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Convergence Structures in Numerical Analysis

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Abstract. The paper deals – under the viewpoint of topology – with discrete Cauchy spaces, which are spaces where a discrete Cauchy structure (t, C) (with t being a discrete convergence and C being a discrete pre-Cauchy structure) is defined. More precisely, let $E_1, E_2, \ldots$ and E be arbitrary sets and let $\mathcal{S}$ denote the set of all discrete sequences $(x_n)_{n\in N'}$ with $x_n \in E_n$ $(n \in N')$ and with N' being an infinite subset of $\mathbb{N} = \{1, 2, \ldots\}$. Then t and C are certain subsets of $(\mathcal{S}, E)$ respectively of $\mathcal{S}$, which in a certain sense are assumed to be compatible. The paper gives properties of t and C and among others is devoted to the problem of completion of discrete Cauchy spaces $\big(((E_1, E_2, \ldots), E); (t, C)\big)$. The construction of a completion of a discrete Cauchy space differs (in some sense essentially) from the construction of a completion of a usual sequential Cauchy space and is even more simple.

An essential part of the paper is devoted to certain metric discrete Cauchy spaces, where – among others assuming that E is equipped with a metric d and that there exist mappings $q_n : E_n \to E$ $(n \in \mathbb{N})$ – the discrete Cauchy structure (t, C) is defined by

$$\big((x_n)_{N'}, x\big) \in t \iff \big(d(q_n(x_n), x)\big)_{N'} \longrightarrow 0$$

$$(x_n)_{N'} \in C \iff \big(q_n(x_n)\big)_{N'} \text{ is a Cauchy sequence in } (E, d).$$

It turns out that such a metric discrete Cauchy space is complete if and only if (E, d) is complete and that also the completion is metric.

A further subject of the paper are metric discrete Cauchy spaces of mappings between metric discrete Cauchy spaces, where simple characterizations of the corresponding discrete convergence and discrete pre-Cauchy structure of such a discrete Cauchy space as well as a necessary and sufficient condition for its completeness are given.

Keywords: *Discrete sequences, discrete convergence, discrete (pre-)Cauchy sequences, discrete Cauchy spaces, metric discrete Cauchy spaces, discrete Cauchy spaces of mappings*

AMS subject classification: 54 A 99, 54 C 99, 65 D 99, 65 J 99

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1. Introduction

In numerical analysis the notion of discrete convergence plays an important role. It appears in papers of several authors in different degrees of generality. For instance, the approximation schemes used by Browder [1], Petryshyn [8, 9] and others yield discrete convergence spaces. Special types of discrete convergence spaces among others also are used in publications of Grigorieff [7], Stummel [10, 11] and Vainikko [12]. Under restrictions to the family of index sets the notion of a discrete limit space introduced and studied by Stummel in [11] is identical with the notion of a separated discrete convergence space which fulfills the Urysohn property.

The notion of discrete convergence is a type of sequential convergence, where – differently to the usual sequential convergence – the elements x_n of the converging sequences as well as the limits are contained in arbitrary fixed sets E_n and E, respectively, which all may be different. Discrete convergence is used in discretization methods to solve approximately equations $f(x) = y$.

The theory of discrete convergence spaces is very interesting from the viewpoint of topology. In this theory often related topological notions are used and helpful. For instance such a related topological notion is that of a discrete Cauchy space, which has been introduced in [5] and studied in more detail in [6]. The present paper mainly deals with a special type of such spaces, where always the discrete convergence as well as the discrete Cauchy structure are defined by means of a metric d on E and mappings $q_n : E_n \longrightarrow E$ and which in the sense of [4] (and [6]) is a special metric discrete Cauchy space.

2. The general notion of a discrete Cauchy space

2.1. Let $\mathcal{N}$ be the set of all infinite subsets of $\mathbb{N} = \{1, 2, \ldots\}$ and let $\leq$ denote the partial ordering in $\mathcal{N}$ given by set inclusion. Let $E_1, E_2, \ldots$ be arbitrary sets and $\mathbf{E} = (E_1, E_2, \ldots)$. Every $\mathbf{x} = (x_n)_{N'}$ $(= (x_n)_{n \in N'})$ with $x_n \in E_n$ and $N' \in \mathcal{N}$ is called a *discrete sequence in* $\mathbf{E}$ and every $\mathbf{y} = (x_n)_{N''}$ with $N'' \in \mathcal{N}$ and $N'' \leq N'$ is called a *subsequence of* $\mathbf{x}$. If a discrete sequence $\mathbf{y}$ is a subsequence of a discrete sequence $\mathbf{x}$, then we write $\mathbf{y} \leq \mathbf{x}$.

For arbitrary discrete sequences $\mathbf{x} = (x_n)_{N'}$ $(N' \in \mathcal{N})$ and $\mathbf{y} = (y_n)_{N''}$ $(N'' \in \mathcal{N})$ let

$$m(\mathbf{x}, \mathbf{y}) = \left\{ (z_n)_{N' \cup N''} \;\middle|\; \begin{array}{l} \text{there exist sets } N_1 \leq N' \text{ and } N_2 \leq N'' \\ \text{with } N_1 \cap N_2 = \emptyset \text{ and } N_1 \cup N_2 = N' \cup N'' \\ \text{such that } z_n = x_n \text{ on } N_1 \text{ and } z_n = y_n \text{ on } N_2 \end{array} \right\}$$

and

$$\overline{m}(\mathbf{x}, \mathbf{y}) = \left\{ (z_n)_{N' \cup N''} \;\middle|\; \begin{array}{l} \text{there exist sets } N_1 \subseteq N' \text{ and } N_2 \subseteq N'' \\ \text{with } N_1 \cap N_2 = \emptyset \text{ and } N_1 \cup N_2 = N' \cup N'' \\ \text{such that } z_n = x_n \text{ on } N_1 \text{ and } z_n = y_n \text{ on } N_2 \end{array} \right\}.$$

Remark that the difference between $m(\mathbf{x},\mathbf{y})$ and $\overline{m}(\mathbf{x},\mathbf{y})$ consists in the fact that N_1 and N_2 in the first case are infinite and in the second case more generally may be finite or infinite. Hence $m(\mathbf{x},\mathbf{y}) \subseteq \overline{m}(\mathbf{x},\mathbf{y})$. Obviously, $\overline{m}(\mathbf{x},\mathbf{y})$ may also be defined to be the set of all discrete sequences $(z_n)_{N' \cup N''}$, where

$$z_n = \begin{cases} x_n & \text{on } N' \setminus N'' \\ y_n & \text{on } N'' \setminus N' \\ x_n \text{ or } y_n & \text{on } N' \cap N''. \end{cases}$$

2.2. Let E be an arbitrary set and let t denote a set of pairs $(\mathbf{x}, x)$ with $\mathbf{x}$ being a discrete sequence in $\mathbf{E}$ and $x \in E$, such that the following two properties are fulfilled:

$$(\mathbf{x},x) \in t \text{ and } \mathbf{y} \le \mathbf{x} \quad \Longrightarrow \quad (\mathbf{y},x) \in t. \tag{1}$$

For all $x \in E$ there exists a discrete sequence $\mathbf{x}$ in $\mathbf{E}$ such that $(\mathbf{x}, x) \in t$. (2)

Instead of $(\mathbf{x},x) \in t$ we mostly write $\mathbf{x} \longrightarrow_t x$ and say that $\mathbf{x}$ *converges discretely* to x. Moreover t is called a *discrete convergence* on $(\mathbf{E}, E)$ and $((\mathbf{E},E),t)$ is said to be a *discrete convergence space.* A discrete convergence space $((\mathbf{E},E),t)$ and also the discrete convergence t are said to be *separated* if $\mathbf{x} \longrightarrow_t x$ and $\mathbf{x} \longrightarrow_t y$ together imply $x = y$.

2.3. Let C be a set of discrete sequences in $\mathbf{E}$ with implication properties

$$\mathbf{x} \in C \text{ and } \mathbf{y} \le \mathbf{x} \quad \Longrightarrow \quad \mathbf{y} \in C \tag{3}$$

$$(x_n)_{N'}, (x_n)_{N''} \in C \text{ and } N' \cap N'' \in \mathcal{N} \quad \Longrightarrow \quad (x_n)_{N' \cup N''} \in C. \tag{4}$$

Every $\mathbf{x} \in C$ is called a *discrete pre-Cauchy sequence* in $\mathbf{E}$, C a *discrete pre-Cauchy structure* on $\mathbf{E}$ and $(\mathbf{E}, C)$ a *discrete pre-Cauchy space.*

Theorem 1 (see [5: Theorem 1]). *$m(\mathbf{x},\mathbf{y}) \subseteq C$ implies $\overline{m}(\mathbf{x},\mathbf{y}) \subseteq C$, hence also $\mathbf{x},\mathbf{y} \in C$.*

Theorem 2. *In C there is given an equivalence relation $\sim$ by*

$$\mathbf{x} \sim \mathbf{y} \quad \Longleftrightarrow \quad m(\mathbf{x},\mathbf{y}) \subseteq C$$

or even more simple by

$$\mathbf{x} \sim \mathbf{y} \quad \Longleftrightarrow \quad m(\mathbf{x},\mathbf{y}) \cap C \ne \emptyset. \tag{5}$$

Proof. From [5: Theorem 2] we know that

$$\mathbf{x} \sim \mathbf{y} \quad \Longleftrightarrow \quad m(\mathbf{x},\mathbf{y}) \subseteq C$$

defines an equivalence relation $\sim$ on C. This implies that the arrow $\Longrightarrow$ in definition (5) is true.

To prove the validity of the arrow $\Longleftarrow$, let

$$\mathbf{x} = (x_n)_{N'} \in \mathcal{C} \qquad\qquad \mathbf{z} = (z_n)_{N' \cup N''} \in m(\mathbf{x}, \mathbf{y}) \cap \mathcal{C}$$

and

$$\mathbf{y} = (y_n)_{N''} \in \mathcal{C} \qquad\qquad \mathbf{z}' = (z'_n)_{N' \cup N''} \in m(\mathbf{x}, \mathbf{y})$$

be given. There exist sets

$$N_1 \leq N' \text{ and } N_2 \leq N'' \quad \text{with } N_1 \cap N_2 = \emptyset \text{ and } N_1 \cup N_2 = N' \cup N''$$

such that

$$z_n = \begin{cases} x_n & \text{if } n \in N_1 \\ y_n & \text{if } n \in N_2. \end{cases}$$

Moreover there exist sets

$$N'_1 \leq N' \text{ and } N'_2 \leq N'' \quad \text{with } N'_1 \cap N'_2 = \emptyset \text{ and } N'_1 \cup N'_2 = N' \cup N''$$

such that

$$z'_n = \begin{cases} x_n & \text{if } n \in N'_1 \\ y_n & \text{if } n \in N'_2. \end{cases}$$

Let us consider at first the case where $N_2 \cap N'_2$ is infinite. Define $(z_n^+)_{N' \cup N''}$ by

$$z_n^+ = \begin{cases} x_n & \text{if } n \in N_1 \cup N'_1 \\ y_n & \text{if } n \in N_2 \cap N'_2 \quad (= N_2 \setminus N'_1 = N'_2 \setminus N_1). \end{cases}$$

Since $(x_n)_{N_1 \cup N'_1} \in \mathcal{C}$, $(z_n)_{N_1 \cup (N_2 \cap N'_2)} \in \mathcal{C}$ and $(N_1 \cup N'_1) \cap (N_1 \cup (N_2 \cap N'_2)) = N_1$, $(N_1 \cup N'_1) \cup (N_1 \cup (N_2 \cap N'_2)) = N' \cup N''$, by means of (4) we get $(z_n^+)_{N' \cup N''} \in \mathcal{C}$, hence $(z_n^+)_{N'_1 \cup (N_2 \cap N'_2)} \in \mathcal{C}$. Consequently, using $(y_n)_{N'_2} \in \mathcal{C}$ and the fact that $N_2 \cap N'_2$ is infinite, it follows that $\mathbf{z}' = (z'_n)_{N' \cup N''} \in \mathcal{C}$.

Now assume that $N_2 \cap N'_2$ is finite, hence $N_2 \cap N'_1$ infinite. Fixing disjoint infinite subsets M_1 and M_2 of $N_2 \cap N'_1$, we define $(z_n^\star)_{N' \cup N''}$ and $(z_n^{\star\star})_{N' \cup N''}$ by

$$z_n^\star = \begin{cases} x_n & \text{if } n \in (N_1 \cup N'_1) \setminus M_1 \\ y_n & \text{if } n \in (N_2 \cap N'_2) \cup M_1 \end{cases} \quad \text{and} \quad z_n^{\star\star} = \begin{cases} x_n & \text{if } n \in N'_1 \setminus M_1 \\ y_n & \text{if } n \in N'_2 \cup M_1. \end{cases}$$

Since $(x_n)_{(N_1 \cup N'_1) \setminus M_1} \in \mathcal{C}$, $(z_n)_{N_1 \cup (N_2 \cap N'_2) \cup M_1} \in \mathcal{C}$ and $N_1 \subseteq (N_1 \cup N'_1) \setminus M_1$ we get $(z_n^\star)_{N' \cup N''} \in \mathcal{C}$, hence $(z_n^\star)_{(N_1 \cap N'_1) \cup N_2} \in \mathcal{C}$ and together with $(y_n)_{N'_2 \cup M_1} \in \mathcal{C}$ and $M_1 \subseteq (N_1 \cap N'_1) \cup N_2$ conseqently $(z_n^{\star\star})_{N' \cup N''} \in \mathcal{C}$, thus $(z_n^{\star\star})_{(N' \cup N'') \setminus M_1} \in \mathcal{C}$. Because of $(x_n)_{N'_1} \in \mathcal{C}$ and $M_2 \subseteq (N' \cup N'') \setminus M_1$ this gives finally $\mathbf{z}' = (z'_n)_{N' \cup N''} \in \mathcal{C}$ which we wanted to prove ∎

Corollary. *The following assertions are true:*

1. *If* $\mathbf{x} \in C$ *and* $\mathbf{y} \leq \mathbf{x}$, *then* $\mathbf{x} \sim \mathbf{y}$.
2. *If for* $\mathbf{x}, \mathbf{y} \in C$ *there exists a* $\mathbf{z}$ *with* $\mathbf{z} \leq \mathbf{x}$ *and* $\mathbf{z} \leq \mathbf{y}$, *then* $\mathbf{x} \sim \mathbf{y}$.

2.4. Let t denote a discrete convergence on $(\mathbf{E}, E)$ and C a discrete pre-Cauchy structure on $\mathbf{E}$ with properties

$$\mathbf{x} \longrightarrow_t x \quad \Longrightarrow \quad \mathbf{x} \in C \tag{6}$$

$$\text{If } \mathbf{x} \longrightarrow_t x \text{ and } \mathbf{y} \in C, \text{ then } \mathbf{x} \sim \mathbf{y} \quad \Longleftrightarrow \quad \mathbf{y} \longrightarrow_t x. \tag{7}$$

Then t and C are called *compatible* and every $\mathbf{x} \in C$ is called in this case a *discrete Cauchy sequence*. Moreover (t, C) is said to be a *discrete Cauchy structure* on $(\mathbf{E}, E)$ and $\big((\mathbf{E}, E), (t, C)\big)$ to be a *discrete Cauchy space*.

In a discrete Cauchy space obviously the implications

$$\mathbf{x} \longrightarrow_t x \text{ and } \mathbf{y} \in C \text{ such that } \mathbf{x} \leq \mathbf{y} \quad \Longrightarrow \quad \mathbf{y} \longrightarrow_t x$$

and moreover

$$\left.\begin{array}{l} \mathbf{x} \longrightarrow_t x \\ \mathbf{y} \longrightarrow_t x \end{array}\right\} \quad \Longrightarrow \quad \mathbf{z} \longrightarrow_t x \text{ for all } \mathbf{z} \in \overline{m}(\mathbf{x}, \mathbf{y}) \tag{8}$$

are fulfilled.

2.5. A discrete Cauchy space $X = \big((\mathbf{E}, E), (t, C)\big)$ is said to be *separated* if the underlying discrete convergence space $\big((\mathbf{E}, E), t\big)$ is separated, and X is said to be *complete* if every of its discrete Cauchy sequences converges discretely.

Theorem 3 [6]. *Let* $X = \big((\mathbf{E}, E), t\big)$ *be a separated discrete convergence space fulfilling property* (8). *Then*

$$C = \bigcup_{x \in E} \{(x_n)_{N'} \mid (x_n)_{N'} \longrightarrow_t x\}$$

is a discrete pre-Cauchy structure on $\mathbf{E}$ *compatible with* t *and the discrete Cauchy space* $X' = \big((\mathbf{E}, E), (t, C)\big)$ *is complete.*

Theorem 4 [5]. *Let* X *be a discrete pre-Cauchy space* $(\mathbf{E}, C)$ *or a separated discrete Cauchy space* $\big((\mathbf{E}, E), (t, C)\big)$. *Moreover let* $E^\wedge = C/{\sim}$ *and* $t^\wedge = \{(\mathbf{x}, x) \mid x \in E^\wedge \text{ and } \mathbf{x} \in x\}$. *Then* $X^\wedge = \big((\mathbf{E}, E^\wedge), (t^\wedge, C)\big)$ *is a complete separated discrete Cauchy space.*

Remark. Let especially X be a separated discrete Cauchy space $\big((\mathbf{E}, E), (t, C)\big)$. For every $n \in \mathbb{N}$ let 1_{E_n} denote the identity mapping on E_n and let $\nu : E \to E^\wedge$ be defined by $\nu(x) = \big\{(x_n)_{N'} \mid (x_n)_{N'} \longrightarrow_t x\big\}$. Then $\mathbf{x} \longrightarrow_t x$ implies $\mathbf{x} \longrightarrow_{t^\wedge} \nu(x)$ and $X^\wedge$ together with $\big((1_{E_1}, 1_{E_2}, \ldots), \nu\big)$ may be considered to be the *completion* of X. This can also be motivated more precisely by category-theoretical considerations (see [4]; also [6]).

3. A special type of (metric) discrete Cauchy spaces

3.1. $E_1, E_2, \ldots$ and E being arbitrary sets and $\mathbf{E}$ as in Subsection 2.1, let us assume that for every $n \in \mathbb{N}$ there is given a mapping $p_n : E \to E_n$ and an injective mapping $q_n : E_n \to E$ with property

(A) $p_n q_n = 1_{E_n}$

which implies that every mapping p_n is surjective. Let a metric d on E be given. Then for every $n \in \mathbb{N}$

$$d_n(x_n, y_n) = d\big(q_n(x_n), q_n(y_n)\big)$$

defines a metric d_n on E_n such that obviously the mapping $q_n : (E_n, d_n) \longrightarrow (E, d)$ is continuous. Let us assume that condition

(B) $\big(d(q_n p_n(x), x)\big)_{\mathbf{N}} \longrightarrow 0$ for every $x \in E$

is fulfilled.

Theorem 5. *The following assertions are true:*

1. $d_n\big(p_n(x), p_n(y)\big) = d(x, y)$ *for all* $x, y \in q_n[E_n]$.

2. $\big(d_n(p_n(x), p_n(y))\big)_{\mathbf{N}} \longrightarrow d(x, y)$ *for all* $x, y \in E$.

Proof. Assertion 1 is obvious. Assertion 2 follows by means of condition (B) from the fact that, for arbitrary $x, y \in E$,

$$d_n\big(p_n(x), p_n(y)\big) = d\big(q_n p_n(x), q_n p_n(y)\big) \leq d\big(q_n p_n(x), x\big) + d(x, y) + d\big(y, q_n p_n(y)\big)$$

and

$$d(x, y) \leq d\big(x, q_n p_n(x)\big) + d_n\big(p_n(x), p_n(y)\big) + d\big(q_n p_n(y), y\big).$$

Thus the theorem is proved ∎

Additionally to conditions (A) and (B) let us assume the following condition

(C) There exists a zero sequence $(\varepsilon_n)_{\mathbf{N}}$ of non-negative reals ε_n such that $d_n\big(p_n(x), p_n(y)\big) \leq d(x, y) + \varepsilon_n$ for every $n \in \mathbb{N}$ and every $x, y \in E$.

If especially $\varepsilon_n = 0$, then the mapping $p_n : (E, d) \longrightarrow (E_n, d_n)$ is continuous.

In the following subsections the notions, notations and assumptions of the underlying subsection are always used without mentioning this explicitely. For simplicity let us use also the notations $p = (p_1, p_2, \ldots)$ and $q = (q_1, q_2, \ldots)$.

3.2. On $(\mathbf{E}, E)$ a discrete convergence $t = t(p, q, d)$ is given by

$$\big((x_n)_{N'}, x\big) \in t \iff \big(d(q_n(x_n), x)\big)_{N'} \longrightarrow 0.$$

Condition (2) is fulfilled, since for every $x \in E$ we have $\big((p_n(x))_{\mathbf{N}}, x\big) \in t$.

Theorem 6. *For $t = t(p,q,d)$ the following assertions are true:*

1. $((x_n)_{N'}, x) \in t$ *if and only if* $\big(d_n(x_n, p_n(x))\big)_{N'} \longrightarrow 0$.

2. $\mathbf{x} \to x$ *and* $\mathbf{x} \to y$ *imply* $x = y$.

3. *Given a discrete sequence* $\mathbf{x}$ *in* $\mathbf{E}$ *and an* $x \in E$ *such that for all* $\mathbf{y} \leq \mathbf{x}$ *there exists a* $\mathbf{z} \leq \mathbf{y}$ *with* $\mathbf{z} \longrightarrow_t x$, *then* $\mathbf{x} \longrightarrow_t x$.

The **Proof** is obvious ∎

Statements 2 and 3 of Theorem 6 mean that $t = t(p,q,d)$ is separated respectively fulfills the so-called *Urysohn property* (see [5]).

3.3. On $\mathbf{E}$ a discrete pre-Cauchy structure $\mathcal{C} = \mathcal{C}(q,d)$ is given by

$$(x_n)_{N'} \in \mathcal{C} \iff \big(q_n(x_n)\big)_{N'} \text{ is a Cauchy sequence in } (E,d).$$

Theorem 7. *For $\mathcal{C} = \mathcal{C}(q,d)$, the equivalence*

$$(x_n)_{N'} \in \mathcal{C} \iff \big(\sup_{m(\geq n)\in N'} d_m(x_m, p_m q_n(x_n))\big)_{n\in N'} \longrightarrow 0$$

is true.

Proof. The part $\Longrightarrow$ of the statement follows easily by means of conditions (A) and (C). Now let the supposition of the part $\Longleftarrow$ be fulfilled. Then for every $\varepsilon > 0$ there exists an $n' \in N'$ such that

$$\sup_{m\in N', m\geq n} d\big(q_m(x_m), q_m p_m q_{n'}(x_{n'})\big) \leq \varepsilon.$$

By means of

$$d\big(q_m(x_m), q_{n'}(x_{n'})\big) \leq d\big(q_m(x_m), q_m p_m q_{n'}(x_{n'})\big) + d\big(q_m p_m q_{n'}(x_{n'}), q_{n'}(x_{n'})\big)$$

and condition (B) we see that there exists an $n^* \in N'$ with $n^* \geq n'$ such that

$$d\big(q_m(x_m), q_{n'}(x_{n'})\big) \leq 2\varepsilon$$

for all $m \in N'$ with $m \geq n^*$, from which it follows that

$$d\big(q_m(x_m), q_n(x_n)\big) \leq 4\varepsilon$$

for all $m, n \in N'$ with $m, n \geq n^*$. This proves also the part $\Longleftarrow$ of the statement ∎

Theorem 8. *For $\mathcal{C} = \mathcal{C}(q,d)$ the following assertions are true:*

1. $(p_n(x))_{\mathbf{N}} \in \mathcal{C}$ *for every* $x \in E$.

2. *If* $\mathbf{x} = (x_n)_{N'}$ *is a discrete sequence in* $\mathbf{E}$ *such that*
   (i) *for every* $\mathbf{y} \leq \mathbf{x}$ *there exists a* $\mathbf{z} \leq \mathbf{y}$ *with* $\mathbf{z} \in \mathcal{C}$
   (ii) *for* $(x_n)_{N^+}, (x_n)_{N^*} \in \mathcal{C}$ *with* N^+, $N^* \leq N'$ *there follows* $(x_n)_{N^+\cup N^*} \in \mathcal{C}$,

then $\mathbf{x} \in \mathcal{C}$.

Proof. Statement 1 follows by means of condition (B). To prove statement 2 let its assumptions for a fixed $\mathbf{x} = (x_n)_{N'}$ be fulfilled and assume $\mathbf{x} \notin \mathcal{C}$. Then there exist an $\varepsilon > 0$, an $N^+ \leq N'$ and for every $n \in N^+$ an $m(n) \in N'$ with $m(n) > n$ such that $n < n'$ always implies $m(n) < m(n')$ and moreover

$$d\big(q_n(x_n), q_{m(n)}(x_{m(n)})\big) \geq \varepsilon \tag{9}$$

for all $n \in N^+$. Because of assumption (i) we may assume that $(x_n)_{N^+} \in \mathcal{C}$ and that there exists an $N^* \leq N^+$ such that $(x_{m(n)})_{n \in N^*} \in \mathcal{C}$. Thus because of assumption (ii) we get $(x_n)_{N^* \cup m[N^*]} \in \mathcal{C}$, which contradicts inequality (9) ∎

Using a notation of [5], statement 2 of Theorem 8 means that $\mathcal{C} = \mathcal{C}(q,d)$ fulfills the *Urysohn property.*

3.4. In this subsection let always be $t = t(p,q,d)$ and $\mathcal{C} = \mathcal{C}(q,d)$.

Theorem 9. *For every* $\mathbf{x} = (x_n)_{N'}$ *and* $\mathbf{y} = (y_n)_{N''}$ *the following assertions are equivalent:*

1. $\mathbf{x},\ \mathbf{y} \in \mathcal{C}$ *and* $\mathbf{x} \sim \mathbf{y}$.

2. $\overline{m}(\mathbf{x},\mathbf{y}) \subseteq \mathcal{C}$.

3. $\big(q_n(x_n)\big)_{N'}$ *and* $\big(q_n(y_n)\big)_{N''}$ *are equivalent Cauchy sequences in* (E,d).

4. *There exist discrete sequences* $(x_n)_{N' \cup N''} \in \mathcal{C}$ *and* $(y_n)_{N' \cup N''} \in \mathcal{C}$ *such that* $\big(d_n(x_n, y_n)\big)_{N' \cup N''} \longrightarrow 0$.

Proof. Since $m(\mathbf{x},\mathbf{y}) \subseteq \mathcal{C}$ entails $\overline{m}(\mathbf{x},\mathbf{y}) \subseteq \mathcal{C}$, the equivalence of assertions 1 - 3 is obvious. Concerning their equivalence to assertion 4 we follow the proof of [5: Theorem 7]. If assertion 4 is fulfilled, then $\big(q_n(x_n)\big)_{N' \cup N''}$ and $\big(q_n(y_n)\big)_{N' \cup N''}$ are equivalent Cauchy sequences in (E,d), hence we have $\mathbf{x} \in \mathcal{C}$, $\mathbf{y} \in \mathcal{C}$ and $\mathbf{x} \sim (x_n)_{N' \cup N''} \sim (y_n)_{N' \cup N''} \sim \mathbf{y}$, which shows that assertions 1 - 3 are fulfilled.

Conversely, if assertions 1 - 3 are fulfilled, then

$$\big(q_n(x_n)\big)_{N' \cup N''} \qquad \text{with } x_n = y_n \text{ for all } n \in N'' \setminus N'$$

and

$$\big(q_n(y_n)\big)_{N' \cup N''} \qquad \text{with } y_n = x_n \text{ for all } n \in N' \setminus N''$$

are equivalent Cauchy sequences in (E,d), hence $(x_n)_{N' \cup N''} \in \mathcal{C}$ and $(y_n)_{N' \cup N''} \in \mathcal{C}$ and

$$\big(d_n(x_n, y_n)\big)_{N' \cup N''} = \big(d(q_n(x_n), q_n(y_n))\big)_{N' \cup N''} \longrightarrow 0.$$

This proves assertion 4 ∎

Corollary 1. *Let* $\mathbf{x} = (x_n)_{N'}$ *and* $\mathbf{y} = (y_n)_{N'}$ *be discrete sequences with* $\mathbf{x} \in \mathcal{C}$. *Then the equivalence*

$$\mathbf{y} \in \mathcal{C} \ \textit{and} \ \mathbf{x} \sim \mathbf{y} \quad \Longleftrightarrow \quad \big(d_n(x_n, y_n)\big)_{N'} \to 0$$

is true.

Corollary 2. *If* $\mathbf{x}$, $\mathbf{y} \in \mathcal{C}$ *and* $\mathbf{x} \sim \mathbf{y}$, *then* $\mathbf{z} \sim \mathbf{z}'$ *for every* $\mathbf{z}, \mathbf{z}' \in \overline{m}(\mathbf{x}, \mathbf{y})$.

Theorem 10. $(t, \mathcal{C})$ *is a discrete Cauchy structure on* $(\mathbf{E}, E)$.

Proof. The validity of property (6) is obvious. To prove property (7), let $\mathbf{x} = (x_n)_{N'} \longrightarrow_t x$ and $\mathbf{y} = (y_n)_{N''} \in \mathcal{C}$ be given. If $\mathbf{y} \longrightarrow_t x$, then $\big(q_n(x_n)\big)_{N'}$ and $\big(q_n(y_n)\big)_{N''}$ are equivalent Cauchy sequences in (E, d), hence from Theorem 9 we know that $\mathbf{x} \sim \mathbf{y}$. If conversely $\mathbf{x} \sim \mathbf{y}$, then Theorem 9 yields that $\big(q_n(x_n)\big)_{N'}$ and $\big(q_n(y_n)\big)_{N''}$ are equivalent Cauchy sequences in (E, d), and by means of $\mathbf{x} \longrightarrow_t x$ thus we get $\mathbf{y} \longrightarrow_t x$ ∎

Using notations of [5], Corollary 1 of Theorem 9 means that $\mathcal{C}$ is *compatible* with $\delta = (d_1, d_2, \ldots)$ and because of Theorem 10 hence $\big((\mathbf{E}, E), (t, \mathcal{C})\big)$ (with $t = t(p, q, d)$ and $\mathcal{C} = \mathcal{C}(q, d)$) - denoted in [5] by $\big((\mathbf{E}, E), (\delta, t, \mathcal{C})\big)$ - is a (special) *metric discrete Cauchy space.*

3.5. Let X denote the metric discrete Cauchy space $\big((\mathbf{E}, E), (t, \mathcal{C})\big)$ with $t = t(p, q, d)$ and $\mathcal{C} = \mathcal{C}(q, d)$.

Theorem 11. X *is complete if and only if* (E, d) *is complete.*

Proof. Let (E, d) be complete and let $\mathbf{x} = (x_n)_{N'}$ denote a discrete Cauchy sequence in X. Then $\big(q_n(x_n)\big)_{N'}$ is a Cauchy sequence in (E, d), hence it converges to a point $x \in E$, and thus because of $\big(d(q_n(x_n), x)\big)_{N'} \longrightarrow 0$ we get $(x_n)_{N'} \longrightarrow_t x$, which proves that X is complete.

Now assume that X is complete and let $(x_n)_{\mathbb{N}}$ be a Cauchy sequence in (E, d). For every $n \in \mathbb{N}$ we have $\big(p_m(x_n)\big)_{m \in \mathbb{N}} \longrightarrow_t x_n$. Let $n_1, m_1 \in \mathbb{N}$ be so that $d(x_k, x_l) \le \frac{1}{2}$ for every $k, l \ge n_1$ and moreover $d\big(q_{m_1}(y_{m_1}), x_{n_1}\big) \le \frac{1}{2}$ for $y_{m_1} = p_{m_1}(x_{n_1})$. Let $n_2, m_2 \in \mathbb{N}$ with $n_2 > n_1$ and $m_2 > m_1$ be so that $d(x_k, x_l) \le \frac{1}{2^2}$ for every $k, l \ge n_2$ and moreover $d\big(q_{m_2}(y_{m_2}), x_{n_2}\big) \le \frac{1}{2^2}$ for $y_{m_2} = p_{m_2}(x_{n_2})$. If we continue analogously, we get a sequence $(y_n)_{N'}$ with $N' = (m_1, m_2, \ldots)$ such that

$$d\big(q_{m_i}(y_{m_i}), q_{m_j}(y_{m_j})\big) \le d\big(q_{m_i}(y_{m_i}), x_{n_i}\big) + d(x_{n_i}, x_{n_j}) + d\big(x_{n_j}, q_{m_j}(y_{m_j})\big) \le \frac{3}{2^i}$$

for every $i \in \mathbb{N}$ and every $j \ge i$. Hence $(y_n)_{N'} \in \mathcal{C}$ and, since X is complete, there exists a point $x \in E$ with $(y_n)_{N'} \longrightarrow_t x$. This proves $\big(d(q_{m_i}(y_{m_i}), x)\big)_{i \in \mathbb{N}} \longrightarrow 0$, which together with $d\big(q_{m_i}(y_{m_i}), x_{n_i}\big) \le \frac{1}{2^i}$ implies $\big(d(x_{n_i}, x)\big)_{i \in \mathbb{N}} \longrightarrow 0$ and therefore $(x_n)_{\mathbb{N}} \to x$. Thus (E, d) is complete ∎

If X is not complete, then its completion may be identified with a complete Cauchy space of the same type as X. To see this, let $(\hat{E}, \hat{d})$ be the completion of (E, d). For every $x \in \hat{E} \setminus E$ let a sequence $(z_n(x))_{\mathbb{N}}$ in E be fixed such that always $\hat{d}(z_n(x), x) \leq \frac{1}{n}$. For every $n \in \mathbb{N}$ define $\hat{p}_n : \hat{E} \to E_n$ by

$$\hat{p}_n(x) = \begin{cases} p_n(x) & \text{if } x \in E \\ p_n z_n(x) & \text{if } x \in \hat{E} \setminus E. \end{cases}$$

Let $\hat{p} = (\hat{p}_1, \hat{p}_2, \ldots)$, and let $\hat{\varepsilon}_n = \varepsilon_n + \frac{2}{n}$ for every $n \in \mathbb{N}$.

Theorem 12. *With respect to $\hat{E}, \hat{d}, \hat{p}, \hat{\varepsilon}_n$ instead of E, d, p, ε_n $(n \in \mathbb{N})$, respectively, the conditions* (A) - (C) *are fulfilled.*

Proof. Condition (A) is obvious.

For every $x \in \hat{E} \setminus E$ and every $n, m \in \mathbb{N}$ with $n \geq m$ we have

$$\begin{aligned} d(q_n \hat{p}_n(x), q_n p_n z_m(x)) &= d_n(p_n z_n(x), p_n z_m(x)) \\ &\leq d(z_n(x), z_m(x)) + \varepsilon_n \\ &\leq \frac{2}{m} + \varepsilon_n \end{aligned}$$

and hence

$$\begin{aligned} \hat{d}(q_n \hat{p}_n(x), x) &\leq d(q_n \hat{p}_n(x), q_n p_n z_m(x)) + d(q_n p_n z_m(x), z_m(x)) + \hat{d}(z_m(x), x) \\ &\leq d(q_n p_n z_m(x), z_m(x)) + \frac{3}{m} + \varepsilon_n. \end{aligned}$$

Fixing m, consequently for all sufficiently great n we get

$$\hat{d}(q_n \hat{p}_n(x), x) \leq \frac{4}{m} + \varepsilon_n$$

by means of which it follows that $(\hat{d}(q_n \hat{p}_n(x), x))_{\mathbb{N}} \longrightarrow 0$. Thus condition (B) is fulfilled.

To prove condition (C), we have to show that with respect to $\hat{E}, \hat{d}, \hat{p}, \hat{\varepsilon}_n$ the inequality in condition (C) is true for every $n \in \mathbb{N}$ and every $x \in E$, $y \in \hat{E} \setminus E$ respectively every x, $y \in \hat{E} \setminus E$. Since the proofs in both cases are similar, we restrict our considerations to the case $x \in E$ and $y \in \hat{E} \setminus E$. For any such elements and $n, m \in \mathbb{N}$ then

$$\begin{aligned} \hat{d}_n(\hat{p}_n(x), \hat{p}_n(y)) &= d_n(p_n(x), p_n z_n(y)) \\ &\leq \hat{d}(x, y) + \hat{d}(y, z_n(y)) + \varepsilon_n \\ &\leq \hat{d}(x, y) + \hat{\varepsilon}_n. \end{aligned}$$

Hence also condition (C) is true ∎

Let $\hat{t}, \hat{C}$ be defined by $\hat{p}, q, \hat{d}$ analogously as t, C are defined by p, q, d. Obviously $t \subseteq \hat{t}$ and $C = \hat{C}$. An immediate consequence of Theorems 11 and 12 is the following.

Theorem 13. $\hat{X} = ((\mathbf{E}, \hat{E}), (\hat{t}, C))$ *is a complete metric discrete Cauchy space.*

Remark. Let us identify every $x \in \hat{E}$ with the equivalence class $[x]$ of all discrete sequences $\mathbf{x}$ in $\hat{X}$ converging discretely with respect to $\hat{t}$ to x. Then $\hat{E} = C/\sim$ and $\hat{t} = \{(\mathbf{x}, x) |\ x \in \hat{E} \text{ and } \mathbf{x} \in x\}$. Hence by this identification, $\hat{X}$ coincides with the completion $X^\wedge$ of X defined in Subsection 2.5.

4. The linear case

Let X be a metric discrete Cauchy space as in Subsection 3.5, where additionally the following properties are true:

1. Every E_n $(n \in \mathbb{N})$ and E are equipped with a linear structure.
2. Every q_n $(n \in \mathbb{N})$ is linear.
3. The metric d is generated by a norm $\|\cdot\|$ on E.

Then especially for every $n \in \mathbb{N}$, the metric d_n is generated by the norm $\|\cdot\|_n$ on E_n given by $\|x_n\|_n = \|q_n(x_n)\|$, and $p_n|_{q_n[E_n]}$ is linear. By means of the triangular inequality there easily follows:

(i) t is linear, i.e. for arbitrary discretely converging discrete sequences $(x_n)_{N'} \longrightarrow_t x$, $(y_n)_{N'} \longrightarrow_t y$ and arbitrary converging sequences $(\alpha_n)_{N'} \to \alpha$, $(\beta_n)_{N'} \to \beta$ of real numbers, $(\alpha_n x_n + \beta_n y_n)_{N'} \longrightarrow_t \alpha x + \beta y$.

(ii) $\mathcal{C}$ is linear, i.e. for arbitrary discrete Cauchy sequences $(x_n)_{N'}$ and $(y_n)_{N'}$ in $\mathbf{E}$ and arbitrary Cauchy sequences $(\alpha_n)_{N'}$ and $(\beta_n)_{N'}$ of real numbers, $(\alpha_n x_n + \beta_n y_n)_{N'} \in \mathcal{C}$.

Example. Let E be the set $C[0,1]$ of all continuous functions on $[0,1]$, and for every $n \in \mathbb{N}$ let E_n be the set of all functions on $S_n = \{\frac{0}{n}, \frac{1}{n}, \ldots \frac{n}{n}\}$. Moreover let $p_n : E \to E_n$ be defined by $p_n(x) = x|_{S_n}$ and $q_n : E_n \to E$ be such that $q_n x_n(t) = x_n(t)$ for every $t \in S_n$ and that $q_n x_n$ is linear on every interval $[\frac{i}{n}, \frac{i+1}{n}]$, $i \in \{0, \ldots, n-1\}$. Let $\|\cdot\|$ be the maximum norm on E. Then conditions (A) - (C) are fulfilled with $\varepsilon_n = 0$ for every $n \in \mathbb{N}$, hence $X = ((\mathbf{E}, E), (t, \mathcal{C}))$ with $t = t(p,q,d)$ and $\mathcal{C} = \mathcal{C}(q,d)$ is a discrete Cauchy space of the considered special type.

5. Discrete Cauchy spaces of mappings

5.1. Let

$$X = ((\mathbf{E}, E), (t, \mathcal{C})) \qquad t = t(p,q,d) \qquad \mathcal{C} = \mathcal{C}(q,d)$$
$$X' = ((\mathbf{E}', E'), (t', \mathcal{C}')) \qquad t' = t'(p', q', d') \qquad \mathcal{C}' = \mathcal{C}'(q', d')$$

(using symbols with $'$ with obvious meaning) be metric discrete Cauchy spaces as in Subsection 3.5. Assume that (E,d) is compact and that for every $n \in \mathbb{N}$ the mappings

$$p_n : (E,d) \longrightarrow (E_n, d_n) \qquad \text{as well as} \qquad p'_n : (E', d') \longrightarrow (E'_n, d'_n)$$

are continuous. For every $n \in \mathbb{N}$ let F_n denote the set of all continuous mappings $f_n : (E_n, d_n) \longrightarrow (E'_n, d'_n)$ and let F be the set of all continuous mappings $f : (E,d) \longrightarrow (E', d')$. Obviously, $f \in F$ implies $p'_n f q_n \in F_n$, and $f_n \in F_n$ implies $q'_n f_n p_n \in F$. For every $n \in \mathbb{N}$ define mappings

$$p_n^\star : F \to F_n \qquad \text{by } p_n^\star(f) = p'_n f q_n$$
$$q_n^\star : F_n \to F \qquad \text{by } q_n^\star(f_n) = q'_n f_n p_n.$$

If $q_n^\star(f_n) = q_n^\star(g_n)$, hence $q'_n f_n p_n = q'_n g_n p_n$, then because of the injectivity of q'_n and the surjectivity of p_n it follows that $f_n = g_n$, which shows that $q_n^\star$ is injective. Let

$$\mathbf{F} = (F_1, F_2, \ldots), \qquad p^\star = (p_1^\star, p_2^\star, \ldots), \qquad q^\star = (q_1^\star, q_2^\star, \ldots).$$

Since for arbitrary $f, g \in F$ the function $d'\big(f(\cdot), g(\cdot)\big)$ is continuous on (E, d),

$$d^\star(f, g) = \sup_{x \in E} d'\big(f(x), g(x)\big)$$

is finite and thus defines a metric $d^\star$ on F.

Theorem 14. *With respect to* $\mathbf{F}, F, p^\star, q^\star, d^\star, \varepsilon'_n$ *instead of* $\mathbf{E}, E, p, q, d, \varepsilon_n$ $(n \in \mathbb{N})$, *respectively, the conditions* (A) - (C) *from Subsection* 3.1 *are fulfilled.*

Proof. For every $n \in \mathbb{N}$ and $f_n \in F_n$,

$$p_n^\star q_n^\star(f_n) = p_n^\star(q'_n f_n p_n) = p'_n q'_n f_n p_n q_n = f_n$$

which proves condition (A).

To prove condition (B) let us at first show that

$$\big(\sup_{x \in E} d(q_n p_n(x), x)\big)_{\mathbb{N}} \longrightarrow 0. \tag{10}$$

If this would not be true, then there would exist an $\varepsilon > 0$, an $N' \in \mathcal{N}$ and a converging sequence $(x_n)_{N'} \longrightarrow x$ in (E, d) such that

$$d\big(q_n p_n(x_n), x_n\big) > \varepsilon \tag{11}$$

for all $n \in N'$. By means of

$$d\big(q_n p_n(x_n), q_n p_n(x)\big) = d_n\big(p_n(x_n), p_n(x)\big) \le d(x_n, x) + \varepsilon_n$$

we get $\big(d(q_n p_n(x_n), q_n p_n(x))\big)_{N'} \longrightarrow 0$ and because of $\big(d(q_n p_n(x), x)\big)_{N'} \longrightarrow 0$ hence $\big(d(q_n p_n(x), x)\big)_{N'} \longrightarrow 0$, which together with $\big(d(x_n, x)\big)_{N'} \longrightarrow 0$ contradicts inequality (11). Hence the convergence (10) is true. Analogously it can be shown that

$$\big(\sup_{x \in E} d'(q'_n p'_n f(x), f(x))\big)_{\mathbb{N}} \longrightarrow 0,$$

and since f is equicontinuous on (E, d), from (10) we get

$$\big(\sup_{x \in E} d'(f q_n p_n(x), f(x))\big)_{\mathbb{N}} \longrightarrow 0.$$

Because of

$$\begin{aligned} d'\big(q'_n p'_n f q_n p_n(x), f(x)\big) &\le d'\big(q'_n p'_n f q_n p_n(x), q'_n p'_n f(x)\big) + d'\big(q'_n p'_n f(x), f(x)\big) \\ &\le d'\big(f q_n p_n(x), f(x)\big) + d'\big(q'_n p'_n f(x), f(x)\big) + \varepsilon'_n \end{aligned}$$

consequently

$$\big(d^\star(q_n^\star p_n^\star(f), f)\big)_{\mathbb{N}} = \big(\sup_{x \in E} d'\big(q'_n p'_n f q_n p_n(x), f(x)\big)\big)_{\mathbb{N}} \longrightarrow 0$$

which proves condition (B).

Since

$$\begin{aligned} d_n^\star\big(p_n^\star(f), p_n^\star(g)\big) &= d^\star\big(q_n^\star p_n^\star(f), q_n^\star p_n^\star(g)\big) \\ &= \sup_{x \in E} d'\big(q'_n p'_n f q_n p_n(x), q'_n p'_n g q_n p_n(x)\big) \\ &\le \sup_{x \in E} d'\big(f q_n p_n(x), g q_n p_n(x)\big) + \varepsilon'_n \\ &\le \sup_{x \in E} d'\big(f(x), g(x)\big) + \varepsilon'_n \\ &= d^\star(f, g) + \varepsilon'_n \end{aligned}$$

also condition (C) is true ∎

Let $t^*, \mathcal{C}^*$ be defined by means of p^*, q^*, d^* analogously as $t, \mathcal{C}$ are defined by means of p, q, d. An immediate consequence of Theorem 14 is the following.

Theorem 15. *$Y = ((\mathbf{F}, F), (t^*, \mathcal{C}^*))$ is a discrete Cauchy space.*

5.2. Let $Y = ((\mathbf{F}, F), (t^*, \mathcal{C}^*))$ be as in Subsection 5.1. By definition

$$\begin{aligned} ((f_n)_{N'}, f) \in t^* \quad &\Longleftrightarrow \quad (d^*(q_n^*(f_n), f))_{N'} \longrightarrow 0 \\ &\Longleftrightarrow \quad (\sup_{x\in E} d'(q_n' f_n p_n(x), f(x)))_{N'} \longrightarrow 0. \end{aligned}$$

Theorem 16. *For every discrete sequence $(f_n)_{N'}$ in $\mathbf{F}$ and every $f \in F$ the equivalence*

$$((f_n)_{N'}, f) \in t^* \quad \Longleftrightarrow \quad (x_n)_{N'} \longrightarrow_t x \text{ implies } (f_n(x_n))_{N'} \longrightarrow_t f(x)$$

is true.

Proof. Let $((f_n)_{N'}, f) \in t^*$ and $(x_n)_{N'} \longrightarrow_t x$. Then

$$(\sup_{x\in E} d'(q_n' f_n p_n(x), f(x)))_{N'} \longrightarrow 0 \qquad \text{and} \qquad (d(q_n(x_n), x))_{N'} \longrightarrow 0$$

from which it follows that

$$(d'(q_n' f_n p_n q_n(x_n), f q_n(x_n)))_{N'} \longrightarrow 0 \qquad \text{and} \qquad (d'(f q_n(x_n), f(x)))_{N'} \longrightarrow 0 .$$

hence

$$(d'(q_n' f_n(x_n), f(x)))_{N'} \longrightarrow 0$$

which means $(f_n(x_n))_{N'} \longrightarrow_t f(x)$. Thus the part $\Longrightarrow$ of the assertion of the theorem is true.

Now assume that $(x_n)_{N'} \longrightarrow_t x$ always implies $(f_n(x_n))_{N'} \longrightarrow_t f(x)$. Then for every $x \in E$, because of $(p_n(x))_{N'} \longrightarrow_t x$ we have

$$(d'(q_n' f_n p_n(x), f(x)))_{N'} \longrightarrow 0.$$

We shall prove that even

$$(\sup_{x\in E} d'(q_n' f_n p_n(x), f(x)))_{N'} \longrightarrow 0. \tag{12}$$

If this would not be true, then there would exist an $\varepsilon > 0$, an $N^+ \leq N'$ and a converging sequence $(x_n)_{N^+} \longrightarrow x$ in (E, d) such that

$$d'(q_n' f_n p_n(x_n), f(x_n)) > \varepsilon \tag{13}$$

for all $n \in N^+$. From $(x_n)_{N^+} \longrightarrow x$ we get $(p_n(x_n))_{N^+} \longrightarrow_t x$ and thus

$$(d'(q_n' f_n p_n(x_n), f(x)))_{N^+} \longrightarrow 0$$

which together with $(d'(f(x_n), f(x)))_{N^+} \longrightarrow 0$ contradicts inequality (13). Hence the convergence (12) is true and thus we have $((f_n)_{N'}, f) \in t^*$. This proves the part $\Longleftarrow$ of the assertion of the theorem ∎

By definition

$$\begin{aligned}(f_n)_{N'} \in \mathcal{C}^\star &\iff \left(q_n^\star(f_n)\right)_{N'} \text{ is a Cauchy sequence in } (\mathbf{F}, d^\star)\\ &\iff \left(\sup_{m(\geq n)\in N'} d^\star(q_m^\star(f_m), q_n^\star(f_n))\right)_{n\in N'} \longrightarrow 0\\ &\iff \left(\sup_{m(\geq n)\in N'} \sup_{x\in E} d'(q_m' f_m p_m(x), q_n' f_n p_n(x))\right)_{n\in N'} \longrightarrow 0.\end{aligned}$$

Theorem 17. *For every discrete sequence $(f_n)_{N'}$ in $\mathbf{F}$ the equivalence*

$$(f_n)_{N'} \in \mathcal{C}^\star \quad\iff\quad (x_n)_{N'} \in \mathcal{C} \text{ implies } \left(f_n(x_n)\right)_{N'} \in \mathcal{C}'$$

is true.

Proof. Let at first $(f_n)_{N'} \in \mathcal{C}^\star$ and $(x_n)_{N'} \in \mathcal{C}$. Since (E,d) is compact and thus complete, from Theorem 11 we know that X is complete. Consequently there exists a point $x \in E$ with $(x_n)_{N'} \longrightarrow_t x$, hence $\left(q_n(x_n)\right)_{N'} \longrightarrow x$. Because of $(f_n)_{N'} \in \mathcal{C}^\star$ we have

$$\left(\sup_{m(\geq n)\in N'} \sup_{y\in E} d'(q_m' f_m p_m(y), q_n' f_n p_n(y))\right)_{n\in N'} \longrightarrow 0$$

and thus for arbitrary $\varepsilon > 0$ there exists an $n^\star \in N'$ such that

$$d'\left(q_m' f_m(x_m), q_{n^\star}' f_{n^\star} p_{n^\star} q_m(x_m)\right) \leq \varepsilon \tag{14}$$

for all $m \in N'$ with $m \geq n^\star$. From $\left(q_n(x_n)\right)_{N'} \longrightarrow x$ we get

$$\left(q_{n^\star}' f_{n^\star} p_{n^\star} q_m(x_m)\right)_{m\in N'} \longrightarrow q_{n^\star}' f_{n^\star} p_{n^\star}(x).$$

Hence there exists an $m^\star \in N'$ such that

$$d'\left(q_{n^\star}' f_{n^\star} p_{n^\star} q_m(x_m), q_{n^\star}' f_{n^\star} p_{n^\star} q_n(x_n)\right) \leq \varepsilon \tag{15}$$

for all $m, n \in N'$ with $m, n \geq m^\star$. Inequalities (14) and (15) imply

$$d'\left(q_m' f_m(x_m), q_n' f_n(x_n)\right) \leq 3\varepsilon$$

for all $m, n \in N'$ with $m, n \geq \max\{m^\star, n^\star\}$. This proves $\left(f_n(x_n)\right)_{N'} \in \mathcal{C}'$ and hence the part $\Longrightarrow$ of the assertion of the theorem.

To prove the part $\Longleftarrow$ without loss of generality we may assume that X' is complete since the set of all discrete Cauchy sequences of a discrete Cauchy space is identical with those of its completion. Let $(x_n)_{N'} \in \mathcal{C}$ always imply $\left(f_n(x_n)\right)_{N'} \in \mathcal{C}'$. If $(f_n)_{N'} \notin \mathcal{C}^\star$, then

$$\left(\sup_{m(\geq n)\in N'} \sup_{x\in E} d'\left(q_m' f_m p_m(x), q_n' f_n p_n(x)\right)\right)_{n\in N'} \longrightarrow 0$$

cannot be fulfilled. Thus there exist an $\varepsilon > 0$, sequences $(k_n)_{\mathbb{N}}, (l_n)_{\mathbb{N}} \in \mathcal{N}$ with $k_n, l_n \in N'$ and $l_n < k_n < l_{n+1}$ for every $n \in \mathbb{N}$, and moreover in E a converging sequence $(x_n)_{\mathbb{N}} \to y$ such that

$$d'\left(q_{k_n}' f_{k_n} p_{k_n}(x_n), q_{l_n}' f_{l_n} p_{l_n}(x_n)\right) > \varepsilon \tag{16}$$

for every $n \in \mathbb{N}$. Let $N^+ = \bigcup_{n\in\mathbb{N}} \{k_n, l_n\}$ and define a sequence $(y_n)_{N^+}$ in E by $y_{k_n} = y_{l_n} = x_n$ for every $n \in \mathbb{N}$. Then obviously $(y_n)_{N^+} \longrightarrow y$, which implies $\left(p_n(y_n)\right)_{N^+} \longrightarrow_t y$, hence $\left(p_n(y_n)\right)_{N^+} \in \mathcal{C}$, thus $\left(f_n p_n(y_n)\right)_{N^+} \in \mathcal{C}'$, and consequently there exists a z in E' such that

$$\left(d'(q_{k_n}' f_{k_n} p_{k_n}(x_n), z)\right)_{n\in\mathbb{N}} \longrightarrow 0 \qquad \text{and} \qquad \left(d'(q_{l_n}' f_{l_n} p_{l_n}(x_n), z)\right)_{n\in\mathbb{N}} \longrightarrow 0$$

which are in contradiction to inequality (16). This proves the part $\Longleftarrow$ of the assertion of the theorem ∎

Theorem 18. *Y is complete if and only if X' is complete.*

Proof. Because of Theorem 11 it suffices to show that (F, d^*) is complete if and only if (E', d') is complete. Assume at first that (F, d^*) is complete and that $(y_n)_{\mathbb{N}}$ is a Cauchy sequence in (E', d'). For every $n \in \mathbb{N}$ define a mapping $f_n : E \to E'$ by $f_n[E] = \{y_n\}$. For every $m, n \in \mathbb{N}$,

$$d^*(f_m, f_n) = \sup_{x \in E} d'\big(f_m(x), f_n(x)\big) = d'(y_m, y_n).$$

This shows that $(f_n)_{\mathbb{N}}$ is a Cauchy sequence in (F, d^*), hence converges to an element $f \in F$. Because of

$$d^*(f_n, f) = \sup_{x \in E} d'\big(y_n, f(x)\big)$$

f is constant on E and for y with $\{y\} = f[E]$ we have $(y_n)_{\mathbb{N}} \longrightarrow y$. Thus (E', d') is complete.

Now assume that (E', d') is complete and $(f_n)_{\mathbb{N}}$ is a Cauchy sequence in (F, d^*). Then

$$\big(\sup_{m(\geq n)} \sup_{x \in E} d'(f_m(x), f_n(x))\big)_{n \in \mathbb{N}} \longrightarrow 0 \tag{17}$$

by means of which we see that $\big(f_n(x)\big)_{\mathbb{N}}$ for every $x \in E$ is a Cauchy sequence in (E', d'), hence converges to an $f(x) \in E'$. From (17) we get

$$\big(\sup_{x \in E} d'(f(x), f_n(x))\big)_{n \in \mathbb{N}} \longrightarrow 0$$

and consequently f turns out to be continuous. Thus (F, d^*) is complete ∎

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