vertex z_0, which has the same symmetry axis and the same defining subset of S^{N-1} as $C_{2R'}(a_0)$. Then:*

1. *If $\beta_1 < \beta_2$, then $\tilde{D}(\beta_2) \subset \tilde{D}(\beta_1)$, and $\tilde{D}(\beta) \to C(z_0)$ as $\beta \to 0$.*
2. *If $x \in D(\beta)$ and $x = (r, \theta)$, then $\beta \le r \le 3\beta^{1-\nu} + \beta$.*
3. *If $x \in \Gamma(\beta)$ and $x = (r, \theta)$, then $\frac{r}{\beta} \to 1$ as $\beta \to 0$.*

Proposition 7.1 tells us that the (Euclidean) distance from x to $\Gamma(\beta)$ tends to the distance from x to $\partial C(z_0)$ as $\beta \to 0$. This means that essentially, we have transformed our problem to the problem of determining the boundary behaviour of $|\nabla u|$ (close to z_0) in the cone $C(z_0)$. This is the rescaling argument of C. Bandle and M. Essén [2], modified to our situation.

Proposition 7.2. *Let C be an open cone in $\mathbb{R}^2$, i.e. a sectorial domain, and let $f(t) = t^p$ $(p > 1)$. Then the boundary behaviour of the gradient of the unique solution $u(r, \theta) = r^{-\frac{2}{p-1}} \cdot \alpha(\theta)$ of problem (0.1) in C is given by*

$$\lim_{(r,\theta)\to\partial C} |\nabla u(r,\theta)| \cdot \frac{p-1}{2a_p} \cdot r^{\frac{p+1}{p-1}} \cdot \delta^{\frac{p+1}{p-1}}(\theta) = 1 \tag{7.4}$$

where $a_p = \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{-\frac{2}{p-1}}$ and $\delta(\theta)$ denotes the (arc)- distance of θ from the end-points of the subarc Ω_1 of the unit circle, which defines C.

Proof. By Theorem 1.2 there exists a unique solution $u(r, \theta) = r^{-\frac{2}{p-1}} \cdot \alpha(\theta)$ of problem (P), where $\alpha(\theta)$ solves the problem

$$\frac{d^2\alpha}{d\theta^2}(\theta) = \alpha^p(\theta) - \left(\frac{2}{p-1}\right)^2 \cdot \alpha(\theta)$$

with $\alpha(\theta) \to \infty$ as $\theta \to \partial\Omega_1$. Corollary 1.3 gives the boundary blow-up control

$$\lim_{\theta\to\partial\Omega_1} \alpha(\theta) \cdot \frac{1}{a_p} \cdot \delta^{\frac{2}{p-1}}(\theta) = 1. \tag{7.5}$$

Theorem 3.2 in [2] says that (clearly, this result is valid in the one-dimensional case too)

$$\lim_{\theta\to\partial\Omega_1} \left|\frac{d\alpha}{d\theta}\right| \cdot \frac{p-1}{2a_p} \cdot \delta^{\frac{p+1}{p-1}}(\theta) = 1. \tag{7.6}$$

A simple calculation shows that

$$\begin{aligned} |\nabla u|^2 &= \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} \cdot \left(\frac{\partial u}{\partial \theta}\right)^2 \\ &= \frac{4}{(p-1)^2} \cdot r^{-2\frac{p+1}{p-1}} \cdot \alpha^2(\theta) + r^{-2\frac{p+1}{p-1}} \cdot \left(\frac{\partial \alpha}{\partial \theta}\right)^2. \end{aligned} \tag{7.7}$$

The equations (7.5) - (7.7) yield the following boundary behaviour of the gradient of u:

$$\begin{aligned} &\lim_{\theta\to\partial\Omega_1} |\nabla u|^2 \cdot \frac{(p-1)^2}{4a_p^2} \cdot r^{2\frac{p+1}{p-1}} \cdot \delta^{2\frac{p+1}{p-1}}(\theta) \\ &= \lim_{\theta\to\partial\Omega_1} \left(\delta^2(\theta) + \left(\frac{d\alpha}{d\theta}\right)^2 \cdot \frac{(p-1)^2}{4a_p^2} \cdot \delta^{2\frac{p+1}{p-1}}(\theta)\right) = 1. \end{aligned} \tag{7.8}$$

The statement of the proposition now follows from (7.8) ∎

One of the main results of the present section follows from Propositions 7.1 and 7.2:

Theorem 7.3. *Assume that the bounded domain $\Omega \subset \mathbb{R}^2$ satisfies a uniform interior and exterior cone condition. Let u be a positive solution of problem* (0.1), *with* $f(t) = t^p$ $(p > 1)$. *Then the boundary behaviour of the gradient of u is given by*

$$\lim_{x \to \partial\Omega} |\nabla u(x)| \cdot \frac{p-1}{2a_p} \cdot \operatorname{dist}^{\frac{p+1}{p-1}}(x, \partial\Omega) = 1 \tag{7.9}$$

where $a_p = \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{-\frac{2}{p-1}}$.

Proof. Copy the proof of [2: Theorem 3.2], but use Propositions 7.1 and 7.2 instead of the corresponding results in [2] ∎

Recall the condition in Theorem 4.4, giving a 'good' lower bound: $p \in (1, \frac{N}{N-2})$. It is an immediate consequence of the proof that this condition may be omitted. Clearly, this is true also when $N > 2$, since the rescaling argument still works for u itself: The only problem in the above proof for $N > 2$ is how to deal with the *gradient* of u. Thus we have

Corollary 7.4. *Theorem* 4.4 *holds for all* $p \in (1, \infty)$.

Remark 7.5. In view of Corollary 7.4, Theorem 7.3 holds in $\mathbb{R}^N$ with $N \geq 3$ too. There is one minor technical difficulty only: In Proposition 7.1, the function $\alpha(\theta)$ $(\theta \in S^{N-1})$ solves a Laplace-Beltrami-type problem (see Theorem 1.2). To control the boundary blow-up of the gradient in this case we use [5: Theorem 3.1].

Using the same technique as in the proof of Theorem 7.3, we may easily prove the following

Theorem 7.6. *Assume that the bounded domain $\Omega \subset \mathbb{R}^2$ satisfies a uniform interior and exterior cone condition and condition* (I). *Let u be a solution of the problem* (0.1), *with* $f(t) = e^t$. *Then the boundary behaviour of the gradient of u is given by*

$$\lim_{x \to \partial\Omega} |\nabla u(x)| \cdot \operatorname{dist}(x, \partial\Omega) = 2. \tag{7.10}$$

Of course, the proof of Theorem 7.6 gives a corresponding corollary as in the case $f(t) = t^p$:

Corollary 7.7. *Theorem* 5.8 *holds without assuming the condition* (I) *to hold.*

Theorem 7.6 can not be proved in $\mathbb{R}^N$ with $N \geq 3$ without some extra effort. This is due to the fact that in [5], C. Bandle and M. Marcus did generalize the gradient boundary-blow-up results in [2] for the case $f(t) = t^p$ $(p > 1)$ only. We have not investigated the possibility to prove Theorem 7.6 for the case $f(t) = e^t$ in higher dimensions.

8. A planar domain with fractal boundary

As an application of the results in the previous sections we study our boundary-blow-up problem in the bounded von Koch snowflake domain. Our approach is based on the observation that the domain in each step of the construction of the snowflake satisfies a uniform interior and exterior cone condition.

The snowflake domain of von Koch is constructed from a sequence of polygonal domains. The sequence starts with an equilateral triangle Δ of sidelength 1. In the first step ($n = 0$), we add an equilateral triangle of sidelength $\frac{1}{3} \cdot 1$ to each middle-third of the edges of Δ. The result is a regular polygonal domain Ω_0 with edges of sidelength $\frac{1}{3}$. In the next step ($n = 1$), we add an equilateral triangle of sidelength $(\frac{1}{3})^2$ to each middle-third of the edges of Ω_0 and get a regular polygonal domain Ω_1 with edges of sidelength $(\frac{1}{3})^2$. Repeat this process for $n = 2, 3, 4, \ldots$ to get a sequence $\{\Omega_n\}_{n=0}^{\infty}$ of regular polygonal domains. The snowflake domain Ω is defined as $\Omega = \lim_{n\to\infty} \Omega_n$.

Thus, in the n-th step of the construction of the snowflake domain we add $3 \cdot 4^n$ equilateral triangles of sidelength $3^{-(n+1)}$ to Ω_{n-1}. The distance from a corner z of Δ (which is also a corner of Ω_n) to the closest corner $w \neq z$ of Ω_n is $d_n = 3^{-(n+1)}$.

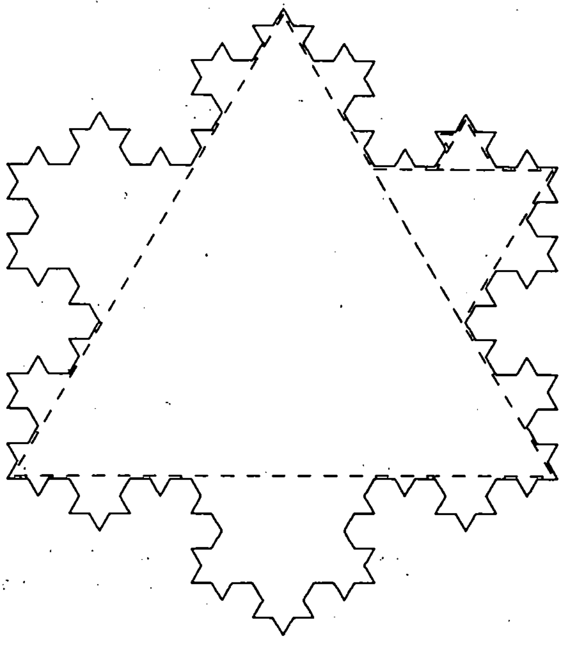

Figure 2

We shall construct a sequence $\{\Omega'_n\}_{n=0}^{\infty}$ of polygonal domains $\Omega'_n \subset\subset \Omega_n$ such that $\Omega'_n \subset\subset \Omega'_{n+1}$ and $\lim_{n\to\infty} \Omega'_n = \Omega$:

Pick an $\varepsilon > 0$, with $\varepsilon < \frac{1}{2}$, say. Let $n \in \mathbb{Z}_+$. For each triangle $\Delta_{i_k}^{(k)}$ ($k = 0, 1, \ldots, n$; $i_k = 0, 1, \ldots, 3 \cdot 4^k$) which is used in the definition of Ω_n, we denote by $\tilde{\Delta}_{i_k}^{(k)}(n)$ the unique triangle which has the following four properties:

1. $\tilde{\Delta}_{i_k}^{(k)}(n)$ is congruent to $\Delta_{i_k}^{(k)}$.

2. $\tilde{\Delta}_{i_k}^{(k)}(n)$ has the same center of mass as $\Delta_{i_k}^{(k)}$.

3. The corners of $\tilde{\Delta}_{i_k}^{(k)}(n)$ are separated from the closest corner of $\Delta_{i_k}^{(k)}$ by the distance $\varepsilon \cdot d_n$.

4. $\tilde{\Delta}_{i_k}^{(k)}(n) \subset \Delta_{i_k}^{(k)}$.

The set Ω'_n is obtained by extending two edges of $\tilde{\Delta}_{i_k}^{(k)}(n)$ to connect it with the appropriate $\tilde{\Delta}_{i_{k-1}}^{(k-1)}(n)$ (define $\Delta_{i_{-1}}^{-1} := \Delta$). Figure 2 should explain how Ω'_n is constructed. It is obvious from the construction that the sequence $\{\Omega'_n\}_{n=0}^{\infty}$ has the desired properties, since for every n, $\tilde{\Delta}_{i_k}^{(k)}(n) \subset\subset \tilde{\Delta}_{i_k}^{(k)}(n+1)$ $(k = 0, 1, \ldots, n;\ i_k = 0, 1, \ldots, 3 \cdot 4^k)$, and $d_n \to 0$ as $n \to \infty$.

By Theorem 6.2 there exists a unique solution of the problem

$$\left.\begin{array}{ll} \Delta u_n(x) = f(u_n(x)) & \text{for } x \in \Omega'_n \\ u_n(x) \to \infty & \text{as } x \to \partial\Omega'_n \end{array}\right\} \qquad (8.1)$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is defined either by $f(t) = t^p$ $(p > 1)$ or by $f(t) = e^t$.

We prove two auxiliary results: Lemmas 8.1 and 8.3.

Lemma 8.1. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions such that u_n solves problem* (8.1) *in Ω'_n $(n \geq 1)$. Then, for every $n \in \mathbb{Z}_+$,*

$$u_n(x) \geq u_{n+1}(x) \geq u_{n+2}(x) \geq \ldots \qquad \textit{for all } x \in \Omega'_n. \qquad (8.2)$$

Remark 8.2. Note that for every $n \in \mathbb{Z}_+$ the set Ω'_{n+1} contains $\bar{\Omega}'_n$. Hence the statement of the lemma makes sense.

Proof of Lemma 8.1. Pick a u_n. Close to $\partial\Omega'_n$ we have $u_{n+1} < u_n$ and u_{n+1} is finite on $\partial\Omega'_n$. Lemma 1.5 (or Lemma 5.4) proves that this inequality actually holds throughout Ω'_n. The statement now follows by induction.

The following lemma is obvious when looking at Figure 3.

Lemma 8.3. *Let z_0 be a corner of $\partial\Omega_{n_0}$ for some $n_0 \in \mathbb{Z}_+$. There exists a polygonal domain K_{n_0} which has less than $2n_0+5$ corners, such that $\Omega \subset K_{n_0}$ and $\partial K_{n_0} \cap \partial\Omega = z_0$. Furthermore, it is possible to find a cut-off outer cone (i.e. a sector) $C_{R(n_0)}(z_0) \subset K_{n_0}$ with vertex at z_0, whose opening angle is $\frac{\pi}{6} - \varepsilon$ with $\varepsilon \in (0, \frac{\pi}{6})$ arbitrary. Here, only the cut-off radius of the sector depends on n_0.*

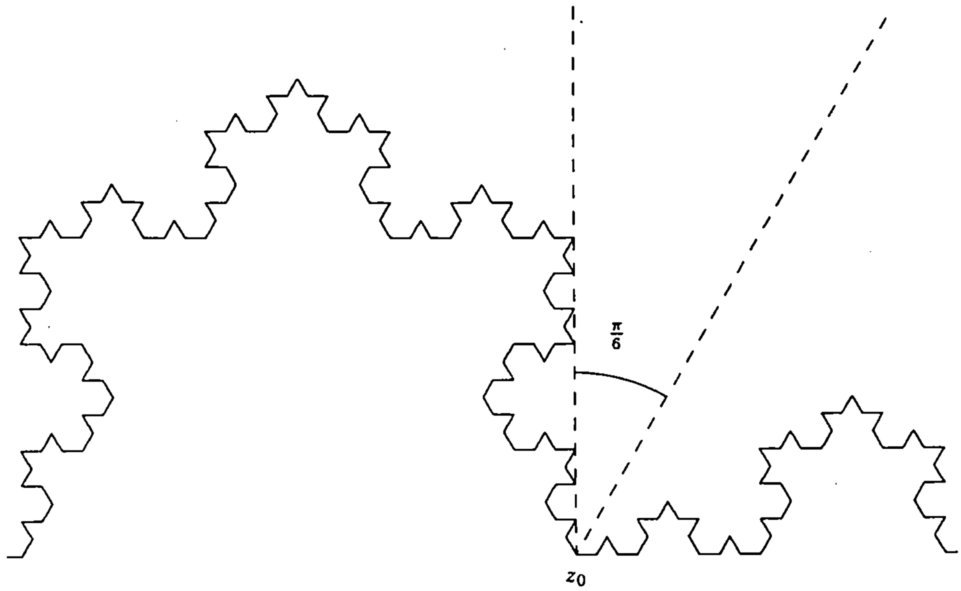

Figure 3

We could say that Lemma 8.3 states that the von Koch snowflake domain satisfies a semi-uniform outer cone condition. This is the key to our existence theorem. The idea is to define $u(x) = \lim_{n\to\infty} u_n(x)$ for every $x \in \Omega$. By Lemma 8.1, the sequence $\{u_n\}_{n=n_0}^{\infty}$ is decreasing on compact subsets of Ω and Lemma 8.3 together with Theorem 6.2 give us a uniform lower bound, with boundary blow-up, of $\{u_n\}_{n=n_0}^{\infty}$ on compact subsets of Ω. Hence there exists a solution of our problem in the snowflake domain.

Theorem 8.4. *Let $\Omega \subset \mathbb{R}^2$ be the bounded von Koch snowflake domain and let either $f(t) = t^p$ $(p > 1)$ or $f(t) = e^t$. Then there exists a positive solution $u \in C^2(\Omega)$ of the problem*

$$\left.\begin{array}{l} \Delta u(x) = f(u(x)) \quad \text{for} \quad x \in \Omega \\ u(x) \to \infty \quad \text{as} \quad x \to \partial\Omega. \end{array}\right\} \tag{8.3}$$

Proof. Let $F \subset \Omega$ be compact. Then there is an integer $n_0 > 0$ such that $F \subset \Omega_n'$ for every integer $n \geq n_0$. Let u_n be the solution of problem (8.1). By Lemma 8.1, the sequence $\{u_n\}_{n=n_0}^{\infty}$ is uniformly bounded on F from above (by u_{n_0}) and decreasing. Now, let $z_m \in \partial\Omega$ be a boundary point of the snowflake domain which is obtained in the construction, i.e. z_m is a corner of $\partial\Omega_m$. Let K_m be a polygonal domain as in Lemma 8.3. Theorem 6.2 says that there exists a unique solution v_m of the boundary-blow-up problem

$$\left.\begin{array}{l} \Delta v_m(x) = f(v_m(x)) \quad \text{for } x \in K_m \\ v_m(x) \to \infty \quad \text{as } x \to \partial K_m. \end{array}\right\}$$

Close to z_m, the function v_m is bounded from below by the solution V_m of the problem

$$\left.\begin{array}{ll}\triangle V_m(x) = f(V_m(x)) & \text{for } x \in C(z_m) \\ V_m(x) \to \infty & \text{as } x \to \partial C(z_m)\end{array}\right\}$$

where $C(z_m)$ is an open sector with vertex at z_m, which is complementary to the outer sector in Lemma 8.3.

But $\Omega'_n \subset\subset K_m$ for every $n \in \mathbb{Z}_+$. Hence, close to the boundary of Ω'_n the inequality $u_n > v_m$ holds and v_m is finite on $\partial\Omega'_n$. Lemma 1.5 (or Lemma 5.4) implies that this inequality is valid throughout Ω'_n, so for $n \geq n_0$ we have $u_n(x) > v_m(x)$ for every $x \in F \subset \Omega'_n$. Hence, $\{u_n\}_{n=n_0}^{\infty}$ is uniformly bounded on F from below by v_m. By the above observations, there is a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ of $\{u_n\}_{n=n_0}^{\infty}$ such that

$$u(x) := \lim_{n_i \to \infty} u_{n_i}(x)$$

is well-defined on F. Our function u solves the problem $\triangle u = f(u)$ on F. Now we may exhaust Ω by an increasing sequence of compact subsets F_i, and define u by taking the limit of a diagonal sequence of functions 'u_{n_i}' as above, to get convergence on each F_i. By construction, the limit function u is of C^2-type on each set F_i and it explodes as x tends to z_m.

The boundary points $\{z_m\}$ form a dense subset of $\partial\Omega$ and according to Theorem 4.4 or Theorem 5.8 (depending on our particular choice of f), there exists a uniform lower blow-up rate of the family of functions $\{V_m\}_{m=1}^{\infty}$ and hence of $\{v_m\}_{m=1}^{\infty}$.

If $z \in \partial\Omega$ is a boundary point which is not a corner of any Ω_n, we use one of the functions

$$W_p(x) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} \cdot |x - z|^{-\frac{2}{p-1}} \tag{8.4}$$

and

$$W_{\exp}(x) = \log 2 - \log|x_1 - z_1|\,|x_2 - z_2| \tag{8.5}$$

as a uniform lower bound for the sequence $\{u_n\}_{n=n_0}^{\infty}$. In (8.5) we use the notation $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Thus there exists a solution of problem (8.3).

The positivity of u is evident for $f(t) = t^p$: $U \equiv 0$ is a subsolution of problem (8.3), so Lemma 1.5 implies $u \geq U$ throughout Ω. For $f(t) = e^t$ we suggest the following argument: Cover Ω by a half disc of radius $R = \frac{2}{\sqrt{3}}$. This is actually a cut-off open sector C_R of opening angle π. Hence $\Omega'_n \subset\subset C_R$ for every $n \in \mathbb{Z}_+$. Furthermore, the function

$$U(r,\theta) = \log\left(\frac{2}{r^2 \sin^2(\theta)}\right) \qquad ((r,\theta) \in (0,R] \times (0,\pi]) \tag{8.6}$$

is the restriction to C_R of the solution of problem $(P)_1$ (Theorem 5.1 and Remark 5.3) in the open sector C which corresponds to C_R. Now, the solution u_n in Ω'_n blows up at $\partial\Omega'_n$ and the restriction of U to Ω'_n is finite on $\partial\Omega'_n$, so Lemma 5.4 implies that $u_n \geq U$ throughout Ω'_n. But from (8.6) it is clear that $U \geq 0$ in C_R. Hence u_n is positive and the limit function u cannot be negative. This concludes the proof of Theorem 8.4 ∎

To estimate the growth of $u(x)$ as x tends to the boundary $\partial\Omega$ of Ω, we use Theorem 4.4 or Theorem 5.8, depending on the choice of $f(t)$.

The following notation is used in Theorems 8.5 and 8.6. If $z = z_n \in \partial\Omega_n$ for some integer n, we use the (local) polar coordinates $x = (r, \theta) = (\text{dist}(x, z_n), \theta)$ for $x \in \Omega$. Let $\delta(\theta)$ denote the Euclidean distance from θ to the closest end point of the subarc of the unit circle which defines the largest interior cut-off open sector $C_R(z_n)$ with vertex at z_n.

Theorem 8.5. *Let $\Omega \subset \mathbb{R}^2$ be the bounded snowflake domain of von Koch and let $f(t) = t^p$ $(p > 1)$. If $u \in C^2(\Omega)$ is a positive solution of problem* (8.3) *and if $z = z_n \in \partial\Omega_n$ for some n, then the estimates*

$$\lim_{r\to 0} \lim_{\delta(\theta)\to 0} u(r,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot \left(r \cdot \delta(\theta)\right)^{\frac{2}{p-1}} \le 2 \tag{8.7}$$

$$\lim_{r\to 0} \lim_{\delta(\theta)\to 0} u(r,\theta) \cdot \left(\frac{p-1}{\sqrt{2(p+1)}}\right)^{\frac{2}{p-1}} \cdot \left(r \cdot \delta(\theta)\right)^{\frac{2}{p-1}} \ge 1 \tag{8.8}$$

hold. For a general $z \in \partial\Omega$ we have the lower bound

$$\lim_{r\to 0} \lim_{\delta(\theta)\to 0} u(r,\theta) \cdot \left(\frac{2}{p-1}\right)^{-\frac{2}{p-1}} \cdot r^{\frac{2}{p-1}} \ge 1. \tag{8.9}$$

Proof. Let $z_n \in \partial\Omega_n$ be a corner of Ω_n, the regular polygonal domain in the n-th step of the construction of the snowflake domain. The z_n form a dense subset of $\partial\Omega$. The upper bound follows from a semi-uniform interior cone condition. There exists an interior cut-off open sector $C_{R_n}(z_n)$ of opening angle $\frac{\pi}{3}$, with its vertex at z_n and cut-off radius $R_n = \frac{1}{2\sqrt{3}} \cdot 3^{-n}$. Let U_n denote the solution of problem (1.7) in $C_{R_n}(z_n)$, given in Proposition 1.8. Lemma 1.5 implies that $u \le U_n$ in $C_{R_n}(z_n)$. The upper bound (8.7) now follows from Proposition 1.8.

To deduce the lower bound we could use Theorem 4.4, but we shall refer to Proposition 4.1, which states that there exists a positive solution v_n of problem (4.1) in $C_{R_n}(z_n)$. By Lemma 1.5, $v_n \le u$ in $C_{R_n}(z_n)$, and the lower bound (8.8) follows if z_n is the corner at an acute angle of Ω_n. If this is not the case, the same argument works for the cut-off open sector $\tilde{C}_{R_n}(z_n)$ of opening angle $\frac{4\pi}{3}$.

Finally, if $z \in \partial\Omega$ is a general boundary point, we use the fact that the restriction to Ω of the function $W_p(x)$ defined in (8.4) is a subsolution of problem (8.3). Thus, Lemma 1.5 implies $u \ge W_p$ which proves (8.9) ∎

Theorem 8.6. *Let $\Omega \subset \mathbb{R}^2$ be the bounded von Koch snowflake domain and let $f(t) = e^t$. If $u \in C^2(\Omega)$ is a positive solution of problem* (8.3) *and if $z = z_n \in \partial\Omega_n$ for some n, then the estimates*

$$\lim_{r\to 0} \lim_{\delta(\theta)\to 0} u(r,\theta) + 4 \cdot \log(r\delta(\theta)) \le \log 4 \tag{8.10}$$

$$\lim_{r\to 0} \lim_{\delta(\theta)\to 0} u(r,\theta) + 2 \cdot \log(r\delta(\theta)) \ge \log 2 \tag{8.11}$$

hold. For a general $z \in \partial\Omega$ we have the lower estimate

$$\lim_{r \to 0} \lim_{\delta(\theta) \to 0} u(r, \theta) + 2 \cdot \log r \geq \log 2. \tag{8.12}$$

Proof. Copy the proof of Theorem 8.5, but use Proposition 5.7, Theorem 5.8, Lemma 5.4 and $W_{exp}(x)$ defined in (8.5) instead of Proposition 1.8, Theorem 4.4, Lemma 1.5 and $W_p(x)$ ∎

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