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 Fourier Multipliers
 Part I: Besov Spaces between Weighted Anisotropic Function Spaces Part I: Besov Spaces

P. Dintelmann

Abstract. We determine classes $M(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1))$ of Fourier multipliers between weighted anisotropic Besov spaces $B_{p_0,q_0}^{s_0}(w_0)$ and $B_{p_1,q_1}^{s_1}(w_1)$ where $p_0 \leq 1$ and w_0 , w_1 are weight functions of, polynomial growth. To this end we use a discrete characterization of the function spaces akin to the φ -transform of Frazier and Jawerth which leads to a unified approach to the multiplier problem. In this way widely generalized versions of known results of Bui, Johnson, Peetre and others are obtained from a single theorem.

Keywords: *Fourier multipliers, weighted Besov spaces, anisotropic spaces* AMS subject classification: Primary 42 B 25, 46 E 35, secondary 46 E 39

1. Introduction

The purpose of this and a subsequent paper is to give a detailed study of the class $M(X_{p_0,q_0}^{s_0}(w_0), Y_{p_1,q_1}^{s_1}(w_1))$ of Fourier multipliers between two anisotropic weighted func-

M($X_{p_0,q_0}^{0}(w_0)$, $Y_{p_1,q_1}^{0}(w_1)$) of rourier mutiphers between two anisotropic weighted function spaces $X_{p_0,q_0}^{s_0}(w_0)$ and $Y_{p_1,q_1}^{s_1}(w_1)$ of Besov and Triebel type in the case of $p_0 \leq 1$.

In this P In this Part I we restrict ourselves 'to the case of Besov spaces. So we have to determine the class of tempered distributions *M* generating bounded operators

$$
T_M: B_{p_0,q_0}^{s_0}(w_0) \to B_{p_1,q_1}^{s_1}(w_1), \qquad T_Mf = \mathcal{F}^{-1}[M\mathcal{F}f] \quad (f \in \mathcal{S}).
$$

The general case (Besov and Triebel spaces) will be considered in a following paper . which will also contain some results concerning the case of Besov spaces. The reason for this splitting is twofold. On the one hand weighted Besov spaces have recently attracted much attention (cf., e.g., $[2, 3, 8, 11, 13]$) so that their study has a right in its own. On the other hand the case of Besov spaces is much simpler to deal with from the technical point of view (e.g. we do not need anisotropic maximal functions). Thus we develope the basic ideas of our method in the ease of Besov spaces and refine them later to deal with the general situation extending the results presented in this paper. The current work is selfcontained and has no reference to the forthcoming Part II except for the proof of a certain characterization of Besov spaces.

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To give an impression of the obtained results we formulate a special case of the main theorem of this paper. Let $B_{p_0,q_0}^{s_0}(w_0)$ and $B_{p_1,q_1}^{s_1}(w_1)$ be two isotropic Besov spaces with weight functions $w_j(x)=(1+|x|)^{d_j}$, $0 \leq d_0 \leq d_1$, and $0 < p_j, q_j < \infty$, $s_j \in \mathbb{R}$ $(j = 0, 1)$. If $p_0 \leq \min\{1,p_1\}$, then $M(B_{p_0,q_0}^{s_0}(w_0),\ldots, B_{p_1,q_1}^{s_1}(w_1))$ is two isotromond $B_{p_1,q_1}^{s_1}(w_1)$ be two isotrof $\mathcal{D} = (1+|x|)^{d_j}, 0 \leq d_0 \leq d_1,$ and $0 < p_j, q_j < \infty$
 $M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1)) \implies \mathcal{F}[B_{p_1,r}^{s}(w_1)]$
 $\text{$ m of this paper.

functions $w_j(x)$

min $\{1, p_1\}$, the
 Λ
 $\frac{1}{r} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_{+}$

sotropic spaces and $\begin{array}{l} \text{pression} \ \text{aper.} \ \text{Let} \ w_j(x) = 0, \ \text{then} \ \text{M}(I) \ \text{a} \ \text{a} \ \text{a} \ \text{a} \ \text{a} \ \text{and} \ \text{a} \ \text{c} \ \text{a} \ \text{b} \ \text{b} \ \text{b} \ \text{b} \ \text{c} \ \text{c} \ \text{d} \ \text{c} \ \text{c} \ \text{d} \ \text{c} \ \text{$

$$
\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0),B^{s_1}_{p_1,q_1}(w_1)\big) \leftrightarrow \mathcal{F}[B^{s}_{p_1,r}(w_1)]
$$

where $\frac{1}{r} = \left(\frac{1}{a}\right)^2$ and $s = n(\frac{1}{p_0} - 1) + s_1 - s_0$. A corresponding theorem is proved $\frac{1}{q_1}$ – $\frac{1}{q_2}$
ic sp for anisotropic spaces and will be sharpened in Part II

This extends earlier work of the author [6, 7] and generalizes in particular the following results to weighted spaces and extends them even in the unweighted case:

$$
\frac{1}{r} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+ \text{ and } s = n\left(\frac{1}{p_0} - 1\right) + s_1 - s_0. \text{ A corresponding theorem is a positive space, and will be sharpened in Part II.}
$$
\n
$$
\frac{1}{r} = \left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+ \text{ and } s = n\left(\frac{1}{p_0} - 1\right) + s_1 - s_0. \text{ A corresponding theorem is a positive space, and will be sharpened in Part II.}
$$
\n
$$
\text{This extends earlier work of the author [6, 7] and generalizes in particular, and we can use a particular way.}
$$
\n
$$
\mathbf{M}\left(B_{p,q}^s, B_{p,q}^s\right) \implies \mathcal{F}\left[B_{p,\infty}^{n\left(\frac{1}{p}-1\right)}\right] \quad (0 < p < 1) \qquad \text{(Peetre 1976)}
$$
\n
$$
\mathbf{M}\left(B_{1,q_0}^{s_0}, B_{p,q_1}^{s_1}\right) \implies \mathcal{F}\left[B_{p,\infty}^{s_1 - s_0}\right] \quad (q_0 \leq q_1, 1 \leq p < \infty) \qquad \text{(Johnson 1978)}
$$
\nmethod of proof is based on a discrete characterization of Bessel groups (proof.)

Our method of proof is based on a discrete characterization of Besov spaces (proved in Part II) which will be introduced in the next section. This leads to the study of matrix operators between sequence spaces instead of the original Fourier multipliers between function spaces. These matrix operators are discussed in Sections 3 and 4. The results are applied to the study of Fourier multipliers in the final Section 5.

2. Besov spaces

Let \mathbb{R}^n be the *n*-dimensional Euclidean space and let us start with a real $(n \times n)$ -matrix *P* the eigenvalues of which have positive real parts. We define

$$
\nu=\operatorname{trace} P.
$$

With the group $\langle A_t \rangle_{t>0}$ of the *dilation matrices*

$$
\nu = \text{trace } P
$$

the dilation matrices

$$
A_t = \exp(P \cdot \ln t) \qquad (t > 0)
$$

we associate a *positive A^t -homogeneous distance function g,* i.e. a continuous function $\varrho : \mathbb{R}^n \to \mathbb{R}$ with properties $A_t = \exp(P \cdot \ln t)$ $(t > 0)$
 positive A_t -homogeneous distance function ϱ , i.e. a continually properties
 $\varrho(A_t x) = t \varrho(x)$ $(t > 0)$ and $\varrho(x) > 0$ $(x \neq 0)$.

at any two A_t -homogeneous distance functions are point:

$$
\varrho(A_t x) = t \varrho(x) \quad (t > 0) \quad \text{and} \quad \varrho(x) > 0 \quad (x \neq 0).
$$

It is known that any two A_t -homogeneous distance functions are pointwise equivalent and that there exist constants $C, C', C'' > 0$ and $0 < a \leq b < \infty$ so that the estimates

$$
\varrho(x+y)\leq C\cdot\big(\varrho(x)+\varrho(y)\big)
$$

and

$$
C' \cdot \min\left\{|x|^a, |x|^b\right\} \le \varrho(x) \le C'' \cdot \max\left\{|x|^a, |x|^b\right\}
$$

hold for all $x, y \in \mathbb{R}^n$. For proofs, examples and further details concerning this concept we refer to [5: 14, 221. Stein and Waigner [22) proved that for a given matrix *P* there always exists an A_t -homogeneous distance function $\rho \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$ and in the sequel g denotes always such a fixed *A*t-homogeneous distance function. The adjoint matrices A_t^* of A_t form a group of dilation matrices generated by P^* via $\begin{aligned} \text{[c] [22] proved that} \text{function } &\varrho \in C^\infty \text{ is} \text{geous distance for} \text{is generated by } P \ \text{[c] In t}. \end{aligned}$

$$
A_t^* = \exp(P^* \cdot \ln t) \qquad (t > 0).
$$

In the sequel ϱ^* always denotes a fixed A^* -homogeneous distance function from $C^\infty(\mathbb{R}^n\setminus\mathbb{R}^n)$ **{O})** satisfying the additional condition

$$
\{\xi\in\mathbb{R}^n:\,\varrho^*(\xi)\leq 2\}\subseteq[-\pi,\pi]^n
$$

which will be used in the proof of the discrete characterization of Besov spaces.

The Fourier transform ${\mathcal F}$ and its inverse ${\mathcal F}^{-1}$ are defined by

proof of the discrete characteristic
\n
$$
f \text{ and its inverse } \mathcal{F}^{-1} \text{ are defined}
$$
\n
$$
\mathcal{F}^{\pm 1} f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{\mp ix\xi} dx
$$

on the Schwartz space $S = \mathcal{S}(\mathbb{R}^n)$. Since $\det A_t = t^{\nu}$ we have

$$
\mathcal{F}^{\pm 1}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x)e^{\mp ix\xi} dx
$$

\n2 Schwartz space $S = S(\mathbb{R}^n)$. Since det $A_t = t^{\nu}$ we have
\n
$$
\mathcal{F}^{\pm 1}[f(A_{t^{\bullet}})] = t^{-\nu} \cdot (\mathcal{F}^{\pm 1}f)(A_{1/t^{\bullet}}^*) \quad \text{and} \quad ||f(A_{t^{\bullet}})||_{p} = t^{-\frac{\nu}{p}}||f||_{p}
$$

for all suitable f and $0 < p \leq \infty$ where $\|\cdot\|_p$ denotes the usual L_p quasinorm for measurable functions on \mathbb{R}^n . will be used in the proof of the discrete characterization of Besov

e Fourier transform \mathcal{F} and its inverse \mathcal{F}^{-1} are defined by
 $\mathcal{F}^{\pm 1}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{\mp i x \xi} dx$

Schwartz space S $S(\mathbb{R}^n)$. Since det $A_t = t^{\nu}$ we have
 $\langle (f^{\pm 1}f)(A_{1/t}^*) \rangle$ and $||f(A_{t^*})||_p = t^{-\frac{\nu}{p}}||f||_p$
 $\langle p \rangle \leq \infty$ where $||\cdot||_p$ denotes the usual L_p quas L_p
 W_d of weight functions contains all continuous functio

For $d \geq 0$ the class W_d of weight functions contains all continuous functions w:

$$
0
$$

for a suitable constant $C_w > 0$ and all $x, y \in \mathbb{R}^n$. The typical example of such a weight function is

$$
w(x)=(1+|x|)^d.
$$

The class *W* of *admissible weight functions is* defined by

$$
W=\bigcup_{d\geq 0}W_d.
$$

It has been studied by several authors in connection with Besov and Triebel spaces (cf., e.g., $[8, 11, 13, 18]$. The following estimate is very useful in connection with the matrices *A_t*. Denote by lub₂ the matrix norm associated with $|\cdot|$. From $A_t = \exp(P \cdot \ln t)$ we get by several authors in connection with Bes

The following estimate is very useful in co

the matrix norm associated with $|\cdot|$. From
 $\text{lub}_2 A_{2^{-j}} \leq \exp(\ln 2^{-j} \cdot \text{lub}_2 P) \leq C_0$ $\begin{aligned} \text{occ is very useful in} \ \text{occiated with } |\cdot|. \ \text{C} \cdot (1 + |x|)^d \end{aligned}$ Besov and Triebel spaces (cf.,

n connection with the matrices

From $A_t = \exp(P \cdot \ln t)$ we get
 $(j \in \mathbb{N})$
 $(d \ge 0).$ (1)

$$
\mathrm{lub}_2 A_{2^{-j}} \leq \exp(\ln 2^{-j} \cdot \mathrm{lub}_2 P) \leq C_0 \qquad (j \in \mathbb{N})
$$

which leads to

$$
{}_{2}A_{2^{-j}} \le \exp(\ln 2^{-j} \cdot \text{lub}_{2}P) \le C_{0} \qquad (j \in \mathbb{N})
$$

$$
(1 + |A_{2^{-j}}x|)^{d} \le C \cdot (1 + |x|)^{d} \qquad (d \ge 0).
$$
 (1)

be a fixed bump function with properties supp $\phi \subseteq [0,2]$ and $\phi|_{[0,1]} \equiv 1$. Define

To give the definition of Besov spaces we need a resolution of unity. Let
$$
\phi \in C^{\infty}(\mathbb{R}_{+})
$$

\ni fixed bump function with properties $\text{supp }\phi \subseteq [0,2]$ and $\phi|_{[0,1]} \equiv 1$. Define
\n
$$
\phi_0(\xi) = \phi(\varrho^*(\xi))
$$
\n
$$
\phi_j(\xi) = \phi(2^{-j}\varrho^*(\xi)) - \phi(2^{-j+1}\varrho^*(\xi)) \qquad (j \ge 1)
$$

for $\xi \in \mathbb{R}^n$. These functions are of C^{∞} -type with

the functions are of
$$
C^{\infty}
$$
-type with

\n
$$
\operatorname{supp} \phi_0 \subseteq \left\{ \xi \in \mathbb{R}^n : \varrho^*(\xi) \le 2 \right\}
$$
\n
$$
\operatorname{supp} \phi_j \subseteq \left\{ \xi \in \mathbb{R}^n : 2^{j-1} \le \varrho^*(\xi) \le 2^{j+1} \right\} \quad (j \ge 1).
$$

$$
\operatorname{supp} \phi_0 \subseteq \left\{ \xi \in \mathbb{R}^n : \varrho^*(\xi) \le 2 \right\}
$$
\n
$$
\operatorname{supp} \phi_j \subseteq \left\{ \xi \in \mathbb{R}^n : 2^{j-1} \le \varrho^*(\xi) \le 2^{j+1} \right\} \quad (j \ge 1).
$$
\n\nFurthermore, we have\n
$$
\sum_{r=-1}^1 \phi_{j+r}(\xi) = 1 \quad (\xi \in \operatorname{supp} \phi_j) \quad \text{and} \quad \sum_{j=0}^\infty \phi_j(\xi) = 1 \quad (\xi \in \mathbb{R}^n)
$$
\n
$$
\text{where } \phi_{-1} = 0.
$$
\nFor $0 < p, q < \infty$, $s \in \mathbb{R}$ and $w \in W$ the (anisotropic in homogeneous).

For $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $w \in W$ the (anisotropic inhomogeneous) *Besov space* $B_{p,q}^s(\mathbb{R}^n;P,w)$ (denoted by $B_{p,q}^s(w)$ for short) contains all $f \in \mathcal{S}'$ (the space of tempered distributions) with finite quasinorm

$$
||f|B_{p,q}^s(w)|| = ||\langle 2^{js}||w \cdot \mathcal{F}^{-1}[\phi_j \mathcal{F}f]||_p \rangle_{j \in \mathbb{N}} |\ell_q||.
$$

These are quasi-Banach spaces which are independent of the choice of ϕ and ϱ^* (cf. [19]; the proof is analogous to [24: p. 46]). In the isotropic case, i.e. $P = I$ (the unit matrix) the theory of these spaces is extensively studied in [24, 25] (unweighted case) and (18] (weighted case). Anisotropic spaces were used by Dappa [4], Dappa and Trebels [5], Seeger [19] and Marschall [14].

To give a discrete characterization of the Besov space $B_{p,q}^s(w)$ we use the sequence space $b_{p,q}^s(\mathbb{R}^n;P,w)$ which is denoted by $b_{p,q}^s(w)$ for short $(0 < p,q \leq \infty, s \in \mathbb{R}$ and $w \in W$) containing all complex sequences $\alpha = \langle \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}$ with finite quasinorm

$$
\left\|\alpha|b_{p,q}^s(w)\right\| = \left\|\left\langle 2^{j(s-\frac{\nu}{p})}\right\|\left\langle w_k^j \alpha_k^j\right\rangle_{k\in\mathbb{Z}^n} |\ell_p|\right\}\right\}_{j\in\mathbb{N}} |\ell_q\right\|
$$

where $w_k^j = w(A_{2^{-j}}k)$. Note the analogy in the structure of the two norms of $b_{p,q}^s(w)$ and $B_{p,q}^s(w)$. These sequence spaces are quasi-Banach spaces and we remark that the finite sequences are dense in $b_{p,q}^s(w)$ in the case of $0 < p, q < \infty$. From the embedding $\ell_u \hookrightarrow \ell_v$ $(0 < u \le v \le \infty)$ we obtain the two embeddings *b*. Note the analogy in the structure of equence spaces are quasi-Banach space ense in $b_{p,q}^s(w)$ in the case of $0 < p, q < \leq \infty$) we obtain the two embeddings $b_{p,q_0}^s(w) \hookrightarrow b_{p,q_1}^s(w)$ $(0 < q_0 \leq q_1 \leq 1,$ $\left\{\begin{aligned}\nu_k' \alpha_k' &\rangle_k \in \mathbb{Z}^n |\ell_p||\rangle_{j \in \mathbb{N}}\n\end{aligned}\right\}$ he structure of the
i-Banach spaces anse of $0 < p, q < \infty$.
embeddings
 $(0 < q_0 \leq q_1 \leq \infty)$
 $(s_0 - \frac{\nu}{p_0} = s_1 - \frac{\nu}{p_1})$

ence, in
$$
b_{p,q}^s(w)
$$
 in the case of $0 < p, q < \infty$.

\n $\leq \infty$ we obtain the two embeddings

\n $b_{p,q_0}^s(w) \hookrightarrow b_{p,q_1}^s(w)$ $(0 < q_0 \leq q_1 \leq \infty)$

\n $b_{p_0,q}^s(w) \hookrightarrow b_{p_1,q}^{s_1}(w)$ $(s_0 - \frac{\nu}{p_0} = s_1 - \frac{\nu}{p_1})$

where $0 < p, p_0, p_1, q \le \infty$, $s, s_0, s_1 \in \mathbb{R}$ and $w \in W$, like in the case of Besov spaces.
The unweighted spaces (i.e. $w \equiv 1$) are denoted by $B_{p,q}^s$ and $b_{p,q}^s$ as usual.

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where $0 < p, p_0, p_1, q \le \infty$, $s, s_0, s_1 \in \mathbb{R}$ and $w \in W$, like in the case of Besov spaces.
The unweighted spaces (i.e. $w \equiv 1$) are denoted by $B_{p,q}^s$ and $b_{p,q}^s$ a To establish the connection between these sequence spaces and Besov spaces we furthermore need the two special functions ψ_0 and ψ_1 defined as

Fourier Multipliers between Fu
\n
$$
\leq \infty
$$
, $s, s_0, s_1 \in \mathbb{R}$ and $w \in W$, like in the co
\nplaces (i.e. $w \equiv 1$) are denoted by $B_{p,q}^s$ and
\nconnection between these sequence spaces
\ntwo special functions ψ_0 and ψ_1 defined as
\n
$$
\psi_0(\xi) = \begin{cases}\n\frac{\phi_0(\xi)}{\phi_0(\xi)^2 + \phi_1(\xi)^2} & \text{for } \varrho^*(\xi) \leq 3 \\
0 & \text{otherwise}\n\end{cases}
$$
\n
$$
\begin{cases}\n\frac{\phi_1(A_2^*\xi)}{\phi_1(\xi)^2 + \phi_1(A_2^*\xi)^2 + \phi_1(A_4^*\xi)^2} & \text{for } \frac{1}{3} \leq 1\n\end{cases}
$$

and

Fourier Multipliers between Function Spa
\n
$$
p_0, p_1, q \le \infty, s, s_0, s_1 \in \mathbb{R}
$$
 and $w \in W$, like in the case of Be
\nreighthed spaces (i.e. $w \equiv 1$) are denoted by $B_{p,q}^s$ and $b_{p,q}^s$ as u
\ndisplay the connection between these sequence spaces and Bes
\nneed the two special functions ψ_0 and ψ_1 defined as
\n
$$
\psi_0(\xi) = \begin{cases}\n\frac{\phi_0(\xi)}{\phi_0(\xi)^2 + \phi_1(\xi)^2} & \text{for } \rho^*(\xi) \le 3 \\
0 & \text{otherwise}\n\end{cases}
$$
\n
$$
\psi_1(\xi) = \begin{cases}\n\frac{\phi_1(A_2^*\xi)}{\phi_1(\xi)^2 + \phi_1(A_2^*\xi)^2 + \phi_1(A_4^*\xi)^2} & \text{for } \frac{1}{3} \le \rho^*(\xi) \le 3 \\
0 & \text{otherwise.} \end{cases}
$$
\nons are of C^{∞} -type with compact supports
\n $\subseteq \{\xi \in \mathbb{R}^n : \rho^*(\xi) \le 2\}$ and $\text{supp } \psi_1 \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \le \rho^*(\xi)\}$
\neorem contains the discrete characterization of $B_{p,q}^s(w)$ with

These functions are of C^{∞} -type with compact supports

 $\text{supp}\,\psi_0\subseteq \{\xi\in\mathbb{R}^n:\,\varrho^*(\xi)\leq 2\}\quad\text{and}\quad \text{supp}\,\psi_1\subseteq \{\xi\in\mathbb{R}^n:\,\frac{1}{2}\leq \varrho^*(\xi)\leq 2\}.$ The next theorem contains the discrete characterization of $B_{p,q}^s(w)$ with the help of $b_{p,q}^s(w)$ and is the basis of our work. It will be proved at the end of Part II of this paper. Note that the unweighted case was already proved in [8]. These functions are of C^{∞} -type with compact supports

supp $\psi_0 \subseteq \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2\}$ and supp $\psi_1 \subseteq \{\xi \in \mathbb{R}^n : \frac{1}{2} \leq \varrho^*(\xi) \leq 2\}$.

The next theorem contains the discrete characterization of ctions are of C^{∞} -type with compact support
 $\psi_0 \subseteq \{ \xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2 \}$ and $\sup p \psi_1 \subseteq$

theorem contains the discrete characterizat

d is the basis of our work. It will be proved a

the unweighted case w

Theorem 2.1 (Discrete characterization of Besov spaces). For $f \in S'$ define the *sequence* sef *by*

$$
\mathbf{sef} = \left\langle (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}[\phi_j \mathcal{F}f](A_{2^{-j}}k) \right\rangle_{k \in \mathbf{Z}^n}^{j \in \mathbf{N}}
$$

For finite sequences $\alpha = (\alpha_k^j)_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}$ of complex numbers define the function fu α by

the unweighted case was already proved in [8].
\n**rem 2.1** (Discrete characterization of Besov spaces). For
$$
f \in S'
$$

\n
$$
\mathbf{se}f = \left\langle (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}[\phi_j \mathcal{F}f](A_{2^{-j}}k) \right\rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}
$$
\n
$$
\mathbf{se}gueences \alpha = \langle \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} \text{ of complex numbers define the function function}
$$
\n
$$
\mathbf{fu} \alpha = \sum_{k \in \mathbb{Z}^n} \alpha_k^0 \cdot (\mathcal{F}^{-1}\psi_0)(\cdot - k) + \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \alpha_k^j \cdot (\mathcal{F}^{-1}\psi_1)(A_{2^{j}} \cdot - k).
$$
\n
$$
\langle p, q \langle \infty, s \in \mathbb{R} \text{ and } w \in W. \text{ Then the operators}
$$
\n
$$
\mathbf{se} : B_{2^{j}}^s(w) \to b_{2^{j}}^s(w) \text{ and } \mathbf{fu} : b_{2^{j}}^s(w) \to B_{2^{j}}^s(w)
$$

 $Assume\ 0 < p, q < \infty$, $s \in \mathbb{R}$ and $w \in W$. Then the operators

$$
\mathbf{se}: B^s_{p,q}(w) \to b^s_{p,q}(w) \quad and \quad \text{fu}: b^s_{p,q}(w) \to B^s_{p,q}(w)
$$

are bounded (the unique extension of fu to $b_{p,q}^s(w)$ is denoted by fu, too). Furthermore, ${\bf f u} \circ {\bf s e} = id \text{ on } B^s_{p,q}(w) \text{ and }$ *s*
 *s*_{*sn*,*q*}(*w*) *and* **fu** : b_p^s ,
 *s*_p_,*q*(*w*) *and* **fu** : b_p^s ,
 sef $|b_{p,q}^s(w)| \sim ||f|B_{p,q}^s(w)||$

$$
||\text{sef}|b_{p,q}^s(w)|| \sim ||f|B_{p,q}^s(w)||
$$

holds for all $f \in S'$.

This characterization is akin to the ϕ -transform of Frazier and Jawerth [10] for isotropic unweighted homogeneous spaces. Similar results for the isotropic unweighted inhomogeneous case were proved by Sickel using splines [20] and wavelets [21]. holds for all $f \in S'$.

This characterization is akin to the ϕ -transform

isotropic unweighted homogeneous spaces. Similar r

inhomogeneous case were proved by Sickel using spl

The following important corollary is imm

The following important corollary is immidiate from the above theorem.

Corollary 2.2 (Discrete characterization of linear operators). Assume $0 < p_0, p_1$, $q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$. The equivalence **Corollary 2.2** (Discrete characteriz
 $q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$.
 $||T|B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1)||$

holds for all linear operators $T : S \to S'$.
 $||T||_{q_0} ||T||_{q_0} \leq ||T||_{q_0}$

$$
||T|B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1)|| \sim ||\mathbf{se}T\,\mathbf{fu}|b_{p_0,q_0}^{s_0}(w_0),b_{p_1,q_1}^{s_1}(w_1)||
$$

Here $||A|X, Y||$ denotes the operator quasinorm of the linear operator $A: X \to Y$. To apply this corollary to operators T_M with $T_Mf = \mathcal{F}^{-1}[M\mathcal{F}f]$ we first study the $\begin{aligned} \text{opera} \ \text{e} \ \text{q}_0(w_0) \ \text{the line} \ \text{if} \ \mathcal{F}^{-1} | \end{aligned}$

3. Boundedness of matrix operators

For a matrix $A = \langle A_{k,m}^{j,l} \rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}$ mann

ness of matrix of
 $= \langle A_{k,m}^{j,l} \rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}$ and **perators**

a sequence $\alpha = \langle \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}$ we define the product
 $A_{k,m}^{j,l} \cdot \alpha_k^j$ $(l \in \mathbb{N}, m \in \mathbb{Z}^n)$. *Ao* via

$$
\begin{aligned}\n\text{ans} \quad & \text{of matrix operators} \\
\langle A_{k,m}^{j,l} \rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}} \text{ and a sequence } \alpha = \langle \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} \text{ we} \\
\langle A\alpha \rangle_m^l &= \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} A_{k,m}^{j,l} \cdot \alpha_k^j \qquad (l \in \mathbb{N}, m \in \mathbb{Z}^n).\n\end{aligned}
$$

By the following lemma we can restrict ourselves to the case of matrix operators between unweighted sequence spaces.

Lemma 3.1 (Boundedness of *A* and $A(w_0, w_1)$). Assume $0 < p_0, p_1, q_0, q_1 \le \infty$,
 $1 \in \mathbb{R}$ and $w_0, w_1 \in W$. For a matrix $A = \langle A_k^1, b_k^1, b_k^1 \rangle \in \mathbb{R}$. The $A(w_0, w_1)$ by $s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$. For a matrix $A = (A_{k,m}^{j,l})_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}$ define $A(w_0, w_1)$ by

underness of A and
$$
A(w_0, w_1)
$$
. Assume

\n
$$
W. For a matrix A = \langle A_k^{j,l} \rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}
$$

\n
$$
A(w_0, w_1) = \left\langle (w_1)^l_m A_{k,m}^{j,l} \frac{1}{(w_0)^l_k} \right\rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}
$$

with $(w_0)^l_m = w_0(A_{2^{-l}}m)$ and $(w_1)^j_k = w_1(A_{2^{-j}}k)$ (like in the definition of $b_{p,q}^s$). Then
the relation $||A|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)|| = ||A(w_0, w_1)|b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1}||$ *the relation* $\left\{ \begin{array}{l} \left\langle k,m\in\mathbb{Z}^n\right\rangle \ \left\langle k,m\in\mathbb{Z}^n\right\rangle \ \left\langle k\right\rangle \left\langle k$

$$
||A|b_{p_0,q_0}^{s_0}(w_0),b_{p_1,q_1}^{s_1}(w_1)|| = ||A(w_0,w_1)|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}||
$$

holds for all A.

The next theorem contains a boundedness criterion for matrix operators in the case *of* $q_0 \leq q_1$ *.*

Theorem 3.2 (First boundedness criterion for matrix operators). Assume 0 < $p_0, p_1, q_0, q_1 < \infty$ and $s_0, s_1 \in \mathbb{R}$. For a matrix $A = \langle A_k^j \rangle$ next theorem contains a boundedness contains and solution of q_1 is the *S* of *P C* $q_1 < \infty$ and $s_0, s_1 \in \mathbb{R}$. For a matrix $B(A; s_0, p_0, b_{p_1, q_1}^s) = \left\| \left\langle 2^{j(\frac{s}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \right\| \right\|$ \hat{P}_{max}
 \hat{P}_{max} a matrix $A = \langle A_{k,m}^{j,l} \rangle_{k,m \in \mathbb{Z}^n}^{j,l \in \mathbb{N}}$
 $\sum_{k=2}^{\infty}$
 $\sum_{k=2}^{\infty} || \langle A_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1,q_1}^{s_1}|$ $\begin{array}{l} \text{ators)} \ \text{define} \ \text{all} \end{array}$ $\begin{aligned} \text{ness criterion for matrix opera}\ \text{interion for matrix operators}\ \text{interion for matrix operators}\ \text{matrix}\ A &= \langle A^{j,l}_{k,m}\rangle^{j,l\in\mathbb{N}}_{k,m\in\mathbb{Z}^n}\ \text{define}\ \text{sup}\ \left\|\langle A^{j,l}_{k,m}\rangle^{l\in\mathbb{N}}_{m\in\mathbb{Z}^n}\left|b^{s_1}_{p_1,q_1}\right|\right\}_{j\in\mathbb{N}} \end{aligned}$

$$
B(A; s_0, p_0, b_{p_1, q_1}^{s_1}) = \left\| \left\langle 2^{j(\frac{\nu}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \left\| \left\langle A_{k,m}^{j,l} \right\rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1, q_1}^{s_1}| \right\| \right\rangle_{j \in \mathbb{N}} \right\| \ell_{\infty} \right\|
$$
\n7e always have the estimate

\n
$$
B(A; s_0, p_0, b_{p_1, q_1}^{s_1}) \leq \|A| b_{p_0, q_0}^{s_0}, b_{p_1, q_1}^{s_1} \|.
$$
\ni the case of $\max\{p_0, q_0\} \leq \min\{1, p_1, q_1\}$ the equivalence

\n
$$
\|A| b_{p_0, q_0}^{s_0}, b_{p_1, q_1}^{s_1} \| \sim B(A; s_0, p_0, b_{p_1, q_1}^{s_1})
$$
\nall A.

\n7299 of s_0 , s_0 is proved by the following of s_0 .

Then:

a) We always have the estimate $B(A; s_0, p_0, b_{p_1, q_1}^{s_1})$

the estimate
\n
$$
B(A; s_0, p_0, b_{p_1,q_1}^{s_1}) \leq ||A| b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1} ||.
$$

b) *In the case of* $max\{p_0, q_0\} \leq min\{1, p_1, q_1\}$ *the equivalence b*) *In the case of* $max\{p_0, q_0\} \leq min\{1, p_1, q_1\}$ *the equivalence*

$$
\|A|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}\| \sim B(A;s_0,p_0,b_{p_1,q_1}^{s_1})
$$

$$
\|A|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}\| \sim B(A;s_0,p_0,b_{p_1,q_1}^{s_1})
$$

holds for all A.

The case of $q_0 > q_1$ is covered by the following theorem.

Theorem 3.3 (Second boundedness criterion for matrix operators). Assume 0 < $p_0, p_1 < \infty$, $0 < q_1 < q_0 < \infty$ and $s_0, s_1 \in \mathbb{R}$. Additionally assume that all coefficients **A**_{k,m} (*A*_{k</sup>,m} with $|j - l| > 1$ of the matrix $A = \begin{cases} A_k^{j,l} < m \ A_{k,m}^{j,l} < m \end{cases}$
 *A*_k^{*n*}_{*M*} with $|j - l| > 1$ of the matrix $A = \langle A_k^{j,l}, h_k^{j,l} \rangle_{k,m}^{j,l \in \mathbb{R}}$
 $N_l(A; p_0, p_1) = \sup \frac{\left|\left|\left(\left(A\alpha\right)_{m}^l\right)_{m \in \mathbb$ *vanish. We define N*₁ $\langle q_0 \rangle \langle \infty \rangle$ and $s_0, s_1 \in \mathbb{R}$. *Additionally as*
 N₁ (*A*_{*i*} *A*_{*i*} *A*_{*i*} *A*_{*i*}^{*I*_{*i*}^{*I*}} *A*_{*i*}^{*I*_{*i*}^{*I*} *A*_{*i*}*I*^{*I*} *A*_{*i*}*II*^{*I*} *A*_{*i*}*IIP*_{*I*} *M*_{*I*}*IA}*

$$
N_l(A; p_0, p_1) = \sup \frac{\left\| \langle (A\alpha)^l_m \rangle_{m \in \mathbb{Z}^n} | \ell_{p_1} \right\|}{\left\| \langle \alpha^{l+1}_m \rangle_{m \in \mathbb{Z}^n}^{t=0,\pm 1} | \ell_{p_0} \right\|} \qquad (l \in \mathbb{N})
$$

where the supremum is taken over all sequences α for which the denominator does not *vanish. The numbers* $N_l(A; p_0, p_1)$ *have the following properties:*
 a) *The equivalence*
 $||A|b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1}|| \sim ||\langle 2^{ls} N_l(A; p_0, p_1) \rangle_{l \in \mathbb{N}}|l,$ *ken over al*
A; *p*₀, *p*₁) *h*₀
*p*_{0, *q*0}, *b*^{*s*₁}_{*n*, *q*₁]} $N_l(A; p_0, p_1) = \sup \frac{||\langle \langle \cdot \rangle \rangle}{||\langle \cdot \rangle \rangle}$

where the supremum is taken over all sequentsh. The numbers $N_l(A; p_0, p_1)$ have i

a) The equivalence
 $||A|b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}|| \sim |$

holds for all $A, s = \nu(\frac{1}{p_0} - \frac{1$

a) The equivalence

$$
||A|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}|| \sim ||\langle 2^{ls} N_l(A;p_0,p_1) \rangle_{l \in \mathbb{N}}| \ell_r||
$$

and $\frac{1}{n} = \frac{1}{n} - \frac{1}{n}$ $V_l(A; p_0, p_1)$
 $\frac{1}{r} = \frac{1}{q_1} - \frac{1}{q_0}$

b) The estimate

$$
\begin{aligned}\n\sup_{k \in \mathbb{Z}^n} \sup_{t=0, \pm 1} \| \langle A_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} | \ell_{p_1} \| &\leq N_l(A; p_0, p_1) \\
\text{for all } k \in \mathbb{Z}^n \end{aligned}
$$

holds for all $l \in \mathbb{N}$ *. In particular there exists a constant* $C > 0$ *by part a) so that*

$$
||A|b_{p_{0},q_{0}}^{s_{0}},b_{p_{1},q_{1}}^{s_{1}}|| \sim ||\langle 2^{ls}N_{l}(A;p_{0},p_{1})\rangle_{l\in\mathbb{N}}| \ell_{r}||
$$

all A, $s = \nu(\frac{1}{p_{0}} - \frac{1}{p_{1}}) + s_{1} - s_{0}$ and $\frac{1}{r} = \frac{1}{q_{1}} - \frac{1}{q_{0}}$.
we estimate

$$
\sup_{k\in\mathbb{Z}^{n}} \sup_{t=0,\pm 1} ||\langle A_{k,m}^{l+t,l}\rangle_{m\in\mathbb{Z}^{n}}|\ell_{p_{1}}|| \leq N_{l}(A;p_{0},p_{1})
$$

all $l \in \mathbb{N}$. In particular there exists a constant $C > 0$ by part a) so

$$
||\langle 2^{ls} \sup_{k\in\mathbb{Z}^{n}} \sup_{t=0,\pm 1} ||\langle A_{k,m}^{l+t,l}\rangle_{m\in\mathbb{Z}^{n}}|\ell_{p_{1}}|| \rangle_{l\in\mathbb{N}}| \ell_{r}|| \leq C \cdot ||A|b_{p_{0},q_{0}}^{s_{0}},b_{p_{1},q_{1}}^{s_{1}}||
$$

all A (s and r as in part a)).

holds for all A (s and r as in part a)).

c) In the case of $0 < p_0 \le \min\{1,p_1\}$ the first inequality of part b) can be reversed *and by part a) the equivalence*

$$
\sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \|(A^{+\ldots}_{k,m})_{m \in \mathbb{Z}^n} |\ell_{p_1}\| \leq N_l(A;p_0,p_1)
$$
\n
$$
l \, l \in \mathbb{N}.
$$
\nIn particular there exists a constant $C > 0$ by part a) so
\n
$$
\left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \|(A^{l+t,l}_{k,m})_{m \in \mathbb{Z}^n} |\ell_{p_1}\| \right\rangle_{l \in \mathbb{N}} |\ell_r| \leq C \cdot \|A| b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1}
$$
\n
$$
l \, A \text{ (s and r as in part a)).
$$
\nwhere $0 \leq p_0 \leq \min\{1, p_1\}$ the first inequality of part b) can a) the equivalence

\n
$$
\|A| b^{s_0}_{p_0,q_0}, b^{s_1}_{p_1,q_1} \| \sim \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \|(A^{l+t,l}_{k,m})_{m \in \mathbb{Z}^n} |\ell_{p_1}\| \right\rangle_{l \in \mathbb{N}} |\ell_r| \right\}
$$
\nand

\n
$$
l \, A.
$$

holds for all A.

In the proof of these two theorems we use the special sequences

= { 1 for *i* = land *k = m (jl* E N, *k, in* E V') *(2)* 0 otherwise *I I e b° ii = 2* 30

They satisfy the two relations

$$
\|\varepsilon_k^j|b_{p_0,q_0}^{s_0}\| = 2^{j(s_0 - \frac{\nu}{p_0})} \qquad (j \in \mathbb{N}, \, k \in \mathbb{Z}^n)
$$

and

the two relations
\n
$$
\| \varepsilon_k^j | b_{p_0,q_0}^{s_0} \| = 2^{j(s_0 - \frac{r}{p_0})} \qquad (j \in \mathbb{N}, k \in \mathbb{Z}^n)
$$
\n
$$
(A\varepsilon_k^j)_{m}^l = \sum_{u=0}^{\infty} \sum_{v \in \mathbb{Z}^n} A_{v,m}^{u,l} \cdot (\varepsilon_k^j)_{v}^{u} = A_{k,m}^{j,l} \qquad (l \in \mathbb{N}, m \in \mathbb{Z}^n).
$$

Proof of Theorem 3.2. Step 1. From the above relations we conclude that

$$
B(A; s_0, p_0, b_{p_1,q_1}^{s_1}) = \sup_{j \in \mathbb{N}} \sup_{k \in \mathbb{Z}^n} \frac{\left\| A \varepsilon_k^1 | b_{p_1,q_1}^{s_1} \right\|}{\left\| \varepsilon_k^j | b_{p_0,q_0}^{s_0} \right\|} \leq \left\| A | b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1} \right\|
$$

which proves assertion a) of the theorem.

Step 2. Now we show the converse of this inequality to obtain the equivalences of assertions b) and c) of the theorem. Without loss of generality we may restrict to finite sequences α . Put $r = \min\{1, p_1, q_1\}$. From the embedding $\ell_r \hookrightarrow \ell_1$ we obtain

$$
\left\| A \left(\sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \alpha_k^j \varepsilon_k^j \right) \left| b_{p_1, q_1}^{s_1} \right\| \leq \left\| \left\langle 2^{l(s_1 - \frac{r}{p_1})} \right\| \left\langle \left\| \left\langle \left(\alpha_k^j \cdot A \varepsilon_k^j \right)_{m}^l \right\rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\ell_1| \right\| \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1}| \right\| \right\rangle_{l \in \mathbb{N}} \right\| \leq \left\| \left\langle 2^{l(s_1 - \frac{r}{p_1})} \right\| \left\langle \left\| \left\langle \left(\alpha_k^j \cdot A \varepsilon_k^j \right)_{m}^l \right\rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\ell_r| \right\| \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1}| \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{q_1} \right\| \right\|
$$

Applying the generalized Minkowski inequality twice this can be estimated by

$$
\begin{split}\n\left\| \left\langle 2^{l(s_{1}-\frac{\nu}{p_{1}})} \right\| \left\langle \left\| \left\langle \left(\alpha_{k}^{j} \cdot A \varepsilon_{k}^{j} \right)_{m}^{l} \right\rangle_{k \in \mathbb{Z}^{n}}^{j \in \mathbb{N}} \left| \ell_{r} \right\| \right\rangle_{n \in \mathbb{Z}^{n}} \left| \ell_{p_{1}} \right\| \right\rangle_{l \in \mathbb{N}} \right| \ell_{q_{1}} \\
&= \left\| \left\langle 2^{l(s_{1}-\frac{\nu}{p_{1}})} \right\| \left\langle \left\| \left\langle \left| \left(\alpha_{k}^{j} \cdot A \varepsilon_{k}^{j} \right)_{m}^{l} \right|^{r} \right\rangle_{k \in \mathbb{Z}^{n}}^{j \in \mathbb{N}} \left| \ell_{1} \right\| \right\rangle_{m \in \mathbb{Z}^{n}} \left| \ell_{p_{1}/r} \right\| \right\rangle_{l \in \mathbb{N}} \right| \ell_{q_{1}/r} \right\| \right\}^{\frac{1}{r}} \\
&\leq \left\| \left\langle \left\| \left\langle 2^{l(s_{1}-\frac{\nu}{p_{1}})} \right\| \left\langle \left| \left(\alpha_{k}^{j} \cdot A \varepsilon_{k}^{j} \right)_{m}^{l} \right|^{r} \right\rangle_{m \in \mathbb{Z}^{n}} \left| \ell_{p_{1}/r} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{q_{1}/r} \right\| \right\rangle \right\rangle_{k \in \mathbb{Z}^{n}} \left| \ell_{1} \right\| \right\}^{\frac{1}{r}} \\
&\leq \left\| \left\langle \left| \alpha_{k}^{j} \right| \cdot \left\| A \varepsilon_{k}^{j} \right| b_{p_{1},q_{1}}^{s_{1}} \right\| \right\rangle_{k \in \mathbb{Z}^{n}}^{j \in \mathbb{N}} \left| \ell_{r} \right\| \right\|.\n\end{split}
$$

Now using the definition of $B(A; s_0, p_0, b_{p_1,q_1}^{s_1})$ and the embeddings

$$
b_{p_0,q_0}^{s_0} \hookrightarrow b_{p_0,r}^{s_0} \hookrightarrow b_{r,r}^{\sigma} \qquad (\sigma = s_0 + \nu \left(\frac{1}{r} - \frac{1}{p_0} \right))
$$

 $\epsilon_{\rm{max}}$

we finally get

$$
\begin{aligned}\n&\left\|\left\langle |\alpha_k^j| \cdot \|A \varepsilon_k^j| b_{p_1,q_1}^{s_1} \|\right\rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\ell_r\right\| \\
&\leq \sup_{j \in \mathbb{N}} \sup_{k \in \mathbb{Z}^n} \frac{\left\|A \varepsilon_k^j| b_{p_1,q_1}^{s_1} \right\|}{\left\| \varepsilon_k^j| b_{p_0,q_0}^{s_0} \right\|} \cdot \left\| \left\langle |\alpha_k^j| \cdot \| \varepsilon_k^j| b_{p_0,q_0}^{s_0} \|\right\rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\ell_r\right\| \\
&= B(A; s_0, p_0, b_{p_1,q_1}^{s_1}) \cdot \left\| \alpha | b_{r,r}^{\sigma} \right\| \\
&\leq C \cdot B(A; s_0, p_0, b_{p_1,q_1}^{s_1}) \cdot \left\| \alpha | b_{p_0,q_0}^{s_0} \in \mathbb{N}\n\end{aligned}
$$

which proves the theorem \blacksquare

Proof of Theorem 3.3. We write $N_l = N_l(A; p_0, p_1)$ for short. $\ddot{}$ $\frac{1}{2}$ \ddotsc \sim \sim \sim \sim \sim \sim \sim

Part a) Using the Holder inequality with
$$
1 = \frac{u_1}{r} + \frac{u_1}{q_0}
$$
 we get the estimate $||A\alpha|b_{p_1,q_1}^{s_1}||$

$$
\begin{split}\n&= \left\| \left\langle 2^{l(s_{1}-\frac{\nu}{p_{1}})} \right\| \left\langle (A\alpha)_{m}^{l} \right\rangle_{m\in\mathbb{Z}^{n}} |e_{p_{1}}| \right\rangle_{l\in\mathbb{N}} |e_{q_{1}}| \right\| \\
&\leq \left\| \left\langle 2^{l(s_{1}-\frac{\nu}{p_{1}})} N_{l}(A;p_{0},p_{1}) \right\| \left\langle \alpha_{m}^{l+1} \right\rangle_{m\in\mathbb{Z}^{n}}^{t=0, \pm 1} |e_{p_{0}}| \right\rangle_{l\in\mathbb{N}} |e_{q_{1}}| \right\| \\
&= \left\| \left\langle \left| 2^{l(s_{1}-\frac{\nu}{p_{1}})} N_{l} \right\| \left\langle \alpha_{m}^{l+1} \right\rangle_{m\in\mathbb{Z}^{n}}^{t=0, \pm 1} |e_{p_{0}}| \right\|^{q_{1}} \right\rangle_{l\in\mathbb{N}} |e_{1}| \right\|^{1/q_{1}} \\
&\leq \left\| \left\langle |2^{ls} N_{l}|^{q_{1}} \right\rangle_{l\in\mathbb{N}} |e_{r/q_{1}}| \right\|_{q_{1}} \cdot \left\| \left\langle \left| 2^{l(s_{0}-\frac{\nu}{p_{0}})} \right\| \left\langle \alpha_{m}^{l+1} \right\rangle_{m\in\mathbb{Z}^{n}}^{t=0, \pm 1} |e_{p_{0}}| \right\|^{q_{1}} \right\rangle_{l\in\mathbb{N}} |e_{q_{0}/q_{1}}| \right\|_{q_{1}} \\
&\leq C_{0} \cdot \left\| \left\langle 2^{ls} N_{l} \right\rangle_{l\in\mathbb{N}} |e_{r}| \right\| \cdot \left\| \alpha| b_{p_{0},q_{0}}^{s_{0}} \right\| \n\end{split}
$$

which yields

$$
||A|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}|| \leq C_0 \cdot ||\langle 2^{l_3} N_l \rangle_{l \in \mathbb{N}}| \ell_r||.
$$

To prove the converse inequality we split up the set of l -values in three disjoint parts which will be put together in the end. Therefore define

$$
I_u = \{3j + u : j \in \mathbb{N}\} \qquad (u = 0, 1, 2).
$$

There exists a sequence $a = (a_j)_{j \in \mathbb{N}}$ of complex numbers satisfying the two conditions

$$
\|\langle 1_{I_{\mathbf{u}}}(j)a_{j}\rangle_{j\in\mathbf{N}}|\ell_{q_{0}/q_{1}}\|=1
$$

and

$$
\left\| \langle 2^{ls} N_l \rangle_{l \in I_u} | \ell_r \right\| = \left\| \langle 2^{ls} N_l |^{q_1} \rangle_{l \in I_u} \right| \ell_{(q_0/q_1)'} \left\| \frac{1}{q_1} \right\| = \left(\sum_{l \in I_u} [2^{ls} N_l]^{q_1} \cdot |a_l| \right)^{\frac{1}{q_1}} \tag{3}
$$

where

$$
\frac{1}{(q_0/q_1)'}=1-\frac{q_1}{q_0}=\frac{q_1}{r}.
$$

The second condition states that a is a maximal element for the converse of the Hölder inequality. Since the coefficients $A_{k,m}^{j,l}$ with $|j - l| > 1$ vanish we can find a sequence $\tilde{\alpha}$ satisfying the condition

$$
N_l \cdot \left\| \langle \tilde{\alpha}_m^{l+t} \rangle_{m \in \mathbb{Z}^n}^{t=0,\pm 1} |\ell_{p_0}| \right\| \leq 2 \cdot \left\| \langle (A\tilde{\alpha})_m^l \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |\ell_{p_1}| \right\| \tag{4}
$$

for all $l \in I_u$. This follows from the definition of N_l and the structure of the set I_u . Note that the numbers $l + t$ and $(l + 1) + t$ with $t = 0, \pm 1$ are pairwise different for each two successive values of *l*. Since this inequality stays, true if we dilate the sequence $\tilde{\alpha}$ by a positive constant we can adjust $\tilde{\alpha}$ to satisfy the additional condition

$$
2^{l(s_0 - \frac{\nu}{p_0})} \left\| \langle \tilde{\alpha}_m^{l+t} \rangle_{m \in \mathbb{Z}^n}^{t = 0, \pm 1} |\ell_{p_0}| \right\| = |a_l|^{\frac{1}{q_1}} \tag{5}
$$

for all $l \in I_u$. We show that $\tilde{\alpha}$ is an element of the space $b_{p_0,q_0}^{s_0}$. Formula (5) leads to

$$
\begin{split}\n\|\tilde{\alpha}|b_{p_{0},q_{0}}^{s_{0}}\| &= \left(\sum_{l=0}^{\infty} \left[2^{l(s_{0}-\frac{\nu}{p_{0}})}\left\|\langle\tilde{\alpha}_{m}^{l}\rangle_{m\in\mathbb{Z}^{n}}|\ell_{p_{0}}\right\|\right]^{q_{0}}\right)^{\frac{1}{q_{0}}} \\
&\leq C_{1} \cdot \left(\sum_{l\in I_{u}} \left[2^{l(s_{0}-\frac{\nu}{p_{0}})}\left\|\langle\tilde{\alpha}_{m}^{l+t}\rangle_{m\in\mathbb{Z}^{n}}^{t=0,\pm1}|\ell_{p_{0}}\right\|\right]^{q_{0}}\right)^{\frac{1}{q_{0}}} \\
&= C_{1} \cdot \left(\sum_{l\in I_{u}} |a_{l}|^{\frac{q_{0}}{q_{1}}}\right)^{\frac{1}{q_{0}}} \\
&= C_{1} \cdot \left\|\langle1_{I_{u}}(l)a_{l}\rangle_{l\in\mathbb{N}}|\ell_{q_{0}/q_{1}}\right|^{\frac{1}{q_{1}}} \\
&= C_{1} \cdot \left\|\langle1_{I_{u}}(l)a_{l}\rangle_{l\in\mathbb{N}}|\ell_{q_{0}/q_{1}}\right|^{\frac{1}{q_{1}}} \\
&= C_{1} \cdot \left\|\langle\langle1_{I_{u}}(l)a_{l}\rangle_{l\in\mathbb{N}}|\ell_{q_{0}/q_{1}}\right|^{\frac{1}{q_{1}}} \\
&= C_{1} \cdot \left\|\langle1_{u}(|a_{l}|)^{\frac{1}{q_{1}}}\rangle_{l\in\mathbb{N}}\right\|^{q_{0}}\right\|^{1/2}.\n\end{split}
$$

Now we use $\tilde{\alpha}$ to estimate the operator quasinorm of A from below. To this end we use the estimates from (3) - (5) to obtain

$$
\begin{split} \left\| \langle 2^{ls} N_l \rangle_{l \in I_u} |\ell_r \right\| &= \left(\sum_{l \in I_u} [2^{ls} N_l]^{q_1} \cdot |a_l| \right)^{\frac{1}{q_1}} \\ &= \left\| \left\langle 2^{l(s_1 - \frac{\nu}{p_1})} N_l \cdot \left\| \left\langle \tilde{\alpha}_m^{l+t} \right\rangle_{m \in \mathbb{Z}^n}^{t=0, \pm 1} |\ell_{p_0}| \right\rangle \right\rangle_{l \in I_u} |\ell_{q_1}| \right\| \\ &\leq 2 \cdot \left\| \left\langle 2^{l(s_1 - \frac{\nu}{p_1})} \right\| \left\langle \left(A \tilde{\alpha} \right)_m^l \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1}| \right\rangle_{l \in \mathbb{N}} |\ell_{q_1}| \right\| \\ &= 2 \cdot \left\| A \tilde{\alpha} |b_{p_1, q_1}^{s_1} \right\| \\ &\leq 2C_1 \cdot \left\| A |b_{p_0, q_0}^{s_0}, b_{p_1, q_1}^{s_1} \right\| .\end{split}
$$

Summing up in u yields the desired estimate

$$
\left\| \langle 2^{ls} N_l \rangle_{l \in \mathbb{N}} | \ell_r \right\| \leq \sum_{u=0}^2 \left\| \langle 2^{ls} N_l \rangle_{l \in I_u} | \ell_r \right\| \leq C_2 \cdot \left\| A | b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1} \right\|.
$$

 \hat{z}

Part b) The assertion follows from the simple inequality

$$
\sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A_{k,m}^{l+t,l}\rangle_{m\in\mathbb{Z}^n} | \ell_{p_1} \right\| = \sup_{k\in\mathbb{Z}^n} \sup_{t=0,\pm 1} \frac{\left\| \langle (A\epsilon_{k}^{l+t})_{m}^{l} \rangle_{m\in\mathbb{Z}^n} | \ell_{p_1} \right\|}{\left\| \langle (\epsilon_{k}^{l+t})_{m}^{l+s} \rangle_{m\in\mathbb{Z}^n}^{s=0,\pm 1} | \ell_{p_0} \right\|} \leq N_l
$$

Note that the denominator always equals 1.

Part c) It remains to show the estimate

$$
N_l \leq \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\|.
$$

Put $v = \min\{1, p_1\}$. Applying the same technique as in the proof of Theorem 3.2 we use $(A \varepsilon_k^{l+t})_m^l = A_{k,m}^{l+t,l}$, the embedding $\ell_v \hookrightarrow \ell_1$ and the generalized Minkowski inequality to obtain

$$
\begin{split}\n\left\| \left\langle (A\alpha)^l_m \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\| &= \left\| \left\langle \sum_{t=-1}^1 \sum_{k \in \mathbb{Z}^n} A^{l+t,l}_{k,m} \alpha_k^{l+t} \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\| \\
&\leq \left\| \left\langle \| \left\langle \alpha_k^{l+t} \cdot (A \varepsilon_k^{l+t}) \right\rangle_{m}^{t=0,\pm 1} |\ell_1| \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\| \\
&\leq \left\| \left\langle \| \left\langle \alpha_k^{l+t} \cdot (A \varepsilon_k^{l+t}) \right\rangle_{m}^{l} \right\rangle_{k \in \mathbb{Z}^n}^{-1} |\ell_1| \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\| \\
&\leq \left\| \left\langle \| \alpha_k^{l+t} \cdot (A \varepsilon_k^{l+t}) \right\rangle_{m}^{l} \right\rangle_{k \in \mathbb{Z}^n} |\ell_{p_1} \right\| \right\rangle_{k \in \mathbb{Z}^n}^{-1} |\ell_{v} \right\| \\
&\leq \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \left\langle (A \varepsilon_k^{l+t}) \right\rangle_{m}^{l} \right\rangle_{m \in \mathbb{Z}^n} |\ell_{p_1} \right\| \cdot \left\| \left\langle \alpha_k^{l+t} \right\rangle_{k \in \mathbb{Z}^n}^{\epsilon=0,\pm 1} |\ell_{v} \right\| \n\end{split}
$$

Because of $\ell_{p_0} \hookrightarrow \ell_v$ and $(A \varepsilon_k^{l+t})_m^l = A_{k,m}^{l+t,l}$ we have

$$
\left|\left\langle \left(A\alpha\right)^l_m\right\rangle_{m\in\mathbb{Z}^n}\left|\ell_{p_1}\right|\right|\leq \sup_{k\in\mathbb{Z}^n}\sup_{t=0,\pm 1}\left\|\left\langle A^{l+t,l}_{k,m}\right\rangle_{m\in\mathbb{Z}^n}\left|\ell_{p_1}\right|\right|\cdot\left\|\left\langle \alpha^{l+t}_{k}\right\rangle^{t=0,\pm 1}_{k\in\mathbb{Z}^n}\left|\ell_{p_0}\right|\right|.
$$

Dividing this by $\|\langle \alpha_k^{l+t} \rangle_{k \in \mathbb{Z}^n}^{t=0,\pm 1} |\ell_{p_0}\|$ and taking the supremum in α completes the proof

e4. Matrices associated with Fourier multipliers

For $M \in \mathcal{S}'$ the operator T_M is given by

$$
T_M f = \mathcal{F}^{-1}[M\mathcal{F}f] \qquad (f \in \mathcal{S})
$$

and the class of Fourier multipliers between the two spaces $B_{p_0,q_0}^{s_0}(w_0)$ and $B_{p_1,q_1}^{s_1}(w_1)$ is defined by

$$
\mathbf{M}(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1)) = \left\{ M \in \mathcal{S}' \Big| T_M : B_{p_0,q_0}^{s_0}(w_0) \to B_{p_1,q_1}^{s_1}(w_1) \text{ is bounded} \right\}
$$

equipped with the quasinorm

$$
||M|M(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1))|| = ||T_M|B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1)||.
$$

The matrix operator associated with T_M is

$$
\widetilde{M} = \mathbf{se} \, T_M \, \mathbf{fu}
$$

and a simple calculation shows that its coefficients are

$$
\widetilde{M}_{k,m}^{j,l} := (\text{se } T_M \text{ fu})_{k,m}^{j,l}
$$
\n
$$
= \begin{cases}\n\left(\text{se}(T_M\left[(\mathcal{F}^{-1} \psi_0)(\cdot - k) \right])\right)_m' & \text{for } j = 0 \\
\left(\text{se}(T_M\left[(\mathcal{F}^{-1} \psi_1)(A_{2j}\cdot - k) \right])\right)_m' & \text{for } j \ge 1\n\end{cases}
$$
\n
$$
= \frac{1}{(2\pi)^{\frac{n}{2}}} \begin{cases}\n\mathcal{F}^{-1}\left[\phi_l \psi_0 M \right](A_{2^{-l}}m - k) & \text{for } j = 0 \\
\mathcal{F}^{-1}\left[\phi_l (A_{2j}^*, \cdot) \psi_1 M(A_{2j}^*, \cdot) \right](A_{2j-l}m - k) & \text{for } j \ge 1.\n\end{cases}
$$

We make the important observation that due to the overlapping of the supports of **90** P. Dintelmann
Ve make the important observation that due to the overlapping of $\iota(A_{2j}^*, \cdot)$ and $\psi_{0,1}$ all coefficients $\widetilde{M}_{k,m}^{j,l}$ with $|j - l| > 1$ vanish and thus
 $\widetilde{M}_{k,m}^{j,l} = 0$ (*Ii I*l > 1 vani Figure $M_{k,m}^{i,l}$ with $|j - l| > 1$ vanish and $\widetilde{M}_{k,m}^{i,l}$ with $|j - l| > 1$ vanish and $\widetilde{M}(w_0, w_1)_{k,m}^{j,l} = 0$ ($|j - l| > 1$; $w_0, w_1 \in W$).

$$
M(w_0, w_1)_{k,m}^{j,l} = 0 \qquad (|j-l| > 1; w_0, w_1 \in W).
$$

 ${\rm Recall~that~} \widetilde{M}(w_0,w_1)$ is the corresponding operator for unweighted spaces (see Lemma 3.1).

Combining Corollary 2.2 and Lemma 3.1 yields the relation

$$
\psi_{0,1}
$$
 all coefficients $\widetilde{M}_{k,m}^{j,l}$ with $|j-l| > 1$ vanish and thus
\n
$$
\widetilde{M}(w_0, w_1)_{k,m}^{j,l} = 0 \qquad (|j-l| > 1; w_0, w_1 \in W).
$$
\n
$$
\widetilde{M}(w_0, w_1)
$$
 is the corresponding operator for unweighted spaces
\n
$$
\text{ug Corollary 2.2 and Lemma 3.1 yields the relation}
$$
\n
$$
||M|M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))|| \sim ||\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)||
$$
\n
$$
= ||\widetilde{M}(w_0, w_1)|b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1}||
$$
\n
$$
\text{ut quasinorm on the right side is equivalent to } B(\widetilde{M}(w_0, w_1);
$$

where the last quasinorm on the right side is equivalent to $B(\widetilde{M}(w_0,w_1);s_0,p_0,b_{p_1,q_1}^{s_1})$ under certain restrictions on the parameters p, *q* and *s* by the results of the previous section. So we are interested in a characterization of $B(\tilde{M}(w_0,w_1);s_0,p_0,b_{p_1,q_1}^{s_1})$ in terms of *M.* This is done with the help of the following theorem. *HM* $|M(D_{p_0,q_0}^{*}(w_0), B_{p_1,q_1}^{*})|$

where the last quasinorm on the right

under certain restrictions on the para

section. So we are interested in a cl

terms of *M*. This is done with the hel
 Theorem 4.1 (Charact

Theorem 4.1 (Characterization of the matrices \widetilde{M}). Assume $0 < p < \infty$, $0 < q \le$ ∞ *and* $s, \sigma \in \mathbb{R}$.

Theorem 4.1 (Characterization of the matrices \widetilde{M}). Assume $0 < p < \infty$, $0 < q \le$
 and $s, \sigma \in \mathbb{R}$.
 a) If $w_1 \in W_d$ ($d \ge 0$) and $w_0 \in W$ satisfy the condition $\left\| \frac{(1+|\bullet|)^d}{w_0} \right\|_{\infty} < \infty$, then equival the $equivalence$

For example,
$$
M
$$
 is a non-
to M is a non-
to M is a non-
to S , $\sigma \in \mathbb{R}$.

\nIf $w_1 \in W_d$ ($d \geq 0$) and $w_0 \in W$ satisfy the condition $\left\| \frac{(1+|\bullet|)^d}{w_0} \right\|_{\infty} < \infty$, w_0 and $w_0 \in W$ satisfy the condition $\left\| \frac{(1+|\bullet|)^d}{w_0} \right\|_{\infty} < \infty$, w_0 and $w_0 \in W$ with $\left\| \frac{1}{M_{k,m}^{j,l}} \right\| \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \right\| \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \right\| \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \right\| \left\| \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \right\| \left\| \left\| \left\| \frac{1}{M_{k,m}^{j,l}} \right\|_{\infty} \right\| \left\| \left\| \left\| \frac{1}{M_{k,m}$

holds for all $M \in S'$.

b) *If* $w_0 \in W$ with $\left\| \frac{1}{w_0} \right\|_{\infty} = \infty$, then

$$
\in \mathcal{S}'.
$$
\n
$$
W \text{ with } ||\frac{1}{w_0}||_{\infty} = \infty, \text{ then}
$$
\n
$$
\left| \left\langle 2^{j\sigma} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \left\| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \right\| \right\rangle_{j \in \mathbb{N}} \left| \ell_q \right| \right| < \infty
$$

implies $M \equiv 0$.

The following two corollaries specialize this theorem which will be proved at the end of this section to the conditions appearing in the boundedness results for matrix operators (Theorems 3.2 and 3.3). *If* $\left\{ \left\langle 2^{j\sigma} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \left\| \left\langle \widetilde{M}_{k,m}^{j,l} \right\rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \right\} \right\}_{j \in \mathbb{N}} \left| \ell_q \right| \right\} < \infty$
 Ites $M \equiv 0$.

The following two corollaries specialize this theorem wh

Corollary 4.2 (First characterization of $\widetilde{M}(w_0, w_1)$). Assume $0 < p_0, p_1 < \infty$, $0 < q_1 \leq \infty$ and $s_0, s_1 \in \mathbb{R}$.

equivalence $B(\widetilde{M}(w_0, w_1); s_0, p_0, b_{p_1, q_1}^{s_1}) \sim$

$$
B\big(\widetilde{M}(w_0,w_1);s_0,p_0,b_{p_1,q_1}^{s_1}\big) \sim \|M|\mathcal{F}\big[B^\sigma_{p_1,\infty}(w_1)\big]\|
$$

with $\sigma = \nu(\frac{1}{p_0} - 1) + s_1 - s_0$ holds for all $M \in S'$. b) If $w_0 \in W$ with $\left\| \frac{1}{w_0} \right\|_{\infty} = \infty$, then $B(\widetilde{M}(w_0, 1); s_0, p_0, b_{p_1, q_1}^{s_1}) < \infty$

implies $M \equiv 0$.

Proof. We only prove part a) because the second assertion follows in exactly the same manner.

Since the coefficients $\widetilde{M}(w_0, w_1)_{k,m}^{j,l}$ with $|j - l| > 1$ vanish only three terms in l occur for a fixed j. Thus we can change the parameter q_1 to p_1 and by the definition of $M(w_0, w_1)$ we are lead to the equivalence

$$
\begin{aligned} \left\| \langle \widetilde{M}(w_0, w_1)_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1,q_1}^{s_1}| \right\| &\sim \left\| \langle \widetilde{M}(w_0, w_1)_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1,p_1}^{s_1}| \right\| \\ &= \frac{1}{(w_0)_k^l} \left\| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1,p_1}^{s_1}(w_1) \right\| . \end{aligned}
$$

An application of Theorem 4.1 yields the relation

$$
B(\widetilde{M}(w_0, w_1); s_0, p_0, b_{p_1, q_1}^{s_1})
$$
\n
$$
= \left\| \left\langle 2^{j(\frac{r}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \| \langle \widetilde{M}(w_0, w_1)_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1, q_1}^{s_1}| \right\rangle_{j \in \mathbb{N}} \right\| \left\langle \infty \right\|
$$
\n
$$
\sim \left\| \left\langle 2^{j(\frac{r}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_{k}^{j}} \| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p_1, p_1}^{s_1}(w_1) \| \right\rangle_{j \in \mathbb{N}} \right\| \left\langle \infty \right\|
$$
\n
$$
\sim \left\| M \right| \mathcal{F} \left[B_{p_1, \infty}^{\sigma}(w_1) \right] \right\|
$$

which proves the corollary \blacksquare

í,

Corollary 4.3 (Second characterization of $\widetilde{M}(w_0, w_1)$). Assume $0 < p < \infty$, $0 < r \leq \infty$ and $s \in \mathbb{R}$. If $w_1 \in W_d$ $(d \geq 0)$ and $w_0 \in W$ satisfy the condition $\left\| \frac{(1+|e|)^d}{w_0} \right\|_{\infty} < \infty$, then the equivalence

$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \left\langle \widetilde{M}(w_0, w_1)_{k,m}^{l+t,l} \right\rangle_{m \in \mathbb{Z}^n} |\ell_p| \right\rangle_{l \in \mathbb{N}} \right| \ell_r \right\| \sim \left\| M |\mathcal{F}[B_{p,r}^{\sigma}(w_1)] \right\|
$$

with $\sigma = \nu(\frac{1}{p} - 1) + s$ holds for all $M \in S'$.

Proof. We write $A_{k,m}^{j,l}$ for $\widetilde{M}(w_0, w_1)_{k,m}^{j,l}$. Note that the sequence space $b_{p,p}^{\nu/p}$ is identical with the ℓ_p -space on $\mathbb{N} \times \mathbb{Z}^n$. We have the estimate

$$
2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} |\ell_p| \right\| \leq 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A_{k,m}^{l+t,j} \rangle_{m \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\delta_{p,p}^{\nu/p} \right\|
$$

$$
\leq 2^{ls} \sum_{t=-1}^l \sup_{k \in \mathbb{Z}^n} \left\| \langle A_{k,m}^{l+t,j} \rangle_{m \in \mathbb{Z}^n}^{j \in \mathbb{N}} |\delta_{p,p}^{\nu/p} \right\|.
$$

Now applying the **£r** quasinorm in *1* yields

P. Dintelman
\napplying the
$$
\ell_r
$$
 quasinorm in *l* yields
\n
$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A^{l+t,l}_{k,m} \rangle_{m \in \mathbb{Z}^n} | \ell_p \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_r \right\| \right\|
$$
\n
$$
\leq C_0 \cdot \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \left\| \langle A^{l,j}_{k,m} \rangle_{m \in \mathbb{Z}^n} | b^{l/p}_{p,p} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_r \right\|.
$$
\n
$$
\text{same way we show the reverse inequality starting with}
$$
\n
$$
2^{ls} \sup_{k \in \mathbb{Z}^n} \left\| \langle A^{l,j}_{k,m} \rangle_{m \in \mathbb{Z}^n} | b^{l/p}_{p,p} \right\| \leq C_1 \cdot 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{u=0,\pm 1} \left\| \langle A^{l,l+u}_{k,m} \rangle_{m \in \mathbb{Z}^n} | \ell_p \right\|
$$
\n
$$
\leq C_1 \cdot \sum_{k=1}^1 2^{ls} \sup_{k \in \mathbb{Z}^n} \left\| \langle A^{l,l+u}_{k,m} \rangle_{m \in \mathbb{Z}^n} | \ell_p \right\|
$$

In the same way we show the reverse inequality starting with

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\nNow applying the
$$
\ell_r
$$
 quasinorm in l yields
\n
$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t = 0, \pm 1} \left\| (A_{k,m}^{l+t,l})_{m \in \mathbb{Z}^n} |\ell_r \right\rangle \right\|_{\ell \in \mathbb{N}} \right\|_{\ell \in \mathbb{N}} \left\| \ell_r \right\|_{\ell \in \mathbb{N}} \right\|_{\ell \in \mathbb{N}} \left\| \ell_r \right\|_{\ell \in \mathbb{N}} \left\| \ell_r \right\|_{\ell \in \mathbb{N}} \right\|_{\ell \in \mathbb{N}} \left\| \ell_r \right\|_{\ell \in \mathbb{N}} \left\| (A_{k,m}^{l,l})_{m \in \mathbb{Z}^n} |\ell_r^{l,l} \right\|_{\ell \in \mathbb{Z}^n} \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l+1})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l+1})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l+1})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{N}} \left\| (A_{k,m}^{l,l+1})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}^n} \left\| (A_{k,m}^{l,l+1})_{m \in \mathbb{Z}^n} |\ell_r \right\|_{\ell \in \mathbb{Z}
$$

This leads to

$$
\leq C_{1} \cdot \sum_{u=-1}^{n} 2^{ls} \sup_{k \in \mathbb{Z}^{n}} \left\| \langle A_{k,m}^{l, l+u} \rangle_{m \in \mathbb{Z}^{n}} | \ell_{p} \right\|
$$
\n
$$
\leq C_{1} \cdot \sum_{u=-1}^{1} 2^{ls} \sup_{t=0, \pm 1} \sup_{k \in \mathbb{Z}^{n}} \left\| \langle A_{k,m}^{l+1, l+u} \rangle_{m \in \mathbb{Z}^{n}} | \ell_{p} \right\|.
$$
\n
$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^{n}} \left\| \langle A_{k,m}^{l,j} \rangle_{m \in \mathbb{Z}^{n}} | b_{p,p}^{\nu/p} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{r} \right\| \right\|
$$
\n
$$
\leq C_{2} \cdot \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^{n}} \sup_{t=0, \pm 1} \left\| \langle A_{k,m}^{l+1, l} \rangle_{m \in \mathbb{Z}^{n}} | \ell_{p} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{r} \right\| \right|.
$$
\non of formulae (6) and (7) finally results in

\n
$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^{n}} \sup_{t=0, \pm 1} \left\| \langle \widetilde{M}(w_{0}, w_{1})_{k,m}^{l+1, l} \rangle_{m \in \mathbb{Z}^{n}} | \ell_{p} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{r} \right\| \right\|
$$
\n
$$
\sim \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^{n}} \left\| \langle \widetilde{M}(w_{0}, w_{1})_{k,m}^{j, l} \rangle_{m \in \mathbb{Z}^{n}} | \ell_{p} \right\| \right\rangle_{l \in \mathbb{N}} \left| \ell_{r} \right\| \right\|
$$

A combination of formulae (6) and (7) finally results in

$$
\left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \left\| \langle A_{k,m}^{l,j} \rangle_{m \in \mathbb{Z}^n}^{j \in \mathbb{N}} \left\| b_{p,p}^{l/p} \right\| \right\rangle_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle A_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} \left\| e_p \right\| \right\rangle_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \sup_{t=0,\pm 1} \left\| \langle \widetilde{M}(w_0, w_1)_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} \left\| e_p \right\| \right\rangle_{l \in \mathbb{N}} \right| \left\langle e_r \right\| \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \left\| \langle \widetilde{M}(w_0, w_1)_{k,m}^{l+t,l} \rangle_{m \in \mathbb{Z}^n} \left\| e_p \right\| \right\rangle_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \left\| e_r \right\|_{l \in \mathbb{Z}^n} \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_l^l} \left\| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n} \left\| b_{p,p}^{\nu/p} (w_1) \right\| \right\rangle_{j \in \mathbb{N}} \right| e_r \right\|_{l \in \mathbb{N}} \right\|_{l \in \mathbb{N}} \left\| \left\langle 2^{ls} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_l^l} \left\| \langle \widetilde{
$$

and the assertion follows from Theorem **4.11**

To prove Theorem 4.1 we need the following lemma.

Lemma 4.4 (simple Fourier multipliers for L_p). Asume $0 < p < \infty$ and $w \in W_d$ $(d \ge 0)$. Then there exists a constant $C > 0$ such that heorem 4.1 we
4 (simple Fourther exists)
then there exists
 $\mathcal{F}^{-1}[fg]$, \leq

$$
\left\|w \cdot \mathcal{F}^{-1}[fg]\right\|_p \leq C \cdot \left\|(1+|\bullet|)^d \cdot \mathcal{F}^{-1}[f(A_{2^j}^*,\bullet)]\right\|_{\vec{p}} \cdot \left\|w \cdot \mathcal{F}^{-1}g\right\|_p
$$

with $\tilde{p} = \min\{1, p\}$ *holds for all* $j \in \mathbb{N}$ *and all* $f \in S$ *and* $q \in S'$ *satisfying the condition* $\text{supp } f, \text{supp } g \subseteq \{ \xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2^j \}.$

Proof. Since the supports of $f(A_2^*, \cdot)$ and $g(A_2^*, \cdot)$ are both contained in the com-
 $\text{Set } \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 1\}$ and the relation $w(x) \leq C_0 \cdot w(y)(1 + |x - y|)^d$ holds
 $\text{e exists a constant } C_1 > 0$ by [18: Theorem 1.7.2] for **Proof.** Since the supports of $f(A_2, \cdot)$ and $g(A_2, \cdot)$ are both contained in the compact set $\{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 1\}$ and the relation $w(x) \leq C_0 \cdot w(y)(1 + |x - y|)^d$ holds there exists a constant $C_1 > 0$ by [18: Theo there exists a constant $C_1 > 0$ by [18: Theorem 1.7.2] for which

$$
\|w \cdot \mathcal{F}^{-1}[fg]\|_p = 2^{j\nu(1-\frac{1}{p})} \|w(A_{2^{-j}} \cdot) \cdot \mathcal{F}^{-1}[f(A_{2^j}^* \cdot)g(A_{2^j}^* \cdot)]\|_p
$$

\n
$$
\leq C_0 C_1 \cdot 2^{j\nu(1-\frac{1}{p})} \|(1+|A_{2^{-j}} \cdot) \cdot \mathcal{F}^{-1}[f(A_{2^j}^* \cdot)]\|_p
$$

\n
$$
\times \|w(A_{2^{-j}} \cdot) \cdot \mathcal{F}^{-1}[g(A_{2^j}^* \cdot)]\|_p.
$$

\nate the $L_{\bar{p}}$ quasinorm we apply formula (1) which in connection with
\nds to the desired estimate
\n
$$
\|w \cdot \mathcal{F}^{-1}[fg]\|_p \leq C_0 C_1 C_2 \cdot \|(1+|\cdot|)^d \cdot \mathcal{F}^{-1}[f(A_{2^j}^* \cdot)]\|_{\bar{p}} \cdot \|w \cdot \mathcal{F}^{-1}g\|_p
$$

\nassertion is proved \blacksquare

To estimate the $L_{\tilde{p}}$ quasinorm we apply formula (1) which in connection with a substitution leads to the desired estimate

$$
\|w \cdot \mathcal{F}^{-1}[fg]\|_p \leq C_0 C_1 C_2 \cdot \left\|(1+|\bullet|)^d \cdot \mathcal{F}^{-1}[f(A_{2^j}^{\bullet} \bullet)]\right\|_{\tilde{p}} \cdot \left\|w \cdot \mathcal{F}^{-1}g\right\|_p
$$

and the assertion is proved \blacksquare

Proof of Theorem 4.1. The proof is divided into the following four steps:

- 1. First we derive an equivalence for
- 2. Using this equivalence and Lemma 4.4 we estimate the discrete quasinorm in the **Proof of Theorem 4.1.**
First we derive an equivalence and
Using this equivalence and
theorem by $||M|\mathcal{F}[B^{s+\sigma-}_{p,q}]$
We prove the estimate from
- 3. We prove the estimate from below using a similar technique as in the previous step. This completes the proof of part a).
- 4. Part b) is derived from the equivalence in the first step.

$$
||w \cdot \mathcal{F} [Jg]||_p \leq C_0C_1C_2 \cdot ||(1+|\bullet|) \cdot \mathcal{F} [J(A_{2j} \bullet)]||_p \cdot ||w \cdot \mathcal{F} g||_p
$$

and the assertion is proved **Example**
Proof of Theorem 4.1. The proof is divided into the following four steps:
First we derive an equivalence for $||(\widetilde{M}_{k,m}^{j,l})_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}}|b_{p,p}^s(w_1)||$.
Using this equivalence and Lemma 4.4 we estimate the discrete quasinorm in the theorem by $||M|\mathcal{F}[B_{p,q}^{s+\sigma-\nu}(w_1)]||$ from above.
We prove the estimate from below using a similar technique as in the previous ste
This completes the proof of part a).
Part b) is derived from the equivalence in the first step.
Step 1. Recall that the coefficients of \widetilde{M} are given by

$$
\widetilde{M}_{k,m}^{j,l} = \begin{cases} \left(se(T_M[(\mathcal{F}^{-1}\psi_0)(\bullet - k)]) \right)_m^l & \text{for } j = 0 \\ \left(se(T_M[(\mathcal{F}^{-1}\psi_1)(A_{2j}\bullet - k)] \right) \right)_m^l & \text{for } j \geq 1. \end{cases} \quad (j, l \in \mathbb{N}; k, m \in \mathbb{Z}^n).
$$

see $||se||b_{p,p}^s(w_1)||$ is an equivalent quasinorm on $B_{p,p}^s(w_1)$ we have

$$
||(\widetilde{M}_{k,m}^{j,l})_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}}|b_{p,p}^s(w_1)|| \sim \begin{cases} ||T_M[(\mathcal{F}^{-1}\psi_0)(\bullet - k)]|B_{p,p}^s(w_1)|| & \text{for } j = 0 \\ ||T_M[(\mathcal{F}^{-1}\psi_1)(A_{2j}\bullet - k)]|B_{p,p}^s(w_1)|| & \text{for } j \geq 1. \end{cases}
$$

dead with the terms on the right-hand side we use the identity

Since $\left\| \mathbf{se} \bullet | b_{\bm{p},\bm{p}}^{\bm{s}}(w_1) \right\|$ is an equivalent quasinorm on $B_{\bm{p},\bm{p}}^{\bm{s}}(w_1)$ we have

$$
\begin{aligned}\n\text{where } \left\| \mathbf{S} \mathbf{e} \cdot | b_{p,p}^s(w_1) \right\| \text{ is an equivalent quasinorm on } B_{p,p}^s(w_1) \text{ we have} \\
\left\| \left(\widetilde{M}_{k,m}^{j,l} \right)_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} | b_{p,p}^s(w_1) \right\| &\sim \begin{cases} \left\| T_M \left[\left(\mathcal{F}^{-1} \psi_0 \right) \left(\bullet - k \right) \right] | B_{p,p}^s(w_1) \right\| & \text{for } j = 0 \\
\left\| T_M \left[\left(\mathcal{F}^{-1} \psi_1 \right) \left(A_{2j} \bullet - k \right) \right] | B_{p,p}^s(w_1) \right\| & \text{for } j \ge 1.\n\end{cases}\n\end{aligned}
$$

To deal with the terms on the right-hand side we use the identity

$$
\left\{\|T_M\|(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\|B_{p,p}^s(\psi_1)\| \text{ for } j \ge 1.\right\}
$$

deal with the terms on the right-hand side we use the identity

$$
\mathcal{F}^{-1}\Big[\phi_u \mathcal{F}\Big[T_M\big[(\mathcal{F}^{-1}\psi_1)(A_{2^j}\bullet - k)\big]\Big]\Big] = \mathcal{F}^{-1}\big[\phi_u \cdot 2^{-j\nu}M\psi_1(A_{2^{-j}}^*)\big](\bullet - A_{2^{-j}}k)
$$

which holds for $j \ge 1$ and all $u \in \mathbb{N}$. Due to the location of the supports of ψ_1 and ϕ_u only the terms with $u = j + r$ ($r = 0, \pm 1$) are of interest. This leads to

$$
\left\{ \left\|T_M\left[\left(\mathcal{F}^{-1}\psi_1\right)(A_{2^j}\bullet - k)\right]\right|B_{p,p}^s(w_1)\right\| \text{ for } j \geq 0 \right\}
$$

with the terms on the right-hand side we use the identity

$$
\phi_u \mathcal{F}\left[T_M\left[\left(\mathcal{F}^{-1}\psi_1\right)(A_{2^j}\bullet - k)\right]\right] = \mathcal{F}^{-1}\left[\phi_u \cdot 2^{-j\nu}M\psi_1(A_{2^{-j}}^*)\right] \left(\bullet - A_2\right)
$$

ds for $j \geq 1$ and all $u \in \mathbb{N}$. Due to the location of the supports of ψ_1
terms with $u = j + r$ $(r = 0, \pm 1)$ are of interest. This leads to

$$
\left\|T_M\left[\left(\mathcal{F}^{-1}\psi_1\right)(A_{2^j}\bullet - k)\right]\right|B_{p,p}^s(w_1)\right\|
$$

$$
= \left\|\left\langle 2^{us}\right\|w_1(\bullet + A_{2^{-j}}k) \cdot \mathcal{F}^{-1}\left[\phi_u 2^{-j\nu}M\psi_1(A_{2^{-j}}^*)\right]\right\|_p\right\rangle_{u \in \mathbb{N}}\left\|\ell_p\right\|
$$

$$
\sim \sup_{r=0,\pm 1} 2^{j(s-\nu)}\left\|w_1(\bullet + A_{2^{-j}}k) \cdot \mathcal{F}^{-1}\left[\phi_{j+r}M\psi_1(A_{2^{-j}}^*)\right]\right\|_p
$$

and a similar equivalence holds in the case of $j = 0$. Thus we arrive at

1. a similar equivalence holds in the case of
$$
j = 0
$$
. Thus we arrive at $\left\| \langle \tilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \right\|$ \n $\sim 2^{j(s-\nu)} \sup_{r=0,\pm 1} \left\{ \left\| w_1(\cdot + k) \cdot \mathcal{F}^{-1} [\psi_0 \phi_r M] \right\|_p \right\}$ for $j = 0$ (8)\n $\sim 2^{j(s-\nu)} \sup_{r=0,\pm 1} \left\{ \left\| w_1(\cdot + A_{2^{-j}} k) \cdot \mathcal{F}^{-1} [\psi_1 (A_{2^{-j}}^* \cdot) \phi_{j+r} M] \right\|_p \right\}$ for $j \geq 1$.
\nStep 2. To prove the estimate from above we use the inequality\n
$$
w_1(x + A_{2^{-j}} k) \leq C_0 \cdot w_1(x)(1 + |A_{2^{-j}} k|)^d
$$
\n $\text{ch in combination with (8) yields}$ \n
$$
\sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \leq C_1 \cdot 2^{j(s-\nu)} \sup_{k \in \mathbb{Z}^n} \frac{(1 + |A_{2^{-j}} k|)^d}{w_0(A_{2^{-j}} k)} \cdot I_j
$$
\n $\leq C_2 \cdot 2^{j(s-\nu)} I_j$ (9)

Step 2. To prove the estimate from above we use the inequality

$$
w_1(x+A_{2-i}k) \leq C_0 \cdot w_1(x)(1+|A_{2-i}k|)^d
$$

which in combination with (8) yields

Step 2. To prove the estimate from above we use the inequality
\n
$$
w_1(x + A_{2^{-j}}k) \leq C_0 \cdot w_1(x)(1 + |A_{2^{-j}}k|)^d
$$
\nwhich in combination with (8) yields
\n
$$
\sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|(\widetilde{M}_{k,m}^{j,l})_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \leq C_1 \cdot 2^{j(s-\nu)} \sup_{k \in \mathbb{Z}^n} \frac{(1 + |A_{2^{-j}}k|)^d}{w_0(A_{2^{-j}}k)} \cdot I_j
$$
\nwhere
\n
$$
I_j = \sup_{r=0,\pm 1} \left\{ \left\| w_1 \cdot \mathcal{F}^{-1} [\psi_0 \phi_r M] \right\|_p \quad \text{for } j = 0
$$
\nbecause the fraction on the right is bounded uniformly in j and k . Now we apply Lemma

where

$$
I_j = \sup_{r=0,\pm 1} \left\{ \begin{aligned} \left\| w_1 \cdot \mathcal{F}^{-1} \left[\psi_0 \phi_r M \right] \right\|_p & \text{for } j = 0 \\ \left\| w_1 \cdot \mathcal{F}^{-1} \left[\psi_1 (A_{2^{-j}}^*) \phi_{j+r} M \right] \right\|_p & \text{for } j \ge 1 \end{aligned} \right.
$$

because the fraction on the right is bound.

4.4 to estimate the *I_j* and we obtain
 $||w_1 \cdot \mathcal{F}^{-1}[\psi_1(A_{2-i}^*, \cdot) \phi_{j+r} M]||_p$

$$
I_j = \sup_{r=0,\pm 1} \left\{ \begin{aligned} &\| \cdot r^{j+1} \cdot \left[\left(\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \right) \right] \Big|_p \quad \text{for } j \ge 1 \\ &\| w_1 \cdot \mathcal{F}^{-1} \left[\psi_1(A_{2^{-j}}^*) \phi_{j+r} M \right] \Big|_p \quad \text{for } j \ge 1 \end{aligned} \right.
$$
\nThe fraction on the right is bounded uniformly in j and k . Now we apply

\nmake the I_j and we obtain

\n
$$
\| w_1 \cdot \mathcal{F}^{-1} \left[\psi_1(A_{2^{-j}}^*) \phi_{j+r} M \right] \Big|_p
$$
\n
$$
\leq C_3 \cdot \left\| (1+|\cdot|)^d \cdot \mathcal{F}^{-1} \left[\psi_1(A_4^*) \right] \right\|_{\tilde{p}} \cdot \left\| w_1 \cdot \mathcal{F}^{-1} \left[\phi_{j+r} M \right] \right\|_p
$$
\nsee of $j \geq 1$ and $r = 0, \pm 1$. Similar estimates can be proved in the i .

in the case of $j \ge 1$ and $r = 0, \pm 1$. Similar estimates can be proved in the remaining cases. Since $\psi_0, \psi_1 \in \mathcal{S}$ the $L_{\bar{p}}$ -terms are bounded and we get

1 and
$$
r = 0, \pm 1
$$
. Similar estimates can be proved
\n $\in S$ the $L_{\bar{p}}$ -terms are bounded and we get
\n $I_j \leq C_4 \cdot \sup_{r=0,\pm 1} ||w_1 \cdot \mathcal{F}^{-1}[\phi_{j+r}M]||_p$ $(j \in \mathbb{N}).$

From this we obtain by (9) the inequality

$$
\leq C_3 \cdot \left\| (1+|\bullet|)^d \cdot \mathcal{F}^{-1} \left[\psi_1(A_4^*) \right] \right\|_{\bar{p}} \cdot \left\| w_1 \cdot \mathcal{F}^{-1} \left[\phi_1(A_4^*) \right] \right\|_{\bar{p}}.
$$
\n1 and $r = 0, \pm 1$. Similar estimates can be proved
\n $l \in S$ the $L_{\bar{p}}$ -terms are bounded and we get
\n $I_j \leq C_4 \cdot \sup_{r=0,\pm 1} \left\| w_1 \cdot \mathcal{F}^{-1} \left[\phi_{j+r} M \right] \right\|_p$ $(j \in \mathbb{N})$.
\n \sin by (9) the inequality
\n
$$
\left\| \left\langle 2^{j\sigma} \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \left\| \left\langle \widetilde{M}_{k,m}^{j,l} \right\rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} \left| b_{p,p}^*(w_1) \right| \right\rangle_{j \in \mathbb{N}} \right\| \ell_q \right\|
$$
\n
$$
\leq C_5 \cdot \left\| \left\langle 2^{j(s+\sigma-\nu)} \left\| w_1 \cdot \mathcal{F}^{-1} [\phi_j M] \right\|_p \right\rangle_{j \in \mathbb{N}} \left\| e_q \right\|
$$
\n
$$
= C_5 \cdot \left\| \mathcal{F}^{-1} M | B_{p,q}^{s+\sigma-\nu}(w_1) \right\|.
$$

Step 3. To prove the estimate from below we use the inequality

$$
(1+|A_{2^{-j}}k|)^{-d}w_1(x)\leq C_0\cdot w_1(x+A_{2^{-j}}k)
$$

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which in combination with (8) yields

Fourier Multipliers between Function Spaces 1 595
\nwhich in combination with (8) yields
\n
$$
\sup_{k \in \mathbb{Z}^n} \frac{(1+|A_{2^{-j}}k|)^{-d}}{(w_0)_k^j} \cdot 2^{j(s-\nu)} I_j \leq C_6 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \|
$$
\nwhere I_j is defined as above. The supremum on the left side can be estimated from
\nbelow by choosing $k = 0$ which leads to
\n
$$
2^{j(s-\nu)} I_j \leq C_7 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \| \langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \|.
$$
\n(10)
\nTo estimate the I_j from below we use the partition of unity

below by choosing $k = 0$ which leads to

Fourier Multipliers between Function Spaces 1 595
\ntion with (8) yields
\n
$$
\frac{A_{2-j} k|)^{-d}}{w_0)_k^j} \cdot 2^{j(s-\nu)} I_j \leq C_6 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|(\widetilde{M}_{k,m}^{j,l})_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)|\|
$$
\ned as above. The supremum on the left side can be estimated from
\n
$$
g \cdot k = 0
$$
 which leads to
\n
$$
2^{j(s-\nu)} I_j \leq C_7 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|(\widetilde{M}_{k,m}^{j,l})_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)|\|.
$$
 (10)
\n
$$
y_j
$$
 from below we use the partition of unity

To estimate the I_j from below we use the partition of unity

h (8) yields
\n
$$
\frac{-d}{dt} - 2^{j(s-\nu)}I_j \leq C_6 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} ||\langle \widetilde{M}_k^j \rangle
$$
\nsove. The supremum on the left side
\nwhich leads to
\n
$$
I_j \leq C_7 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} ||\langle \widetilde{M}_k^j \rangle_{m=1}^{j-1} |b_{p,j}^s|
$$
\nbelow we use the partition of unity
\n
$$
\phi_0(\xi)\psi_0(\xi) + \sum_{j=1}^{\infty} \phi_j(\xi)\psi_1(A_{2^{-j}}^*\xi) = 1
$$
\nition of ψ_0 and ψ_1) and Lemma 4.4.

(following from the definition of ψ_0 and ψ_1) and Lemma 4.4. In this way we obtain

$$
2^{j(s-\nu)}I_j \leq C_7 \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|\langle \widetilde{M}_{k,m}^{j,l} \rangle_{m \in \mathbb{Z}^n}^{l \in \mathbb{N}} |b_{p,p}^s(w_1)| \|.
$$

\nwe the I_j from below we use the partition of unity
\n
$$
\phi_0(\xi)\psi_0(\xi) + \sum_{j=1}^{\infty} \phi_j(\xi)\psi_1(A_{2^{-j}}^* \xi) = 1
$$

\nfrom the definition of ψ_0 and ψ_1) and Lemma 4.4. In this way we
\n
$$
||w_1 \cdot \mathcal{F}^{-1}[\phi_j M]||_p = ||w_1 \cdot \mathcal{F}^{-1}[\phi_j \sum_{t=-1}^1 \phi_{j+t}\psi_1(A_{2^{-(j+t)}}^*)M]||_p
$$
\n
$$
\leq C_8 \cdot ||(1+|\cdot|)^d \cdot \mathcal{F}^{-1}[\phi_j(A_{2^{j+2}}^*)]||_p
$$
\n
$$
\times \sum_{t=-1}^1 ||w_1 \cdot \mathcal{F}^{-1}[\phi_{j+t}\psi_1(A_{2^{-(j+t)}}^*)M]||_p
$$
\n
$$
\leq C_9 \cdot \sup_{t=0, \pm 1} ||w_1 \cdot \mathcal{F}^{-1}[\psi_1(A_{2^{-(j+t)}}^*)\phi_{j+t}M]||_p
$$
\nne $L_{\tilde{p}}$ quasinorms are uniformly bounded in j . Similar estimates can
\nnaining cases and we get the relation
\n
$$
||w_1 \cdot \mathcal{F}^{-1}[\phi_j M]||_p \leq C_{10} \cdot \sup_{t=0, \pm 1} I_{j+t} \quad (j \in \mathbb{N})
$$

because the $L_{\bar{p}}$ quasinorms are uniformly bounded in j . Similar estimates can be proved in the remaining cases and we get the relation

$$
||w_1 \cdot \mathcal{F}^{-1}[\phi_j M]||_p \leq C_{10} \cdot \sup_{t=0,\pm 1} I_{j+t}
$$
 $(j \in \mathbb{N})$

with $I_{-1} = 0$. By (10) it follows that

$$
\times \sum_{t=-1}^{\infty} \|\mathbf{w}_1 \cdot \mathcal{F}^{-1}[\phi_{j+t}\psi_1(A_{2-(j+t)}^*)M]\|_p
$$

\n
$$
\leq C_9 \cdot \sup_{t=0,\pm 1} \|\mathbf{w}_1 \cdot \mathcal{F}^{-1}[\psi_1(A_{2-(j+t)}^*)\phi_{j+t}M]\|_p
$$

\nuse the $L_{\bar{p}}$ quasinorms are uniformly bounded in j . Similar estimates can
\nwe remaining cases and we get the relation
\n
$$
\|\mathbf{w}_1 \cdot \mathcal{F}^{-1}[\phi_j M]\|_p \leq C_{10} \cdot \sup_{t=0,\pm 1} I_{j+t} \qquad (j \in \mathbb{N})
$$

\n $I_{-1} = 0.$ By (10) it follows that
\n
$$
\|\mathcal{F}^{-1}M[B_{p,q}^{s+\sigma-\nu}(w_1)]\|_p
$$

\n
$$
= \left\|\left\langle 2^{j(s+\sigma-\nu)}\|\mathbf{w}_1 \cdot \mathcal{F}^{-1}[\phi_j M]\|_p\right\rangle_{j\in\mathbb{N}} |\ell_q|\right\|_p
$$

\n
$$
\leq C_{11} \cdot \left\|\left\langle 2^{j\sigma} \sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|(\widetilde{M}_{k,m}^{j,l})_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} |b_{p,p}^* (w_1)|\| \right\rangle_{j\in\mathbb{N}} |\ell_q|\right\|_p.
$$

\nStep 4. If $w_1 \equiv 1$, then (8) implies
\n
$$
\sup_{k\in\mathbb{Z}^n} \frac{1}{(w_0)_k^j} \|(\widetilde{M}_{k,m}^{j,l})_{m\in\mathbb{Z}^n}^{l\in\mathbb{N}} |b_{p,p}^*||
$$

\n
$$
\leq C_{12} \cdot \left\|\mathcal{F}^{-1}[\psi_0 \phi_r M]\|_p \qquad \text{for } j = 0
$$

\n
$$
\mathbf{p}(s-\nu) = \mathbf{p}(s-\nu) = \mathbf{p}(s-\nu) = \mathbf{p}(s-\nu)
$$

$$
\leq C_{11} \left\| \left\langle 2^{J^{2}} \sup_{k \in \mathbb{Z}^{n}} \frac{1}{(w_{0})_{k}^{J}} \left\| \left\langle M_{k,m}^{J}\right\rangle_{m \in \mathbb{Z}^{n}} | \theta_{p,p}(w_{1})| \right\vert \right\rangle_{j \in \mathbb{N}} |\epsilon_{q}| \right\|.
$$

Step 4. If $w_{1} \equiv 1$, then (8) implies

$$
\sup_{k \in \mathbb{Z}^{n}} \frac{1}{(w_{0})_{k}^{J}} \left\| \left\langle \widetilde{M}_{k,m}^{j,l} \right\rangle_{m \in \mathbb{Z}^{n}} | \theta_{p,p}^{s} \right\|
$$

$$
\sim 2^{j(s-\nu)} \sup_{k \in \mathbb{Z}^{n}} \frac{1}{(w_{0})_{k}^{j}} \sup_{r=0,\pm 1} \left\{ \left\| \mathcal{F}^{-1} \left[\psi_{0}\phi_{r} M \right] \right\|_{p} \text{ for } j \geq 1. \right\}
$$
(11)

By definition of *W* we have

596 P. Dintelmann
\nBy definition of W we have
\n
$$
(w_0)_k^j = w_0(A_{2^{-j}}k) \leq C_{12} \cdot w_0(x)(1+|A_{2^{-j}}k-x|)^{d'}
$$
\nfor a suitable $d' > 0$. For $x \in \diamond_k^j := A_{2^{-j}}(k + [-\frac{1}{2}, \frac{1}{2})^n)$ the estimate

$$
(1+|A_{2-i}|k-x|)^{d'}\leq C_{13}
$$

holds uniformly in *j* and *k,* leading to

2.
$$
\text{Dintelmann}
$$

\n2. $(w_0)_k^j = w_0(A_{2^{-j}}k) \leq C_{12} \cdot w_0(x)(1 + |A_{2^{-j}}k - x|)^{d'}$

\n2. $w_0(k) = w_0(A_{2^{-j}}k) \leq C_{12} \cdot w_0(x)(1 + |A_{2^{-j}}k - x|)^{d'}$

\n2. $(w_0)_k^j \geq (1 + |A_{2^{-j}}k - x|)^{d'} \leq C_{13}$

\n2. $(w_0)_k^j \geq (C_{12}C_{13})^{-1} \cdot \sup_{k \in \mathbb{Z}^n} \sup_{x \in \sigma_k^j} \frac{1}{w_0(x)} = (C_{12}C_{13})^{-1} \cdot \left\| \frac{1}{w_0} \right\|_{\infty}$

\n2. $(w_0)_k^j \geq (C_{12}C_{13})^{-1} \cdot \sup_{k \in \mathbb{Z}^n} \sup_{x \in \sigma_k^j} \frac{1}{w_0(x)} = (C_{12}C_{13})^{-1} \cdot \left\| \frac{1}{w_0} \right\|_{\infty}$

\n2. $(w_0)_k^j \geq (C_{12}C_{13})^{-1} \cdot \sup_{k \in \mathbb{Z}^n} \frac{1}{w_0(k)}$

and the theorem is proved \blacksquare

5. Fourier. Multipliers

This section contains the main results of this paper. We start with the following two propositions which will be proved at the end of this section.

Proposition 5.1 (Change of *s*). Assume $0 < p_0, p, q_0, q_1 < \infty, s_0, s_1 \in \mathbb{R}$ and $w_0, w_1 \in W$. Then the relation **MODE 19.13** (Change of s). Assume $0 < p_0, p, q_0, q_1 < \infty$,
 M($B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1)$) \hookrightarrow *M*($B_{p_0,q_0}^{s_1-s_0}(w_0), B_{p_1,r_1}^{0}(w_1)$)

$$
\mathbf{M}(B^{\bullet\circ}_{p_0,q_0}(w_0),B^{\bullet\circ}_{p_1,q_1}(w_1))\rightleftarrows\mathbf{M}(B^{\bullet\circ-\bullet\circ}_{p_0,q_0}(w_0),B^0_{p_1,r_1}(w_1))
$$

holds.

This is usually proved with the help of Bessel potential operators. These operators are quite difficult to deal with in our'anisotropic weighted setting. Our proof using the discrete characterization is much simpler. This is usually proved with the
are quite difficult to deal with in α
discrete characterization is much:
Proposition 5.2 (Change of
 $w_0, w_1 \in W$. If $\left(\frac{1}{q_1} - \frac{1}{q_0}\right)_+ = \left(\frac{1}{r_1}\right)$
 $M(P^{s_0} - (w_1) - P^{s_1})$

Proposition 5.2 (Change of *q*). Assume $0 < p_0, p_1q_0, q_1, r, r_1 < \infty, s_0, s_1 \in \mathbb{R}$ and $-\frac{1}{r_0}$ ₊, then the relation *I*(*B*) *I*(*B*) *I*(*B*) *I*(*B*) *I*(*B*₂)_{*I*}(*B*)_{*I*}(*B*₇)_{*I*}(*B*₇)_{*I*}(*B*_{7⁰)_{*I*}(*B*_{7⁰₁</sup>(*W*₀)}*I*(*B*_{7⁰₁</sup>(*W*₀)}*I*(*B*_{7⁰₁</sup>(*W*₀)*I*(*B*_{7⁰₁</sup>(*w*₀)*I*(*B*_{7⁰₁,(*w}}}}*

$$
\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0),B^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,r_0}(w_0),B^{s_1}_{p_1,r_1}(w_1)\big)
$$

holds.

Note that the assertion for $q_0 > q_1$ was already proved by Orlovskij [16] for unweighted isotropic spaces in the case of $p_0, p_1 > 1$.

Now we come to the main theorem of this paper.

Theorem 5.3 (Fourier multipliers between Besov spaces). Assume $0 < p_0, p_1, q_0, q_1$ \leq ∞ and s₀, s₁ \in **R**. *If* $w_1 \in W_d$ (d \geq 0) and $w_0 \in W$ satisfy, the condition in the condition *(I* $\begin{array}{l} t \ s_0, s_1 \ \in \ \mathbb{R}. \end{array}$ *If* $w_1 \in \\ \infty < \infty$, then the relation $\begin{aligned} \textit{Us.}\ \textit{Note that the}\ \textit{in} \ \textit{not} \ \textit{in} \ \textit{$ **5.3** (Fourier multipliers between Besov spaces). Assume $0 < p$
 $s_1 \in \mathbb{R}$. If $w_1 \in W_d$ ($d \ge 0$) and $w_0 \in W$ satisfy the
 ∞ , then the relation
 $B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1)) \rightarrow \mathcal{F}[B_{p_1,r}^{\sigma}(w_1)]$ (p_0

$$
\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0),B^{s_1}_{p_1,q_1}(w_1)\big)\nrightarrow{\sim} \mathcal{F}\big[B^\sigma_{p_1,r}(w_1)\big] \qquad (p_0\leq \min\{1,p_1\})
$$

holds where $\sigma = \nu(\frac{1}{p_0} - 1) + s_1 - s_0$ *and* $\frac{1}{r} = (\frac{1}{q_1} - \frac{1}{q_0})_+$.

This theorem will be completed by negative results in Part II of this paper.

Proof of Theorem 5.3. Step 1. First assume $q_0 \leq q_1$, i.e. $r = \infty$. We apply position 5.2, Lemma 3.1, Theorem 3.2 and Theorem 4.2 to obtain the equivalence $||M|M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))|| \sim ||M|M(B_{p_0,p_0}^{s_0}(w$ Proposition 5.2, Lemma 3.1, Theorem 3.2 and Theorem 4.2 to obtain the equivalence

Fourier Multipliers between Function Spaces 1

\nProof of Theorem 5.3. Step 1. First assume
$$
q_0 \leq q_1
$$
, i.e. $r = \infty$. Proposition 5.2, Lemma 3.1, Theorem 3.2 and Theorem 4.2 to obtain the equation $||M|M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))|| \sim ||M|M(B_{p_0,p_0}^{s_0}(w_0), B_{p_1,p_1}^{s_1}(w_1))|| \sim ||\widetilde{M}(w_0, w_1)|b_{p_0,p_0}^{s_0}, b_{p_1,p_1}^{s_1}|| \sim B(\widetilde{M}(w_0, w_1); s_0, p_0, b_{p_1,p_1}^{s_1})$

\n
$$
\sim ||M|\mathcal{F}[B_{p_1,\infty}^{\sigma}(w_1)]||.
$$
\nStep 2. Now let $q_0 > q_1$. We apply Lemma 3.1, Theorem 3.3 and Theorem obtain the equivalence

\n
$$
||M|M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))|| \sim ||\widetilde{M}(w_0, w_1)|b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1}|| \sim ||\left\langle 2^{ls} \sup \sup \sup \left\| \langle \widetilde{M}(w_0, w_1)|^{l+t,l}_{r,l} \rangle_{r,l} \leq \sum_{i=1}^n |\ell_{p_i}|| \right\rangle
$$

Step 2. Now let $q_0 > q_1$. We apply Lemma 3.1, Theorem 3.3 and Theorem 4.3 to obtain the equivalence

$$
\|M|M(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1))\|
$$

\n
$$
\sim \|\widetilde{M}(w_0,w_1)|b_{p_0,q_0}^{s_0},b_{p_1,q_1}^{s_1}\|
$$

\n
$$
\sim \left\|\left\langle 2^{ls}\sup_{k\in\mathbb{Z}^n}\sup_{t=0,\pm 1}\left\|\left\langle \widetilde{M}(w_0,w_1)_{k,m}^{l+t,l}\right\rangle_{m\in\mathbb{Z}^n}|e_{p_1} \right\|\right\rangle_{l\in\mathbb{N}}|e_r\right\|
$$

\n
$$
\sim \|M|\mathcal{F}[B_{p_1,r}^{\sigma}(w_1)]\|
$$

which proves the theorem \blacksquare

Remarks. The first theorem of this type goes back to Taibleson [23: Part II/p. 827] and asserts *M* \blacksquare
 M $(B_{1,1}^s(\mathbb{R}^n; I, 1)) \Leftrightarrow \mathcal{F}[B_{1,\infty}^0(\mathbb{R}^n; I, 1)] \quad (s \in \mathbb{R}).$

$$
\mathbf{M}\big(B^s_{1,1}(\mathbb{R}^n;I,1)\big) \Leftrightarrow \mathcal{F}\big[B^0_{1,\infty}(\mathbb{R}^n;I,1)\big] \quad (s \in \mathbb{R}).
$$

It can be obtained by choosing $p_0 = p_1 = q_0 = q_1 = 1$ and $s = s_0 = s_1$ in Theorem 5.3. Peetre [17: p. 249] proved the supplement

$$
M(B_{1,1}^{s}(\mathbb{R}^{n};I,1)) \xrightarrow{\sim} \mathcal{F}[B_{1,\infty}^{n}(\mathbb{R}^{n};I,1)] \quad (s \in \mathbb{R}).
$$

It can be obtained by choosing $p_0 = p_1 = q_0 = q_1 = 1$ and $s = s_0 = s_1$ in Theorem 5.3.
Peetre [17: p. 249] proved the supplement

$$
M(B_{p,q}^{s}(\mathbb{R}^{n};I,1)) \xrightarrow{\sim} \mathcal{F}[B_{p,\infty}^{n(\frac{1}{p}-1)}(\mathbb{R}^{n};I,1)] \quad \begin{pmatrix} 0 < p < 1 \\ 0 < q < \infty \\ s & \in \mathbb{R} \end{pmatrix}
$$
which can be obtained by choosing $p = p_0 = p_1$, $q = q_0 = q_1$ and $s = s_0 = s_1$ in Theorem 5.3. Johnson [12: Theorem 6] proved

Theorem 5.3. Johnson [12: Theorem 61 proved

$$
M(B_{1,1}^{s}(\mathbb{R}^{n};I,1)) \Leftrightarrow \mathcal{F}[B_{1,\infty}^{0}(\mathbb{R}^{n};I,1)] \quad (s \in \mathbb{R}).
$$

can be obtained by choosing $p_{0} = p_{1} = q_{0} = q_{1} = 1$ and $s = s_{0} = s_{1}$ in Theorem 5.
center [17: p. 249] proved the supplement

$$
M(B_{p,q}^{s}(\mathbb{R}^{n};I,1)) \Leftrightarrow \mathcal{F}[B_{p,\infty}^{n(\frac{1}{p}-1)}(\mathbb{R}^{n};I,1)] \quad \begin{pmatrix} 0 < p < 1 \\ 0 < q < \infty \\ s \in \mathbb{R} \end{pmatrix}
$$

which can be obtained by choosing $p = p_{0} = p_{1}, q = q_{0} = q_{1}$ and $s = s_{0} = s_{1}$ i
theorem 5.3. Johnson [12: Theorem 6] proved

$$
M(B_{1,q_{0}}^{s_{0}}(\mathbb{R}^{n};I,1), B_{p_{1},q_{1}}^{s_{1}}(\mathbb{R}^{n};I,1)) \Leftrightarrow \mathcal{F}[B_{p_{1},\infty}^{s_{1}-s_{0}}(\mathbb{R}^{n};I,1)] \quad \begin{pmatrix} 1 \le p_{1}\infty \\ 1 \le q_{0} \le q_{1} < \infty \\ s_{0}, s_{1} \in \mathbb{R} \end{pmatrix}
$$

or homogeneous spaces with the help of a characterization of Besov spaces in terms of
realu-Weierstrab Kernel and preliminary work of Tableson. Due to technical reason

for homogeneous spaces with the help of a characterization of Besov spaces in terms of the Gau-Weierstral3 kernel and preliminary work of Taibleson. Due to technical reasons the characterization of Fourier multipliers is usually much simpler in the homogeneous deal with inhomogeneous spaces and obtained the counterpart of Johnson's result, i.e.

case than in the inhomogeneous one. Bui [1: Theorem 2] modified Johnson's method to deal with inhomogeneous spaces and obtained the counterpart of Johnson's result, i.e.
$$
M(B_{1,q_0}^{s_0}(\mathbb{R}^n;I,1),B_{p_1,q_1}^{s_1}(\mathbb{R}^n;I,1)) \Leftrightarrow \mathcal{F}[B_{p_1,\infty}^{s_1-s_0}(\mathbb{R}^n;I,1)] \left(1 \leq q_0 \leq q_1 < \infty \atop s_0, s_1 \in \mathbb{R} \right)
$$

which can be proved by choosing $p_0 = 1$ in Theorem 5.3. Bui recently generalized his method in order to deal with weighted spaces (cf. [2, 3]) and proved embedding theorems for classes of Fourier multipliers between isotropic Besov spaces with power weights.

The unweighted version of the above theorem was already proved in [6] $(q_0 \leq q_1)$ and [7] $(q_0 > q_1)$. Similar results in the periodic setting (inhomogeneous, unweighted case) were proved by Mizuhara [15]. weights.

The unweighted version of the above theorem was already proved in [6] $(q_0 \leq q_1)$

and [7] $(q_0 > q_1)$. Similar results in the periodic setting (inhomogeneous, unweighted

case) were proved by Mizuhara [15].
 P

Proof of Proposition 5.2. First we consider the case of $q_0 \leq q_1$. If $M \in$ the sequences α_j defined as **and** $\begin{aligned}\n\mathbf{a}_1 &= \mathbf{a}_2 \mathbf{a}_3 + \mathbf{a}_4 \mathbf{a}_5 + \mathbf{a}_5 \mathbf{a}_6\n\end{aligned}$
 Proof of Proposition 5.2. First we consider the case of $q_0 \leq q_1$. If $M \in \mathbf{M}(B_{p_0,q_0}^{\mathfrak{so}}(w_0), B_{p_1,q_1}^{\mathfrak{so}}(w_1))$, then \widetilde{M}

$$
(\alpha_j)_m^l = 1_{[-1,1]}(j-l) \cdot \alpha_m^l
$$

associated with the sequence α . Since the coefficients $\widetilde{M}_{k,m}^{j,l}$ for $|j - l| > 1$ vanish we
have $(\widetilde{M}\alpha)_k^j = (\widetilde{M}\alpha_j)_k^j$ and thus
 $2^{j(s_1 - \frac{\nu}{p_1})}||(w_1)^j \cdot (\widetilde{M}\alpha)^j|$ $|l|$

associated with the sequence
$$
\alpha
$$
. Since the coefficients $\widetilde{M}_{k,m}^{j,l}$ for $|j - l| > 1$ vanish
\nhave $(\widetilde{M}\alpha)^j_k = (\widetilde{M}\alpha)^j_k$ and thus
\n
$$
2^{j(s_1 - \frac{\nu}{p_1})} \|\langle (w_1)^j_k \cdot (\widetilde{M}\alpha)^j_k \rangle_{k \in \mathbb{Z}^n} |\ell_{p_1}\|
$$
\n
$$
\leq \|\widetilde{M}\alpha_j|b_{p_1,q_1}^{s_1}(w_1)\|
$$
\n
$$
\leq \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\| \cdot \|\alpha_j|b_{p_0,q_0}^{s_0}(w_0)\|
$$
\n
$$
\leq C_0 \cdot \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\| \cdot \sum_{l=j-1}^{j+1} 2^{l(s_0 - \frac{\nu}{p_0})} \|\langle (w_0)^l_m \alpha^l_m \rangle_{m \in \mathbb{Z}^n} |\ell_{p_0}\|.
$$
\nApplying the ℓ_{r_1} quasinorm in j and using the embedding $\ell_{r_0} \hookrightarrow \ell_{r_1}$ we obtain
\n
$$
\|\widetilde{M}\alpha|b_{p_1,r_1}^{s_1}(w_1)\| \leq C_1 \cdot \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\| \cdot \|\alpha|b_{p_0,r_0}^{s_0}(w_0)\|.
$$

Applying the ℓ_{r_1} quasinorm in *j* and using the embedding $\ell_{r_0} \hookrightarrow \ell_{r_1}$ we obtain

the
$$
\ell_{r_1}
$$
 quasinorm in j and using the embedding $\ell_{r_0} \hookrightarrow \ell_{r_1}$ we obtain $\|\widetilde{M}\alpha|b_{p_1,r_1}^{s_1}(w_1)\| \leq C_1 \cdot \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\| \cdot \|\alpha|b_{p_0,r_0}^{s_0}(w_0)\|.$ $\mathbf{M}\left(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1)\right) \hookrightarrow \mathbf{M}\left(B_{p_0,r_0}^{s_0}(w_0), B_{p_1,r_1}^{s_1}(w_1)\right).$ the roles of q and r yields the assertion

Therefore

$$
\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0),B^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,r_0}(w_0),B^{s_1}_{p_1,r_1}(w_1)\big).
$$

Reversing the roles of q and *r* yields the assertion.

Now let $q_0 > q_1$. Since $\widetilde{M}(w_0, w_1)$ satisfies the hypothesis of Theorem 3.3 and Reversing the roles of q and r yields the assertion.

Now let $q_0 > q_1$. Since $\widetilde{M}(w_0, w_1)$ satisfies the hypothesis of Th

because of the relation $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_0}$ we obtain the equi

log the roles of
$$
q
$$
 and r yields the assertion.

\nlet $q_0 > q_1$. Since $\widetilde{M}(w_0, w_1)$ satisfies the hypothesis of Theorem of the relation $\frac{1}{q_1} - \frac{1}{q_0} = \frac{1}{r} = \frac{1}{r_1} - \frac{1}{r_0}$ we obtain the equivalence

\n $||M||M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))|| \sim ||\widetilde{M}(w_0, w_1)|b_{p_0,q_0}^{s_0}, b_{p_1,q_1}^{s_1}||$

\n $\sim ||\langle 2^{l_s} N_l(\widetilde{M}(w_0, w_1); p_0, p_1) \rangle_{l \in \mathbb{N}} |\ell_r||$

\n $\sim ||\widetilde{M}(w_0, w_1)|b_{p_0,r_0}^{s_0}, b_{p_1,r_1}^{s_1}||$

\n $\sim ||M||M(B_{p_0,r_0}^{s_0}(w_0), B_{p_1,r_1}^{s_1}(w_1))||$

\nmma 3.1 and Theorem 3.3

from Lemma 3.1 and Theorem **3.31**

Another way to prove the assertion of this proposition for the case of $q_0 \n\t\leq q_1$ is the method of real interpolation.

Proof of Proposition 5.1. For $M \in M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))$ the operator \widetilde{M} : $b_{p_0,q_0}^{s_0}(w_0) \rightarrow b_{p_1,q_1}^{s_1}(w_1)$ is bounded. Note that the mapping Fourier Mul

sition 5.1. For $M \in M(I_1)$
 I_n: $\mathbb{C}^{N \times Z^n} \to \mathbb{C}^{N \times Z^n}$,

rphism *oosition 5.1. For* $M \in M(B_{p_0,q_0}^{s_0}(w_0), B_{p_1,q_1}^{s_1}(w_1))$ *the* $w_1)$ *is bounded. Note that the mapping
* $I_{\sigma}: \mathbb{C}^{N \times \mathbb{Z}^n} \to \mathbb{C}^{N \times \mathbb{Z}^n}$ *,* $I_{\sigma} \alpha = \langle 2^{j\sigma} \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}$ *

<i>norphism***
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 $(q_1(w_1))$ the operator \widetilde{M} :
 $\in \mathbb{R}$
 $\rightarrow b^{s_1-\sigma}_{p_1,q_1}(w_1).$
 $(w_1).$ (12)
 $j \in \mathbb{N}$.

$$
I_{\sigma}: \mathbb{C}^{N \times Z^{n}} \to \mathbb{C}^{N \times Z^{n}}, \qquad I_{\sigma} \alpha = \langle 2^{j\sigma} \alpha_{k}^{j} \rangle_{k \in Z^{n}}^{j \in N}
$$

is an isometric isomorphism

$$
I_{\sigma}: \mathbb{C}^{N \times \mathbb{Z}^n} \to \mathbb{C}^{N \times \mathbb{Z}^n}, \qquad I_{\sigma} \alpha = \langle 2^{j\sigma} \alpha_k^j \rangle_{k \in \mathbb{Z}^n}^{j \in \mathbb{N}}
$$

\nric isomorphism
\n
$$
I_{\sigma}: b_{p_0,q_0}^{s_0}(w_0) \to b_{p_0,q_0}^{s_0-\sigma}(w_0), \qquad I_{\sigma}: b_{p_1,q_1}^{s_1}(w_1) \to b_{p_1,q_1}^{s_1-\sigma}(w_1).
$$

\n
$$
P_{\sigma}: \alpha_k^{s_0} \to \alpha_k^{s_0} \quad \text{and} \quad I_{\sigma}: \alpha_k^{s_0} \to \alpha_k^{s_0} \quad \text{and}
$$

We show the boundedness of

$$
\widetilde{M}_{s_0} := I_{-s_0} \circ \widetilde{M} \circ I_{s_0} : b_{p_0,q_0}^{s_0}(w_0) \to b_{p_1,q_1}^{s_1}(w_1).
$$
\n(12)

To this end we decompose $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where

$$
(\alpha_r)_k^j = 1_{J_r}(j) \cdot \alpha_k^j \quad \text{with} \quad J_r = \{3j + r : j \in \mathbb{N}\}.
$$

To this end we decompose $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where
 $(\alpha_r)^j_k = 1_{J_r}(j) \cdot \alpha^j_k$ with $J_r = \{3j + r : j \in \mathbb{N}\}$.

Observe that $\|\alpha_r\|_{p_0, q_0}(w_0)\| \le \|\alpha\|_{p_0, q_0}(w_0)\|$. Since the $\widetilde{M}^{j,l}_{k,m}$ vanish for $|j - l| > 1$

we ar we are lead to

$$
I_{\sigma}: b_{p_0,q_0}^{s_0}(w_0) \to b_{p_0,q_0}^{s_0-\sigma}(w_0), \qquad I_{\sigma}: b_{p_1,q_1}^{s_1}(w_1) \to b_{p_1,q_1}^{s_1-\sigma}(w_1).
$$
\nshow the boundedness of

\n
$$
\widetilde{M}_{s_0} := I_{-s_0} \circ \widetilde{M} \circ I_{s_0}: b_{p_0,q_0}^{s_0}(w_0) \to b_{p_1,q_1}^{s_1}(w_1).
$$
\nhis end we decompose $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where

\n
$$
(\alpha_r)_k^j = 1_{J_r}(j) \cdot \alpha_k^j \qquad \text{with } J_r = \{3j + r : j \in \mathbb{N}\}.
$$
\nerve that $||\alpha_r | b_{p_0,q_0}^{s_0}(w_0)|| \le ||\alpha| b_{p_0,q_0}^{s_0}(w_0)||$. Since the $\widetilde{M}_{k,m}^{j,l}$ vanish for $|j - l|$ are lead to

\n
$$
(\widetilde{M}_{s_0}\alpha_r)_m^l = 2^{-l_{s_0}} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \widetilde{M}_{k,m}^{j,l} \cdot 2^{js_0} (\alpha_r)_k^j = \sum_{j=l-1}^{l+1} \sum_{k \in \mathbb{Z}^n} 2^{(j-l)s_0} \widetilde{M}_{k,m}^{j,l} \cdot (\alpha_r)_k^j.
$$
\nwe sum over j only one term appears due to the definition of α_r and so

\n
$$
|(\widetilde{M}_{s_0}\alpha_r)_m^l| \le 2^{|s_0|} \cdot \left|\sum_{j=l-1}^{l+1} \sum_{k \in \mathbb{Z}^n} \widetilde{M}_{k,m}^{j,l} \cdot (\alpha_r)_k^l| = 2^{|s_0|} \cdot |(\widetilde{M}\alpha_r)_m^l|.
$$
\nwe we obtain

\n
$$
||\widetilde{M}_{s_0}\alpha_r|| \le C \int_{\widetilde{M}_{s_0}^{s_0}(w)} \widetilde{M}_{s_0}^{
$$

In the sum over *j* only one term appears due to the definition of α_r and so

$$
\left|(\tilde{M}_{s_0}\alpha_r)^l_m\right| \leq 2^{|s_0|} \cdot \left|\sum_{j=l-1}^{l+1}\sum_{k\in\mathbb{Z}^n} \widetilde{M}_{k,m}^{j,l}\cdot(\alpha_r)^j_k\right| = 2^{|s_0|} \cdot \left|(\tilde{M}\alpha_r)^l_m\right|.
$$

Hence we obtain

In the sum over *j* only one term appears due to the definition of
$$
\alpha_r
$$
 and so
\n
$$
\left| (\tilde{M}_{s_0} \alpha_r)^l_m \right| \leq 2^{|s_0|} \cdot \left| \sum_{j=l-1}^{l+1} \sum_{k \in \mathbb{Z}^n} \widetilde{M}_{k,m}^{j,l} \cdot (\alpha_r)^l_k \right| = 2^{|s_0|} \cdot \left| (\tilde{M} \alpha_r)^l_m \right|.
$$
\nHence we obtain
\n
$$
\left\| \widetilde{M}_{s_0} \alpha | b_{p_1, q_1}^{s_1}(w_1) \right\| \leq C_0 \cdot \sum_{r=0}^2 \left\| \widetilde{M} \alpha_r | b_{p_1, q_1}^{s_1}(w_1) \right\|
$$
\n
$$
\leq C_0 \cdot \left\| \widetilde{M} | b_{p_0, q_0}^{s_0}(w_0), b_{p_1, q_1}^{s_1}(w_1) \right\| \cdot \sum_{r=0}^2 \left\| \alpha_r | b_{p_0, q_0}^{s_0}(w_0) \right\|
$$
\n
$$
\leq 3C_0 \cdot \left\| \widetilde{M} | b_{p_0, q_0}^{s_0}(w_0), b_{p_1, q_1}^{s_1}(w_1) \right\| \cdot \left\| \alpha | b_{p_0, q_0}^{s_0}(w_0) \right\|
$$
\nwhich shows the boundedness of \widetilde{M}_{s_0} from (12) with bound
\n
$$
\left\| \widetilde{M}_{s_0} | b_{p_0, q_0}^{s_0}(w_0), b_{p_1, q_1}^{s_1}(w_1) \right\| \leq 3C_0 \cdot \left\| \widetilde{M} | b_{p_0, q_0}^{s_0}(w_0), b_{p_1, q_1}^{s_1}(w_1) \right\|.
$$
\nSince the I_{σ} are isometric isomorphisms the mapping
\n
$$
\widetilde{M} = I_{s_0} \circ \widetilde{M}_{s_0} \circ I_{-s_0} : b_{p_0, q_0}^0(w_0) \rightarrow b_{p_1, q_1}^{s_1 - s_0}(w_1)
$$
\nis bounded

s the boundedness of
$$
M_{s_0}
$$
 from (12) with bound
\n
$$
\|\widetilde{M}_{s_0}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\| \leq 3C_0 \cdot \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0), b_{p_1,q_1}^{s_1}(w_1)\|.
$$
\nare isometric isomorphisms the mapping
\n
$$
\widetilde{M} = I_{s_0} \circ \widetilde{M}_{s_0} \circ I_{-s_0} : b_{p_0,q_0}^0(w_0) \to b_{p_1,q_1}^{s_1-s_0}(w_1)
$$
\nwith bound

Since the I_{σ} are isometric isomorphisms the mapping

$$
\widetilde{M} = I_{s_0} \circ \widetilde{M}_{s_0} \circ I_{-s_0} : b_{p_0,q_0}^0(w_0) \to b_{p_1,q_1}^{s_1-s_0}(w_1)
$$

is bounded with bound
\n
$$
\|\widetilde{M}|b_{p_0,q_0}^{0}(w_0),b_{p_1,q_1}^{s_1-s_0}(w_1)\| \leq 3C_0 \cdot \|\widetilde{M}|b_{p_0,q_0}^{s_0}(w_0),b_{p_1,q_1}^{s_1}(w_1)\|
$$
\nshowing the embedding
\n
$$
M(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1)) \hookrightarrow M(B_{p_0,q_0}^{0}(w_0),B_{p_1,q_1}^{s_1-s_0}(w_1))
$$
\nSince s_0 and s_1 are arbitrary we get the reverse embedding from the same argument

showing the embedding

$$
\mathbf{M}(B_{p_0,q_0}^{s_0}(w_0),B_{p_1,q_1}^{s_1}(w_1)) \hookrightarrow \mathbf{M}(B_{p_0,q_0}^0(w_0),B_{p_1,q_1}^{s_1-s_0}(w_1)).
$$

and the proof is complete \blacksquare

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