A Numerical Range for Nonlinear Operators in Smooth Banach Spaces

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Abstract. The purpose of this note is to define a numerical range for nonlinear operators in smooth Banach spaces and to use this numerical range to localize certain spectral sets of Lipschitz continuous operators.

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1. Introduction

By Φ we always denote an operator, i.e. a continuous mapping of a Banach space X into itself, and by I the identity operator on X. Apparently E. H. Zarantonello [21] was the first who introduced the concept of a numerical range for nonlinear Hilbert space operators Φ and proved that for every (real or complex) number λ at a positive distance d from the numerical range of Φ the operator $(\lambda I - \Phi)^{-1}$ exists and is Lipschitz continuous (see Definition 3 below) with minimal Lipschitz constant less d^{-1}. This means that, if we define the spectrum $\Sigma(\Phi)$ of a nonlinear Hilbert space operator Φ to consist of all scalars λ such that the operator $\lambda I - \Phi$ does not have a Lipschitz continuous inverse operator, then $\Sigma(\Phi)$ is contained in the closure of the numerical range of Φ. This result is coherent with the theory of numerical ranges of linear Banach space operators $\mathcal{A}$, for in [20] J. P. Williams showed that the usual spectrum $\sigma(\mathcal{A})$ of a bounded linear operator in a Banach space X is contained in the closure of its numerical range. For a comprehensive survey of the theory of numerical ranges of bounded linear operators in Banach spaces see [4].

Considering this background, it is quite natural that many attempts have been made to extend the concept of numerical range to nonlinear Banach space operators and relate this numerical range to the spectral set $\Sigma(\Phi)$. We mention here the articles by R. A. Verma [16 - 19], A. Rhodius [12 - 14], and F. Pietschmann and A. Rhodius [11]. An approach different to the ones in the above mentioned articles to define a numerical range for nonlinear Hilbert space operators has been made in [6].

The goal of the present article is to define a numerical range for nonlinear operators in smooth Banach spaces using the fact that in a smooth Banach space X for every non-zero $x \in X$ there is a uniquely determined linear functional f_x of norm one satisfying

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$f_x(x) = \|x\|$ (see Definition 7 below). For Banach spaces this is essentially the definition given by A. Rhodius in [12] for nonlinear operators in locally convex spaces except for our additional assumption that X is smooth. Yet, for our purposes the assumption of smoothness is less a disadvantage but a means to present the obtained results in a more illuminating setting and to reduce the technical apparatus.

A similiar approach like ours has been made by R. A. Verma using semi-inner products that may be defined via the above mentioned functionals whose existence is guaranteed by the Hahn-Banach extension theorem. Yet, if X is not smooth, there may be a whole variety of such support functionals inducing the given norm and one has to distinguish between the numerical ranges corresponding to different choices of support functionals. A disadvantage of the work of R. A. Verma is that he does not do so but for the definition of his numerical range, for every non-zero element x is randomly picking out some support functional f_x.

Further we will show that, for every Lipschitz continuous operator Φ in a real, smooth Banach space X, the above mentioned spectral set $\Sigma(\Phi)$ is contained in the closure of its numerical range. For the case of a complex, smooth Banach space X we will prove that $\Sigma(\Phi)$ is contained in the smallest closed Jordan domain containing the numerical range of Φ. Here, as usual, we call a subset of the complex plane a Jordan domain if its complement is connected. Since every compact and convex subset of the complex plane is a Jordan domain, this improves the results obtained by R. A. Verma in [18: Theorem 2.4] and [19: Theorem 2.4]. In a preceding paper (see [16: Theorem 2.4]) R. A. Verma formulated a theorem which states that the closure of his numerical range contains the spectrum, but the proof contains some mistakes and seems to be uncomplete. Yet, this is true for Fréchet differentiable operators (see [12]) and, as we will show later, when Φ is a compact operator. However, for the lack of a counterexample, the author considers it likely that this holds true for Lipschitz mappings in a complex, smooth Banach space, too.

2. Terminology and basic definitions

The purpose of the following is to collect, for the reader's convenience, some basic definitions that will be needed later on.

Definition 1. Let $B \subset X$ be a bounded set. Then we denote the infimum of all numbers $\varepsilon > 0$ such that B admits a finite covering with sets of diameter less than ε by $\alpha(B)$ and call this number the *Kuratowski measure of non-compactness*.

The measure of non-compactness $\alpha(B)$ of a bounded set $B \subset X$ was first considered by Kuratowski (see [8]). It is clear that $\alpha(B)$ is zero if and only if B is a precompact subset of X.

Definition 2. Let X be a Banach space, $\Phi : X \to X$ an operator and

$$[\Phi]_A = \sup_{0 < \alpha(B) < +\infty} \frac{\alpha(\Phi(B))}{\alpha(B)}.$$

If $[\Phi]_A$ is finite, then we call Φ an *α-condensing operator*.

The basic properties of the Kuratowski measure of non-compactness (see, for example, [1]) imply that $[\cdot]_A$ is a semi-norm on the space of all α-condensing operators. Further we have $[\Phi \circ \Psi]_A \leq [\Phi]_A [\Psi]_A$ for all α-condensing operators Φ and Ψ, and $[\Phi]_A = 0$ if and only if Φ is a compact operator (i.e. Φ maps bounded subsets of X into precompact ones).

In addition to $[\Phi]_A$ we will make use of the following operator numbers that are widely used in nonlinear analysis.

Definition 3. Let X be a Banach space. Then for an operator $\Phi : X \to X$ define

$$[\Phi]_l = \inf_{x \neq y} \frac{\|\Phi(x) - \Phi(y)\|}{\|x - y\|} \quad \text{and} \quad [\Phi]_L = \sup_{x \neq y} \frac{\|\Phi(x) - \Phi(y)\|}{\|x - y\|}.$$

If the operator number $[\Phi]_L$ is finite, we call Φ a *Lipschitz continuous operator.*

If Φ is Lipschitz continuous, then the basic properties of $[\cdot]_A$ (see, for example, [1]) and the fact that $\text{diam}\,\Phi(U) \leq [\Phi]_L \text{diam}\,U$ for every bounded subset U of X imply that Φ is an α-condensing operator with $[\Phi]_A \leq [\Phi]_L$.

It is easily seen that $[\Phi]_l$ is a positive number if and only if Φ has a Lipschitz continuous inverse operator Φ^{-1} defined on $\Phi(X)$. In this case we have $[\Phi]_l^{-1} = [\Phi^{-1}]_{L,\Phi(X)}$. Here, for a subset U of X, $[\Phi]_{L,U}$ denotes the number

$$\sup_{\substack{x,y \in U \\ x \neq y}} \frac{\|\Phi(x) - \Phi(y)\|}{\|x - y\|}.$$

It is also obvious that the positiveness of $[\Phi]_l$ implies that Φ maps closed sets into closed sets. We will need this fact later on.

It is easy to see also that the space $Lip_0(X)$ consisting of all Lipschitz continuous operators $\Phi : X \to X$ with $\Phi(\Theta) = \Theta$ and equipped with the norm $[\cdot]_L$ is a Banach space. Here Θ denotes the zero vector in X.

3. Spectral sets of nonlinear operators

The purpose of this section is to define spectral sets for nonlinear operators. This will be done by means of the operator numbers introduced in the previous section. Further, by $\mathbb{K}$ we will denote the scalar field of the Banach space X, i.e. the field $\mathbb{R}$ of the real or $\mathbb{C}$ of complex numbers.

Definition 4. Let $\Phi : X \to X$ be Lipschitz continuous. Then by $\Sigma(\Phi)$ we denote the set of all scalars $\lambda \in \mathbb{K}$ for which the operator $\lambda I - \Phi$ does not have a Lipschitz continuous inverse operator, and call $\Sigma(\Phi)$ the *spectrum* of Φ.

As best as we can say the spectral set $\Sigma(\Phi)$ was first introduced by R. I. Kachurovskij in [7]. The authors of the more recent paper [9] seemed to be unaware of [7]. In [9] I. J. Maddox and A. W. Wickstead proved the following Theorem 1 on $\Sigma(\Phi)$ for Lipschitz continuous operators Φ with $\Phi(\Theta) = \Theta$. But it is clear that the result still holds if we drop the assumption $\Phi(\Theta) = \Theta$.

Theorem 1. *The spectral set $\Sigma(\Phi)$ is a compact subset of $\mathbb{K}$ which is contained in the closed disk of radius $[\Phi]_L$.*

We are now going to define a certain subset of $\Sigma(\Phi)$.

Definition 5. For a Lipschitz continuous operator $\Phi : X \to X$ we call the set

$$\sigma_l(\Phi) = \big\{ \lambda \in \mathbb{K} \,|\, [\lambda I - \Phi]_l = 0 \big\}.$$

the *approximate point spectrum* of Φ.

In [9] I. J. Maddox and A. W. Wickstead showed that $\sigma_l(\Phi)$ is a closed and hence a compact subset of $\Sigma(\Phi)$, if $\Phi \in Lip_0(X)$. Since the functional $[\,\cdot\,]_l$ is continuous from the space of all Lipschitz continuous operators equipped with the semi-norm $[\,\cdot\,]_L$ into the real numbers, $\sigma_l(\Phi)$ is a compact subset of $\Sigma(\Phi)$ for all Lipschitz continuous operators Φ.

It is clear that $\Sigma(\Phi)$ coincides with the usual spectrum of Φ if Φ is a linear operator. In this case it is easily checked that $\sigma_l(\Phi)$ coincides with the approximate point spectrum $\sigma_{ap}(\Phi)$ of Φ, i.e. the set of all scalars $\lambda \in \mathbb{K}$ for which there is a sequence $(x_n)_{n \in \mathbf{N}} \subset X$ of unit vectors with $(\lambda I - \Phi) x_n \to \Theta$ as $n \to \infty$, justifying the name given to $\sigma_l(\Phi)$ in Definition 5. Further it is well known that for a linear operator Φ the boundary $\partial \Sigma(\Phi)$ of $\Sigma(\Phi)$ is contained in $\sigma_{ap}(\Phi)$. We are going to show that this is true, if Φ is nonlinear and Lipschitz continuous. This answers a question raised in [9].

To that end let $\Phi : X \to X$ be Lipschitz continuous, $\lambda \in \mathbb{K} \setminus \Sigma(\Phi)$ and $R_\lambda := (\lambda I - \Phi)^{-1}$. If $d(\lambda) := \inf_{\zeta \in \Sigma(\Phi)} |\zeta - \lambda|$, then by Theorem 1 we have $d(\lambda) > 0$ and $[R_\lambda]_L \geq \frac{1}{d(\lambda)}$. Indeed, suppose $d(\lambda) < [R_\lambda]_L^{-1}$. Then we can find some $\zeta \in \Sigma(\Phi)$ with $|\zeta - \lambda| < [R_\lambda]_L^{-1}$. Let $\mu := (\lambda - \zeta)^{-1}$. Then $|\mu| > [R_\lambda]_L$ and by the Banach fixed point theorem the equation $\mu x - R_\lambda(x) = y$ has a unique solution for every $y \in X$. Thus the operator $\mu I - R_\lambda$ is one-to-one and onto and its inverse mapping S_μ exists. For every $y \in X$ we have

$$\mu S_\mu(y) = R_\lambda(S_\mu(y)) + y.$$

Using this, a straightforward calculation shows that

$$\|S_\mu(y_1) - S_\mu(y_2)\| \leq \frac{1}{|\mu| - [R_\lambda]_L} \|y_1 - y_2\|$$

for all $y_1, y_2 \in X$. This shows that S_μ is Lipschitz continuous. Since we have

$$\zeta I - \Phi = \big(I - (\lambda - \zeta) R_\lambda\big)(\lambda I - \Phi) = (\lambda - \zeta)(\mu I - R_\lambda)(\lambda I - \Phi),$$

this implies that the operator $\zeta I - \Phi$ has a Lipschitz continuous inverse operator, i.e. $\zeta \in \mathbb{K} \setminus \Sigma(\Phi)$ contradicting the fact that $\zeta \in \Sigma(\Phi)$. Thus we have proved the following

Lemma 1. *If $\lambda \in \mathbb{K} \setminus \Sigma(\Phi)$ and $d(\lambda)$ denotes the distance of λ from $\Sigma(\Phi)$, then we have $[\lambda I - \Phi]_l = [R_\lambda]_L^{-1} \leq d(\lambda)$.*

As an immediate consequence of Lemma 1 we obtain the following

Corollary 1. *Let* $\Phi : X \to X$ *be Lipschitz continuous. Then* $\sigma_l(\Phi)$ *contains the boundary of* $\Sigma(\Phi)$.

As in the linear case it is quite difficult to describe the spectral sets $\Sigma(\Phi)$ or $\sigma_l(\Phi)$. For this reason we are now going to define a numerical range for an operator Φ which is easier to handle, and use this to localize the spectral sets introduced above. As already pointed out we are going to improve results recently obtained by R. A. Verma.

4. A numerical range for nonlinear operators

In order to define a numerical range for a nonlinear operator Φ in a Banach space X we are going to restrict ourselves to the following class of Banach spaces (see, for example, [5]).

Definition 6. We call a Banach space X *smooth* if for every $x \in X \setminus \{\Theta\}$ there is a uniquely determined linear functional f_x in the dual space X^* of X such that $\|f_x\|_{X^*} = 1$ and $f_x(x) = \|x\|$.

If X is a real Banach space, then it is well known that X is smooth if and only if its norm is Gâteaux differentiable. For a proof of this fact see, for example, [5]. For the possibility of equivalently renorming a given Banach space X such that the renormed space $\tilde{X}$ is smooth, the reader is also referred to [5].

In the sequel we will make use of the following known result.

Lemma 2 (see [5]). *Let the dual space* X^* *of a Banach space* X *be strictly convex. Then* X *is smooth.*

Here and in the sequel we will always assume that X is a smooth Banach space. Furthermore, if $x \in X \setminus \{\Theta\}$, then by f_x we will denote the uniquely determined linear functional with $\|f_x\|_{X^*} = 1$ and $f_x(x) = \|x\|$.

We give an example. If $p \in (1, +\infty)$, by Lemma 2 we conclude that the space l_p is smooth. Indeed, if $x = (x_i)_{i \in \mathbb{N}} \in l_p \setminus \{\Theta\}$, then

$$f_x(y) = \sum_{k=1}^{\infty} \alpha_k(x)\, y_k \qquad \text{for all} \ \ y = (y_i)_{i \in \mathbb{N}} \in l_p \tag{1}$$

with

$$\alpha_k(x) = \begin{cases} 0 & \text{if } x_k = 0 \\ \dfrac{|x_k|^{p-2}\overline{x_k}}{\|x\|^{p-1}} & \text{if } x_k \neq 0 \end{cases}$$

is the uniquely determined linear functional with $f_x(x) = \|x\|_p$ and $\|f_x\|_q = 1$ where q denotes the number conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 7. Let $\Phi : X \to X$ be an operator. Then we call the set

$$R(\Phi) = \left\{ \frac{f_{x-y}(\Phi(x) - \Phi(y))}{\|x - y\|} \ \middle| \ x, y \in X \ \text{ with } \ x \neq y \right\}$$

the *numerical range* of Φ.

If X is a Hilbert space, then X is norm isomorphic to X^*, i.e. X^* is strictly convex. By Lemma 2, X is smooth. If $x \neq \Theta$, then f_x is given by $f_x(\cdot) = (\cdot \mid \frac{x}{\|x\|})$, where $(\cdot \mid \cdot)$ denotes the scalar product in X. If in addition Φ is a linear operator, then $R(\Phi)$ coincides with the usual numerical range $R(\Phi) = \{(\Phi(x)\mid x)\mid \|x\| = 1\}$ of Φ. In this case it is well known that $R(\Phi)$ is a convex set (see [15]). As already remarked in the introduction in [21] E. H. Zarantonello introduced a numerical range for nonlinear operators coinciding with our definition, if X is a Hilbert space.

Since Φ is continuous, $R(\Phi)$ is a connected subset of $\mathbb{K}$ (see also [4] and [12]). If Φ is Lipschitz continuous, then $R(\Phi)$ is a bounded set and we have

$$R(\Phi) \subseteq \big\{\lambda \in \mathbb{K}\mid |\lambda| \leq [\Phi]_L\big\}.$$

Suppose that for some $\lambda \in \mathbb{K}$ the distance d_λ of λ to $R(\Phi)$ is positive. If T_λ denotes the operator $\lambda I - \Phi$, then for all $x, y \in X$ with $x \neq y$ we have

$$\big|f_{x-y}\big(T_\lambda(x) - T_\lambda(y)\big)\big| \geq d_\lambda\|x - y\|. \tag{2}$$

Since $\|f_{x-y}\|_{X^*} = 1$, we conclude that $[T_\lambda]_l \geq d_\lambda > 0$, i.e. $\lambda \notin \sigma_l(\Phi)$. This yields the following

Corollary 2. *Let the operator* $\Phi : X \to X$ *be Lipschitz continuous. Then* $R(\Phi)$ *is contained in the closed disk of radius* $[\Phi]_L$ *and*

$$\sigma_l(\Phi) \subseteq \overline{R(\Phi)},$$

where $\overline{R(\Phi)}$ *denotes the closure of* $R(\Phi)$.

We are now going to relate the numerical range of a Lipschitz continuous operator Φ to the spectral set $\Sigma(\Phi)$. For this purpose we need the following

Definition 8. We call a closed subset M of the complex plane $\mathbb{C}$ a *Jordan domain* if its complement $\mathbb{C} \setminus M$ is connected.

It is clear that a Jordan domain $M \subset \mathbb{C}$ is necessarily simply connected. This easily follows from the fact that the connectedness of $\mathbb{C} \setminus M$ in $\mathbb{C}$ implies its connectedness in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Definition 9. Let M be a non-empty, bounded subset of the complex plane. Then we call the smallest Jordan domain containing M the *complementarily connected hull* of M and denote this set by $cc(M)$.

We remark that we always have $\overline{M} \subseteq cc(M) \subseteq \overline{co}(M)$, since one easily checks that a compact, convex subset of the complex plane is a Jordan domain. Here, as usual, by $co(M)$ we denote the convex hull of M.

Now let X be a smooth, real Banach space and $\Phi : X \to X$ a Lipschitz continuous operator. Then we have $\Sigma(\Phi) \subseteq \overline{R(\Phi)}$. Indeed, if $\lambda \in \Sigma(\Phi)$ but $\lambda \notin \overline{R(\Phi)}$, then by the connectedness of $R(\Phi)$ there is a $\mu \in \partial\Sigma(\Phi)$ with $\mu \notin \overline{R(\Phi)}$, contradicting Corollaries 1 and 2. Thus we have the following

Theorem 2. *Let X be a smooth, real Banach space and $\Phi : X \to X$ a Lipschitz continuous operator. Then $\Sigma(\Phi)$ is contained in the closure of $R(\Phi)$.*

We are now going to prove that the spectral set $\Sigma(\Phi)$ of a Lipschitz continuous operator Φ in a smooth, complex Banach space is contained in $cc(R(\Phi))$.

Theorem 3. *Let X be a smooth, complex Banach space and $\Phi : X \to X$ a Lipschitz continuous operator. Then $\Sigma(\Phi)$ is contained in $cc(R(\Phi))$.*

Proof. Since Φ is Lipschitz continuous, $R(\Phi)$ is bounded and $cc(R(\Phi))$ is defined. Suppose $\lambda \in \Sigma(\Phi)$ but $\lambda \notin cc(R(\Phi))$, i.e. $\lambda \in \mathbf{C} \setminus cc(R(\Phi)) =: S$. Since S is open and connected, S is arcwise connected. Let K be a closed disk such that $R(\Phi)$ is contained in K. Since K is a Jordan domain, we have $cc(R(\Phi)) \subseteq K$. As $\Sigma(\Phi)$ is bounded, too, we can find a closed disk $K_1 \supset K$ such that $\Sigma(\Phi)$ is contained in K_1. Thus we have $L := \mathbf{C} \setminus \Sigma(\Phi) \supset \mathbf{C} \setminus K_1$ and $S \supset \mathbf{C} \setminus K_1$. Let $\mu \in \mathbf{C} \setminus K_1$. Then $\mu \in L$ and $\mu \in S$. Since S is arcwise connected, there is a continuous mapping $\gamma : [a,b] \to \mathbf{C}$ with $\lambda = \gamma(a)$ and $\mu = \gamma(b)$ such that $P = \{\gamma(t) | t \in [a,b]\}$ is contained in S. Since $\lambda \in \Sigma(\Phi)$ and $\mu \in L$, there is some $t_0 \in [a,b]$ with $\nu = \gamma(t_0) \in \partial\Sigma(\Phi)$. By Corollary 1 we have $\nu \in \sigma_l(\Phi) \subseteq \overline{R(\Phi)} \subseteq cc(R(\Phi))$ contradicting the fact that $\nu \in S$. Thus we have $\Sigma(\Phi) \subseteq cc(R(\Phi))$ ∎

In [9] I. J. Maddox and A. W. Wickstead showed that if X is finite-dimensional, then $\Sigma(\Phi) = \sigma_l(\Phi)$. This is also true, if $\Phi(\Theta) \neq \Theta$. Clearly, every operator Φ in a finite-dimensional Banach space satisfies $[\Phi]_A = 0$, i.e. is compact. We are going to show that this result holds true if X is infinite-dimensional and Φ is compact. To accomplish this we recall the following

Definition 10. We call an operator Φ in a Banach space X *proper* if the inverse image of every compact set in X is compact, too.

It is well-known that the properness of an operator Φ in a Banach space X is equivalent to the fact that the solution sets $S_y = \{x \in X \,|\, \Phi(x) = y\}$ are compact for every $y \in X$ and Φ is a closed mapping, i.e. maps closed sets into closed ones (see, for example, [3]). Using these facts we can prove the following

Theorem 4. *Let X be a Banach space and $\Phi : X \to X$ a compact, Lipschitz continuous operator. Then we have $\Sigma(\Phi) = \sigma_l(\Phi)$. If in addition X is smooth, then $\Sigma(\Phi)$ is contained in the closure of $R(\Phi)$.*

Proof. We only need to prove the theorem for infinite-dimensional Banach spaces and compact operators therein. Since Φ is compact, by definition Φ cannot be proper and hence we must have $[\Phi]_l = 0$. Thus we have $0 \in \sigma_l(\Phi)$. Now assume $\lambda \notin \sigma_l(\Phi)$. Then we have $\lambda \neq 0$, i.e. $[I - \lambda^{-1}\Phi]_l > 0$. By a result of W. V. Petryshyn (see [10: Theorem 8/p. 732]) the compactness of Φ implies that the operator $I - \lambda^{-1}\Phi$, as well as the operator $\lambda I - \Phi$ are ono-to-one and onto. Hence $\lambda \notin \Sigma(\Phi)$. This shows that $\Sigma(\Phi) = \sigma_l(\Phi)$. If X is a smooth Banach space, an application of Corollary 2 yields $\Sigma(\Phi) \subseteq \overline{R(\Phi)}$ ∎

Let us remark that the proof of Theorem 4 shows that the spectrum of a compact, Lipschitz continuous operator in an infinite-dimensional Banach space contains 0 and thus is always non-empty. This result is completely analogous to the linear case. Yet, in

general, the spectral set $\Sigma(\Phi)$ may be empty (see Example 7 in the forthcoming paper [2]). This answers another question raised in [9: p. 105].

Using (2) and the theorem of W. V. Petryshyn cited in the proof of Theorem 4 we immediatly obtain the following

Corollary 3. *Let X be a smooth Banach space and $\Phi : X \to X$ a Lipschitz continuous operator. If $\lambda \in \mathbb{K}$ is at a positive distance d_λ from $R(\Phi)$ and satisfies $|\lambda| \geq [\Phi]_A$, then the operator $(\lambda I - \Phi)^{-1}$ exists and is Lipschitz continuous with $[(\lambda I - \Phi)^{-1}]_L = [\lambda I - \Phi]_l^{-1} \leq d_\lambda^{-1}$.*

5. Some examples

We are now going to illustrate our results by some examples.

Example 1. Let $p \in (1, +\infty)$. Then, as we already remarked, the space l_p is smooth. Let the operator $\Phi : l_p \to l_p$ be defined by

$$\Phi(x) = \|x\| e_1 + Rx, \tag{3}$$

where $e_1 = (1, 0, \ldots)$ and R denotes the right-shift operator in l_p. Then we have $[\Phi]_L = 2^{1/p}$ and $[\Phi]_l = 1$. An easy calculation shows that every $\lambda \in \sigma_l(\Phi)$ satisfies $[\Phi]_l \leq |\lambda| \leq [\Phi]_L$. Thus we have

$$\sigma_l(\Phi) \subseteq \{\lambda \in \mathbb{K} \mid 1 \leq |\lambda| \leq 2^{\frac{1}{p}}\}.$$

Since $\sigma(R) = \{\lambda \in \mathbb{K} \mid |\lambda| \leq 1\}$ and the boundary of $\sigma(R)$ is a subset of $\sigma_{ap}(R)$, for every $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ there is a sequence $(x_n)_{n \in \mathbb{N}} \subset l_p$ of unit vectors such that $(\lambda I - R)x_n \to \Theta$ as $n \to +\infty$. Setting $\tilde{x}_n = \frac{1}{2}x_n$ und $\tilde{y}_n = -\frac{1}{2}x_n$, by (3) and the definition of $[\cdot]_l$ we obtain

$$[\lambda I - \Phi]_l \leq \lim_{n \to +\infty} \|(\lambda I - R)x_n\| = 0,$$

i.e. $\lambda \in \sigma_l(\Phi)$. Now let $\lambda \in \mathbb{K}$ with $1 < |\lambda| \leq 2^{1/p}$. Then $S_\lambda = (\lambda I - R)^{-1}$ is bounded. If $T_\lambda := \lambda I - \Phi$, then we have

$$S_\lambda\big(T_\lambda(x) - T_\lambda(y)\big) = x - y - (\|x\| - \|y\|)S_\lambda e_1. \tag{4}$$

The equation $x_\lambda = S_\lambda e_1$ implies $x_\lambda = \left(\frac{1}{\lambda^k}\right)_{k \in \mathbb{N}} \in X$ and $\|x_\lambda\| = (|\lambda|^p - 1)^{-1/p} \geq 1$. Now let $\Psi_\lambda : l_p \to l_p$ be defined by $\Psi_\lambda(x) = \|x\| x_\lambda$. Since Ψ_λ is not one-to-one, we have $0 \in \sigma_l(\Psi_\lambda)$. Suppose $\mu \in \mathbb{K}$ satisfies $\Psi_\lambda(x) = \mu x$ for some $x \neq \Theta$. Then $|\mu| = \|x_\lambda\|$. If vice versa $|\mu| = \|x_\lambda\|$ and $x := \frac{x_\lambda}{\mu}$, then $\Psi_\lambda(x) = \mu x$. Thus every $\mu \in \mathbb{K}$ with $|\mu| = \|x_\lambda\|$ is in $\sigma_l(\Psi_\lambda)$. Finally let

$$\mu \in \mathbb{K} \text{ with } 0 < |\mu| < \|x_\lambda\|, \qquad \beta = -\frac{\|x_\lambda\| - |\mu|}{2|\mu|}, \qquad \gamma = 1 + |\beta|.$$

If $x := \overline{\mu}\gamma x_\lambda$ and $y := \overline{\mu}\beta x_\lambda$, then we have

$$\frac{\left\|\left(\mu x - \Psi_\lambda(x)\right) - \left(\mu y - \Psi_\lambda(y)\right)\right\|}{\|x - y\|} = 0,$$

i.e. $\mu \in \sigma_l(\Psi_\lambda)$. Since $[\Psi_\lambda]_L = \|x_\lambda\|$, by Theorem 1 we conclude that

$$\sigma_l(\Psi_\lambda) = \left\{\mu \in \mathbb{K}\mid |\mu| \leq \|x_\lambda\|\right\}.$$

As $\|x_\lambda\| \geq 1$, we have $1 \in \sigma_l(\Psi_\lambda)$ for all $\lambda \in \mathbb{K}$ with $1 < |\lambda| \leq 2^{1/p}$. Thus (4) implies that

$$[\lambda I - \Phi]_l \leq [S_\lambda^{-1}]_L \, [I - \Psi_\lambda]_l = 0,$$

i.e. $\lambda \in \sigma_l(\Phi)$. Thus we have

$$\sigma_l(\Phi) = \left\{\lambda \in \mathbb{K}\mid 1 \leq |\lambda| \leq 2^{1/p}\right\} \tag{5}$$

and by Corollary 2 we conclude that $\overline{R(\Phi)} \supseteq \sigma_l(\Phi)$. Now let $a \in \mathbb{K}$ with $|a| \leq 1$ and $x := \left(\overline{a}, 0, (1 - |a|^p)^{1/p}, 0, \dots\right)$. Then we have $\|x\| = 1$ and

$$f_x(\Phi(x)) = \begin{cases} 0 & \text{if } a = 0 \\ |a|^{p-2}a & \text{if } a \neq 0 \end{cases}.$$

Since a is arbitrary, we conclude that $R(\Phi) \supseteq \{\lambda \in \mathbb{K}\mid |\lambda| \leq 1\}$. This and (5) together with Corollary 2 and Theorem 3 yields

$$cc(R(\Phi)) = \overline{R(\Phi)} = \Sigma(\Phi) = \{\lambda \in \mathbb{K}\mid |\lambda| \leq 2^{1/p}\}.$$

Example 2. Let X be the real Banach space $\mathbb{R}^2$ equipped with the Euclidean norm and $\Phi : X \to X$ be defined by $\Phi(x,y) = (x, \arctan y)$. Then Φ is Lipschitz continuous with $[\Phi]_L = 1$. By definition, a number $\lambda \in \mathbb{R}$ is in $R(\Phi)$ if and only if

$$\lambda = \frac{(x - u)^2 + (\arctan y - \arctan v)(y - v)}{(x - u)^2 + (y - v)^2} \tag{6}$$

for some $(x,y), (u,v) \in X$ with $(x,y) \neq (u,v)$. Since

$$\text{sign}(\arctan y - \arctan v) = \text{sign}(y - v) \qquad \text{for all } y, v \in \mathbb{R},$$

every $\lambda \in R(\Phi)$ is positive. Thus, by (6), one easily checks that $R(\Phi) = (0, 1]$. Since

$$\lambda(x,y) - \Phi(x,y) = \left((\lambda - 1)x, \lambda y - \arctan y\right),$$

we have $\Sigma(\Phi) = \{1\} \cup \Sigma(f)$, where $f(x) = \arctan x$. For every $c > 0$ the number $\lambda_c = \frac{f(c)}{c}$ is in $\sigma_l(\Phi)$, i.e. $(0,1) \subset \Sigma(f)$. Now let $\lambda \in \mathbb{R} \setminus \{0\}$ be a negative number. By the mean value theorem we conclude that $[\lambda I - f]_l \geq |\lambda|$ and thus $\lambda \notin \sigma_l(f)$. Furthermore, $(\lambda I - f)'(x) = \lambda - \frac{1}{1+x^2} < 0$ and $|\lambda x - f(x)| \to +\infty$ as $|x| \to +\infty$ implies that $\lambda I - f$ is bijective, i.e. $\lambda \notin \Sigma(f)$. Since $(I - f)'(0) = 0$, we have $1 \in \Sigma(f)$. Thus we conclude that $\Sigma(\Phi) = \Sigma(f) = [0,1]$ and $\overline{R(\Phi)} = \Sigma(\Phi)$.

The following example is to show that $R(\Phi)$ may be considerably bigger than $\Sigma(\Phi)$ and thus may contain a whole variety of points λ for which the operator $\lambda I - \Phi$ has a Lipschitz continuous inverse.

Example 3. Let $p \in (1, +\infty)$, $A = \{a_k \mid k \in \mathbb{N}\}$ a countable, bounded subset of $\mathbb{K}$ and $\Phi_A : l_p \to l_p$ defined by

$$\Phi_A(x_1, x_2, \ldots) = (a_1 x_1, a_2 x_2, \ldots). \tag{7}$$

Then Φ_A is a bounded, linear mapping with $\|\Phi_A\| = \sup_{k \in \mathbb{N}} |a_k|$. Further, as is well known and easily checked, we have

$$\Sigma(\Phi_A) = \overline{A}. \tag{8}$$

By (1) and (7) we have $\lambda \in R(\Phi_A)$ if and only if

$$\lambda = \|x\|^{-p} \sum_{k=1}^{\infty} |x_k|^p a_k \tag{9}$$

for some $x = (x_k)_{k \in \mathbb{N}} \in l_p \setminus \{\Theta\}$. Now let μ_k $(k = 1, \ldots, m)$ be positive numbers with $\sum_{k=1}^{m} \mu_k = 1$ and

$$x := \left(0, \ldots, 0, \mu_1^{1/p}, 0, \ldots, 0, \mu_k^{1/p}, 0 \ldots, 0, \mu_m^{1/p}, 0, \ldots\right)$$

where $\mu_k^{1/p}$ occurs at the n_k-th position. Then we have $\|x\| = 1$ and by (9) we conclude that

$$f_x(\Phi_A x) = \sum_{k=1}^{m} \mu_k a_{n_k} \in R(\Phi_A).$$

This implies that

$$co(A) \subseteq R(\Phi_A). \tag{10}$$

Now let $\lambda \in R(\Phi_A)$. Then there is some $x \in l_p \setminus \{\Theta\}$ with $\lambda = \sum_{k=1}^{\infty} \mu_k a_k$, where $\mu_k = |x_k|^p \|x\|^{-p}$. By definition we have $\sum_{k=1}^{\infty} \mu_k = 1$. Set $M_k = \mu_k$ $(k = 1, \ldots, n)$ and $M_{n+1} = 1 - \sum_{k=1}^{n} \mu_k$, as well as

$$x_n = \left(M_1^{1/p}, \ldots, M_{n+1}^{1/p}, 0, \ldots\right).$$

Then for all $n \in \mathbb{N}$ we have $\|x_n\| = 1$,

$$\lambda_n := f_{x_n}(\Phi_A x_n) = \sum_{k=1}^{n+1} \mu_k a_k \in co(A)$$

and

$$\lambda - \lambda_n = \sum_{k=n+1}^{\infty} \mu_k a_k - M_{n+1} a_{n+1}. \tag{11}$$

Since $\sum_{k=1}^{\infty} \mu_k a_k$ is convergent and A is bounded by (11) we conclude that $\lambda_n \to \lambda$ as $n \to +\infty$. This implies that $\lambda \in \overline{co}(A)$. Using (10) we obtain

$$co(A) \subseteq R(\Phi_A) \subseteq \overline{co}(A). \tag{12}$$

Regarding the special choice $A = \{\frac{1}{k} \mid k \in \mathbb{N}\}$ one sees that $R(\Phi_A)$ is not closed. Thus by (12) we have $R(\Phi_A) \subset \overline{co}(A)$. Yet, again by (12) we obtain

$$\overline{R(\Phi_A)} = \overline{co}(A). \tag{13}$$

Comparing (8) and (13) we see that in general there is a big discrepancy between $\Sigma(\Phi)$ and $\overline{R(\Phi_A)}$. This becomes extremely obvious, if we choose $A = \{-M, M\}$ for some $M > 0$. Then, by (8) and (13) we have $\Sigma(\Phi_A) = \{-M, M\}$ and $\overline{R(\Phi_A)} = R(\Phi_A) = [-M, +M]$.

Remark. As already alluded to in the introduction, for the lack of a counterexample and in view of the above results the author considers it likely that the spectral set $\Sigma(\Phi)$ of a Lipschitz continuous operator in a smooth, complex Banach space is already contained in the closure of $R(\Phi)$.

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