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Fourier Multipliers between Weighted Anisotropic Function Spaces Part I: Besov Spaces

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Abstract. We determine classes $\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1))$ of Fourier multipliers between weighted anisotropic Besov spaces $B^{s_0}_{p_0,q_0}(w_0)$ and $B^{s_1}_{p_1,q_1}(w_1)$ where $p_0 \leq 1$ and w_0, w_1 are weight functions of polynomial growth. To this end we use a discrete characterization of the function spaces akin to the φ-transform of Frazier and Jawerth which leads to a unified approach to the multiplier problem. In this way widely generalized versions of known results of Bui, Johnson, Peetre and others are obtained from a single theorem.

Keywords: *Fourier multipliers, weighted Besov spaces, anisotropic spaces*

AMS subject classification: Primary 42 B 25, 46 E 35, secondary 46 E 39

1. Introduction

The purpose of this and a subsequent paper is to give a detailed study of the class $\mathbf{M}(X^{s_0}_{p_0,q_0}(w_0), Y^{s_1}_{p_1,q_1}(w_1))$ of Fourier multipliers between two anisotropic weighted function spaces $X^{s_0}_{p_0,q_0}(w_0)$ and $Y^{s_1}_{p_1,q_1}(w_1)$ of Besov and Triebel type in the case of $p_0 \leq 1$.

In this Part I we restrict ourselves to the case of Besov spaces. So we have to determine the class of tempered distributions M generating bounded operators

$$T_M : B^{s_0}_{p_0,q_0}(w_0) \to B^{s_1}_{p_1,q_1}(w_1), \qquad T_M f = \mathcal{F}^{-1}[M\mathcal{F}f] \quad (f \in \mathcal{S}).$$

The general case (Besov and Triebel spaces) will be considered in a following paper which will also contain some results concerning the case of Besov spaces. The reason for this splitting is twofold. On the one hand weighted Besov spaces have recently attracted much attention (cf., e.g., [2, 3, 8, 11, 13]) so that their study has a right in its own. On the other hand the case of Besov spaces is much simpler to deal with from the technical point of view (e.g. we do not need anisotropic maximal functions). Thus we develope the basic ideas of our method in the case of Besov spaces and refine them later to deal with the general situation extending the results presented in this paper. The current work is selfcontained and has no reference to the forthcoming Part II except for the proof of a certain characterization of Besov spaces.

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To give an impression of the obtained results we formulate a special case of the main theorem of this paper. Let $B^{s_0}_{p_0,q_0}(w_0)$ and $B^{s_1}_{p_1,q_1}(w_1)$ be two isotropic Besov spaces with weight functions $w_j(x) = (1+|x|)^{d_j}$, $0 \le d_0 \le d_1$, and $0 < p_j, q_j < \infty$, $s_j \in \mathbb{R}$ $(j = 0,1)$. If $p_0 \le \min\{1, p_1\}$, then

$$\mathrm{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) \rightleftarrows \mathcal{F}[B^{s}_{p_1,r}(w_1)]$$

where $\frac{1}{r} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+$ and $s = n\left(\frac{1}{p_0} - 1\right) + s_1 - s_0$. A corresponding theorem is proved for anisotropic spaces and will be sharpened in Part II.

This extends earlier work of the author [6, 7] and generalizes in particular the following results to weighted spaces and extends them even in the unweighted case:

$$\mathrm{M}(B^{s}_{p,q}, B^{s}_{p,q}) \rightleftarrows \mathcal{F}[B^{n(\frac{1}{p}-1)}_{p,\infty}] \quad (0 < p < 1) \qquad \text{(Peetre 1976)}$$

$$\mathrm{M}(B^{s_0}_{1,q_0}, B^{s_1}_{p,q_1}) \rightleftarrows \mathcal{F}[B^{s_1-s_0}_{p,\infty}] \quad (q_0 \le q_1,\ 1 \le p < \infty) \qquad \text{(Johnson 1978).}$$

Our method of proof is based on a discrete characterization of Besov spaces (proved in Part II) which will be introduced in the next section. This leads to the study of matrix operators between sequence spaces instead of the original Fourier multipliers between function spaces. These matrix operators are discussed in Sections 3 and 4. The results are applied to the study of Fourier multipliers in the final Section 5.

2. Besov spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let us start with a real $(n \times n)$-matrix P the eigenvalues of which have positive real parts. We define

$$\nu = \operatorname{trace} P .$$

With the group $\langle A_t \rangle_{t>0}$ of the *dilation matrices*

$$A_t = \exp(P \cdot \ln t) \qquad (t > 0)$$

we associate a *positive A_t-homogeneous distance function* ϱ, i.e. a continuous function $\varrho : \mathbb{R}^n \to \mathbb{R}$ with properties

$$\varrho(A_t x) = t\varrho(x) \quad (t > 0) \qquad \text{and} \qquad \varrho(x) > 0 \quad (x \neq 0).$$

It is known that any two A_t-homogeneous distance functions are pointwise equivalent and that there exist constants $C, C', C'' > 0$ and $0 < a \le b < \infty$ so that the estimates

$$\varrho(x + y) \le C \cdot (\varrho(x) + \varrho(y))$$

and

$$C' \cdot \min\{|x|^a, |x|^b\} \le \varrho(x) \le C'' \cdot \max\{|x|^a, |x|^b\}$$

hold for all $x, y \in \mathbb{R}^n$. For proofs, examples and further details concerning this concept we refer to [5, 14, 22]. Stein and Waigner [22] proved that for a given matrix P there always exists an A_t-homogeneous distance function $\varrho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and in the sequel ϱ denotes always such a fixed A_t-homogeneous distance function. The adjoint matrices A_t^* of A_t form a group of dilation matrices generated by P^* via

$$A_t^* = \exp(P^* \cdot \ln t) \qquad (t > 0).$$

In the sequel ϱ^* always denotes a fixed A_t^*-homogeneous distance function from $C^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying the additional condition

$$\{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2\} \subseteq [-\pi, \pi]^n$$

which will be used in the proof of the discrete characterization of Besov spaces.

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}^{\pm 1} f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{\mp ix\xi} dx$$

on the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$. Since $\det A_t = t^\nu$ we have

$$\mathcal{F}^{\pm 1}[f(A_t \bullet)] = t^{-\nu} \cdot (\mathcal{F}^{\pm 1} f)(A_{1/t}^* \bullet) \qquad \text{and} \qquad \|f(A_t \bullet)\|_p = t^{-\frac{\nu}{p}} \|f\|_p$$

for all suitable f and $0 < p \leq \infty$ where $\|\bullet\|_p$ denotes the usual L_p quasinorm for measurable functions on $\mathbb{R}^n$.

For $d \geq 0$ the class W_d of weight functions contains all continuous functions $w : \mathbb{R}^n \to \mathbb{R}$ satisfying the estimate

$$0 < w(x) \leq C_w \cdot w(y)(1 + |x - y|)^d$$

for a suitable constant $C_w > 0$ and all $x, y \in \mathbb{R}^n$. The typical example of such a weight function is

$$w(x) = (1 + |x|)^d.$$

The class W of *admissible weight functions* is defined by

$$W = \bigcup_{d \geq 0} W_d.$$

It has been studied by several authors in connection with Besov and Triebel spaces (cf., e.g., [8, 11, 13, 18]). The following estimate is very useful in connection with the matrices A_t. Denote by lub_2 the matrix norm associated with $|\bullet|$. From $A_t = \exp(P \cdot \ln t)$ we get

$$\text{lub}_2 A_{2^{-j}} \leq \exp(\ln 2^{-j} \cdot \text{lub}_2 P) \leq C_0 \qquad (j \in \mathbb{N})$$

which leads to

$$(1 + |A_{2^{-j}} x|)^d \leq C \cdot (1 + |x|)^d \qquad (d \geq 0). \tag{1}$$

To give the definiton of Besov spaces we need a resolution of unity. Let $\phi \in C^\infty(\mathbb{R}_+)$ be a fixed bump function with properties $\operatorname{supp}\phi \subseteq [0,2]$ and $\phi|_{[0,1]} \equiv 1$. Define

$$\phi_0(\xi) = \phi(\varrho^*(\xi))$$
$$\phi_j(\xi) = \phi(2^{-j}\varrho^*(\xi)) - \phi(2^{-j+1}\varrho^*(\xi)) \qquad (j \geq 1)$$

for $\xi \in \mathbb{R}^n$. These functions are of C^∞-type with

$$\operatorname{supp}\phi_0 \subseteq \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2\}$$
$$\operatorname{supp}\phi_j \subseteq \{\xi \in \mathbb{R}^n : 2^{j-1} \leq \varrho^*(\xi) \leq 2^{j+1}\} \quad (j \geq 1).$$

Furthermore, we have

$$\sum_{r=-1}^{1} \phi_{j+r}(\xi) = 1 \quad (\xi \in \operatorname{supp}\phi_j) \qquad \text{and} \qquad \sum_{j=0}^{\infty} \phi_j(\xi) = 1 \quad (\xi \in \mathbb{R}^n)$$

where $\phi_{-1} = 0$.

For $0 < p,q \leq \infty$, $s \in \mathbb{R}$ and $w \in W$ the (anisotropic inhomogeneous) *Besov space* $B^s_{p,q}(\mathbb{R}^n; P, w)$ (denoted by $B^s_{p,q}(w)$ for short) contains all $f \in S'$ (the space of tempered distributions) with finite quasinorm

$$\|f|B^s_{p,q}(w)\| = \left\|\langle 2^{js}\|w \cdot \mathcal{F}^{-1}[\phi_j \mathcal{F} f]\|_p\rangle_{j\in\mathbb{N}}|\ell_q\right\|.$$

These are quasi-Banach spaces which are independent of the choice of ϕ and ϱ^* (cf. [19]; the proof is analogous to [24: p. 46]). In the isotropic case, i.e. $P = I$ (the unit matrix) the theory of these spaces is extensively studied in [24, 25] (unweighted case) and [18] (weighted case). Anisotropic spaces were used by Dappa [4], Dappa and Trebels [5], Seeger [19] and Marschall [14].

To give a discrete characterization of the Besov space $B^s_{p,q}(w)$ we use the sequence space $b^s_{p,q}(\mathbb{R}^n; P, w)$ which is denoted by $b^s_{p,q}(w)$ for short ($0 < p,q \leq \infty$, $s \in \mathbb{R}$ and $w \in W$) containing all complex sequences $\alpha = \langle\alpha^j_k\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}$ with finite quasinorm

$$\|\alpha|b^s_{p,q}(w)\| = \left\|\langle 2^{j(s-\frac{\nu}{p})}\|\langle w^j_k\alpha^j_k\rangle_{k\in\mathbb{Z}^n}|\ell_p\|\rangle_{j\in\mathbb{N}}|\ell_q\right\|$$

where $w^j_k = w(A_{2^{-j}}k)$. Note the analogy in the structure of the two norms of $b^s_{p,q}(w)$ and $B^s_{p,q}(w)$. These sequence spaces are quasi-Banach spaces and we remark that the finite sequences are dense in $b^s_{p,q}(w)$ in the case of $0 < p,q < \infty$. From the embedding $\ell_u \hookrightarrow \ell_v$ $(0 < u \leq v \leq \infty)$ we obtain the two embeddings

$$b^s_{p,q_0}(w) \hookrightarrow b^s_{p,q_1}(w) \qquad (0 < q_0 \leq q_1 \leq \infty)$$
$$b^{s_0}_{p_0,q}(w) \hookrightarrow b^{s_1}_{p_1,q}(w) \qquad (s_0 - \tfrac{\nu}{p_0} = s_1 - \tfrac{\nu}{p_1})$$

where $0 < p, p_0, p_1, q \leq \infty$, $s, s_0, s_1 \in \mathbb{R}$ and $w \in W$, like in the case of Besov spaces.

The unweighted spaces (i.e. $w \equiv 1$) are denoted by $B^s_{p,q}$ and $b^s_{p,q}$ as usual.

To establish the connection between these sequence spaces and Besov spaces we furthermore need the two special functions ψ_0 and ψ_1 defined as

$$\psi_0(\xi) = \begin{cases} \dfrac{\phi_0(\xi)}{\phi_0(\xi)^2 + \phi_1(\xi)^2} & \text{for } \varrho^*(\xi) \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\psi_1(\xi) = \begin{cases} \dfrac{\phi_1(A_2^*\xi)}{\phi_1(\xi)^2 + \phi_1(A_2^*\xi)^2 + \phi_1(A_4^*\xi)^2} & \text{for } \frac{1}{3} \leq \varrho^*(\xi) \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

These functions are of C^∞-type with compact supports

$$\operatorname{supp}\psi_0 \subseteq \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2\} \quad \text{and} \quad \operatorname{supp}\psi_1 \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} \leq \varrho^*(\xi) \leq 2\}.$$

The next theorem contains the discrete characterization of $B^s_{p,q}(w)$ with the help of $b^s_{p,q}(w)$ and is the basis of our work. It will be proved at the end of Part II of this paper. Note that the unweighted case was already proved in [8].

Theorem 2.1 (Discrete characterization of Besov spaces). *For $f \in S'$ define the sequence* $\mathbf{se}f$ *by*

$$\mathbf{se}f = \left\langle (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}[\phi_j \mathcal{F} f](A_{2^{-j}} k) \right\rangle_{k\in\mathbf{Z}^n}^{j\in\mathbf{N}}.$$

For finite sequences $\alpha = \langle \alpha^j_k \rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}$ of complex numbers define the function $\mathbf{fu}\alpha$ *by*

$$\mathbf{fu}\,\alpha = \sum_{k\in\mathbf{Z}^n} \alpha^0_k \cdot (\mathcal{F}^{-1}\psi_0)(\bullet - k) + \sum_{j=1}^{\infty} \sum_{k\in\mathbf{Z}^n} \alpha^j_k \cdot (\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k).$$

Assume $0 < p, q < \infty$, $s \in \mathbb{R}$ and $w \in W$. Then the operators

$$\mathbf{se} : B^s_{p,q}(w) \to b^s_{p,q}(w) \qquad \text{and} \qquad \mathbf{fu} : b^s_{p,q}(w) \to B^s_{p,q}(w)$$

are bounded (the unique extension of $\mathbf{fu}$ *to $b^s_{p,q}(w)$ is denoted by* $\mathbf{fu}$*, too). Furthermore,* $\mathbf{fu} \circ \mathbf{se} = id$ *on $B^s_{p,q}(w)$ and*

$$\|\mathbf{se}f | b^s_{p,q}(w)\| \sim \|f | B^s_{p,q}(w)\|$$

holds for all $f \in S'$.

This characterization is akin to the ϕ-transform of Frazier and Jawerth [10] for isotropic unweighted homogeneous spaces. Similar results for the isotropic unweighted inhomogeneous case were proved by Sickel using splines [20] and wavelets [21].

The following important corollary is immidiate from the above theorem.

Corollary 2.2 (Discrete characterization of linear operators). *Assume $0 < p_0, p_1, q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$. The equivalence*

$$\|T | B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\| \sim \|\mathbf{se}\, T\, \mathbf{fu} | b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\|$$

holds for all linear operators $T : S \to S'$.

Here $\|A|X, Y\|$ denotes the operator quasinorm of the linear operator $A : X \to Y$. To apply this corollary to operators T_M with $T_M f = \mathcal{F}^{-1}[M\mathcal{F}f]$ we first study the boundedness of matrix operators.

3. Boundedness of matrix operators

For a matrix $A = \langle A^{j,l}_{k,m} \rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$ and a sequence $\alpha = \langle \alpha^j_k \rangle^{j \in \mathbb{N}}_{k \in \mathbb{Z}^n}$ we define the product $A\alpha$ via

$$(A\alpha)^l_m = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} A^{j,l}_{k,m} \cdot \alpha^j_k \qquad (l \in \mathbb{N},\, m \in \mathbb{Z}^n).$$

By the following lemma we can restrict ourselves to the case of matrix operators between unweighted sequence spaces.

Lemma 3.1 (Boundedness of A and $A(w_0, w_1)$). *Assume $0 < p_0, p_1, q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$. For a matrix $A = \langle A^{j,l}_{k,m} \rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$ define $A(w_0, w_1)$ by*

$$A(w_0, w_1) = \left\langle (w_1)^l_m A^{j,l}_{k,m} \frac{1}{(w_0)^j_k} \right\rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$$

with $(w_0)^l_m = w_0(A_{2^{-l}} m)$ and $(w_1)^j_k = w_1(A_{2^{-j}} k)$ (like in the definition of $b^s_{p,q}$). Then the relation

$$\left\| A | b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1) \right\| = \left\| A(w_0, w_1) | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \right\|$$

holds for all A.

The next theorem contains a boundedness criterion for matrix operators in the case of $q_0 \le q_1$.

Theorem 3.2 (First boundedness criterion for matrix operators). *Assume $0 < p_0, p_1, q_0, q_1 < \infty$ and $s_0, s_1 \in \mathbb{R}$. For a matrix $A = \langle A^{j,l}_{k,m} \rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$ define*

$$B(A; s_0, p_0, b^{s_1}_{p_1,q_1}) = \left\| \left\langle 2^{j(\frac{\nu}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \left\| \langle A^{j,l}_{k,m} \rangle^{l \in \mathbb{N}}_{m \in \mathbb{Z}^n} | b^{s_1}_{p_1,q_1} \right\| \right\rangle_{j \in \mathbb{N}} | \ell_\infty \right\| .$$

Then:

a) *We always have the estimate*

$$B(A; s_0, p_0, b^{s_1}_{p_1,q_1}) \le \left\| A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \right\| .$$

b) *In the case of $\max\{p_0, q_0\} \le \min\{1, p_1, q_1\}$ the equivalence*

$$\left\| A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \right\| \sim B(A; s_0, p_0, b^{s_1}_{p_1,q_1})$$

holds for all A.

The case of $q_0 > q_1$ is covered by the following theorem.

Theorem 3.3 (Second boundedness criterion for matrix operators). *Assume* $0 < p_0, p_1 < \infty$, $0 < q_1 < q_0 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Additionally assume that all coefficients* $A^{j,l}_{k,m}$ *with* $|j-l|>1$ *of the matrix* $A = \langle A^{j,l}_{k,m}\rangle^{j,l\in\mathbb{N}}_{k,m\in\mathbb{Z}^n}$ *vanish. We define*

$$N_l(A;p_0,p_1) = \sup \frac{\left\|\langle (A\alpha)^l_m\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1}\right\|}{\left\|\langle \alpha^{l+t}_m\rangle^{t=0,\pm1}_{m\in\mathbb{Z}^n} | \ell_{p_0}\right\|} \qquad (l \in \mathbb{N})$$

where the supremum is taken over all sequences α *for which the denominator does not vanish. The numbers* $N_l(A;p_0,p_1)$ *have the following properties:*

a) *The equivalence*

$$\left\|A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}\right\| \sim \left\|\langle 2^{ls} N_l(A;p_0,p_1)\rangle_{l\in\mathbb{N}} | \ell_r\right\|$$

holds for all A, $s = \nu\left(\frac{1}{p_0} - \frac{1}{p_1}\right) + s_1 - s_0$ *and* $\frac{1}{r} = \frac{1}{q_1} - \frac{1}{q_0}$.

b) *The estimate*

$$\sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \left\|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1}\right\| \le N_l(A;p_0,p_1)$$

holds for all $l \in \mathbb{N}$. *In particular there exists a constant* $C > 0$ *by part* a) *so that*

$$\left\|\left\langle 2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \left\|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1}\right\|\right\rangle_{l\in\mathbb{N}} | \ell_r\right\| \le C \cdot \left\|A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}\right\|$$

holds for all A (s *and* r *as in part* a)).

c) *In the case of* $0 < p_0 \le \min\{1, p_1\}$ *the first inequality of part b) can be reversed and by part a) the equivalence*

$$\left\|A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}\right\| \sim \left\|\left\langle 2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \left\|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1}\right\|\right\rangle_{l\in\mathbb{N}} | \ell_r\right\|$$

holds for all A.

In the proof of these two theorems we use the special sequences

$$(\varepsilon^j_k)^l_m = \begin{cases} 1 & \text{for } j = l \text{ and } k = m \\ 0 & \text{otherwise} \end{cases} \qquad (j,l \in \mathbb{N},\ k,m \in \mathbb{Z}^n). \tag{2}$$

They satisfy the two relations

$$\left\|\varepsilon^j_k | b^{s_0}_{p_0,q_0}\right\| = 2^{j(s_0 - \frac{\nu}{p_0})} \qquad (j \in \mathbb{N},\ k \in \mathbb{Z}^n)$$

and

$$(A\varepsilon^j_k)^l_m = \sum_{u=0}^{\infty} \sum_{v\in\mathbb{Z}^n} A^{u,l}_{v,m} \cdot (\varepsilon^j_k)^u_v = A^{j,l}_{k,m} \qquad (l \in \mathbb{N},\ m \in \mathbb{Z}^n).$$

Proof of Theorem 3.2. **Step 1.** From the above relations we conclude that

$$B(A;s_0,p_0,b^{s_1}_{p_1,q_1}) = \sup_{j\in\mathbf{N}}\sup_{k\in\mathbf{Z}^n} \frac{\|A\varepsilon^j_k|b^{s_1}_{p_1,q_1}\|}{\|\varepsilon^j_k|b^{s_0}_{p_0,q_0}\|} \le \|A|b^{s_0}_{p_0,q_0},b^{s_1}_{p_1,q_1}\|$$

which proves assertion a) of the theorem.

Step 2. Now we show the converse of this inequality to obtain the equivalences of assertions b) and c) of the theorem. Without loss of generality we may restrict to finite sequences α. Put $r=\min\{1,p_1,q_1\}$. From the embedding $\ell_r \hookrightarrow \ell_1$ we obtain

$$\begin{aligned}
&\left\| A\left(\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}^n}\alpha^j_k\varepsilon^j_k\right)\middle| b^{s_1}_{p_1,q_1}\right\| \\
&\le \left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\left\|\left\langle \|\langle(\alpha^j_k\cdot A\varepsilon^j_k)^l_m\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}|\ell_1\|\right\rangle_{m\in\mathbf{Z}^n}\middle|\ell_{p_1}\right\|\right\rangle_{l\in\mathbf{N}}\middle|\ell_{q_1}\right\| \\
&\le \left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\left\|\left\langle \|\langle(\alpha^j_k\cdot A\varepsilon^j_k)^l_m\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}|\ell_r\|\right\rangle_{m\in\mathbf{Z}^n}\middle|\ell_{p_1}\right\|\right\rangle_{l\in\mathbf{N}}\middle|\ell_{q_1}\right\|.
\end{aligned}$$

Applying the generalized Minkowski inequality twice this can be estimated by

$$\begin{aligned}
&\left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\left\|\left\langle \|\langle(\alpha^j_k\cdot A\varepsilon^j_k)^l_m\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}|\ell_r\|\right\rangle_{m\in\mathbf{Z}^n}\middle|\ell_{p_1}\right\|\right\rangle_{l\in\mathbf{N}}\middle|\ell_{q_1}\right\| \\
&= \left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\left\|\left\langle \|\langle|(\alpha^j_k\cdot A\varepsilon^j_k)^l_m|^r\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}|\ell_1\|\right\rangle_{m\in\mathbf{Z}^n}\middle|\ell_{p_1/r}\right\|\right\rangle_{l\in\mathbf{N}}\middle|\ell_{q_1/r}\right\|^{\frac{1}{r}} \\
&\le \left\|\left\langle \left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\|\langle|(\alpha^j_k\cdot A\varepsilon^j_k)^l_m|^r\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1/r}\|\right\rangle_{l\in\mathbf{N}}\middle|\ell_{q_1/r}\right\|\right\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}\middle|\ell_1\right\|^{\frac{1}{r}} \\
&\le \left\|\left\langle|\alpha^j_k|\cdot\|A\varepsilon^j_k|b^{s_1}_{p_1,q_1}\|\right\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}\middle|\ell_r\right\|.
\end{aligned}$$

Now using the definition of $B(A;s_0,p_0,b^{s_1}_{p_1,q_1})$ and the embeddings

$$b^{s_0}_{p_0,q_0}\hookrightarrow b^{s_0}_{p_0,r}\hookrightarrow b^{\sigma}_{r,r} \qquad (\sigma = s_0+\nu(\tfrac{1}{r}-\tfrac{1}{p_0}))$$

we finally get

$$\begin{aligned}
&\left\|\left\langle|\alpha^j_k|\cdot\|A\varepsilon^j_k|b^{s_1}_{p_1,q_1}\|\right\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}\middle|\ell_r\right\| \\
&\le \sup_{j\in\mathbf{N}}\sup_{k\in\mathbf{Z}^n}\frac{\|A\varepsilon^j_k|b^{s_1}_{p_1,q_1}\|}{\|\varepsilon^j_k|b^{s_0}_{p_0,q_0}\|}\cdot\left\|\left\langle|\alpha^j_k|\cdot\|\varepsilon^j_k|b^{s_0}_{p_0,q_0}\|\right\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}\middle|\ell_r\right\| \\
&= B(A;s_0,p_0,b^{s_1}_{p_1,q_1})\cdot\|\alpha|b^{\sigma}_{r,r}\| \\
&\le C\cdot B(A;s_0,p_0,b^{s_1}_{p_1,q_1})\cdot\|\alpha|b^{s_0}_{p_0,q_0}\in\mathbb{N}
\end{aligned}$$

which proves the theorem ∎

Proof of Theorem 3.3. We write $N_l = N_l(A; p_0, p_1)$ for short.

Part a) Using the Hölder inequality with $1 = \frac{q_1}{r} + \frac{q_1}{q_0}$ we get the estimate

$$\begin{aligned}
&\left\| A\alpha | b^{s_1}_{p_1,q_1} \right\| \\
&= \left\| \left\langle 2^{l(s_1 - \frac{\nu}{p_1})} \left\| \left\langle (A\alpha)^l_m \right\rangle_{m\in\mathbf{Z}^n} | \ell_{p_1} \right\| \right\rangle_{l\in\mathbf{N}} | \ell_{q_1} \right\| \\
&\le \left\| \left\langle 2^{l(s_1 - \frac{\nu}{p_1})} N_l(A; p_0, p_1) \left\| \langle \alpha^{l+t}_m \rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n} | \ell_{p_0} \right\| \right\rangle_{l\in\mathbf{N}} | \ell_{q_1} \right\| \\
&= \left\| \left\langle \left| 2^{l(s_1 - \frac{\nu}{p_1})} N_l \left\| \langle \alpha^{l+t}_m \rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n} | \ell_{p_0} \right\| \right|^{q_1} \right\rangle_{l\in\mathbf{N}} | \ell_1 \right\|^{1/q_1} \\
&\le \left\| \langle |2^{ls} N_l|^{q_1} \rangle_{l\in\mathbf{N}} | \ell_{r/q_1} \right\|^{\frac{1}{q_1}} \cdot \left\| \left\langle \left| 2^{l(s_0 - \frac{\nu}{p_0})} \left\| \langle \alpha^{l+t}_m \rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n} | \ell_{p_0} \right\| \right|^{q_1} \right\rangle_{l\in\mathbf{N}} | \ell_{q_0/q_1} \right\|^{\frac{1}{q_1}} \\
&\le C_0 \cdot \left\| \langle 2^{ls} N_l \rangle_{l\in\mathbf{N}} | \ell_r \right\| \cdot \left\| \alpha | b^{s_0}_{p_0,q_0} \right\|
\end{aligned}$$

which yields

$$\left\| A | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \right\| \le C_0 \cdot \left\| \langle 2^{ls} N_l \rangle_{l\in\mathbf{N}} | \ell_r \right\|.$$

To prove the converse inequality we split up the set of l-values in three disjoint parts which will be put together in the end. Therefore define

$$I_u = \{3j + u : j \in \mathbb{N}\} \qquad (u = 0, 1, 2).$$

There exists a sequence $a = \langle a_j \rangle_{j\in\mathbf{N}}$ of complex numbers satisfying the two conditions

$$\left\| \left\langle \mathbf{1}_{I_u}(j) a_j \right\rangle_{j\in\mathbf{N}} | \ell_{q_0/q_1} \right\| = 1$$

and

$$\left\| \langle 2^{ls} N_l \rangle_{l\in I_u} | \ell_r \right\| = \left\| \langle [2^{ls} N_l]^{q_1} \rangle_{l\in I_u} | \ell_{(q_0/q_1)'} \right\|^{\frac{1}{q_1}} = \left(\sum_{l\in I_u} [2^{ls} N_l]^{q_1} \cdot |a_l| \right)^{\frac{1}{q_1}} \tag{3}$$

where

$$\frac{1}{(q_0/q_1)'} = 1 - \frac{q_1}{q_0} = \frac{q_1}{r}.$$

The second condition states that a is a maximal element for the converse of the Hölder inequality. Since the coefficients $A^{j,l}_{k,m}$ with $|j - l| > 1$ vanish we can find a sequence $\tilde{\alpha}$ satisfying the condition

$$N_l \cdot \left\| \langle \tilde{\alpha}^{l+t}_m \rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n} | \ell_{p_0} \right\| \le 2 \cdot \left\| \left\langle (A\tilde{\alpha})^l_m \right\rangle^{l\in\mathbf{N}}_{m\in\mathbf{Z}^n} | \ell_{p_1} \right\| \tag{4}$$

for all $l \in I_u$. This follows from the definition of N_l and the structure of the set I_u. Note that the numbers $l + t$ and $(l+1) + t$ with $t = 0, \pm1$ are pairwise different for each two successive values of l. Since this inequality stays true if we dilate the sequence $\tilde{\alpha}$ by a positive constant we can adjust $\tilde{\alpha}$ to satisfy the additional condition

$$2^{l(s_0 - \frac{\nu}{p_0})} \left\| \langle \tilde{\alpha}^{l+t}_m \rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n} | \ell_{p_0} \right\| = |a_l|^{\frac{1}{q_1}} \tag{5}$$

for all $l \in I_u$. We show that $\tilde{\alpha}$ is an element of the space $b^{s_0}_{p_0,q_0}$. Formula (5) leads to

$$\begin{aligned}
\|\tilde{\alpha}|b^{s_0}_{p_0,q_0}\| &= \left(\sum_{l=0}^{\infty}\left[2^{l(s_0-\frac{\nu}{p_0})}\|\langle\tilde{\alpha}^l_m\rangle_{m\in\mathbf{Z}^n}|\ell_{p_0}\|\right]^{q_0}\right)^{\frac{1}{q_0}} \\
&\le C_1\cdot\left(\sum_{l\in I_u}\left[2^{l(s_0-\frac{\nu}{p_0})}\|\langle\tilde{\alpha}^{l+t}_m\rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n}|\ell_{p_0}\|\right]^{q_0}\right)^{\frac{1}{q_0}} \\
&= C_1\cdot\left(\sum_{l\in I_u}|a_l|^{\frac{q_0}{q_1}}\right)^{\frac{1}{q_0}} \\
&= C_1\cdot\|\langle\mathbf{1}_{I_u}(l)a_l\rangle_{l\in\mathbf{N}}|\ell_{q_0/q_1}\|^{\frac{1}{q_1}} \\
&= C_1 .
\end{aligned}$$

Now we use $\tilde{\alpha}$ to estimate the operator quasinorm of A from below. To this end we use the estimates from (3) - (5) to obtain

$$\begin{aligned}
\|\langle 2^{ls}N_l\rangle_{l\in I_u}|\ell_r\| &= \left(\sum_{l\in I_u}[2^{ls}N_l]^{q_1}\cdot|a_l|\right)^{\frac{1}{q_1}} \\
&= \left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}N_l\cdot\|\langle\tilde{\alpha}^{l+t}_m\rangle^{t=0,\pm1}_{m\in\mathbf{Z}^n}|\ell_{p_0}\|\right\rangle_{l\in I_u}|\ell_{q_1}\right\| \\
&\le 2\cdot\left\|\left\langle 2^{l(s_1-\frac{\nu}{p_1})}\|\langle(A\tilde{\alpha})^l_m\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\|\right\rangle_{l\in\mathbf{N}}|\ell_{q_1}\right\| \\
&= 2\cdot\|A\tilde{\alpha}|b^{s_1}_{p_1,q_1}\| \\
&\le 2C_1\cdot\|A|b^{s_0}_{p_0,q_0},b^{s_1}_{p_1,q_1}\| .
\end{aligned}$$

Summing up in u yields the desired estimate

$$\|\langle 2^{ls}N_l\rangle_{l\in\mathbf{N}}|\ell_r\| \le \sum_{u=0}^{2}\|\langle 2^{ls}N_l\rangle_{l\in I_u}|\ell_r\| \le C_2\cdot\|A|b^{s_0}_{p_0,q_0},b^{s_1}_{p_1,q_1}\| .$$

Part b) The assertion follows from the simple inequality

$$\sup_{k\in\mathbf{Z}^n}\sup_{t=0,\pm1}\|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\| = \sup_{k\in\mathbf{Z}^n}\sup_{t=0,\pm1}\frac{\|\langle(A\varepsilon^{l+t}_k)^l_m\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\|}{\|\langle(\varepsilon^{l+t}_k)^{l+s}_m\rangle^{s=0,\pm1}_{m\in\mathbf{Z}^n}|\ell_{p_0}\|} \le N_l .$$

Note that the denominator always equals 1.

Part c) It remains to show the estimate

$$N_l \le \sup_{k\in\mathbf{Z}^n}\sup_{t=0,\pm1}\|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\| .$$

Put $v = \min\{1, p_1\}$. Applying the same technique as in the proof of Theorem 3.2 we use $(A\varepsilon_k^{l+t})_m^l = A_{k,m}^{l+t,l}$, the embedding $\ell_v \hookrightarrow \ell_1$ and the generalized Minkowski inequality to obtain

$$\begin{aligned}\left\|\left\langle (A\alpha)_m^l\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\| &= \left\|\left\langle \sum_{t=-1}^{1}\sum_{k\in\mathbf{Z}^n} A_{k,m}^{l+t,l}\alpha_k^{l+t}\right\rangle_{m\in\mathbf{Z}^n}\left|\ell_{p_1}\right.\right\| \\ &\le \left\|\left\langle \left\|\left\langle \alpha_k^{l+t}\cdot (A\varepsilon_k^{l+t})_m^l\right\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_1\right\|\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\| \\ &\le \left\|\left\langle \left\|\left\langle \alpha_k^{l+t}\cdot (A\varepsilon_k^{l+t})_m^l\right\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_v\right\|\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\| \\ &\le \left\|\left\langle |\alpha_k^{l+t}|\cdot \left\|\left\langle (A\varepsilon_k^{l+t})_m^l\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\|\right\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_v\right\| \\ &\le \sup_{k\in\mathbf{Z}^n}\sup_{t=0,\pm1}\left\|\left\langle (A\varepsilon_k^{l+t})_m^l\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\|\cdot\left\|\left\langle \alpha_k^{l+t}\right\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_v\right\|.\end{aligned}$$

Because of $\ell_{p_0} \hookrightarrow \ell_v$ and $(A\varepsilon_k^{l+t})_m^l = A_{k,m}^{l+t,l}$ we have

$$\left\|\left\langle (A\alpha)_m^l\right\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\| \le \sup_{k\in\mathbf{Z}^n}\sup_{t=0,\pm1}\left\|\langle A_{k,m}^{l+t,l}\rangle_{m\in\mathbf{Z}^n}|\ell_{p_1}\right\|\cdot\left\|\langle\alpha_k^{l+t}\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_{p_0}\right\|.$$

Dividing this by $\left\|\langle\alpha_k^{l+t}\rangle_{k\in\mathbf{Z}^n}^{t=0,\pm1}|\ell_{p_0}\right\|$ and taking the supremum in α completes the proof ∎

4. Matrices associated with Fourier multipliers

For $M \in \mathcal{S}'$ the operator T_M is given by

$$T_M f = \mathcal{F}^{-1}[M\mathcal{F}f] \qquad (f \in \mathcal{S})$$

and the class of *Fourier multipliers* between the two spaces $B^{s_0}_{p_0,q_0}(w_0)$ and $B^{s_1}_{p_1,q_1}(w_1)$ is defined by

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) = \left\{M \in \mathcal{S}'\,\middle|\, T_M : B^{s_0}_{p_0,q_0}(w_0) \to B^{s_1}_{p_1,q_1}(w_1) \text{ is bounded}\right\}$$

equipped with the quasinorm

$$\left\|M|\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big)\right\| = \left\|T_M|B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\right\|.$$

The matrix operator associated with T_M is

$$\widetilde{M} = \mathbf{se}\,T_M\,\mathbf{fu}$$

and a simple calculation shows that its coefficients are

$$\begin{aligned}\widetilde{M}_{k,m}^{j,l} &:= (\mathbf{se}\,T_M\,\mathbf{fu})_{k,m}^{j,l} \\ &= \begin{cases} \left(\mathbf{se}\big(T_M\left[(\mathcal{F}^{-1}\psi_0)(\bullet - k)\right]\big)\right)_m^l & \text{for } j = 0 \\ \left(\mathbf{se}\big(T_M\left[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\right]\big)\right)_m^l & \text{for } j \ge 1 \end{cases} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\begin{cases} \mathcal{F}^{-1}\left[\phi_l\psi_0 M\right](A_{2^{-l}}m - k) & \text{for } j = 0 \\ \mathcal{F}^{-1}\left[\phi_l(A_{2^j}^*\bullet)\psi_1 M(A_{2^j}^*\bullet)\right](A_{2^{j-l}}m - k) & \text{for } j \ge 1. \end{cases}\end{aligned}$$

We make the important observation that due to the overlapping of the supports of $\phi_l(A^*_{2^j}\bullet)$ and $\psi_{0,1}$ all coefficients $\widetilde{M}^{j,l}_{k,m}$ with $|j-l|>1$ vanish and thus

$$\widetilde{M}(w_0,w_1)^{j,l}_{k,m}=0 \qquad (|j-l|>1;\ w_0,w_1\in W).$$

Recall that $\widetilde{M}(w_0,w_1)$ is the corresponding operator for unweighted spaces (see Lemma 3.1).

Combining Corollary 2.2 and Lemma 3.1 yields the relation

$$\begin{aligned}\|M|\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0),B^{s_1}_{p_1,q_1}(w_1))\| &\sim \|\widetilde{M}|b^{s_0}_{p_0,q_0}(w_0),b^{s_1}_{p_1,q_1}(w_1)\|\\ &= \|\widetilde{M}(w_0,w_1)|b^{s_0}_{p_0,q_0},b^{s_1}_{p_1,q_1}\|\end{aligned}$$

where the last quasinorm on the right side is equivalent to $B(\widetilde{M}(w_0,w_1);s_0,p_0,b^{s_1}_{p_1,q_1})$ under certain restrictions on the parameters p,q and s by the results of the previous section. So we are interested in a characterization of $B(\widetilde{M}(w_0,w_1);s_0,p_0,b^{s_1}_{p_1,q_1})$ in terms of M. This is done with the help of the following theorem.

Theorem 4.1 (Characterization of the matrices $\widetilde{M}$). *Assume* $0<p<\infty$, $0<q\le\infty$ *and* $s,\sigma\in\mathbb{R}$.

a) *If* $w_1\in W_d$ $(d\ge 0)$ *and* $w_0\in W$ *satisfy the condition* $\|\frac{(1+|\bullet|)^d}{w_0}\|_\infty<\infty$, *then the equivalence*

$$\left\|\left\langle 2^{j\sigma}\sup_{k\in\mathbf{Z}^n}\frac{1}{(w_0)^j_k}\|\langle\widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbf{N}}_{m\in\mathbf{Z}^n}|b^s_{p,p}\|\right\rangle_{j\in\mathbf{N}}\middle|\ell_q\right\|\sim\|M|\mathcal{F}[B^{s+\sigma-\nu}_{p,q}(w_1)]\|$$

holds for all $M\in\mathcal{S}'$.

b) *If* $w_0\in W$ *with* $\|\frac{1}{w_0}\|_\infty=\infty$, *then*

$$\left\|\left\langle 2^{j\sigma}\sup_{k\in\mathbf{Z}^n}\frac{1}{(w_0)^j_k}\|\langle\widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbf{N}}_{m\in\mathbf{Z}^n}|b^s_{p,p}(w_1)\|\right\rangle_{j\in\mathbf{N}}\middle|\ell_q\right\|<\infty$$

implies $M\equiv 0$.

The following two corollaries specialize this theorem which will be proved at the end of this section to the conditions appearing in the boundedness results for matrix operators (Theorems 3.2 and 3.3).

Corollary 4.2 (First characterization of $\widetilde{M}(w_0,w_1)$). *Assume* $0<p_0,p_1<\infty$, $0<q_1\le\infty$ *and* $s_0,s_1\in\mathbb{R}$.

a) *If* $w_1\in W_d$ $(d\ge 0)$ *and* $w_0\in W$ *satisfy the condition* $\frac{\|(1+|\bullet|)^d}{w_0\|_\infty<\infty}$, *then the equivalence*

$$B(\widetilde{M}(w_0,w_1);s_0,p_0,b^{s_1}_{p_1,q_1})\sim\|M|\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)]\|$$

with $\sigma = \nu(\frac{1}{p_0} - 1) + s_1 - s_0$ *holds for all* $M \in \mathcal{S}'$.

b) *If* $w_0 \in W$ *with* $\|\frac{1}{w_0}\|_\infty = \infty$, *then*

$$B(\widetilde{M}(w_0,1); s_0, p_0, b^{s_1}_{p_1,q_1}) < \infty$$

implies $M \equiv 0$.

Proof. We only prove part a) because the second assertion follows in exactly the same manner.

Since the coefficients $\widetilde{M}(w_0,w_1)^{j,l}_{k,m}$ with $|j-l| > 1$ vanish only three terms in l occur for a fixed j. Thus we can change the parameter q_1 to p_1 and by the definition of $\widetilde{M}(w_0,w_1)$ we are lead to the equivalence

$$\begin{aligned}\|\langle \widetilde{M}(w_0,w_1)^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{s_1}_{p_1,q_1}\| &\sim \|\langle \widetilde{M}(w_0,w_1)^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{s_1}_{p_1,p_1}\| \\ &= \frac{1}{(w_0)^j_k}\|\langle \tilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{s_1}_{p_1,p_1}(w_1)\|.\end{aligned}$$

An application of Theorem 4.1 yields the relation

$$\begin{aligned}&B(\widetilde{M}(w_0,w_1); s_0, p_0, b^{s_1}_{p_1,q_1}) \\ &= \left\| \left\langle 2^{j(\frac{\nu}{p_0}-s_0)} \sup_{k\in\mathbb{Z}^n} \|\langle \widetilde{M}(w_0,w_1)^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{s_1}_{p_1,q_1}\| \right\rangle_{j\in\mathbb{N}} \middle| \ell_\infty \right\| \\ &\sim \left\| \left\langle 2^{j(\frac{\nu}{p_0}-s_0)} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)^j_k} \|\langle \widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{s_1}_{p_1,p_1}(w_1)\| \right\rangle_{j\in\mathbb{N}} \middle| \ell_\infty \right\| \\ &\sim \|M|\mathcal{F}[B^\sigma_{p_1,\infty}(w_1)]\|\end{aligned}$$

which proves the corollary ∎

Corollary 4.3 (Second characterization of $\widetilde{M}(w_0,w_1)$). *Assume* $0 < p < \infty$, $0 < r \le \infty$ *and* $s \in \mathbb{R}$. *If* $w_1 \in W_d$ $(d \ge 0)$ *and* $w_0 \in W$ *satisfy the condition* $\|\frac{(1+|\bullet|)^d}{w_0}\|_\infty < \infty$, *then the equivalence*

$$\left\| \left\langle 2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \|\langle \widetilde{M}(w_0,w_1)^{l+t,l}_{k,m}\rangle_{m\in\mathbb{Z}^n} | \ell_p\| \right\rangle_{l\in\mathbb{N}} \middle| \ell_r \right\| \sim \|M|\mathcal{F}[B^\sigma_{p,r}(w_1)]\|$$

with $\sigma = \nu(\frac{1}{p} - 1) + s$ *holds for all* $M \in \mathcal{S}'$.

Proof. We write $A^{j,l}_{k,m}$ for $\widetilde{M}(w_0,w_1)^{j,l}_{k,m}$. Note that the sequence space $b^{\nu/p}_{p,p}$ is identical with the ℓ_p-space on $\mathbb{N} \times \mathbb{Z}^n$. We have the estimate

$$\begin{aligned}2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \|\langle A^{l+t,l}_{k,m}\rangle_{m\in\mathbb{Z}^n} | \ell_p\| &\le 2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \|\langle A^{l+t,j}_{k,m}\rangle^{j\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{\nu/p}_{p,p}\| \\ &\le 2^{ls} \sum_{t=-1}^{1} \sup_{k\in\mathbb{Z}^n} \|\langle A^{l+t,j}_{k,m}\rangle^{j\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^{\nu/p}_{p,p}\|.\end{aligned}$$

Now applying the ℓ_r quasinorm in l yields

$$\left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \sup_{t=0,\pm1} \left\| \langle A_{k,m}^{l+t,l} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| \right\rangle_{l\in\mathbf{N}} \left| \ell_r \right\| \le C_0 \cdot \left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \left\| \langle A_{k,m}^{l,j} \rangle_{m\in\mathbf{Z}^n}^{j\in\mathbf{N}} | b_{p,p}^{\nu/p} \right\| \right\rangle_{l\in\mathbf{N}} \left| \ell_r \right\| . \tag{6}$$

In the same way we show the reverse inequality starting with

$$\begin{aligned} 2^{ls} \sup_{k\in\mathbf{Z}^n} \left\| \langle A_{k,m}^{l,j} \rangle_{m\in\mathbf{Z}^n}^{j\in\mathbf{N}} | b_{p,p}^{\nu/p} \right\| &\le C_1 \cdot 2^{ls} \sup_{k\in\mathbf{Z}^n} \sup_{u=0,\pm1} \left\| \langle A_{k,m}^{l,l+u} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| \\ &\le C_1 \cdot \sum_{u=-1}^{1} 2^{ls} \sup_{k\in\mathbf{Z}^n} \left\| \langle A_{k,m}^{l,l+u} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| \\ &\le C_1 \cdot \sum_{u=-1}^{1} 2^{ls} \sup_{t=0,\pm1} \sup_{k\in\mathbf{Z}^n} \left\| \langle A_{k,m}^{l+t,l+u} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| . \end{aligned}$$

This leads to

$$\left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \left\| \langle A_{k,m}^{l,j} \rangle_{m\in\mathbf{Z}^n}^{j\in\mathbf{N}} | b_{p,p}^{\nu/p} \right\| \right\rangle_{l\in\mathbf{N}} \left| \ell_r \right\| \le C_2 \cdot \left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \sup_{t=0,\pm1} \left\| \langle A_{k,m}^{l+t,l} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| \right\rangle_{l\in\mathbf{N}} \left| \ell_r \right\| . \tag{7}$$

A combination of formulae (6) and (7) finally results in

$$\begin{aligned} &\left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \sup_{t=0,\pm1} \left\| \langle \widetilde{M}(w_0,w_1)_{k,m}^{l+t,l} \rangle_{m\in\mathbf{Z}^n} | \ell_p \right\| \right\rangle_{l\in\mathbf{N}} \left| \ell_r \right\| \\ &\sim \left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \left\| \langle \widetilde{M}(w_0,w_1)_{k,m}^{j,l} \rangle_{m\in\mathbf{Z}^n} | b_{p,p}^{\nu/p} \right\| \right\rangle_{j\in\mathbf{N}} \left| \ell_r \right\| \\ &= \left\| \left\langle 2^{ls} \sup_{k\in\mathbf{Z}^n} \frac{1}{(w_0)_k^j} \left\| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m\in\mathbf{Z}^n} | b_{p,p}^{\nu/p}(w_1) \right\| \right\rangle_{j\in\mathbf{N}} \left| \ell_r \right\| \\ &\sim \left\| M | \mathcal{F}\left[B_{p,r}^{\sigma}(w_1)\right] \right\| \end{aligned}$$

and the assertion follows from Theorem 4.1 ∎

To prove Theorem 4.1 we need the following lemma.

Lemma 4.4 (simple Fourier multipliers for L_p). *Asume $0 < p < \infty$ and $w \in W_d$ $(d \ge 0)$. Then there exists a constant $C > 0$ such that*

$$\left\| w \cdot \mathcal{F}^{-1}[fg] \right\|_p \le C \cdot \left\| (1 + |\bullet|)^d \cdot \mathcal{F}^{-1}\left[f(A_{2^j}^* \bullet) \right] \right\|_{\tilde{p}} \cdot \left\| w \cdot \mathcal{F}^{-1} g \right\|_p$$

with $\tilde{p} = \min\{1,p\}$ *holds for all* $j \in \mathbb{N}$ *and all* $f \in \mathcal{S}$ *and* $g \in \mathcal{S}'$ *satisfying the condition* $\operatorname{supp} f, \operatorname{supp} g \subseteq \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \le 2^j\}$.

Proof. Since the supports of $f(A^*_{2^j}\bullet)$ and $g(A^*_{2^j}\bullet)$ are both contained in the compact set $\{\xi \in \mathbb{R}^n : \varrho^*(\xi) \le 1\}$ and the relation $w(x) \le C_0 \cdot w(y)(1+|x-y|)^d$ holds there exists a constant $C_1 > 0$ by [18: Theorem 1.7.2] for which

$$\begin{aligned} \left\|w \cdot \mathcal{F}^{-1}[fg]\right\|_p &= 2^{j\nu(1-\frac{1}{p})}\left\|w(A_{2^{-j}}\bullet) \cdot \mathcal{F}^{-1}\left[f(A^*_{2^j}\bullet)g(A^*_{2^j}\bullet)\right]\right\|_p \\ &\le C_0C_1 \cdot 2^{j\nu(1-\frac{1}{p})}\left\|(1+|A_{2^{-j}}\bullet|)^d \cdot \mathcal{F}^{-1}\left[f(A^*_{2^j}\bullet)\right]\right\|_{\tilde{p}} \\ &\quad \times \left\|w(A_{2^{-j}}\bullet) \cdot \mathcal{F}^{-1}\left[g(A^*_{2^j}\bullet)\right]\right\|_p. \end{aligned}$$

To estimate the $L_{\tilde{p}}$ quasinorm we apply formula (1) which in connection with a substitution leads to the desired estimate

$$\left\|w \cdot \mathcal{F}^{-1}[fg]\right\|_p \le C_0C_1C_2 \cdot \left\|(1+|\bullet|)^d \cdot \mathcal{F}^{-1}\left[f(A^*_{2^j}\bullet)\right]\right\|_{\tilde{p}} \cdot \left\|w \cdot \mathcal{F}^{-1}g\right\|_p$$

and the assertion is proved ■

Proof of Theorem 4.1. The proof is divided into the following four steps:

1. First we derive an equivalence for $\left\|\langle \widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^s_{p,p}(w_1)\right\|$.
2. Using this equivalence and Lemma 4.4 we estimate the discrete quasinorm in the theorem by $\left\|M|\mathcal{F}\left[B^{s+\sigma-\nu}_{p,q}(w_1)\right]\right\|$ from above.
3. We prove the estimate from below using a similar technique as in the previous step. This completes the proof of part a).
4. Part b) is derived from the equivalence in the first step.

Step 1. Recall that the coefficients of $\widetilde{M}$ are given by

$$\widetilde{M}^{j,l}_{k,m} = \begin{cases} \left(\operatorname{se}\left(T_M\left[(\mathcal{F}^{-1}\psi_0)(\bullet - k)\right]\right)\right)^l_m & \text{for } j = 0 \\ \left(\operatorname{se}\left(T_M\left[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\right]\right)\right)^l_m & \text{for } j \ge 1. \end{cases} \qquad (j,l \in \mathbb{N};\ k,m \in \mathbb{Z}^n).$$

Since $\left\|\operatorname{se}\bullet|b^s_{p,p}(w_1)\right\|$ is an equivalent quasinorm on $B^s_{p,p}(w_1)$ we have

$$\left\|\langle \widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^s_{p,p}(w_1)\right\| \sim \begin{cases} \left\|T_M\left[(\mathcal{F}^{-1}\psi_0)(\bullet - k)\right]|B^s_{p,p}(w_1)\right\| & \text{for } j = 0 \\ \left\|T_M\left[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\right]|B^s_{p,p}(w_1)\right\| & \text{for } j \ge 1. \end{cases}$$

To deal with the terms on the right-hand side we use the identity

$$\mathcal{F}^{-1}\left[\phi_u\mathcal{F}\left[T_M\left[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\right]\right]\right] = \mathcal{F}^{-1}\left[\phi_u \cdot 2^{-j\nu}M\psi_1(A^*_{2^{-j}}\bullet)\right](\bullet - A_{2^{-j}}k)$$

which holds for $j \ge 1$ and all $u \in \mathbb{N}$. Due to the location of the supports of ψ_1 and ϕ_u only the terms with $u = j + r$ ($r = 0, \pm 1$) are of interest. This leads to

$$\begin{aligned} &\left\|T_M\left[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\right]|B^s_{p,p}(w_1)\right\| \\ &\quad = \left\|\left\langle 2^{us}\left\|w_1(\bullet + A_{2^{-j}}k) \cdot \mathcal{F}^{-1}\left[\phi_u 2^{-j\nu}M\psi_1(A^*_{2^{-j}}\bullet)\right]\right\|_p\right\rangle_{u\in\mathbb{N}}|\ell_p\right\| \\ &\quad \sim \sup_{r=0,\pm1} 2^{j(s-\nu)}\left\|w_1(\bullet + A_{2^{-j}}k) \cdot \mathcal{F}^{-1}\left[\phi_{j+r}M\psi_1(A^*_{2^{-j}}\bullet)\right]\right\|_p \end{aligned}$$

and a similar equivalence holds in the case of $j = 0$. Thus we arrive at

$$\left\|\langle \tilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^s_{p,p}(w_1)\right\| \sim 2^{j(s-\nu)} \sup_{r=0,\pm1} \begin{cases} \left\|w_1(\bullet + k)\cdot \mathcal{F}^{-1}[\psi_0\phi_r M]\right\|_p & \text{for } j=0 \\ \left\|w_1(\bullet + A_{2^{-j}}k)\cdot \mathcal{F}^{-1}\left[\psi_1(A^*_{2^{-j}}\bullet)\phi_{j+r}M\right]\right\|_p & \text{for } j\geq 1. \end{cases} \tag{8}$$

Step 2. To prove the estimate from above we use the inequality

$$w_1(x + A_{2^{-j}}k) \leq C_0 \cdot w_1(x)(1 + |A_{2^{-j}}k|)^d$$

which in combination with (8) yields

$$\begin{aligned} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)^j_k} \left\|\langle \widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^s_{p,p}(w_1)\right\| &\leq C_1 \cdot 2^{j(s-\nu)} \sup_{k\in\mathbb{Z}^n} \frac{(1+|A_{2^{-j}}k|)^d}{w_0(A_{2^{-j}}k)} \cdot I_j \\ &\leq C_2 \cdot 2^{j(s-\nu)} I_j \end{aligned} \tag{9}$$

where

$$I_j = \sup_{r=0,\pm1} \begin{cases} \left\|w_1 \cdot \mathcal{F}^{-1}\left[\psi_0\phi_r M\right]\right\|_p & \text{for } j=0 \\ \left\|w_1 \cdot \mathcal{F}^{-1}\left[\psi_1(A^*_{2^{-j}}\bullet)\phi_{j+r}M\right]\right\|_p & \text{for } j\geq 1 \end{cases}$$

because the fraction on the right is bounded uniformly in j and k. Now we apply Lemma 4.4 to estimate the I_j and we obtain

$$\begin{aligned} &\left\|w_1 \cdot \mathcal{F}^{-1}\left[\psi_1(A^*_{2^{-j}}\bullet)\phi_{j+r}M\right]\right\|_p \\ &\qquad \leq C_3 \cdot \left\|(1+|\bullet|)^d \cdot \mathcal{F}^{-1}\left[\psi_1(A^*_4\bullet)\right]\right\|_{\bar{p}} \cdot \left\|w_1 \cdot \mathcal{F}^{-1}\left[\phi_{j+r}M\right]\right\|_p \end{aligned}$$

in the case of $j \geq 1$ and $r = 0, \pm1$. Similar estimates can be proved in the remaining cases. Since $\psi_0, \psi_1 \in \mathcal{S}$ the $L_{\bar{p}}$-terms are bounded and we get

$$I_j \leq C_4 \cdot \sup_{r=0,\pm1} \left\|w_1 \cdot \mathcal{F}^{-1}\left[\phi_{j+r}M\right]\right\|_p \qquad (j\in\mathbb{N}).$$

From this we obtain by (9) the inequality

$$\begin{aligned} &\left\|\left\langle 2^{j\sigma} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)^j_k} \left\|\langle \widetilde{M}^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n} | b^s_{p,p}(w_1)\right\|\right\rangle_{j\in\mathbb{N}} \Big| \ell_q\right\| \\ &\qquad \leq C_5 \cdot \left\|\left\langle 2^{j(s+\sigma-\nu)} \left\|w_1 \cdot \mathcal{F}^{-1}[\phi_j M]\right\|_p\right\rangle_{j\in\mathbb{N}} \Big| \ell_q\right\| \\ &\qquad = C_5 \cdot \left\|\mathcal{F}^{-1}M | B^{s+\sigma-\nu}_{p,q}(w_1)\right\|. \end{aligned}$$

Step 3. To prove the estimate from below we use the inequality

$$(1 + |A_{2^{-j}}k|)^{-d} w_1(x) \leq C_0 \cdot w_1(x + A_{2^{-j}}k)$$

which in combination with (8) yields

$$\sup_{k\in\mathbb{Z}^n} \frac{(1+|A_{2^{-j}}k|)^{-d}}{(w_0)_k^j} \cdot 2^{j(s-\nu)} I_j \le C_6 \cdot \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|\langle \widetilde{M}_{k,m}^{j,l} \rangle_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} | b_{p,p}^s(w_1)\|$$

where I_j is defined as above. The supremum on the left side can be estimated from below by choosing $k=0$ which leads to

$$2^{j(s-\nu)} I_j \le C_7 \cdot \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|\langle \widetilde{M}_{k,m}^{j,l} \rangle_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} | b_{p,p}^s(w_1)\| . \tag{10}$$

To estimate the I_j from below we use the partition of unity

$$\phi_0(\xi)\psi_0(\xi) + \sum_{j=1}^{\infty} \phi_j(\xi)\psi_1(A_{2^{-j}}^*\xi) = 1$$

(following from the definition of ψ_0 and ψ_1) and Lemma 4.4. In this way we obtain

$$\begin{aligned}
\|w_1 \cdot \mathcal{F}^{-1}[\phi_j M]\|_p &= \left\| w_1 \cdot \mathcal{F}^{-1}\left[\phi_j \sum_{t=-1}^{1} \phi_{j+t}\psi_1(A_{2^{-(j+t)}}^*\bullet)M\right]\right\|_p \\
&\le C_8 \cdot \left\|(1+|\bullet|)^d \cdot \mathcal{F}^{-1}\left[\phi_j(A_{2^{j+2}}^*\bullet)\right]\right\|_{\bar{p}} \\
&\quad \times \sum_{t=-1}^{1} \left\|w_1 \cdot \mathcal{F}^{-1}\left[\phi_{j+t}\psi_1(A_{2^{-(j+t)}}^*\bullet)M\right]\right\|_p \\
&\le C_9 \cdot \sup_{t=0,\pm1} \left\|w_1 \cdot \mathcal{F}^{-1}\left[\psi_1(A_{2^{-(j+t)}}^*\bullet)\phi_{j+t}M\right]\right\|_p
\end{aligned}$$

because the $L_{\bar{p}}$ quasinorms are uniformly bounded in j. Similar estimates can be proved in the remaining cases and we get the relation

$$\left\|w_1 \cdot \mathcal{F}^{-1}[\phi_j M]\right\|_p \le C_{10} \cdot \sup_{t=0,\pm1} I_{j+t} \qquad (j\in\mathbb{N})$$

with $I_{-1}=0$. By (10) it follows that

$$\begin{aligned}
&\left\|\mathcal{F}^{-1}M | B_{p,q}^{s+\sigma-\nu}(w_1)\right\| \\
&\quad = \left\|\left\langle 2^{j(s+\sigma-\nu)} \left\|w_1 \cdot \mathcal{F}^{-1}[\phi_j M]\right\|_p \right\rangle_{j\in\mathbb{N}} | \ell_q \right\| \\
&\quad \le C_{11} \cdot \left\| \left\langle 2^{j\sigma} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|\langle \widetilde{M}_{k,m}^{j,l} \rangle_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} | b_{p,p}^s(w_1)\| \right\rangle_{j\in\mathbb{N}} | \ell_q \right\| .
\end{aligned}$$

Step 4. If $w_1 \equiv 1$, then (8) implies

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|\langle \widetilde{M}_{k,m}^{j,l} \rangle_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} | b_{p,p}^s\| \\
&\sim 2^{j(s-\nu)} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \sup_{r=0,\pm1} \begin{cases} \left\|\mathcal{F}^{-1}\left[\psi_0\phi_r M\right]\right\|_p & \text{for } j=0 \\ \left\|\mathcal{F}^{-1}\left[\psi_1(A_{2^{-j}}^*\bullet)\phi_{j+r}M\right]\right\|_p & \text{for } j\ge 1. \end{cases}
\end{aligned} \tag{11}$$

By definition of W we have

$$(w_0)_k^j = w_0(A_{2^{-j}}k) \le C_{12} \cdot w_0(x)(1 + |A_{2^{-j}}k - x|)^{d'}$$

for a suitable $d' > 0$. For $x \in \diamond_k^j := A_{2^{-j}}\big(k + [-\frac{1}{2}, \frac{1}{2})^n\big)$ the estimate

$$(1 + |A_{2^{-j}}k - x|)^{d'} \le C_{13}$$

holds uniformly in j and k, leading to

$$\sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \ge (C_{12}C_{13})^{-1} \cdot \sup_{k\in\mathbb{Z}^n} \sup_{x\in\diamond_k^j} \frac{1}{w_0(x)} = (C_{12}C_{13})^{-1} \cdot \left\| \frac{1}{w_0} \right\|_\infty.$$

and the theorem is proved ∎

5. Fourier Multipliers

This section contains the main results of this paper. We start with the following two propositions which will be proved at the end of this section.

Proposition 5.1 (Change of s). *Assume* $0 < p_0, p, q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w_0, w_1 \in W$. *Then the relation*

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \leftrightarrows \mathbf{M}\big(B^{s_1-s_0}_{p_0,q_0}(w_0), B^{0}_{p_1,r_1}(w_1)\big)$$

holds.

This is usually proved with the help of Bessel potential operators. These operators are quite difficult to deal with in our anisotropic weighted setting. Our proof using the discrete characterization is much simpler.

Proposition 5.2 (Change of q). *Assume* $0 < p_0, p_1 q_0, q_1, r, r_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w_0, w_1 \in W$. *If* $\big(\frac{1}{q_1} - \frac{1}{q_0}\big)_+ = \big(\frac{1}{r_1} - \frac{1}{r_0}\big)_+$, *then the relation*

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \leftrightarrows \mathbf{M}\big(B^{s_0}_{p_0,r_0}(w_0), B^{s_1}_{p_1,r_1}(w_1)\big)$$

holds.

Note that the assertion for $q_0 > q_1$ was already proved by Orlovskij [16] for unweighted isotropic spaces in the case of $p_0, p_1 > 1$.

Now we come to the main theorem of this paper.

Theorem 5.3 (Fourier multipliers between Besov spaces). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *If* $w_1 \in W_d$ $(d \ge 0)$ *and* $w_0 \in W$ *satisfy the condition* $\big\| \frac{(1+|\bullet|)^d}{w_0} \big\|_\infty < \infty$, *then the relation*

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \leftrightarrows \mathcal{F}\big[B^{\sigma}_{p_1,r}(w_1)\big] \qquad (p_0 \le \min\{1, p_1\})$$

holds where $\sigma = \nu\big(\frac{1}{p_0} - 1\big) + s_1 - s_0$ *and* $\frac{1}{r} = \big(\frac{1}{q_1} - \frac{1}{q_0}\big)_+$.

This theorem will be completed by negative results in Part II of this paper.

Proof of Theorem 5.3. Step 1. First assume $q_0 \le q_1$, i.e. $r = \infty$. We apply Proposition 5.2, Lemma 3.1, Theorem 3.2 and Theorem 4.2 to obtain the equivalence

$$\begin{aligned} \big\| M | \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big)\big\| &\sim \big\| M | \mathbf{M}\big(B^{s_0}_{p_0,p_0}(w_0), B^{s_1}_{p_1,p_1}(w_1)\big)\big\| \\ &\sim \big\| \widetilde{M}(w_0,w_1) | b^{s_0}_{p_0,p_0}, b^{s_1}_{p_1,p_1}\big\| \\ &\sim B\big(\widetilde{M}(w_0,w_1); s_0, p_0, b^{s_1}_{p_1,p_1}\big) \\ &\sim \big\| M | \mathcal{F}\big[B^{\sigma}_{p_1,\infty}(w_1)\big]\big\|. \end{aligned}$$

Step 2. Now let $q_0 > q_1$. We apply Lemma 3.1, Theorem 3.3 and Theorem 4.3 to obtain the equivalence

$$\begin{aligned} &\big\| M | \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big)\big\| \\ &\quad \sim \big\| \widetilde{M}(w_0,w_1) | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}\big\| \\ &\quad \sim \Big\| \Big\langle 2^{ls} \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm1} \big\| \big\langle \widetilde{M}(w_0,w_1)^{l+t,l}_{k,m}\big\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1}\big\| \Big\rangle_{l\in\mathbb{N}} | \ell_r \Big\| \\ &\quad \sim \big\| M | \mathcal{F}\big[B^{\sigma}_{p_1,r}(w_1)\big]\big\| \end{aligned}$$

which proves the theorem ∎

Remarks. The first theorem of this type goes back to Taibleson [23: Part II/p. 827] and asserts

$$\mathbf{M}\big(B^{s}_{1,1}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}\big[B^{0}_{1,\infty}(\mathbb{R}^n; I, 1)\big] \quad (s \in \mathbb{R}).$$

It can be obtained by choosing $p_0 = p_1 = q_0 = q_1 = 1$ and $s = s_0 = s_1$ in Theorem 5.3. Peetre [17: p. 249] proved the supplement

$$\mathbf{M}\big(B^{s}_{p,q}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}\big[B^{n(\frac{1}{p}-1)}_{p,\infty}(\mathbb{R}^n; I, 1)\big] \quad \begin{pmatrix} 0 < p < 1 \\ 0 < q < \infty \\ s \in \mathbb{R} \end{pmatrix}$$

which can be obtained by choosing $p = p_0 = p_1$, $q = q_0 = q_1$ and $s = s_0 = s_1$ in Theorem 5.3. Johnson [12: Theorem 6] proved

$$\mathbf{M}\big(\dot{B}^{s_0}_{1,q_0}(\mathbb{R}^n; I, 1), \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}\big[\dot{B}^{s_1-s_0}_{p_1,\infty}(\mathbb{R}^n; I, 1)\big] \quad \begin{pmatrix} 1 \le p_1 \infty \\ 1 \le q_0 \le q_1 < \infty \\ s_0, s_1 \in \mathbb{R} \end{pmatrix}$$

for homogeneous spaces with the help of a characterization of Besov spaces in terms of the Gau-Weierstraß kernel and preliminary work of Taibleson. Due to technical reasons the characterization of Fourier multipliers is usually much simpler in the homogeneous case than in the inhomogeneous one. Bui [1: Theorem 2] modified Johnson's method to deal with inhomogeneous spaces and obtained the counterpart of Johnson's result, i.e.

$$\mathbf{M}\big(B^{s_0}_{1,q_0}(\mathbb{R}^n; I, 1), B^{s_1}_{p_1,q_1}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}\big[B^{s_1-s_0}_{p_1,\infty}(\mathbb{R}^n; I, 1)\big] \quad \begin{pmatrix} 1 \le p_1 < \infty \\ 1 \le q_0 \le q_1 < \infty \\ s_0, s_1 \in \mathbb{R} \end{pmatrix}$$

which can be proved by choosing $p_0 = 1$ in Theorem 5.3. Bui recently generalized his method in order to deal with weighted spaces (cf. [2, 3]) and proved embedding theorems for classes of Fourier multipliers between isotropic Besov spaces with power weights.

The unweighted version of the above theorem was already proved in [6] ($q_0 \le q_1$) and [7] ($q_0 > q_1$). Similar results in the periodic setting (inhomogeneous, unweighted case) were proved by Mizuhara [15].

Proof of Proposition 5.2. First we consider the case of $q_0 \le q_1$. If $M \in \mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1))$, then $\widetilde{M} : b^{s_0}_{p_0,q_0}(w_0) \to b^{s_1}_{p_1,q_1}(w_1)$ is bounded. Consider the sequences α_j defined as

$$(\alpha_j)^l_m = \mathbf{1}_{[-1,1]}(j-l) \cdot \alpha^l_m$$

associated with the sequence α. Since the coefficients $\widetilde{M}^{j,l}_{k,m}$ for $|j-l| > 1$ vanish we have $(\widetilde{M}\alpha)^j_k = (\widetilde{M}\alpha_j)^j_k$ and thus

$$\begin{aligned}
&2^{j(s_1 - \frac{\nu}{p_1})} \big\| \langle (w_1)^j_k \cdot (\tilde{M}\alpha)^j_k \rangle_{k\in\mathbf{Z}^n} | \ell_{p_1} \big\| \\
&\quad \le \big\| \widetilde{M}\alpha_j | b^{s_1}_{p_1,q_1}(w_1) \big\| \\
&\quad \le \big\| \widetilde{M} | b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1) \big\| \cdot \big\| \alpha_j | b^{s_0}_{p_0,q_0}(w_0) \big\| \\
&\quad \le C_0 \cdot \big\| \widetilde{M} | b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1) \big\| \cdot \sum_{l=j-1}^{j+1} 2^{l(s_0 - \frac{\nu}{p_0})} \big\| \langle (w_0)^l_m \alpha^l_m \rangle_{m\in\mathbf{Z}^n} | \ell_{p_0} \big\| .
\end{aligned}$$

Applying the ℓ_{r_1} quasinorm in j and using the embedding $\ell_{r_0} \hookrightarrow \ell_{r_1}$ we obtain

$$\big\| \widetilde{M}\alpha | b^{s_1}_{p_1,r_1}(w_1) \big\| \le C_1 \cdot \big\| \widetilde{M} | b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1) \big\| \cdot \big\| \alpha | b^{s_0}_{p_0,r_0}(w_0) \big\| .$$

Therefore

$$\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) \hookrightarrow \mathbf{M}(B^{s_0}_{p_0,r_0}(w_0), B^{s_1}_{p_1,r_1}(w_1)) .$$

Reversing the roles of q and r yields the assertion.

Now let $q_0 > q_1$. Since $\widetilde{M}(w_0, w_1)$ satisfies the hypothesis of Theorem 3.3 and because of the relation $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_0}$ we obtain the equivalence

$$\begin{aligned}
\big\| M | \mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) \big\| &\sim \big\| \widetilde{M}(w_0,w_1) | b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \big\| \\
&\sim \big\| \langle 2^{ls} N_l(\widetilde{M}(w_0,w_1); p_0, p_1) \rangle_{l\in\mathbf{N}} | \ell_r \big\| \\
&\sim \big\| \widetilde{M}(w_0,w_1) | b^{s_0}_{p_0,r_0}, b^{s_1}_{p_1,r_1} \big\| \\
&\sim \big\| M | \mathbf{M}(B^{s_0}_{p_0,r_0}(w_0), B^{s_1}_{p_1,r_1}(w_1)) \big\|
\end{aligned}$$

from Lemma 3.1 and Theorem 3.3 ∎

Another way to prove the assertion of this proposition for the case of $q_0 \le q_1$ is the method of real interpolation.

Proof of Proposition 5.1. For $M \in \mathrm{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1))$ the operator $\widetilde{M}$: $b^{s_0}_{p_0,q_0}(w_0) \to b^{s_1}_{p_1,q_1}(w_1)$ is bounded. Note that the mapping

$$I_\sigma : \mathbb{C}^{\mathbb{N}\times\mathbf{Z}^n} \to \mathbb{C}^{\mathbb{N}\times\mathbf{Z}^n}, \qquad I_\sigma\alpha = \langle 2^{j\sigma}\alpha^j_k\rangle^{j\in\mathbb{N}}_{k\in\mathbf{Z}^n}$$

is an isometric isomorphism

$$I_\sigma : b^{s_0}_{p_0,q_0}(w_0) \to b^{s_0-\sigma}_{p_0,q_0}(w_0), \qquad I_\sigma : b^{s_1}_{p_1,q_1}(w_1) \to b^{s_1-\sigma}_{p_1,q_1}(w_1).$$

We show the boundedness of

$$\widetilde{M}_{s_0} := I_{-s_0} \circ \widetilde{M} \circ I_{s_0} : b^{s_0}_{p_0,q_0}(w_0) \to b^{s_1}_{p_1,q_1}(w_1). \tag{12}$$

To this end we decompose $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where

$$(\alpha_r)^j_k = \mathbf{1}_{J_r}(j)\cdot\alpha^j_k \qquad \text{with } J_r = \{3j + r : j \in \mathbb{N}\}.$$

Observe that $\|\alpha_r \,|\, b^{s_0}_{p_0,q_0}(w_0)\| \le \|\alpha \,|\, b^{s_0}_{p_0,q_0}(w_0)\|$. Since the $\widetilde{M}^{j,l}_{k,m}$ vanish for $|j-l|>1$ we are lead to

$$(\widetilde{M}_{s_0}\alpha_r)^l_m = 2^{-ls_0}\sum_{j=0}^{\infty}\sum_{k\in\mathbf{Z}^n}\widetilde{M}^{j,l}_{k,m}\cdot 2^{js_0}(\alpha_r)^j_k = \sum_{j=l-1}^{l+1}\sum_{k\in\mathbf{Z}^n}2^{(j-l)s_0}\widetilde{M}^{j,l}_{k,m}\cdot(\alpha_r)^j_k.$$

In the sum over j only one term appears due to the definition of α_r and so

$$|(\tilde{M}_{s_0}\alpha_r)^l_m| \le 2^{|s_0|}\cdot\left|\sum_{j=l-1}^{l+1}\sum_{k\in\mathbf{Z}^n}\widetilde{M}^{j,l}_{k,m}\cdot(\alpha_r)^j_k\right| = 2^{|s_0|}\cdot|(\tilde{M}\alpha_r)^l_m|.$$

Hence we obtain

$$\begin{aligned}\|\widetilde{M}_{s_0}\alpha|b^{s_1}_{p_1,q_1}(w_1)\| &\le C_0\cdot\sum_{r=0}^{2}\|\widetilde{M}\alpha_r|b^{s_1}_{p_1,q_1}(w_1)\| \\ &\le C_0\cdot\|\widetilde{M}|b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\|\cdot\sum_{r=0}^{2}\|\alpha_r|b^{s_0}_{p_0,q_0}(w_0)\| \\ &\le 3C_0\cdot\|\widetilde{M}|b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\|\cdot\|\alpha|b^{s_0}_{p_0,q_0}(w_0)\|\end{aligned}$$

which shows the boundedness of $\widetilde{M}_{s_0}$ from (12) with bound

$$\|\widetilde{M}_{s_0}|b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\| \le 3C_0\cdot\|\widetilde{M}|b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\|.$$

Since the I_σ are isometric isomorphisms the mapping

$$\widetilde{M} = I_{s_0}\circ\widetilde{M}_{s_0}\circ I_{-s_0} : b^0_{p_0,q_0}(w_0) \to b^{s_1-s_0}_{p_1,q_1}(w_1)$$

is bounded with bound

$$\|\widetilde{M}|b^0_{p_0,q_0}(w_0), b^{s_1-s_0}_{p_1,q_1}(w_1)\| \le 3C_0\cdot\|\widetilde{M}|b^{s_0}_{p_0,q_0}(w_0), b^{s_1}_{p_1,q_1}(w_1)\|$$

showing the embedding

$$\mathrm{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) \hookrightarrow \mathrm{M}(B^0_{p_0,q_0}(w_0), B^{s_1-s_0}_{p_1,q_1}(w_1)).$$

Since s_0 and s_1 are arbitrary we get the reverse embedding from the same argument and the proof is complete ∎

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