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Representation and Approximation of the Solution of an Initial Value Problem for a First Order Differential Equation in Banach Spaces

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Abstract. An initial value problem $x(0) = x_0$ for the first order differential equation $\dot{x}(t) + Ax(t) = g(t)$ with an unbounded operator coefficient A in a Banach space is considered. Using the Cayley transform we give explicit formulas for the solution of this problem in case the operator A is strongly positive. On the basis of these formulas we propose numerical algorithms for the approximate solution of the initial value problem and give error estimates. The main property of these algorithms is the following: the accuracy of the approximate solutions depends automatically on the "smoothness" of the initial data (the initial vector x_0 and the right-hand side g).

Keywords: *Differential equations in Banach spaces, Cayley transform, Laguerre polynomials, explicit representations of the solution*

AMS subject classification: Primary 65 J 10, secondary 35 A 40, 35 C 10, 35 L 10

1. Introduction

The Cayley transform of an operator A

$$T_\gamma = (\gamma I - A)(\gamma I + A)^{-1},$$

where I is the identity operator and γ is an arbitrary complex number, is well-known in operator theory and posesses many useful properties. For example, if A is a densely defined, strictly dissipative unbounded operator in some Hilbert space H, then the operator T_γ is contractive (see [1, 2, 7] and references cited there). In [1] it was found one more application of the Cayley transform, namely: it was used to represent the exact and an approximate solution of the initial value problem

$$\begin{aligned} \dot{x}(t) + Ax(t) &= 0 \\ x(0) &= x_0 \end{aligned} \tag{1}$$

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where $-A$ is a bounded strictly dissipative operator in Hilbert space. The discrete initial value problem

$$\begin{aligned} y_{\gamma,n+1} &= T_\gamma^n y_{\gamma,n} \quad (n = 0,1,\ldots) \\ y_{\gamma,0} &= x_0 \end{aligned} \tag{2}$$

was regarded together with problem (1). It was shown that the solutions of problems (1) and (2) and the corresponding continuous and discrete semigroups $\{T(t)\}_{t\geq 0}$ and $\{T_\gamma^n\}_{n\geq 0}$, respectively, are connected by the formulas

$$x(t) = T(t)x_0 = \sum_{p=0}^{\infty}(-1)^p\varphi_p(2\gamma t)y_{\gamma,p}$$

$$y_{\gamma,p} = T_\gamma^p y_{\gamma,0} = (-1)^{p+1}\left[\int_0^{+\infty} \psi_n(t)\, x\left(\frac{t}{2\gamma}\right) dt + x_0\right]$$

$$T(t) = \sum_{p=0}^{+\infty}(-1)^p\varphi_p(2\gamma t)\, T_\gamma^p$$

$$T_\gamma^p = (-1)^p\left[\int_0^{+\infty} \psi_p(t)\, T\left(\frac{t}{2\gamma}\right) dt + I\right]$$

where

$$\varphi_p(t) = -\frac{t}{p}\, e^{-\frac{t}{2}}L_{p-1}^{(1)}(t), \quad |\varphi_p(t)| \leq 1 \text{ for all } p \geq 0$$

$$\psi_p(t) = -e^{-\frac{t}{2}}L_{p-1}^{(1)}(t) = e^{-\frac{t}{2}}\frac{d}{dt}\, L_p^{(0)}(t)$$

with Laguerre polynomials $L_p^{(\alpha)}$. The approximate solution x^N of Problem (1) defined by

$$x^N(t) = \sum_{p=0}^{N}(-1)^p\varphi_p(2\gamma t)\, y_{\gamma,p}$$

converges uniformly in t to its exact solution $x = x(t)$ as $N \to +\infty$ with convergence rate of geometric progression with denominator $q_\gamma < 1$ depending on the condition number of the operator A. In [8] these results were extended to the case of an unbounded selfadjoint positive definite operator A with dense domain $D(A)$. There was shown that the approximate solution x^N of problem (1) defined by

$$x^N(t) = T^N(t)\, x_0 = e^{-\gamma t}\sum_{p=0}^{N}(-1)^p L_p^{(0)}(2\gamma t)(y_{\gamma,p} + y_{\gamma,p+1}) \tag{3}$$

is a best approximation for the exact solution x in some Hilbert subspace. The convergence rate is determined by the "smoothness" of x_0 and is of order $O(N^{\theta-\sigma})$ in some special weak norm $\|\cdot\|_\theta$, with $\sigma \geq 0$ provided that $x_0 \in D(A^{\sigma-\frac{1}{2}})$. Further essential improvements were made in [2, 8], where various uniform estimates for the approximate solution (3) for an unbounded operator A in Hilbert and Banach spaces have been proved. In order to find the sequence $\{y_{\gamma,p}\}_{p=0}^{N}$ participating in the construction of the approximate solution x^N of problem (1) one has to solve the recurrence operator equations (with the same operator but with different right-hand sides)

$$\begin{aligned}(\gamma I + A)\, y_{\gamma,p+1} &= (\gamma I - A)\, y_{\gamma,p} \quad (p = 0, 1, \ldots)\\ y_{\gamma,0} &= x_0.\end{aligned} \tag{4}$$

The main features of this discretization technique are the following ones:

1) Decomposition of an evolution problem in a sequence of stationary problems ("elimination" of one variable (variable t)).

2) Automatic dependence of the rate of convergence on the "smoothness" of the initial data or the solution ("spectral property").

3) Exclusively contractive operators are used.

4) The approximate solution can be determined in an analytical form by a hybrid numerical/analytical/computer-algebraic method.

There are a lot of papers concerning the discretization-in-time (decomposition) for evolution problems (see, for example, [3, 6, 12, 15, 16]). But the authors know only a few methods (for example, [4, 18, 19]) with accuracy automatically depending on the smoothness of the solution which are suitable for rather limited classes of problems.

The recurrence equations (4) seem to be similar to the classical Crank-Nicolson difference scheme if we interpret γ as step size, which appears, for example, in [3, 15] as a simplest example of schemes based on the Padé approximation of $e^{-\lambda t}$. But the approximation (3) is distinguished principally from approximations of [3, 15] in the following sense. First of all, the Padé approximations from [3, 15] are discrete in time and local whereas our approximation is global on the whole interval $[0, +\infty)$. Second, one can although construct a Padé approximation of arbitrary accuracy order but in contrast to (3) this order is fixed independent of the smoothness of the solution and in addition provided that the complexity of the algorithm grows.

In the present paper we show that the representation

$$x(t) = T(t)\, x_0 = e^{-\gamma t} \sum_{p=0}^{\infty} (-1)^p L_p^{(0)}(2\gamma t)(y_{\gamma,p} + y_{\gamma,p+1}) \tag{5}$$

for the solution of problem (1) is also valid if the problem is regarded in some Banach space E and A is a densely defined, strongly positive operator. In this case we have the same estimates as obtained in [2, 8, 9] for a selfadjoint positive definite operator A but under slightly stronger assumptions with respect to the initial data. The case of Banach space requires a special method of analysis which is completely different from that of [1, 2, 9] and is based on the idea of strong positivity of unbounded operators and on the infinite Dunford integral.

One of the fundamental problems in the theory of operator semigroups $\{T(t)\}_{t\geq 0}$ is the relation between the semigroup and its infinitesimal generator [11, 14]. From the point of view of applications to partial differential equations it is more interesting to obtain $\{T(t)\}_{t\geq 0}$ from its infinitesimal generator $-A$. The reason for this is that, for $x \in D(A)$, $T(t)\,x_0$ is the solution of the initial value problem (1).

In fact, Section 3 is dedicated in particular to the problem of representing the semigroup $\{T(t)\}_{t\geq 0}$ in terms of its infinitesimal generator; namely there will be given the following new solution of this problem:

$$T(t) = e^{-\gamma t}\sum_{k=0}^{+\infty}(-1)^k L_k^{(0)}(2\gamma t)\, T_\gamma^k(I + T_\gamma)$$

$$T_\gamma^k = (-1)^k \left(\int_0^{+\infty} \psi_k(t)\, T\left(\frac{t}{2\gamma}\right) dt + I\right).$$

For the inhomogeneous problem

$$\begin{aligned} \dot{x}(t) + Ax(t) &= g(t) \\ x(0) &= x_0 \end{aligned} \tag{6}$$

we regard the representation of the solution

$$\begin{aligned} x &= x_1 + x_2 \\ x_1(t) &= T(t)\,x_0 \\ x_2(t) &= \int_0^t T(t-s)\,g(s)\,ds \\ &= \sum_{q=0}^{+\infty}(-1)^p \int_0^t e^{-\gamma(t-s)} L_q^{(0)}(2\gamma(t-s))\, T_\gamma^q(I+T_\gamma)\, g(s)\, ds \end{aligned} \tag{7}$$

and the representation of the approximate solution

$$\begin{aligned} x^N &= x_1^N + x_2^N \\ x_1^N(t) &= T^N(t)\,x_0 \\ x_2^N(t) &= \int_0^t T^N(t-s)\,g(s)\,ds \\ &= \sum_{p=0}^{N}(-1)^p \int_0^t e^{-\gamma(t-s)} L_q^{(0)}(2\gamma(t-s))\, T_\gamma^q(I+T_\gamma)\, g(s)\, ds. \end{aligned} \tag{8}$$

Accuracy estimates for the error $x - x^N$ in various normed spaces are given.

It makes sense to use the approximation (8) if the corresponding integrals can be calculated analytically. In the opposite case we propose another approach based on the interpolation of $g(t)$ with accuracy rate automatically depending on the smoothness of the initial data x_0 and the right-hand side g.

Throughout the paper c denotes various constants which are independent of the parameters under consideration. $\mathbb{P}_N$ will denote the set of polynomials of degree less or equal than N.

2. Basic definitions and preliminary results

We consider the problems (1) and (6) in some Banach space E, where A is supposed to be a densely defined closed linear operator with domain $D(A)$, resolvent set $\varrho(A)$ and spectral set $\Sigma(A)$. We begin with the following definition of a solution of problem (6).

Definition 1. A function $x : [0,\infty) \to E$ is called a *solution* of problem (6) if it is continuous for $t \geq 0$, continuous differentiable for $t > 0$, satisfies equations (6) and $x(t) \in D(A)$ for all $t > 0$.

We will use functions of certain unbounded linear operators A, in particular fractional powers of A. For our purposes we need the following definition of *strong positivity* of A (compare with *sectorial* operators [7, 16], *strongly positive* operators in the sense of [3, 17], and *normally positive* operators [10]; see also [14: p. 69]).

Definition 2. We say that an operator A is *positive*, if

$$\Sigma^+ = \left\{z \in \mathbb{C} : 0 < \varphi \leq |\arg z| \leq \pi\right\} \cup \left\{z \in \mathbb{C} : |z| \leq \gamma\right\} \subset \varrho(A)$$

and

$$\|(z - A)^{-1}\| \leq \frac{M}{1 + |z|} \qquad \text{for all } z \in \Sigma^+$$

for some positive constants φ, γ and M. The lower bound of all such φ, for which the relations above hold, is called the *spectral angle* of the positive operator A and will be denoted by $\varphi(A; E)$ or simply $\varphi(A)$.

Definition 3. A positive operator A is called *strongly positive* if $\varphi(A) < \frac{\pi}{2}$.

In what follows we assume the operator A to be strongly positive. Let Γ be a closed path in the complex plane $\mathbb{C}$ which consists of two rays

$$S(\pm\varphi) = \left\{\varrho e^{\pm i\varphi} : \gamma \leq \varrho \leq +\infty\right\}$$

and of the circular arc

$$\left\{z \in \mathbb{C} : |z| = \gamma,\ |\arg z| \leq \varphi,\ \varphi(A) < \varphi < \frac{\pi}{2}\right\}.$$

The domain Ω_Γ bounded by Γ contains the spectrum of A. If $M = 1$ and $\varphi = \frac{\pi}{2}$, then $-A$ is the infinitesimal generator of a C_0-semigroup [14: p. 69]. If $\varphi(A) < \frac{\pi}{2}$, i.e. the operator A is strongly positive, then $-A$ is the infinitesimal generator of an analytic semigroup [14: p. 69]. For an analytic function $f = f(z)$ in Ω_Γ one can define the operator $f(A)$ by

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-A)^{-1} dz$$

where the orientation of Γ is chosen so that the spectrum of A lies on the left. In particular, for $\sigma > 0$ we have

$$A^{-\sigma} = \frac{1}{2\pi i} \int_\Gamma z^{-\sigma}(z-A)^{-1} dz$$

where $z^{-\sigma}$ is taken to be positive for real positive values of z. If $\sigma = n$ is an integer, then using the residue theorem it follows that the integral equals A^{-n}. Thus, for positive integer values of σ the definition of $A^{-\sigma}$ above coincides with the classical definition of $(A^{-1})^n$. The operator A^σ $(\sigma > 0)$ is defined as $(A^{-\sigma})^{-1}$. The domain $D^\sigma = D(A^\sigma)$ of the operator A^σ becomes a Banach space with the norm $\|x\|_{D^\sigma} = \|A^\sigma x\|_E$ (see [17]).

Example. Let $1 < p < +\infty$ and let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. Let

$$A(x,D)\,u = \sum_{|\alpha| \le 2m} a_\alpha(x)\, D^\alpha u$$

be a strongly elliptic differential operator in Ω, i.e. there exists a constant $c > 0$ such that

$$\mathrm{Re}(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\,\xi^\alpha \ge c|\xi|^{2m}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$. The coefficients $a_\alpha = a_\alpha(x)$ are assumed to be sufficiently smooth in $\overline{\Omega}$, for example $a_\alpha \in C^{2m}(\overline{\Omega})$ or $a_\alpha \in C^\infty(\overline{\Omega})$. With a strongly elliptic operator $A(x,D)$ we associate a linear (unbounded) operator A_p in $L^p(\Omega)$ as follows:

$$\begin{aligned} D(A_p) &= W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega) \\ A_p u &= A(x,D)\,u \quad \text{for } u \in D(A_p). \end{aligned}$$

The domain $D(A_p)$ of A_p contains $C_0^\infty(\Omega)$ and is therefore dense in $L^p(\Omega)$. Moreover, from the fundamental inequality

$$\|u\|_{2m,p} \le c\big(\|Au\|_{0,p} + \|u\|_{0,p}\big) \qquad \text{for all } u \in D(A_p)$$

it follows that A_p is a closed operator in $L^p(\Omega)$. From [14: Theorem 3.2] it also follows that A_p is a strongly positive operator and the operator $-A_p$ is the infinitesimal generator of an analytic semigroup on $L^p(\Omega)$ [14: Theorem 3.5]. The same is also true in the cases $p = 1$ and $p = +\infty$ [14: pp. 217 - 218] if we define

$$D(A_\infty) = \left\{ u \,\middle|\, \begin{array}{l} u \in W^{2m,p}(\Omega)\ \forall p > n,\ A(x,D)u \in L^\infty(\Omega) \\ D^\beta u = 0 \ \text{ on } \ \partial\Omega \ \text{ for } \ 0 \le |\beta| < m \end{array} \right\}$$

$$A_\infty u = A(x,D)u \quad \text{for } u \in D(A_\infty)$$

and

$$D(A_1) = \left\{ u \,\middle|\, u \in W^{2m-1,1}(\Omega) \cap W_0^{m,1}(\Omega) \ \text{ and } \ A(x,D)u \in L^1(\Omega) \right\}$$

$$A_1 u = A(x,D)u \quad \text{for } u \in D(A_1)$$

where $A(x,D)u$ is understood in the sense of distributions.

Let $A(x,D)$ be the symmetric second order differential operator given by

$$A(x,D)\,u = -\sum_{k,l=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kl}(x)\,\frac{\partial u}{\partial x_l}\right) \tag{11}$$

where the coefficients $a_{kl} = a_{lk}$ are real-valued and continuously differentiable functions in $\overline{\Omega}$. We assume that $A(x,D)$ is strongly elliptic, i.e. that there is a constant $c_0 > 0$ such that

$$\sum_{k,l=1}^{n} a_{kl}(x)\,\xi_k \xi_l \geq c_0 \sum_{k=1}^{n} \xi_k^2 = c_0 |\xi|^2$$

for all real ξ_k $(k = 1, \ldots, n)$ and $x \in \overline{\Omega}$. Analogously as above we associate with the operator A defined by (11) an operator A_p on $L^p(\Omega)$ $(1 < p < +\infty)$. The operator $-A_p$ is the infinitesimal generator of an analytic semigroup of contractions on $L^p(\Omega)$ and the Hille-Yosida theorem yields that A_p is strongly positive (see [14: pp. 8 and 214 - 215]).

We will make certain estimates in some weak norms which we define below.

Let E be a Banach space and E^* its dual space of continuous linear functionals on E. Let $\mathcal{F} = \{f_k\}_{k=1}^{+\infty} \subset E^*$ be a total family of functionals, i.e. from $f_k(x) = 0$ $(k \in \mathbb{N})$ for some $x \in E$ it follows that $x = 0$. In every separable Banach space there is a complete minimal family $\{e_k\}_{k\in\mathbb{N}}$ such that the corresponding biorthogonal functionals form a total family $\{f_k\}_{k\in\mathbb{N}}$. Without loss of generality one can assume that

$$(F_p)^p = \sum_{k=1}^{+\infty} \|f_k\|_{E^*}^p < +\infty \qquad (p \geq 1). \tag{12}$$

We define the normed space G_p^σ by

$$G_p^\sigma = \left\{ x \in D^\sigma \,\middle|\, \|x\|_{G_p^\sigma} = \left\{ \sum_{k=1}^{\infty} |\langle A^\sigma x, f_k\rangle|^p \right\}^{\frac{1}{p}} < +\infty \right\}$$

where $\langle \cdot,\cdot \rangle$ denotes the bracket representing the duality between E and E^*. From this definition it follows easily that

$$\|x\|_{G_p^\sigma} \leq F_p \, \|x\|_{D^\sigma},$$

i.e. D^σ is imbedded into G_p^σ. If $E = H$ is a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$, $p = 2$ and $f_k(x) = (x, e_k)$ $(x \in H)$, then one can omit condition (12). In this case, we have due to the Parseval identity

$$\|x\|_{G_2^\sigma} = \left\{ \sum_{k=1}^{\infty} |(A^\sigma x, f_k)|^2 \right\}^{\frac{1}{2}} = \|A^\sigma x\|_H = \|x\|_{D^\sigma},$$

i.e. $G_2^\sigma = D^\sigma$.

We denote by $\mathcal{E}_p^\theta$ the normed space of all functions $x = x(t)$ with values in $D(A^\theta)$ with finite norm

$$\|x\|_{\mathcal{E}_p^\theta} = \left\{ \sum_{j,k=1}^{+\infty} \left| \int_0^{+\infty} \langle A^\theta x, f_j \rangle \sqrt{2\gamma}\, e^{-\gamma t} L_{k-1}(2\gamma t)\, dt \right|^p \right\}^{\frac{1}{p}}.$$

Let

$$x^\theta(s) = A^\theta x(s) \qquad \text{and} \qquad x_k^\theta = \sqrt{2\gamma} \int_0^{+\infty} e^{-\gamma s} L_k^{(0)}(2\gamma s)\, x^\theta(s)\, ds,$$

i.e. x_k^θ are the Fourier coefficients with respect to the orthonormal family

$$\left\{ \sqrt{2\gamma}\, e^{-\gamma s} L_{k-1}^{(0)}(2\gamma s) \right\}_{k=1}^{+\infty}.$$

We denote by $\bar{\mathcal{E}}_p^\theta$ the space of all functions $x = x(t)$ with values in $D(A^\theta)$ with finite norm

$$\|x\|_{\bar{\mathcal{E}}_p^\theta} = \left\{ \sum_{k=1}^{+\infty} \|x_k^\theta\|_E^p \right\}^{1/p}.$$

It is easy to see that $\bar{\mathcal{E}}_p^\theta$ is embedded into $\mathcal{E}_p^\theta$ and

$$\|x\|_{\mathcal{E}_p^\theta} \le F_p \|x\|_{\bar{\mathcal{E}}_p^\theta}.$$

If $E = H$ is a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$, $p = 2$ and $f_k(x) = (x, e_k)$ $(x \in H)$, then one can omit again condition (12). In this case we have due to the Parseval identity

$$\|x\|_{\mathcal{E}_2^\theta} = \left\{ \sum_{k=1}^{+\infty} \left\| \int_0^{+\infty} \sqrt{2\gamma}\, e^{-\gamma t} L_{k-1}^{(0)}(2\gamma t)\, A^\theta x(t)\, dt \right\|_H^2 \right\}^{\frac{1}{2}} = \left\{ \sum_{k=1}^{+\infty} \|x_k^\theta\|_H^2 \right\}^{\frac{1}{2}} = \|x\|_{\bar{\mathcal{E}}_2^\theta},$$

i.e. $\mathcal{E}_2^\theta = \bar{\mathcal{E}}_2^\theta = \mathcal{H}^\theta$, where $\mathcal{H}^\theta$ is the space with the scalar product

$$(x, y)_{\mathcal{H}^\theta} = \int_0^{+\infty} \left(A^\theta x(t), A^\theta y(t) \right)_H dt.$$

We define for any $\kappa \geq 0$ the following weighted spaces:

$$H^0_{e^{-\kappa t}} = L_{e^{-\kappa t}}$$

$$= \left\{ \varphi : [0,\infty) \to \mathbb{R} \text{ measurable} \,\middle|\, \|\varphi\|_{0,\kappa} = \left\{ \int_0^{+\infty} e^{-\kappa t}\varphi^2(t)\,dt \right\}^{\frac{1}{2}} < +\infty \right\}$$

$$H^m_{e^{-\kappa t}} = \left\{ \varphi \in H^0_{e^{-\kappa t}} \,\middle|\, \|\varphi\|_{m,\kappa} = \left\{ \sum_{k=0}^{m} \|\varphi^{(k)}\|^2_{0,\kappa} \right\}^{\frac{1}{2}} < +\infty \right\}$$

$$\mathcal{E}^{\mu,0}_\kappa = \left\{ g : [0,\infty) \to H \,\middle|\, \|g\|_{\mathcal{E}^{\mu,0}_\kappa} = \int_0^\infty e^{-\kappa t}\|A^\mu g\|^2_H\,dt < +\infty \right\}$$

$$\mathcal{E}^{\mu,m}_\kappa = \left\{ g \,\middle|\, \|g\|^2_{\mathcal{E}^{\mu,m}_\kappa} = \sum_{k=0}^{m} \left\| \frac{d^k g(t)}{dt^k} \right\|^2_{\mathcal{E}^{\mu,0}_\kappa} < +\infty \right\}$$

where H is a Hilbert space and A is an operator in H. Via interpolation one can define $H^p_{e^{-\kappa t}}$ with norm $\|\cdot\|_{p,\kappa}$ for real $p \geq 0$ [13].

3. Representation of the solution of a homogeneous initial value problem

In this section we will justify the representation (5) for the solution of the homogeneous problem (1). Simultaneously, we will consider the series

$$\tilde{x}(t) = e^{-\gamma t} \sum_{p=0}^{\infty} (-1)^p L^{(0)}_p(2\gamma t)(y_{\gamma,k+1} - y_{\gamma,k}) \tag{13}$$

which one obtains by formal differentiation of (5) using the formula (see [5])

$$\frac{d}{dt}\left(L^{(\alpha)}_k(t) - L^{(\alpha)}_{k-1}(t) \right) = -L^{(\alpha)}_{k-1}(t).$$

First of all, we collect some properties of the series (5) and (13) in the next auxiliary statement.

Lemma 1. *Let A be a densely defined, strongly positive linear operator in some Banach space E and $x_0 \in D(A^\sigma)$. Then:*

1) *$\sigma > 0$ implies the uniform convergence of the representation* (5) *of $x = x(t)$ with respect to $t \in [0,+\infty)$ and x is continuous on $[0,+\infty)$.*

2) *$\sigma > 1$ implies the uniform convergence of the representation* (13) *of $\tilde{x} = \tilde{x}(t)$ in E with respect to $t \in [0,\infty)$, $\tilde{x}$ is continuous on $[0,\infty)$ and $\tilde{x}(t) = \dot{x}(t)$ for all $t \geq 0$.*

3) *$\sigma > 1$ implies $x(t) \in D(A)$ for all $t \geq 0$.*

Proof. We have

$$\begin{aligned} y_{\gamma,k}+y_{\gamma,k+1} &= \frac{1}{2\pi i}\int_\Gamma \left(\frac{\gamma-z}{\gamma+z}\right)^k \left(1+\frac{\gamma-z}{\gamma+z}\right)(z-A)^{-1}x_0\,dz \\ &= \frac{\gamma}{\pi i}\int_\Gamma \left(\frac{\gamma-z}{\gamma+z}\right)^k \frac{1}{(\gamma+z)\,z^\sigma}(z-A)^{-1}x_0^\sigma\,dz \end{aligned} \tag{14}$$

and

$$y_{\gamma,k}-y_{\gamma,k+1} = \frac{1}{\pi i}\int_\Gamma \left(\frac{\gamma-z}{\gamma+z}\right)^k \frac{1}{(\gamma+z)\,z^{\sigma-1}}(z-A)^{-1}x_0^\sigma\,dz$$

where $x_0^\sigma = A^\sigma x_0$. Using the strong positivity of the operator A we get from (14)

$$\begin{aligned} &\|y_{\gamma,k}+y_{\gamma,k+1}\| \\ &\le \frac{2\gamma}{\pi}\Bigg\| \mathrm{Im}\, e^{i\varphi}\int_\gamma^{+\infty}\left(\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right)^k \frac{1}{(\gamma+\varrho e^{i\varphi})(\varrho e^{i\varphi})^\sigma}(\varrho e^{i\varphi}-A)^{-1}x_0^\sigma\,d\varrho \\ &\quad + \mathrm{Re}\, i\int_0^\varphi \left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^k \frac{1}{(1+e^{i\theta})(\gamma e^{i\theta})^\sigma}(\gamma e^{i\theta}-A)^{-1}x_0^\sigma\,d\theta\Bigg\| \\ &= \frac{2\gamma}{\pi}\left(\int_\gamma^\infty \left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^k \frac{1}{|\gamma+\varrho e^{i\varphi}|\,\varrho^\sigma(1+\varrho)}\,d\varrho \right. \\ &\quad \left. + \int_0^\varphi \left|\frac{1-e^{i\theta}}{1+e^{i\theta}}\right|^k \frac{1}{|1+e^{i\theta}|(1+\gamma)}\,d\theta\right)\|x_0^\sigma\|. \end{aligned} \tag{15}$$

Simple computations show that

$$\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^2 = \frac{\gamma^2+\varrho^2-2\gamma\varrho\cos\varphi}{\gamma^2+\varrho^2+2\gamma\varrho\cos\varphi} \le \frac{\varrho-\gamma\cos\varphi}{\varrho+\gamma\cos\varphi}. \tag{16}$$

The function

$$\psi(\varrho) = \left[\frac{\varrho-\gamma\cos\varphi}{\varrho+\gamma\cos\varphi}\right]^k \varrho^{-\tau} \qquad (\varrho\ge\gamma,\ \tau>0)$$

satisfies for large k the inequality

$$\begin{aligned} \max_{\varrho\in[\gamma,\infty)}\psi(\varrho) &= \left(\frac{\tau^{-1}(k+\sqrt{k^2+\tau^2})-1}{\tau^{-1}(k+\sqrt{k^2+\tau^2})+1}\right)^k \left(\frac{\gamma\cos\varphi}{\tau}\left(k+\sqrt{k^2+\tau^2}\right)\right)^{-\tau} \\ &\le \frac{c(\gamma,\tau,\varphi)}{k^\tau}. \end{aligned} \tag{17}$$

Using (16) and (17) we get from (15)

$$
\begin{aligned}
&\|y_{\gamma,k} + y_{\gamma,k+1}\| \\
&\le \frac{2\gamma}{\pi}\left(\int_{\gamma}^{+\infty} \left(\frac{\varrho - \gamma\cos\varphi}{\varrho + \gamma\cos\varphi}\right)^{\frac{k}{2}} \frac{1}{\sqrt{\gamma^2 + \varrho^2(1+\varrho)\varrho^{\sigma}}}\, d\varrho \right. \\
&\quad \left. + \frac{1}{2(1+\gamma)} \int_0^{\varphi} \tan^k \frac{\theta}{2} \cos^{-1}\frac{\theta}{2}\, d\theta \right) \|x_0^{\sigma}\| \\
&\le c\left(\int_{\gamma}^{+\infty} \left(\frac{\varrho - \gamma\cos\varphi}{\varrho + \gamma\cos\varphi}\right)^{\frac{k}{2}} \varrho^{-(2+\sigma)} d\varrho + \tan^k \frac{\varphi}{2} \right) \|x_0^{\sigma}\| \\
&\le c\left(\max_{\varrho\in[\gamma,+\infty)} \left(\frac{\varrho - \gamma\cos\varphi}{\varrho + \gamma\cos\varphi}\right)^{\frac{k}{2}} \varrho^{-(1+\sigma-\delta)} \int_{\gamma}^{+\infty} \frac{d\varrho}{\varrho^{1+\delta}} + \tan^k \frac{\varphi}{2} \right) \|x_0^{\sigma}\| \\
&\le \frac{c}{k^{1+\sigma-\delta}} \|x_0^{\sigma}\|
\end{aligned}
\tag{18}
$$

where δ is an arbitrary small number from the interval $(0,\sigma)$. Analogously, one obtains

$$
\begin{aligned}
\|y_{\gamma,k} - y_{\gamma,k+1}\| &\le \frac{2\gamma}{\pi}\left(\int_{\gamma}^{+\infty} \left(\frac{\varrho - \gamma\cos\varphi}{\varrho + \gamma\cos\varphi}\right)^{\frac{k}{2}} \frac{1}{\sqrt{\gamma^2 + \varrho^2(1+\varrho)\,\varrho^{\sigma-1}}}\, d\varrho \right. \\
&\quad \left. + \frac{1}{2(1+\gamma)} \int_0^{\varphi} \tan^k \frac{\theta}{2} \cos^{-1}\frac{\theta}{2}\, d\theta \right) \|x_0^{\sigma}\| \\
&\le \frac{c}{k^{\sigma-\delta}} \|x_0^{\sigma}\|.
\end{aligned}
\tag{19}
$$

If $\sigma > 1$, then

$$
Ax^N(t) = e^{-\gamma t} \sum_{k=0}^{N} (-1)^k L_k^{(0)}(2\gamma t)\, T_\gamma^k (I + T_\gamma)\, Ax_0.
$$

From the estimates

$$
\begin{aligned}
\|T_\gamma^k (I + T_\gamma)\, Ax_0\| &= \left\| \frac{1}{\pi i} \int_{\Gamma} \left(\frac{\gamma - z}{\gamma + z}\right)^k \frac{1}{(\gamma + z)\, z^{\sigma-1}} (z - A)^{-1} dz \right\| \\
&\le \frac{c}{k^{\sigma-\delta}} \|x_0^{\sigma}\|
\end{aligned}
$$

and (see [5])

$$
e^{-\frac{t}{2}} |L_k^{(0)}(t)| \le 1 \qquad \text{for all } \; t \in [0,\infty) \tag{20}
$$

it follows that the series

$$x_A(t) = e^{-\gamma t}\sum_{k=0}^{\infty}(-1)^k L_k^{(0)}(2\gamma t)T_\gamma^k(I+T_\gamma)Ax_0 = \lim_{N\to\infty} Ax^N(t) \tag{21}$$

converges uniformly with respect to $t \in [0,+\infty)$ provided that $\sigma > 1$. The uniform convergence of the series (5) on $[0,+\infty)$ and the continuity of its sum $x = x(t)$ follow from the estimates (18) and (20) provided that $\sigma > 0$. If $\sigma > 1$, then the estimates (19) and (20) imply the uniform convergence of the series (13) with respect to $t \in [0,+\infty)$, the continuity of its sum $\tilde{x} = \tilde{x}(t)$ and $\dot{x}(t) = \tilde{x}(t)$ for all $t \geq 0$. The uniform convergence of the series (5) and (21) under the assumption $\sigma > 1$ and the closedness of the strongly positive operator A yield $x(t) \in D(A)$ for all $t \in [0,+\infty)$. The proof is complete ∎

The assumptions of Lemma 1 can be weakened if we consider a finite interval $[\varepsilon,\omega] \subset (0,+\infty)$ instead of $[0,+\infty)$.

Lemma 2. *Let A be a densely defined strongly positive linear operator and $x_0 \in D(A^\sigma)$. Then:*

1) *The series (5) converges in E uniformly in $t \in [\varepsilon,\omega]$ and its sum $x = x(t)$ is continuous on $[\varepsilon,\omega]$ provided that $\sigma \geq -\frac{1}{4}$.*

2) *The series (13) converges in E uniformly in $t \in [\varepsilon,\omega]$, its sum $\tilde{x} = \tilde{x}(t)$ is continuous on $[\varepsilon,\omega]$ and $\tilde{x}(t) = \dot{x}(t)$ provided that $\sigma > \frac{3}{4}$.*

3) *$x(t) \in D(A)$ for all $t \in [\varepsilon,\omega]$ if $\sigma > \frac{3}{4}$.*

Proof. The proof is similar to that of Lemma 1 if one takes into account the expansion (see [5])

$$L_k^{(\alpha)}(t) = \pi^{-\frac{1}{2}} e^{\frac{t}{2}} t^{-\frac{\alpha}{2}-\frac{1}{4}} k^{\frac{\alpha}{2}-\frac{1}{4}} \left(\cos\left[2(kt)^{\frac{1}{2}} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + (nt)^{-\frac{1}{2}}O(1)\right) \tag{22}$$

where $\alpha > -1$, $ck^{-1} \leq t \leq \omega$ and $c = \text{const} > 0$. It follows from (22) that

$$\left|e^{-\frac{t}{2}} L_k^{(0)}(t)\right| \leq ck^{-\frac{1}{4}} \tag{23}$$

uniformly in $t \in [\varepsilon,\omega]$. Hence the series (5), (13) and (21) are majorized by the number series $c\sum_{k=1}^{+\infty} k^{-(\frac{5}{4}+\sigma-\delta)}$ and $c\sum_{k=1}^{\infty} k^{-(\frac{1}{4}+\sigma-\delta)}$ uniformly on $[\varepsilon,\omega]$ and the statements of the lemma follow ∎

We are now in a position to show that the series (5) represents the solution of problem (1):

Theorem 1. *Let A be a densely defined, strongly positive linear operator in some Banach space E and $x_0 \in D(A^\sigma)$ with $\sigma > \frac{3}{4}$. Then the function $x = x(t)$ given by (5) is the only solution for the Cauchy problem (1).*

Proof. It follows from Lemmas 1 and 2 that under our assumptions the function $x : [0,+\infty) \to E$ given by (5) is continuous for $t \geq 0$, continuous differentiable for $t > 0$

and $x(t) \in D(A)$ for all $t > 0$. It remains to show that x satisfies equation (1). We have

$$\begin{aligned}\dot{x}(t) - Ax(t) &= e^{-\gamma t} \sum_{k=0}^{+\infty} (-1)^k L_k^{(0)}(2\gamma t) \int_\Gamma z^{1-\sigma} \left(\frac{\gamma - z}{\gamma + z}\right)^k \\ &\quad \times \left(\gamma - \gamma \frac{\gamma - z}{\gamma + z} - z - z\,\frac{\gamma - z}{\gamma + z}\right) (z - A)^{-1} x_0^\sigma \, dz \\ &= 0.\end{aligned}$$

Since $-A$ is the infinitesimal generator of an analytical semigroup we get from [14: Theorem 1.4] the uniqueness of the solution ∎

Remark 1. Because of Lemma 1 it makes sense to consider the series (5) also for $x_0 \in D(A^\sigma)$ with $\sigma > -\frac{1}{4}$. The solution $x = x(t)$ given by (5) for $\sigma \in (-\frac{1}{4}, \frac{3}{4}]$ is a generalized solution.

4. Approximation of the solution of a homogeneous initial value problem

In this section we study the truncated sum (3) as an approximate solution of problem (1), exactly speaking, the convergence of x^N to the exact solution x as $N \to \infty$ in various norms. We start with the following algorithm.

Algorithm 1 (Numerical approach to the solution of problem (1) based on the approximation (3)).

1. Input N and set $y_{\gamma,0} = x_0$.
2. For $k = 1$ to $k = N + 1$ solve the operator equations (with the same operator but with various right-hand sides)

$$(\gamma I + A)\,\overline{y}_{\gamma,k} = y_{\gamma,k-1}$$

and find

$$y_{\gamma,k} = (\gamma I - A)\,\overline{y}_{\gamma,k}.$$

3. Input t and find $x^N(t)$ in accordance with (3).

The next theorem states the accuracy of this approximation as $N \to \infty$.

Theorem 2. *Let A be a densely defined, strongly positive operator and $x_0 \in D(A^\sigma)$ for $\sigma > 0$. Then*

$$\|x^N(t) - x(t)\| \le c\, N^{-\sigma+\delta} \|x_0^\sigma\|$$

uniformly in $t \in [0, \infty)$, where $x_0^\sigma = A^\sigma x_0$ and δ is an arbitrary number from the interval $(0, \sigma)$.

Proof. Using the estimates (18) and (20) we get

$$\begin{aligned}\|x^N(t) - x(t)\| &= \left\| e^{-\gamma t} \sum_{k=N+1}^{+\infty} (-1)^k L_k^{(0)}(2\gamma t)(y_{\gamma,k} + y_{\gamma,k+1}) \right\| \\ &\le c \sum_{k=N+1}^{+\infty} k^{-(1+\sigma-\delta)} \|x_0^\sigma\| \\ &\le c N^{-\sigma+\delta} \|x_0^\sigma\|.\end{aligned}$$

and the assertion is proved ∎

Making use of estimates (18) and (23) one can analogously prove the following result.

Theorem 3. *Let A be a densely defined, strongly positive operator and $x_0 \in D(A^\sigma)$ for $\sigma > -\frac{1}{4}$. Then*

$$\|x(t) - x(t)\| \le c N^{-\sigma-\frac{1}{4}+\delta} \|x_0^\sigma\|$$

uniformly in $t \in [\varepsilon, \omega]$ where $[\varepsilon, \omega]$ is an arbitrary closed finite subinterval in $(0, +\infty)$.

As we have mentioned before the domain $D(A^\theta)$ of the operator A^θ becomes a Banach space D^θ with the norm $\|x\|_{D^\theta} = \|A^\theta x\|_E$. In this space we have, for example,

$$\begin{aligned}\|y_{\gamma,k} + y_{\gamma,k+1}\|_{D_\theta} &= \left| \frac{\gamma}{\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z} \right)^k \frac{1}{(\gamma + z) z^{\sigma-\theta}} (z - A)^{-1} x_0^\sigma \, dz \right\| \\ &\le \frac{c}{k^{1+\sigma-\theta-\delta}} \|x_0^\sigma\|\end{aligned}$$

and

$$\begin{aligned}\|y_{\gamma,k} - y_{\gamma,k-1}\|_{D^\theta} &= \left\| \frac{1}{\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z} \right)^k \frac{1}{(\gamma + z) z^{\sigma-\theta-1}} (z - A)^{-1} x_0^\sigma \, dz \right\| \\ &\le \frac{c}{k^{\sigma-\theta-\delta}} \|x_0^\sigma\|\end{aligned}$$

where δ is an arbitrary small positive number. Using these estimates we get in an analogous way as in the proofs of Theorem 1 – 3 the following estimates in the norm of D^θ.

Theorem 4. *Let A be a densely defined, strongly positive operator and $x_0 \in D(A^\sigma)$ for $\sigma > 0$. Then, for an arbitrary small positive δ,*

$$\|x^N(t) - x(t)\|_{D^\theta} \le c N^{-(\sigma-\theta-\delta)} \|x_0^\sigma\|$$

uniformly in $t \in [0, \infty)$ and

$$\|x^N(t) - x(t)\|_{D^\theta} \le c N^{-\sigma-\frac{1}{4}+\theta+\delta} \|x_0^\sigma\|$$

uniformly in $t \in [\varepsilon, \omega] \subset (0, \infty)$ where $[\varepsilon, \omega]$ is an arbitrary closed finite subintervall in $(0, +\infty)$.

We conclude this section with estimates in some weak norm, namely in the norm of $\mathcal{E}_p^\theta$.

Theorem 5. *Let A be a densely defined, strongly positive operator in a separable Banach space E and $x_0 \in D(A^\sigma)$. Then the function x given by (5) belongs to $\mathcal{E}_p^{\bar\sigma}$ with $\bar\sigma = \sigma + \frac{p-1}{p} - \delta$ and*

$$\|x^N - x\|_{\mathcal{E}_p^\theta} \le cN^{-(\bar\sigma-\theta)}\|x_0^\sigma\| \tag{24}$$

where $p \in (1, +\infty)$ and δ is an arbitrary small positive number.

Proof. The family $\{\sqrt{2\gamma}\, e^{-\gamma t} L_{k-1}^{(0)}(2\gamma t)\}_{k=1}^{+\infty}$ is orthonormal on $(0, +\infty)$. Therefore, we have for $\mu \ge 0$ fixed, $p \in (1, \infty)$, $\delta > 0$ arbitrary small and $\bar\sigma = \sigma + \frac{p-1}{p} - \delta$

$$\begin{aligned}
\|x\|_{\mathcal{E}_p^\sigma}^p &= \frac{1}{(2\gamma)^{\frac{p}{2}}} \sum_{j,k=1}^{+\infty} |\langle A^{\bar\sigma}(y_{\gamma,k-1} + y_{\gamma,k}), f_j\rangle|^p \\
&= (2\gamma)^{\frac{p}{2}} \sum_{j,k=1}^{+\infty} \left| \frac{1}{2\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z}\right)^{k-1} \frac{z^{\bar\sigma-\sigma}}{\gamma + z} \langle (z - A)^{-1} x_0^\sigma, f_j\rangle\, dz \right|^p \\
&= \left(\frac{\sqrt{2\gamma}}{\pi}\right)^p \sum_{j,k=1}^{+\infty} \Bigg| \operatorname{Im} \frac{e^{i\varphi}}{i} \int_\gamma^{+\infty} \left(\frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}}\right)^{k-1} \\
&\quad \times \frac{\varrho^{\bar\sigma-\sigma} e^{i(\bar\sigma-\sigma)\varphi}}{\gamma + \varrho e^{i\varphi}} \langle (\varrho e^{i\varphi} - A)^{-1} x_0^\sigma, f_j\rangle\, d\varrho \\
&\quad + \operatorname{Re} \int_0^\varphi \left(\frac{1 - e^{i\theta}}{1 + e^{i\theta}}\right)^{k-1} \frac{\gamma^{\bar\sigma-\sigma} e^{i(\bar\sigma-\sigma)\theta}}{1 + e^{i\theta}} \langle (\gamma e^{i\theta} - A)^{-1} x_0^\sigma, f_j\rangle\, d\theta \Bigg|^p \\
&\le \left(\frac{\sqrt{2\gamma}}{\pi} M F_p\right)^p \sum_{k=1}^{+\infty} \Bigg\{ \int_\gamma^{+\infty} \left|\frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}}\right|^{k-1} \frac{\varrho^{\bar\sigma-\sigma}}{|\gamma + \varrho e^{i\varphi}|} \frac{d\varrho}{1 + \varrho} \\
&\quad + \frac{\gamma^{\bar\sigma-\sigma}}{1 + \gamma} \int_0^\varphi \left|\frac{1 - e^{i\theta}}{1 + e^{i\theta}}\right|^{k-1} \frac{1}{|1 + e^{i\theta}|} d\theta \Bigg\}^p \|x_0^\sigma\|^p \\
&\le c \sum_{k=1}^{+\infty} \Bigg\{ \int_\gamma^{+\infty} \left|\frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}}\right|^{p(k-1)} \frac{\varrho^{p-1-\delta}}{|\gamma + \varrho e^{i\varphi}|^p (1 + \varrho)} d\varrho \left(\int_\gamma^{+\infty} \frac{\varrho^{-\delta}}{1 + \rho} d\varrho\right)^{\frac{p}{p'}} \\
&\quad + \left(\int_0^\varphi \tan^{k-1}\frac{\theta}{2} \cos^{-1}\frac{\theta}{2}\, d\theta\right)^p \Bigg\} \|x_0^\sigma\|^p \\
&\le c \Bigg\{ \int_\gamma^{+\infty} \frac{\varrho^{p-2-\delta}}{|\gamma + \varrho e^{i\varphi}|^p - |\gamma - \varrho e^{i\varphi}|^p} d\varrho \\
&\quad + \sum_{k=1}^{+\infty} \left(\int_0^\varphi \tan^{p(k-1)}\frac{\theta}{2}\, d\theta\right) \left(\int_0^\varphi \frac{d\theta}{\cos^{p'}\frac{\theta}{2}}\right)^{\frac{p}{p'}} \Bigg\} \|x_0^\sigma\|^p
\end{aligned} \tag{25}$$

where $p \in (1,+\infty)$ and $\frac{1}{p}+\frac{1}{p'}=1$. Since $\tan\frac{\theta}{2} \le \tan\frac{\varphi}{2} < 1$ and

$$\lim_{\varrho\to+\infty}\left((\gamma^2+\varrho^2+2\gamma\varrho\cos\varphi)^{\frac{p}{2}}-(\gamma^2+\varrho^2-2\gamma\varrho\cos\varphi)^{\frac{p}{2}}\right)\varrho^{1-p}=\mathrm{const} \qquad (26)$$

we further get from (25)

$$\|x\|_{\mathcal{E}_p^\theta} \le c\left(\int_\gamma^{+\infty}\varrho^{-1-\delta}d\varrho+1\right)\|x_0^\sigma\|^p,$$

i.e. $x \in \mathcal{E}_p^{\bar\sigma}$ provided that $x_0 \in D(A^\sigma)$. For the approximate solution x^N we have the following estimate in the norm of $\mathcal{E}_p^\theta$:

$$\begin{aligned}
\|x^N-x\|_{\mathcal{E}_p^\theta}^p &\le c\,\|x_0^\sigma\|\left\{\sum_{k=N+1}^{+\infty}\left(\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{k-1}\frac{\varrho^{\theta-\sigma}}{|\gamma+\varrho e^{i\varphi}|(1+\varrho)}\,d\varrho\right.\right.\\
&\qquad\left.\left.+\frac{\gamma^{\theta-\sigma}}{1+\gamma}\int_0^\varphi\left|\frac{1-e^{i\xi}}{1+e^{i\xi}}\right|^{k-1}\frac{1}{|1+e^{i\xi}|}\,d\xi\right)^p\right\}\\
&\le c\,\|x_0^\sigma\|\left\{\sum_{k=N+1}^{+\infty}\left(\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{p(k-1)}\right.\right.\\
&\qquad\times\frac{\varrho^{p(\theta-\sigma)+(p-1)\delta}}{|\gamma+\varrho e^{i\varphi}|^p(1+\varrho)}\,d\varrho\cdot\left(\int_\gamma^{+\infty}\frac{\varrho^{-\delta}}{1+\varrho}\,d\varrho\right)^{\frac{p}{p'}}\\
&\qquad\left.\left.+\int_0^\varphi\tan^{p(k-1)}\frac{\xi}{2}\,d\xi\left(\int_0^\varphi\frac{d\xi}{\cos^{p'}\frac{\xi}{2}}\right)^{\frac{p}{p'}}\right)\right\}\\
&\le c\,\|x_0^\sigma\|^p\left\{\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{pN}\right.\\
&\qquad\left.\times\frac{\varrho^{p(\theta-\sigma)+(p-1)\delta-p}}{\left(|\gamma+\varrho e^{i\varphi}|^p-|\gamma-\varrho e^{i\varphi}|^p\right)\varrho^{1-p}}\,d\varrho+\tan^{pN}\frac{\varphi}{2}\right\}.
\end{aligned}$$

Using (16), (17) and (26) one obtains further

$$\begin{aligned}
&\|x^N-x\|_{\mathcal{E}_p^\theta}^p\\
&\quad\le c\,\|x_0^\sigma\|^p\left\{\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{pN}\varrho^{-p(\sigma-\theta+\frac{p-1}{p}-\delta)}\varrho^{-1-\delta}d\varrho+\tan^{pN}\frac{\varphi}{2}\right\}\\
&\quad\le c\,N^{-p(\sigma-\theta+\frac{p-1}{p}-\delta)}\|x_0^\sigma\|^p\left\{\int_\gamma^{+\infty}\varrho^{-1-\delta}\,d\varrho+1\right\}
\end{aligned}$$

completing the proof of Theorem 5 ∎

Remark 2. In the case $p = 2$ and $E = H$ being a Hilbert space with an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ and functionals $f_k(x) = (x, e_k)$ the spaces $\mathcal{E}_2^\theta$ and $\bar{\mathcal{E}}_2^\theta$ coincide with $\mathcal{H}^\theta$ and (24) takes the form

$$\|x^N - x\|_{\mathcal{H}^\theta} = \left(\int_0^{+\infty} \|x^N(t) - x(t)\|_H^2 \, dt \right)^{\frac{1}{2}} \leq cN^{-\sigma-\frac{1}{2}+\delta+\theta} \|x_0^\sigma\|.$$

Remark 3. If $\sigma = +\infty$, then the convergence rate of the approximate solution x^N to the exact solution of problem (1) is exponential, i.e. for every $r > 0$ it holds

$$\lim_{N\to+\infty} N^r \|x^N - x\| = 0 \tag{27}$$

for any of the norms considered above. Really, we have for $\max_{\varrho\in[\gamma,+\infty)} \psi(\varrho)$ from (17)

$$\lim_{k\to+\infty} k^r \max_{\varrho\in[\gamma,+\infty)} \psi(\varrho) = \left(\frac{\tau}{2e\gamma\cos\varphi}\right)^\tau \lim_{k\to+\infty} \frac{1}{k^{\tau-r}} = 0 \qquad \text{for all } \tau > r$$

what implies (27). An interpretation of the case $\sigma = +\infty$ for a Cauchy problem for a homogeneous parabolic partial differential equation is to assume the initial function to be infinitely differentiable.

5. Representation and approximation of the solution of an inhomogeneous initial value problem

In this section we study the inhomogeneous problem (6). We show that under appropriate assumptions the representation (7) of its solution x is valid. We will also be interested in imposing conditions on the right-hand side g so that the solution x belongs to corresponding spaces. Further we give various estimates of the approximate solution x^N as defined by (8). We will assume throughout this section that A is a densely defined, strongly positive operator so that the corresponding homogeneous equation has a unique solution.

Let $L^1(0, T_0; E)$ be the Banach space of Bochner integrable functions $g : [0, T_0] \to E$ ($T_0 \leq +\infty$) with norm

$$\|g\|_{L^1} = \int_0^{T_0} \|g(s)\|_E \, ds.$$

If $g \in L^1(0, T_0; E)$, then for every $x_0 \in E$ the initial value problem (6) has at most one solution, and if it has a solution, then the solution is given by (see [14: pp. 105 - 106])

$$x(t) = T(t)\, x_0 + \int_0^t T(t-s)\, g(s)\, ds.$$

We have shown that in (7)

$$x_1(t) = T(t)\,x_0,$$

where T given by (5) is the solution of the homogeneous equation provided that $x_0 \in D(A^\sigma)$ with $\sigma > \frac{3}{4}$. Thus it is sufficient to consider the second summand $x_2(t)$ in (7). Together with (7) we consider the series

$$\begin{aligned}\tilde{x}_2(t) &= \sum_{k=0}^{\infty}(-1)^k T_\gamma^k (I + T_\gamma)\, g(t) \\ &\quad + \sum_{k=0}^{+\infty}(-1)^k \int_0^t e^{-\gamma(t-s)} L_k^{(0)}(2\gamma(t-s))\, T_\gamma^k (I - T_\gamma)\, g(s)\, ds \\ &=: x_{2,1}(t) + x_{2,2}(t)\end{aligned} \tag{28}$$

which one obtains by formal differentiation of the series (7). If $g = g(t)$ is a function with values in $D(A^\sigma)$ $(\sigma > 0)$, then analogous as in Section 3 one can prove that the series representing $x_2(t)$ and $x_{2,1}(t)$ converge uniformly in $t \in [0, T_0]$. Therefore,

$$\begin{aligned}x_{2,1}(t) &= \sum_{k=0}^{+\infty}(-1)^k T_\gamma^k g(t) + \sum_{k=0}^{+\infty}(-1)^k T_\gamma^{k+1} g(t) \\ &= \sum_{k=0}^{+\infty}(-1)^k T_\gamma^k g(t) - \sum_{k=0}^{+\infty}(-1)^k T_\gamma^k g(t) + g(t) \\ &= g(t).\end{aligned} \tag{29}$$

The series representing $x_{2,2}(t)$ can be studied analogous to statement 2 of Lemma 1. Then we get that this series converges uniformly in $t \in [0, T_0]$ provided that

$$g(t) \in D(A^\sigma) \quad \text{and} \quad A^\sigma g(t) \in L^1(0, T_0; E) \qquad (t \in [0, T_0];\ \sigma > 1).$$

Similary it can be shown that $x_2(t) \in D(A)$ $(t \in [0, T_0])$ if $\sigma > 1$. Thus the following statement holds true.

Lemma 3. *Let A be a densely defined, strongly positive linear operator in a Banach space E and*

$$g(t) \in D(A^\sigma) \quad \text{and} \quad A^\sigma g(t) \in L^1(0, T_0; E) \qquad (t \in [0, T_0]).$$

Then the following assertions are true:

1) *The series (7) for $x_2(t)$ converges in E uniformly in $t \in [0, T_0]$ and x_2 is continuous on $[0, T_0]$ provided that $\sigma > 0$.*

2) *The series (28) for $\tilde{x}_2(t)$ converges in E uniformly in $t \in [0, T_0]$, $\tilde{x}_2$ is continuous on $[0, T_0]$ and $\dot{x}_2 = \tilde{x}_2$ provided that $\sigma > 1$.*

3) *If $\sigma > 1$, then $x_2(t) \in D(A)$ for all $t \in [0, T_0]$.*

The assumptions of Lemma 3 can be weakened if we consider our series on an interval $[\varepsilon, \omega]$ with arbitrary ε and ω such that $0 < \varepsilon < \omega < +\infty$ and if we use the estimates (22) and (23). As a consequence, we get

Lemma 4. *Let A be a densely defined, strongly positive linear operator and*

$$g(t) \in D(A^\sigma) \quad and \quad A^\sigma g(t) \in L^1(0,T_0;E) \qquad (t \in [0,T_0]).$$

Then the following assertions are true:

1) *The series* (7) *for $x_2(t)$ converges in E uniformly in $t \in [\varepsilon,\omega]$ and x_2 is continuous on $[\varepsilon,\omega]$ provided that $\sigma > -\frac{1}{4}$.*

2) *The series* (28) *for $\tilde{x}_2(t)$ converges in E uniformly in $t \in [\varepsilon,\omega]$, $\tilde{x}_2$ is continuous on $[\varepsilon,\omega]$ and $\dot{x}_2 = \tilde{x}_2$ provided that $\sigma > \frac{3}{4}$.*

3) *$x_2(t) \in D(A)$ for all $t \in [\varepsilon,\omega]$ provided that $\sigma > \frac{3}{4}$.*

Now we turn to conditions on the initial data x_0 and the right-hand side g which will ensure that the solution x of problem (6) can be represented by (7).

Let J be an interval. A function $g : J \to E$ is *Hölder continuous with exponent* $\theta \in (0,1)$ on J if there is a constant L such that

$$\|g(t) - g(s)\| \le L\,|t-s|^\theta \qquad \text{for all } \ s,t \in J.$$

It is *locally* Hölder continuous if every $t \in J$ has a neighbourhood in which g is Hölder continuous. It is easy to check that if J is compact, then g is Hölder continuous on J if it is locally Hölder continuous. The family of all Hölder continuous functions with exponent θ is denoted by $C^\theta(J;E)$.

Theorem 6. *Let A be a densely defined, strongly positive linear operator in a Banach space E and*

$$g(t) \in D(A^\sigma) \quad and \quad A^\sigma g(t) \in L^1(0,T_0;E) \qquad (t \in [0,T_0])$$

with $\sigma > \frac{3}{4}$. Then the function x given by (7) *is a solution of problem* (6). *If g is locally Hölder continuous on $(0,T]$, then this solution is unique.*

Proof. Obviously, it is sufficient to show that x_2 satisfies $\dot{x}_2(t) + Ax_2(t) = g(t)$ and $x_2(0) = 0$. Using (7), (28) and (29) we get

$$\begin{aligned}
\dot{x}_2(t) - Ax_2(t) &= g(t) + \sum_{k=0}^{+\infty}(-1)^k \int_0^t e^{-\gamma(t-s)} L_k^{(0)}(2\gamma(t-s)) \\
&\quad \times \int_\Gamma \left(\frac{\gamma - z}{\gamma + z}\right)^k \frac{1}{(\gamma+z)\,z^{\sigma-1}} (z-A)^{-1} A^\sigma g(s)\,dz\,ds \\
&\quad - \sum_{k=0}^{+\infty}(-1)^k \int_0^t e^{-\gamma(t-s)} L_k^{(0)}(2\gamma(t-s)) \\
&\quad \times \int_\Gamma \left(\frac{\gamma - z}{\gamma + z}\right)^k \frac{1}{(\gamma+z)\,z^{\sigma-1}} (z-A)^{-1} A^\sigma g(s)\,dz\,ds \\
&= g(t),
\end{aligned}$$

i.e. $x = x_1 + x_2$ is a solution of problem (6). The uniqueness follows from Corollary 3.3 (see [14: p. 113]). The proof is complete ∎

Let us now consider the approximate solution x^N of problem (6) given by (8). The following two statements can be proved analogously to Theorems 2 - 4 and their proofs are therefore omitted.

Theorem 7: *Let A be a densely defined, strongly positive linear operator in a Banach space E and*

$$g(t) \in D(A^\sigma) \quad \text{and} \quad A^\sigma g(t) \in L^1(0,T_0;E) \qquad (t \in [0,T_0])$$

with $\sigma > \frac{3}{4}$. Then, for the exact solution x and approximate solution x^N of problem (6),

$$\|x^N(t) - x(t)\|_{D^\theta} \le cN^{-\sigma+\theta+\delta}(\|A^\sigma x_0\| + \|A^\sigma g\|_{L^1})$$

uniformly in $t \in [0,T_0]$, where δ is an arbitrary small number such that $\sigma - \delta \ge \theta \ge 0$.

Theorem 8. *Let A be a densely defined, strongly positive linear operator in a Banach space and*

$$g(t) \in D(A^\sigma) \quad \text{and} \quad A^\sigma g(t) \in L^1(0,T_0;E) \qquad (t \in [0,T_0])$$

with $\sigma > -\frac{3}{4}$. Then, for the exact solution x and approximate solution x^N of problem (6),

$$\|x^N(t) - x(t)\|_{D^\theta} \le cN^{-\sigma+\theta-\frac{1}{4}+\delta}(\|A^\sigma x_0\| + \|A^\sigma g\|_{L^1})$$

uniformly in $t \in [\varepsilon,\omega]$ with $\sigma + \frac{1}{4} - \delta \ge \theta \ge 0$, where $[\varepsilon,\omega]$ is an arbitrary closed finite subintervall in $(0,T_0]$.

A regularity result and the error estimates for the case when the right-hand side g in problem (6) belongs to some space with weak norm come next.

Theorem 9. *Let A be a densely defined, strongly positive operator in some separable Banach space E, $x_0 \in D(A^\sigma)$ and $g \in \bar{\mathcal{E}}_p^\beta$ where $\beta = \sigma - \frac{1}{p}$ with $p \in (1,+\infty)$. Then $x \in \mathcal{E}_p^{\bar\sigma}$ and the estimate*

$$\|x^N - x\|_{\mathcal{E}_p^\theta} \le c\left(N^{\theta-\bar\sigma}\|A^\sigma x_0\| + N^{\theta-\bar\sigma+\frac{1}{p}}\right)\|g\|_{\bar{\mathcal{E}}_p^\beta}$$

holds where $\bar\sigma = \sigma + \frac{p-1}{p} - \delta$ and $\bar\sigma \ge \theta \ge 0$ with an arbitrary small positive number δ.

Proof. In Theorem 5 we have shown that $x_1 \in \mathcal{E}_p^{\bar\sigma}$ and

$$\|x_1^N - x_1\|_{\mathcal{E}_p^\theta} \le cN^{\theta-\bar\sigma}\|A^\sigma x_0\|.$$

So it remains only to show that $x_2 \in \mathcal{E}_p^{\bar\sigma}$ and

$$\|x_2^N - x_2\|_{\mathcal{E}_p^\theta} \le c\,N^{\theta-\bar\sigma+\frac{1}{p}}\|g\|_{\bar{\mathcal{E}}_p^\beta}. \tag{30}$$

Let

$$g(s) = \sum_{l=0}^{+\infty} g_l\sqrt{2\gamma}\,e^{-\gamma s}L_l^{(0)}(2\gamma s)$$

where

$$g_l = \sqrt{2\gamma} \int_0^{+\infty} e^{-\gamma s} L_l^{(0)}(2\gamma\xi)\, g(\xi)\, d\xi$$

are the Fourier coefficients of g. Then we have

$$\begin{aligned} \|x_2\|^p_{\mathcal{E}^\varrho_p} = \sum_{k,j=1}^{+\infty} \Bigg| \sum_{l,q=0}^{+\infty} (-1)^q \int_0^{+\infty}\int_0^t e^{-\gamma(t-s)} L_q^{(0)}(2\gamma(t-s)) \\ \times \sqrt{2\gamma}\, e^{-\gamma t} L_{k-1}^{(0)}(2\gamma t) \sqrt{2\gamma}\, e^{-\gamma s} L_l^{(0)}(2\gamma s)\, ds dt \\ \times \left\langle A^{1-\delta} T_\gamma^q (I + T_\gamma)\, A^\beta g_l, f_j \right\rangle \Bigg|^p \end{aligned} \tag{31}$$

where $\{f_k\}_{k\in\mathbb{N}}$ is a total family of functionals from E^*. It is easy to verify that

$$\begin{aligned} I_{q,k,l} &:= \int_0^{+\infty}\int_0^t e^{-2\gamma t} L_q^{(0)}(2\gamma(t-s))\, L_{k-1}^{(0)}(2\gamma t)\, L_l^{(0)}(2\gamma s)\, ds dt \\ &= \frac{1}{2\gamma} \int_0^{+\infty} e^{-\tilde t} L_{k-1}^{(0)}(\tilde t) \int_0^{\tilde t} L_q^{(0)}(\tilde t - \tilde s) L_l^{(0)}(\tilde s)\, d\tilde s d\tilde t \\ &= \frac{1}{2\gamma} \int_0^{+\infty} e^{-\tilde t} L_{k-1}^{(0)}(\tilde t) \big(L_{q+l}^{(0)}(\tilde t) - L_{q+l+1}^{(0)}(\tilde t) \big)\, d\tilde t \\ &= \frac{1}{2\gamma} \left[\delta_{k-1,q+l} - \delta_{k-1,q+l+1} \right] \end{aligned}$$

where δ_{ij} is the Kronecker symbol. Denoting $g^\alpha = A^\alpha g$ we have further from (31)

$$\begin{aligned} \|x_2\|^p_{\mathcal{E}^\varrho_p} &= \sum_{k,j=1}^{+\infty} \left| \left\langle A^{1-\delta} (T_\gamma^{k-1} + T_\gamma^k)\, g_{k-1}^\beta, f_j \right\rangle \right|^p \\ &= \sum_{k,j=1}^{+\infty} \left| \frac{1}{2\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z} \right)^{k-1} \frac{2\gamma z^{1-\delta}}{\gamma + z} \left\langle (z - A)^{-1} g_{k-1}^\beta, f_j \right\rangle dz \right|^p \\ &\le M^p F_p^p \left(\frac{2\gamma}{\pi} \right)^p \sum_{k=1}^{+\infty} \Bigg| \int_\gamma^{+\infty} \left| \frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}} \right|^{k-1} \frac{\varrho^{1-\delta}}{|\gamma + \varrho e^{i\varphi}|} \frac{1}{1+\varrho}\, d\varrho \\ &\quad + \int_0^\varphi \left| \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right|^{k-1} \frac{\gamma^{1-\delta}}{1+\gamma} \frac{1}{|1 + e^{i\theta}|}\, d\theta \Bigg|^p \|g_{k-1}^\beta\|_E^p \\ &\le c \sum_{k=1}^{\infty} \left(\int_\gamma^{+\infty} \frac{\varrho^{1-\delta}}{\varrho^2}\, d\varrho + 1 \right)^p \|g_{k-1}^\beta\|_E^p \\ &\le c\, \|g\|^p_{\tilde{\mathcal{E}}^\beta_p}. \end{aligned} \tag{32}$$

Therefore, we have proved that $x_2 \in \mathcal{E}_p^{\bar{\sigma}}$ provided that $g \in \bar{\mathcal{E}}_p^{\beta}$.

Now we can prove the estimate (30). Analogously to (32), we get

$$
\begin{aligned}
&\|x_2^N - x_2\|_{\mathcal{E}_p^{\theta}}^p \\
&= \sum_{j=1}^{+\infty} \sum_{k=N+1}^{+\infty} \left| \left\langle A^{\theta-\beta}(T_\gamma^{k-1} + T_\gamma^k) g_{k-1}^{\beta}, f_j \right\rangle \right|^p \\
&= \sum_{j=1}^{+\infty} \sum_{k=N+1}^{+\infty} \left| \frac{1}{2\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z} \right)^{k-1} \frac{2\gamma z^{\theta-\beta}}{\gamma + z} \left\langle (z - A)^{-1} g_{k-1}^{\beta}, f_j \right\rangle dz \right|^p \\
&\le M^p F_p^p \left(\frac{\gamma}{\pi} \right)^p \sum_{k=N+1}^{+\infty} \left(\int_\Gamma \left| \frac{\gamma - z}{\gamma + z} \right|^{k-1} \frac{|z|^{\theta-\beta}}{|\gamma + z|} \frac{1}{1 + |z|} |dz| \right)^p \|g_{k-1}^{\beta}\|_E^p \\
&\le c \|g\|_{\bar{\mathcal{E}}_p^{\beta}}^p \sum_{k=N+1}^{+\infty} \left(\int_\Gamma \left| \frac{\gamma - z}{\gamma + z} \right|^{k-1} \frac{|z|^{\theta-\beta}}{|\gamma + z|} \frac{1}{1 + |z|} |dz| \right)^p \\
&\le c \|g\|_{\bar{\mathcal{E}}_p^{\beta}}^p \sum_{k=N+1}^{+\infty} \left\{ \left(\int_\gamma^{+\infty} \left| \frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}} \right|^{k-1} \frac{\varrho^{\theta-\bar{\sigma}+1-\delta}}{|\gamma + \varrho e^{i\varphi}|} \frac{d\varrho}{1 + \varrho} \right)^p \right. \\
&\quad \left. + \left(\int_0^{\varphi} \left| \frac{1 - e^{i\theta}}{1 + e^{i\theta}} \right|^{k-1} \frac{\gamma^{\theta-\bar{\sigma}+1-\delta}}{|1 + e^{i\theta}|} \frac{d\theta}{1 + \gamma} \right)^p \right\} \qquad (33) \\
&\le c \|g\|_{\bar{\mathcal{E}}_p^{\beta}}^p \sum_{k=N+1}^{+\infty} \left\{ \int_\gamma^{+\infty} \left| \frac{\gamma - \varrho e^{i\varphi}}{\gamma + \varrho e^{i\varphi}} \right|^{p(k-1)} \frac{\varrho^{p(\theta-\bar{\sigma})+p-\delta}}{|\gamma + \varrho e^{i\varphi}|^p (1 + \varrho)} d\varrho \right. \\
&\quad \left. \times \left(\int_\gamma^{+\infty} \frac{\varrho^{-\delta}}{1 + \varrho} d\varrho \right)^{\frac{p}{p'}} + \int_0^{\varphi} \left(\tan \frac{\theta}{2} \right)^{p(k-1)} d\theta \left(\int_0^{\varphi} \frac{d\theta}{(1 + \cos\theta)^{p'/2}} \right)^{\frac{p}{p'}} \right\} \\
&\le c \|g\|_{\bar{\mathcal{E}}_p^{\beta}}^p \left\{ \int_\gamma^{+\infty} \left(\frac{\varrho - \gamma \cos\varphi}{\varrho + \gamma \cos\varphi} \right)^{p\frac{N}{2}} \frac{\varrho^{p(\theta-\bar{\sigma})+p}}{\varrho^{p-1}} \frac{d\varrho}{\varrho^{1+\delta}} + \tan^{pN} \frac{\varphi}{2} \right\} \\
&\le c N^{p(\theta-\bar{\sigma})+1} \|g\|_{\bar{\mathcal{E}}_p^{\beta}}^p .
\end{aligned}
$$

The proof is complete ∎

Remark 4. If $E = H$ is a Hilbert space with an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$, $p = 2$

and $f_k(x) = (x, e_k)$, then we have instead of (32) using the Parseval identity

$$\begin{aligned}
\|x_2\|^2_{\mathcal{E}_2^\sigma} &= \int_0^{+\infty} \|A^{\bar\sigma} x(t)\|^2_H \, dt \\
&= \sum_{k,j=1}^{+\infty} \left| \left\langle A^{1-\delta}(T_\gamma^{k-1} + T_\gamma^k)\, g^\beta_{k-1}, f_j \right\rangle \right|^2 \\
&= \sum_{k=1}^{+\infty} \|A^{1-\delta}(T_\gamma^{k-1} + T_\gamma^k)\, g^\beta_{k-1}\|^2_H \\
&= \sum_{k=1}^{+\infty} \left\| \frac{1}{2\pi i} \int_\Gamma \left(\frac{\gamma - z}{\gamma + z}\right)^{k-1} \frac{2\gamma z^{1-\delta}}{\gamma + z} (z - A)^{-1} dz \, g^\beta_{k-1} \right\|^2_H \\
&\le \sum_{k=1}^{+\infty} \left\{ \frac{1}{2\pi} \int_\Gamma \left| \frac{\gamma - z}{\gamma + z} \right|^{k-1} \frac{2\gamma |z|^{1-\delta}}{|\gamma + z|} \frac{M}{1 + |z|} |dz| \, \|g^\beta_{k-1}\|_H \right\}^2 \\
&\le c \sum_{k=1}^{+\infty} \left(\int_\gamma^{+\infty} \frac{\varrho^{1-\delta}}{\varrho^2} \, d\varrho + 1 \right)^2 \|g^\beta_{k-1}\|^2_H \\
&\le c\, \|g\|^2_{\mathcal{E}_2^\beta}.
\end{aligned}$$

Therefore, Theorem 9 holds under the assumptions $g \in \mathcal{E}_2^\beta$ instead of $g \in \bar{\mathcal{E}}_2^\beta$ ∎

The rate of convergence in Theorem 9 can be improved under slightly stronger assumptions with respect to the right-hand side g of problem (6).

Theorem 10. *Let A be a densely defined, strongly positive operator in a separable Banach space E, $x_0 \in D(A^\sigma)$, $g \in \bar{\mathcal{E}}^\alpha_{pq}$ with*

$$\alpha = \bar\sigma - \frac{pq' - 1}{pq'} + \delta, \qquad \bar\sigma = \sigma + \frac{p-1}{p} - \delta, \qquad p \ge 1, \qquad r = pq' \ge 1,$$

where δ is an arbitrary small positive number and $\frac{1}{q} + \frac{1}{q'} = 1$. Then the approximate solution x^N in (8) converges to the exact solution x of the inhomogeneous problem (6) as $N \to \infty$ and the estimate

$$\|x - x^N\|_{\mathcal{E}_p^\theta} \le c\, N^{\theta - \bar\sigma} \left(\|A^\sigma x_0\| + \|g(\cdot)\|_{\bar{\mathcal{E}}^\alpha_{pq}} \right) \qquad (0 \le \theta \le \bar\sigma) \tag{34}$$

holds with a positive constant c independent of N, x_0 and g.

Proof. By analogy with (33), we get

$$\begin{aligned}
\|x_2^N - x_2\|^p_{\mathcal{E}_p^\theta} &\le M^p F_p^p \left(\frac{\gamma}{\pi}\right)^p \\
&\quad \times \sum_{k=N+1}^{+\infty} \left(\int_\Gamma \left| \frac{\gamma - z}{\gamma + z} \right|^{k-1} \frac{|z|^{\theta - \alpha}}{|\gamma + z|} \frac{1}{1 + |z|} |dz| \right)^p \|g^\alpha_{k-1}\|^p_E.
\end{aligned}$$

Using the Hölder inequality with exponents q, q' such that $\frac{1}{q}+\frac{1}{q'}=1$ and r, r' such that $\frac{1}{r}+\frac{1}{r'}=1$, the inequalities $(a+b)^r \le 2^{r-1}(a^r+b^r)$ $(r\ge 1)$ and

$$\phi(N)=\left\{\sum_{k=N+1}^{+\infty}\|g_{k-1}^{\alpha}\|_E^{pq}\right\}^{\frac{1}{q}} \le \|g\|^p_{\bar{\mathcal{E}}^{\alpha}_{pq}},$$

we further deduce

$$\begin{aligned}
&\|x_2-x_2^N\|^p_{\mathcal{E}^{\theta}_p}\\
&\le c\,\phi(N)\left\{\sum_{k=N+1}^{+\infty}\left|\int_\Gamma\left|\frac{\gamma-z}{\gamma+z}\right|^{k-1}\frac{|z|^{\theta-\bar\sigma-\frac{r-1}{r}-\delta}}{|\gamma+z|(1+|z|)}\right|^r|dz|\right\}^{\frac{1}{q'}}\\
&\le c\,\phi(N)\left\{\sum_{k=N+1}^{+\infty}\left[\left(\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{k-1}\frac{\varrho^{\theta-\alpha}}{|\gamma+\varrho e^{i\varphi}|}\frac{d\varrho}{\varrho}\right)^r\right.\right.\\
&\qquad\left.\left.+\left(\int_0^{\varphi}\left|\frac{1-e^{i\theta}}{1+e^{i\theta}}\right|^{k-1}\frac{\gamma^{\theta-\alpha}}{|1+e^{i\theta}|}\frac{d\theta}{1+\gamma}\right)^r\right]\right\}^{\frac{1}{q'}}\\
&\le c\,\phi(N)\left\{\sum_{k=N+1}^{+\infty}\left[\int_\gamma^{+\infty}\left|\frac{\gamma-\varrho e^{i\varphi}}{\gamma+\varrho e^{i\varphi}}\right|^{r(k-1)}\frac{\varrho^{r(\theta-\bar\sigma+\frac{r-1}{r})-\delta}}{|\gamma+\varrho e^{i\varphi}|^r}\frac{d\varrho}{1+\varrho}\right.\right.\\
&\qquad\left.\left.\times\left(\int_\gamma^{+\infty}\frac{\varrho^{-\delta}}{1+\varrho}d\varrho\right)^{\frac{r}{r'}}+\int_0^{\varphi}\left(\tan\frac{\theta}{2}\right)^{r(k-1)}d\theta\left(\int_0^{\varphi}\frac{d\theta}{(1+\cos\theta)^{\frac{r'}{2}}}\right)^{\frac{r}{r'}}\right]\right\}^{\frac{1}{q'}}\\
&\le c\,\phi(N)\left\{\int_\gamma^{+\infty}\left(\frac{\gamma-\varrho\cos\varphi}{\gamma+\varrho\cos\varphi}\right)^{r\frac{N}{2}}\frac{\varrho^{r(\theta-\bar\sigma)}}{\varrho^{1+\varepsilon}}d\varrho+\left(\tan\frac{\varphi}{2}\right)^{rN}\right\}^{\frac{1}{q'}}\\
&\le c\,\phi(N)\,N^{p(\theta-\bar\sigma)}
\end{aligned}$$

where ε is an arbitrary small positive number. Now, the proof follows from this estimate and Theorem 5 ∎

Remark 5. If we choose $q=\frac{p+\delta_1}{p}$ and $\delta_1\to 0$, then

$$\begin{aligned}
&q'=\frac{p+\delta_1}{\delta_1}, \qquad p<r=pq'=\frac{p(p+\delta_1)}{\delta_1}\\
&\beta<\alpha=\bar\sigma-1+\frac{\delta_1}{p(p+\delta_1)}+\delta\to\ \beta=\bar\sigma-1+\delta\\
&pq=p+\delta_1\to p,
\end{aligned}$$

i.e. if $x_0\in D(A^{\sigma})$ and $g\in\bar{\mathcal{E}}_{p+\delta_1}^{\bar\sigma-1+\frac{\delta_1}{p(p+\delta_1)}+\delta}$ ("almost $\bar{\mathcal{E}}_p^{\bar\sigma-1+\delta}$"), then as $N\to+\infty$, the error $x-x^N$ decreases with the rate $O(N^{\theta-\bar\sigma})$ $(0\le\theta\le\bar\sigma)$ in the norm of $\mathcal{E}^{\theta}_p$.

The approach (8) can be useful if the integrals in (8) can be calculated analytically. Otherwise, we propose another approach which will be considered in the next section.

Remark 6. If we multiply (6) by $e^{-\kappa t}$ with $\kappa \geq 0$, then we get for the function $x_*(t) = e^{-\kappa t}x(t)$

$$\dot{x}_*(t) + A_\kappa x_*(t) = g_*(t)$$
$$x_*(0) = x_0$$

where $g_*(t) = e^{-\kappa t}g(t)$ and $A_\kappa = A + \kappa I$. Analogously as above, we get

$$x_* = x_{*1} + x_{*2}$$

$$x_{*1}(t) = T_*(t)\,x_0 = e^{-\gamma t}\sum_{q=0}^{+\infty}(-1)^q L_q^{(0)}(2\gamma t)T_{*\gamma}^q(I + T_{*\gamma})\,x_0$$

$$\begin{aligned} x_{*2}(t) &= \int_0^t T_*(t-s)\,g_*(s)\,ds \\ &= \sum_{q=0}^{+\infty}(-1)^q \int_0^t e^{-\gamma(t-s)}L_q^{(0)}(2\gamma t)\,T_{*\gamma}^q(I + T_{*\gamma})\,g_*(s)\,ds \end{aligned} \tag{35}$$

$$T_{*\gamma} = (\gamma I - A_\kappa)(\gamma I + A_\kappa)^{-1} \quad \text{for all } \gamma > 0.$$

One obtains an approximate solution $\hat{x}^n$ of problem (6) as

$$\hat{x}^N(t) = e^{\kappa t}x_*^N(t) \equiv e^{\kappa t}\big(x_{*1}^N(t) + x_{*2}^N(t)\big) \tag{36}$$

where x_{*1}^N and x_{*2}^N are the partial sums of (35). Similary to (34) we get

$$\left\|e^{-\kappa\cdot}\big(x(\cdot) - \hat{x}^N(\cdot)\big)\right\|_{\mathcal{E}_p^\theta} \leq c\,N^{\theta-\bar{\sigma}}\big(\|A_\kappa^\sigma x_0\| + \|e^{-\kappa\cdot}g(\cdot)\|_{\bar{\mathcal{E}}_{pq}^\alpha}\big) \tag{37}$$

for all $\kappa \geq 0$ provided that A is densely defined, strongly positive, $x_0 \in D(A^\sigma)$ and $e^{-\kappa t}g(t) \in \bar{\mathcal{E}}_{pq}^\alpha$ with p, q and α defined as in Theorem 10.

6. Approximation based on the discretization of non-homogenity

In this section we consider the inhomogeneous initial value problem (6). We will assume throughout the section that A is a strongly positive operator, so that problem (6) has a solution for every initial value $x_0 \in D(A^\sigma)$ and for every right-hand side $g \in \bar{\mathcal{E}}_p^\beta$, with $\beta = \bar{\sigma} - 1 + \delta = \sigma - \frac{1}{p}$ and $\bar{\sigma} = \sigma + \frac{p-1}{p} - \delta$, where δ is an arbitrary small positive number and $p > 1$.

Let us first assume that g is some polynomial, i.e.

$$g(t) = \sum_{p=0}^{n} t^p \tilde{g}_p \qquad \text{with } \tilde{g}_p \in D(A^\beta). \tag{38}$$

One can look for a particular solution of the form

$$\hat{x}(t) = \sum_{p=0}^{n} t^p a_p. \tag{39}$$

Substituting (39) into (6) and comparing the coefficients, we obtain the reccurence relation

$$a_p = A^{-1}\left[\tilde{g}_p - (p+1)\,a_{p+1}\right] \qquad (p = n-1, n-2, \dots, 0) \tag{40}$$

$$a_n = A^{-1}\tilde{g}_n \tag{41}$$

which has the solution

$$a_{n-p} = \sum_{\nu=0}^{p} (-1)^{p+\nu} \frac{(n-\nu)!}{(n-p)!} A^{-p+\nu-1}\tilde{g}_{n-\nu}. \tag{42}$$

Thus, in order to find the particular solution (39) explicitely one has to invert the operator A (the strong positivity of A yields the existence of the inverse A^{-1}) and to use formula (42). Next, we set $w = x - \hat{x}$ and get

$$\begin{aligned} \dot{w}(t) + Aw(t) &= 0 \\ w(0) &= x_0 - a_0. \end{aligned} \tag{43}$$

It is obvious that $a_0 \in D(A^{\sigma+\frac{p-1}{p}})$ and $w(0) \in D(A^\sigma)$. Thus, the unique solution of the homogeneous problem (43) exists for $x_0 \in D^\sigma$ and can be found by (5). We come to the following algorithm.

Algorithm 2 (Approximate solution of the inhomogeneous problem (6) with a polynomial right-hand side (38)).

1. Input t and find $\hat{x}(t)$ in accordance with (39) – (42).
2. Input N and find the numerical approach w^N to the solution of (43) by Algorithm 1.
3. Find the approximate solution of problem (6) as $x^N = \hat{x} + w^N$.

It follows from Theorems 4 and 5 that the estimates

$$\sup_{t\in[0,+\infty)} \left\|A^\theta\big(x(t) - x^N(t)\big)\right\| = \sup_{t\in[0,+\infty)} \left\|A^\theta\big(w(t) - w^N(t)\big)\right\| \le c\,N^{-(\sigma-\theta)+\delta}\left\|A^\sigma(x_0 - a_0)\right\|$$

$$\sup_{t\in[\varepsilon,\omega]} \left\|A^\theta\big(x(t) - x^N(t)\big)\right\| = \sup_{t\in[\varepsilon,\omega]} \left\|A^\theta\big(w(t) - w^N(t)\big)\right\| \le c\,N^{-(\sigma-\theta)-\frac{1}{4}+\delta}\left\|A^\sigma(x_0 - a_0)\right\|$$

and

$$\|x - x^N\|_{\mathcal{E}_p^\theta} = \|w - w^N\|_{\mathcal{E}_p^\theta} \leq c\, N^{-(\bar{\sigma}-\theta)} \|A^\sigma (x_0 - a_0)\|$$

hold where δ is an arbitrary small positive number.

We turn now to the case that the right-hand side g of problem (6) is not a polynomial and $E = H$ is a Hilbert space with orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$. We consider $N+1$ points $t_j \ (0 \leq j \leq N)$:

$$t_0 = 0$$

$$\frac{d}{dt} L_{N+1}^{(0)}(t_j) \equiv -L_N^{(1)}(t_j) = 0 \quad (1 \leq j \leq N)$$

where $L_n^{(\alpha)}$ are the Laguerre polynomials. For each continuous function u on $[0, +\infty)$ let $I_N u \in \mathbb{P}_N$ be the interpolation polynomial of u at the points $t_j \ (0 \leq j \leq N)$. The Gauss-Radau quadrature formula [13]

$$\int_0^{+\infty} u(t)\, e^{-t}\, dt \approx \sum_{i=0}^{N} \omega_i\, u(t_i)$$

is exact for $u \in \mathbb{P}_{2n}$ what yields that

$$I_N u = \sum_{k=0}^{N} \hat{a}_k L_k^{(0)}(t),$$

where

$$\hat{a}_k = \sum_{i=0}^{N} \omega_i\, L_k^{(0)}(t_i)\, u(t_i)$$

is the interpolation polynomial of u at the points $t_j \ (0 \leq j \leq N)$.

Let P_N be the operator of the orthogonal projection in $L^2_{e^{-\kappa t}}$ upon $\mathbb{P}_N$. It was proved in [13] that, for all $\varepsilon > 0$,

$$\|u - P_N u\|_{\mu,1} \leq c\, N^{\mu - \frac{m}{2}} \|u\|_{m,1-\varepsilon} \qquad (0 \leq \mu \leq m)$$

and

$$\|u - I_N u\|_{\mu,1} \leq c_\varepsilon N^{\mu - \frac{m-1}{2}} \|u\|_{m,1-\varepsilon} \quad (0 \leq \mu \leq m;\ m > \tfrac{1}{2}). \tag{44}$$

Besides, it holds

$$\sum_{i=0}^{N} \omega_i\, e^{(1-\alpha)\, t_i} \leq \frac{1}{\alpha} \qquad \text{for all } \alpha \in (0, 1].$$

We approximate problem (6) by

$$\begin{aligned} \dot{\tilde{x}}(t) + A\tilde{x} &= I_N g \\ \tilde{x}(0) &= x_0 \end{aligned} \tag{45}$$

where

$$I_N g = \sum_{k=0}^{N} \hat{g}_k L_k^{(0)}(2\gamma t) \qquad \text{with} \qquad \hat{g}_k = \sum_{i=0}^{N} \omega_i L_k^{(0)}(t_i)\, g(t_i) \tag{46}$$

and $I_N g$ is the interpolation polynomial of $g : [0, +\infty) \to H$ with respect to the points $\frac{t_j}{2\gamma}$ $(0 \le j \le N)$. One can find the approximate solution $\hat{x}^N$ of problem (45) analogously to (36), namely

$$\hat{x}^N = \hat{x}_1^N + \hat{x}_2^N \tag{47}$$

$$\hat{x}_1^N(t) = e^{(\kappa-\gamma)t} \sum_{q=0}^{N} (-1)^q L_q^{(0)}(2\gamma t)\, T_{*\gamma}^q (I + T_{*\gamma})\, x_0 \tag{48}$$

$$\hat{x}_2^N(t) = \sum_{q=0}^{N} (-1)^q \int_0^t e^{-\gamma_*(t-s)} L_q^{(0)}(2\gamma(t-s))\, T_{*\gamma}^q (I - T_{*\gamma})\, I_N g(s)\, ds \tag{49}$$

$$\gamma_* = \gamma + \kappa.$$

Using (46), we further get

$$\hat{x}_2^N(t) = \sum_{k,q=0}^{N} (-1)^q \tau_{q,k}(t)\, T_{*\gamma}^q (I + T_{*\gamma})\, \hat{g}_k \tag{50}$$

where

$$\begin{aligned} \tau_{q,k}(t) &= \int_0^t e^{-\gamma_*(t-s)} L_q^{(0)}(2\gamma(t-s))\, L_k^{(0)}(2\gamma s)\, ds \\ &= \int_0^t e^{-\gamma_* \tilde{s}} L_q^{(0)}(2\gamma \tilde{s})\, L_k^{(0)}(2\gamma(t-\tilde{s}))\, d\tilde{s}. \end{aligned}$$

By partial integration, using the formulas

$$\begin{aligned} L_{-1}^{(\alpha)}(\xi) &= 0 \\ L_0^{(\alpha)}(\xi) &= 1 \\ n L_n^{(\alpha)}(\xi) &= (-\xi + 2n + \alpha - 1)\, L_{n-1}^{(\alpha)}(\xi) - (n + \alpha - 1)\, L_{n-2}^{(\alpha)}(\xi) \quad (n \ge 1) \end{aligned}$$

and

$$\frac{d}{d\xi} L_k^{(0)}(\xi) = k \xi^{-1} \big(L_k^{(0)}(\xi) - L_{k-1}^{(0)}(\xi) \big) = - \sum_{\nu=0}^{k-1} L_\nu^{(0)}(\xi)$$

we get

$$\tau_{0,k}(t) = \frac{e^{-\gamma_* t}}{2\gamma} \int_0^{2\gamma t} e^{\frac{\gamma_*}{2\gamma}\xi} L_k^{(0)}(\xi)\, d\xi$$

$$= \frac{e^{-\gamma_* t}}{2\gamma}\left(\frac{2\gamma}{\gamma_*} e^{\gamma_* t} L_k^{(0)}(2\gamma t) - 1 + \frac{2\gamma}{\gamma_*}\int_0^{2\gamma t} e^{\frac{\gamma_*}{2\gamma}\xi} \sum_{\nu=0}^{k-1} L_\nu^{(0)}(\xi)\,d\xi\right) \tag{51}$$

$$= \frac{1}{\gamma_*} L_k^{(0)}(2\gamma t) - \frac{e^{-\gamma_* t}}{2\gamma} + \frac{2\gamma}{\gamma_*}\sum_{\nu=0}^{k-1}\tau_{0,\nu}(t) \qquad (0 \le k \le N)$$

$$\tau_{0,0}(t) = \frac{1}{\gamma_*}(1 - e^{-\gamma_* t})$$

$$\tau_{q,k}(t) = e^{-\gamma_* t}\int_0^t e^{\gamma_* s} L_q^{(0)}(2\gamma(t-s))\, L_k^{(0)}(2\gamma s)\,ds$$

$$= e^{-\gamma_* t}\int_0^t e^{\gamma_* s}\frac{1}{q}\Big\{\big(-2\gamma(t-s) + 2q - 1\big)\, L_{q-1}^{(0)}(2\gamma(t-s))$$

$$- (q-1)\, L_{q-2}^{(0)}(2\gamma(t-s))\Big\}\, L_k^{(0)}(2\gamma s)\,ds$$

$$= e^{-\gamma_* t}\int_0^t e^{\gamma_* s}\frac{1}{q}\Big\{\Big[(-2\gamma t + 2q - 1)\, L_{q-1}^{(0)}(2\gamma(t-s)) \tag{52}$$

$$- (q-1) L_{q-2}^{(0)}(2\gamma(t-s))\Big]\, L_k^{(0)}(2\gamma s) + L_{q-1}^{(0)}(2\gamma(t-s))$$

$$\times\Big[-(k+1)\, L_{k+1}^{(0)}(2\gamma s) + (2k+1)\, L_k^{(0)}(2\gamma s) - k L_{k-1}^{(0)}(2\gamma s)\Big]\Big\}\,ds$$

$$= \frac{1}{q}\Big[(-2\gamma t + 2q - 1)\,\tau_{q-1,k}(t) - (q-1)\,\tau_{q-2,k}(t) - (k+1)\,\tau_{q-1,k+1}(t)$$

$$+ (2k+1)\,\tau_{q-1,k}(t) - k\tau_{q-1,k-1}(t)\Big]$$

for $1 \le q \le N$ and $0 \le k \le N$.

Thus, we can formulate the following algorithm to calculate the approximate solution (47) of the inhomogeneous problem (6).

Algorithm 3 (Approximate solution of the inhomogeneous problem (6) with a non-polynomial right-hand side g).

1. Input N, calculate the coefficients $z_{k,0} = \hat{g}_k$ $(0 \le k \le N)$ of the interpolating polynomial $I_N g$ by (46) and set $y_0 = x_0$.
2. For $q = 0$ to $q = N - 1$:
   - **2.1.** Solve the operator equations (with the same operator but with various right-hand sides)

     $$(\gamma I + A_\kappa)\,\bar{y}_{q+1} = y_q$$

     and find

     $$y_{q+1} = (\gamma I - A_\kappa)\,\bar{y}_{q+1}.$$

2.2 For $k = 0$ to $k = N - 1$ solve the operator equations

$$(\gamma I + A_\kappa)\,\bar{z}_{k+1,q} = z_{k,q}$$

and find

$$z_{k+1,q} = (\gamma I - A_\kappa)\,\bar{z}_{k+1,q}.$$

3. Input t, calculate $\sigma_q = e^{(\kappa-\gamma)t}L_q^{(0)}(2\gamma t)$ $(0 \le q \le N)$, $\tau_{q,k} \equiv \tau_{q,k}(t)$ by (51) and (52) and $\hat{x}^N$ in accordance with (47) - (50):

$$\hat{\tilde{x}}_1^N(t) = \sum_{q=0}^{N}(-1)^q \sigma_q y_q, \quad \hat{\tilde{x}}_2^N(t) = \sum_{k,q=0}^{N}(-1)^q \tau_{qk} z_{kq}, \quad \hat{\tilde{x}}^N = \hat{x}_1^N + \hat{x}_2^N.$$

It follows from (37) that

$$\left\|e^{-\kappa\cdot}\left(\tilde{x} - \hat{\tilde{x}}^N\right)\right\|_{\mathcal{E}_p^\theta} \le c\,N^{\theta-\bar{\sigma}}\left(\|A^\sigma x_0\|_H + \left\|e^{-\kappa\cdot} I_N g\right\|_{\bar{\mathcal{E}}_{pq}^\alpha}\right) \qquad (\kappa \ge 0).$$

Specifically, for $p = 2$, $q = 1$ and $\kappa = \frac{1}{2}$ we have $q' = +\infty$,

$$\bar{\sigma} = \sigma + \frac{1}{2} - \delta, \quad \alpha = \bar{\sigma} - 1 + \delta = \sigma - \frac{1}{2}, \quad \mathcal{E}_2^\theta = \mathcal{H}^\theta, \quad \bar{\mathcal{E}}_2^\alpha = \mathcal{E}_2^\alpha = \mathcal{H}^\alpha$$

and

$$\|\tilde{x} - \hat{\tilde{x}}^N\|_{\mathcal{E}_1^{\theta,0}} \le c\,N^{\theta-\bar{\sigma}}\left(\|A^\sigma x_0\|_H + \|I_N g\|_{\mathcal{E}_1^{\alpha,0}}\right) \tag{53}$$

provided that $x_0 \in D(A^\sigma)$ and $I_N g \in \mathcal{E}_1^{\alpha,0}$. In order to estimate

$$\|I_N g\|_{\mathcal{E}_1^{\alpha,0}}^2 = \int_0^{+\infty} e^{-t}\|A^\alpha I_N g(t)\|_H^2\,dt$$

we need the following statement.

Lemma 5. *Assume that $g \in \mathcal{E}_{1-\varepsilon}^{\alpha,m}$ for some $\varepsilon \in (0,1)$ and $1 \le m \le N$. Then*

$$\|r_N\|_{\mathcal{E}_1^{\alpha,0}} \equiv \|g - I_N g\|_{\mathcal{E}_1^{\alpha,0}} \le c(m)\,N^{-\frac{m-1}{2}}\|g\|_{\mathcal{E}_{1-\varepsilon}^{\alpha,m}}.$$

Proof. Let $s_j(t) = \left(A^\alpha g(t), e_j\right)$. Then, due to the Parseval identity

$$\|g - I_N g\|_{\mathcal{E}_1^{\alpha,0}}^2 = \sum_{j=1}^{+\infty}\int_0^{+\infty} e^{-t}\left(s_j(t) - I_N s_j(t)\right)^2 dt = \sum_{j=1}^{+\infty}\|s_j - I_N s_j\|_{0,1}^2. \tag{54}$$

It follows from (44) that

$$\|s_j - I_N s_j\|_{0,1}^2 \le c_\varepsilon N^{-m+1}\|s_j\|_{m,1-\varepsilon}^2 \qquad (\varepsilon \in (0,1)) \tag{55}$$

provided that $s_j \in H^m_{e^{-(1-\epsilon)t}}$. Substituting (55) into (54), we get

$$\begin{aligned}
\|g - I_N g\|^2_{\mathcal{E}_1^{\alpha,0}} &\le c_\epsilon N^{-m+1} \sum_{j=1}^{+\infty} \|s_j\|^2_{m,1-\epsilon} \\
&= c_\epsilon N^{-m+1} \sum_{j=1}^{+\infty} \sum_{k=0}^{m} \int_0^{+\infty} e^{-(1-\epsilon)t} \left(A^\alpha \frac{d^k g(t)}{dt^k}, e_j \right)^2 dt \\
&= c_\epsilon N^{-m+1} \sum_{k=0}^{m} \int_0^{+\infty} \left\| e^{-\frac{1-\epsilon}{2} t} A^\alpha \frac{d^k g(t)}{dt^k} \right\|^2_H dt \\
&= c_\epsilon N^{-m+1} \|g\|_{\mathcal{E}^{\alpha,m}_{1-\epsilon}}.
\end{aligned} \tag{56}$$

The proof is complete ∎

Thus, if $m \ge 1$, then it follows from Lemma 5 that

$$\|I_N g\|_{\mathcal{E}_1^{\alpha,0}} \le c\left(\|g\|_{\mathcal{E}_1^{\alpha,0}} + \|g\|_{\mathcal{E}^{\alpha,m}_{1-\epsilon}}\right) \le c\,\|g\|_{\mathcal{E}^{\alpha,m}_{1-\epsilon}}$$

and we get from (53)

$$\|\tilde{x} - \hat{\tilde{x}}^N\|_{\mathcal{E}_1^{\theta,0}} \le c\, N^{\theta-\tilde{\sigma}}\left(\|A^\sigma x_0\| + \|g\|_{\mathcal{E}^{\alpha,m}_{1-\epsilon}}\right)$$

provided that

$$x_0 \in D(A^\sigma) \qquad \text{and} \qquad g(t) \in \mathcal{E}^{\alpha,m}_{1-\epsilon} \qquad (\varepsilon \in (0,1),\ m \ge 1). \tag{57}$$

We consider now the difference $z = x - \tilde{x}$ which is obviously the solution of the problem

$$\begin{aligned}
\dot{z}(t) + Az(t) &= r_N(t) \\
z(0) &= 0.
\end{aligned} \tag{58}$$

Theorem 11. *Let A be a densely defined, strongly positive operator and $g \in \mathcal{E}^{\mu,m}_{1-\epsilon}$ for some $\varepsilon \in (0,1)$. Then*

$$\|z\|_{\mathcal{E}^{\mu,0}_{1+\epsilon'}} \le c\, N^{-\frac{m-1}{2}} \|g\|_{\mathcal{E}^{\mu,m}_{1-\epsilon}} \tag{59}$$

for an arbitrary small $\varepsilon' > 0$, with a constant $c = c(\varepsilon, \varepsilon', m)$ independent of N and g.

Proof. We have from (58)

$$z(t) = \int_0^t T(t-\xi)\, r_N(\xi)\, d\xi$$

where $\{T(t)\}_{t\geq 0}$ is the analytic semigroup with the infinitesimal generator $-A$. There exists a constant $\delta > 0$ such that $-A+\delta$ is still an infinitesimal generator of an analytic semigroup and (see [14: p. 70]) $\|T(t)\| \leq Me^{-\delta t}$. Thus, we have

$$\begin{aligned}
\|z\|^2_{\mathcal{E}^{\mu,0}_{1+\varepsilon'}} &= \int_0^{+\infty} e^{-(1+\varepsilon')t} \left\| A^\mu \int_0^t T(t-\xi)\, r_N(\xi)\, d\xi \right\|_H^2 dt \\
&\leq c \int_0^{+\infty} e^{-(1+\varepsilon')t} \left(\int_0^t e^{-\delta(t-\xi)} e^{\frac{\xi}{2}} e^{-\frac{\xi}{2}} \|A^\mu r_N(\xi)\|_H \, d\xi \right)^2 dt \\
&\leq \int_0^{+\infty} e^{-(1+\varepsilon')t} \int_0^t e^{-2\delta(t-\xi)} e^{\xi}\, d\xi \int_0^t e^{-\xi} \|A^\mu r_N(\xi)\|_H^2 \, d\xi dt \\
&= \frac{c}{1+2\delta} \int_0^{+\infty} e^{-(1+\varepsilon'+2\delta)t} \left(e^{(1+2\delta)t} - 1 \right) dt \, \|r_N\|^2_{\mathcal{E}^{\mu,0}_1} \\
&\leq c(\varepsilon', \delta)\, \|r_N\|^2_{\mathcal{E}^{\mu,0}_1}.
\end{aligned}$$

Using Lemma 5, we get the statement of the theorem ∎

We are now in a position to give a characterization of the accuracy of the approach $\hat{\tilde{x}}^N$ calculated by Algorithm 3.

Theorem 12. *Let A be a densely defined, strongly positive operator in H,*

$$x_0 \in D(A^\sigma), \qquad g \in \mathcal{E}^{\alpha,0}_{1-\varepsilon} \quad \textit{for some } \varepsilon \in (0,1), \qquad \alpha = \sigma - \frac{1}{2}.$$

Then

$$\|x - \hat{\tilde{x}}^N\|_{\mathcal{E}^{0,0}_{1+\varepsilon'}} \leq c\left(N^{-\frac{m-1}{2}} \|g\|_{\mathcal{E}^{\alpha,m}_{1-\varepsilon}} + N^{-\sigma-\frac{1}{2}+\varepsilon''} \left(\|A^\sigma x_0\|_H + \|g\|_{\mathcal{E}^{\alpha,m}_{1-\varepsilon}} \right) \right)$$

where $1 \leq m \leq N$, ε' and ε'' are arbitrary small positive numbers, and c is a constant independent of N, x_0 and g.

Proof. First of all we remark that $\mathcal{E}^{\mu,m}_{1+\varepsilon'} \subseteq \mathcal{E}^{0,m}_{1+\varepsilon'}$ for all $\mu \geq 0$. Then, due to (56) and (59), we have

$$\begin{aligned}
\|x - \hat{\tilde{x}}^N\|_{\mathcal{E}^{0,0}_{1+\varepsilon'}} &= \|x - \tilde{x}\|_{\mathcal{E}^{0,0}_{1+\varepsilon'}} + \|\tilde{x} - \hat{\tilde{x}}^N\|_{\mathcal{E}^{0,0}_{1+\varepsilon'}} \\
&\leq c\left(N^{-\frac{m-1}{2}} \|g\|_{\mathcal{E}^{0,m}_{1-\varepsilon}} + N^{-\bar{\sigma}} \left(\|A^\sigma x_0\|_H + \|g\|_{\mathcal{E}^{\alpha,m}_{1-\varepsilon}} \right) \right) \\
&\leq c\left(N^{-\frac{m-1}{2}} \|g\|_{\mathcal{E}^{\alpha,m}_{1-\varepsilon}} + N^{-\sigma-\frac{1}{2}+\varepsilon''} \left(\|A^\sigma x_0\|_H + \|g\|_{\mathcal{E}^{\alpha,m}_{1-\varepsilon}} \right) \right)
\end{aligned}$$

and the assertion is proved ∎

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