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Compactness of an Integro-Differential Operator of Cauchy-Kovalevskaya Theory

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Abstract. The integral operator connected with the Cauchy-Kovalevskaya initial value problem has been shown to be compact between suitable Frechét spaces.

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1. Statement of the problem

M. Nagumo in a celebrated paper [5] initiated a functional-analytic approach [1)] to the Cauchy-Kovalevskaya problem (method of successive approximations, fixed-point methods). In a space of functions depending holomorphically on t and a spacelike variable z, K. Keller and A. Schneider [4] proved a Cauchy-Kovalevskaya theorem using the Schauder-Tikhonov fixed-point theorem. To prove compactness of the operator in an integral rewriting of the problem, the use of the fact that a bounded sequence of holomorphic functions possesses a locally uniformly convergent subsequence is usually made. Using a weighted supremum norm in a space of continuous functions depending continuously on time t and holomorphically on spacelike variables z, W. Walter [10] proved the classical Cauchy-Kovalevskaya theorem by the contraction-mapping principle. W. Walter as also M. Nagumo require continuity in the variable t rather than holomorhy in t as K. Keller and A. Schneider assumed in [4]. We discuss in this short note the compactness of the operator studied in [10].

The initial value problem

$$\partial_t w = f(t, z, w, \partial_z w) \tag{1}$$

$$w(0, z) = w_0(z) \tag{2}$$

[1)] Concerning historical remarks on the abstract linear and non-linear Cauchy-Kovalevskaya Theorem we refer to the appendix of the Russian translation of L. Nirenberg's book [6]; one can find further references to related problems in [6] and [8]. – A generalized formulation and a simplified proof are given in K. Asano's paper [1].

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with suitably chosen initial data, where

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \qquad (z = x + iy)$$

and $w = w(t,z)$ is a desired complex-valued function, can be rewritten in the integral form

$$w(t,z) = w_0(z) + \int_0^t f(\tau, z, w(\tau,z), \partial_z w(\tau,z))d\tau. \tag{3}$$

In the sequel the operator on the right-hand side of (3) will be referred to as *related (integro-differential) operator.*

2. Main result

Let Ω be a bounded domain in the z-plane, $d(z)$ the distance of $z \in \Omega$ from the boundary $\partial\Omega$, and

$$d(t,z) = d(z) - \frac{t}{\eta} \tag{4}$$

where η will be fixed later (cf. W. Walter's paper [10]). Slightly modifying W. Walter's notations, let $M(\Omega,\eta)$ be the conical domain

$$\{(t,z) : z \in \Omega, \quad 0 \le t < \eta d(z)\}$$

and $\mathcal{H}_*(\Omega,\eta)$ the Banach space of those complex-valued functions $w = w(t,z)$ which are

1. defined and continuous in $M(\Omega,\eta)$,
2. holomorphic in z for fixed t and
3. whose norm $\|w\|_*$ is finite,

where

$$\|w\|_* = \sup_{M(\Omega,\eta)} |w(t,z)| d^p(t,z)$$

and p is a fixed positive number [2)]. W. Walter [10] proved, under suitable conditions on the initial data, that the related operator (3) is a continuous operator mapping $\mathcal{H}_*(\Omega,\eta)$ into itself for every choice of η. He further established the existence of a number $\eta_0 > 0$ such that if $\eta < \eta_0$, then the related operator is *contractive.* The number η_0 depends on the data of the initial value problem (1)-(2); its explicit representation will not be needed in the following.

The related operator mapping $\mathcal{H}_*(\Omega,\eta)$ into itself is not, in general, compact as the following example shows:

2) Note that $w(t,z)$ is not necessarily bounded if the $*$-norm is finite.

Let Ω be the unit disk $\{z : |z| < 1\}$ and $p = 1$. Then the function w defined by

$$w(t,z) = \frac{1}{1-z-t} - \frac{1}{1-z} \tag{5}$$

does not belong to $\mathcal{H}_*(\Omega,\eta)$ for any $\eta > 1$ because of its singularitity at $t = 1$. On the other hand, the function under consideration satisfies the differential equation

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z} + \frac{1}{t(1-z)^2}$$

and, therefore, equation (3) has the form

$$w(t,z) = \int_0^t \frac{\partial w}{\partial z}(\tau,z)d\tau + \frac{t}{(1-z)^2}. \tag{6}$$

In view of Holmgren's theorem the corresponding homogeneous equation obtained from (6) has the trivial solution $w_0 = w_0(t,z) \equiv 0$, only. An easy calculation shows that

$$\left\| \frac{t}{(1-z)^2} \right\|_* \leq \frac{1}{4}\eta$$

for each η and, therefore, the inhomogeneous part of equation (6) belongs to $\mathcal{H}_*(\Omega,\eta)$ irrespective of the choice of η. Consequently, assuming the compactness of the operator on the right-hand side of (6), in view of the alternative of Fredholm equation (6) had to be solvable in each $\mathcal{H}_*(\Omega,\eta)$. This, however, is a contradiction for $\eta > 1$.

Define $K_n \subset M(\Omega,\eta)$ by

$$K_n = \left\{ (t,z) : d(t,z) \geq \frac{1}{n} \right\} \qquad (n \geq 1).$$

Then $K_n \subset K_{n+1}$ $(n \geq 1)$ and each K_n is compact. Moreover,

$$M(\Omega,\eta) = \cup_n K_n.$$

We next introduce the topological space $\mathcal{H}_{loc}(\Omega,\eta)$ of complex-valued functions $w = w(t,z)$ continuous in (t,z) and holomorphic in z equipped with a family of semi-norms

$$\|w\|_{*,n} = \sup_{K_n} |w(t,z)| d^p(t,z)$$

where p denotes a fixed positive integer. Then $\mathcal{H}_{loc}(\Omega,\eta)$ is a Fréchet space.

Observe that for $t = \tau$ held fixed, the distance between an arbitrary point $(\tau,z) \in K_n$ and (τ,ζ) belonging to the lateral surface of K_{n+1} can be estimated by

$$|z-\zeta| \geq d(z) - d(\zeta) = d(\tau,z) - d(\tau,\zeta) \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

We next state the main theorem of this note and indicate its proof. The details of the proof follow on rather not too unfamiliar lines and have, therefore, not been provided.

Theorem. *For every choice of η, the related operator in an integral rewriting of the Cauchy-Kovalevskaya problem is compact and maps $\mathcal{H}_*(\Omega,\eta)$ into $\mathcal{H}_{loc}(\Omega,\eta)$.*

Proof. Let the sequence $\{w_m\}_{m\geq 1} \in \mathcal{H}_*(\Omega,\eta)$ be bounded, with $\|w_m\|_* \leq C$ $(m \geq 1)$. For $(t,z) \in K_n$, $|w_m(t,z)| \leq Cn^p$ $(m \geq 1)$. Fix $t = \tau$ and regard $w_m(\tau, z)$ as functions in z in the intersection of the plane $t = \tau$ with the above closed subcone K_n. Since this intersection is a closed subset of the intersection of the plane $t = \tau$ and K_{n+1} and has a positive distance from its boundary, it follows, using the Cauchy integral formula and Nagumo's lemma (see [10]) that w_m and $\partial_z w_m$ $(m \geq 1)$ are equi-continuous on K_n. Therefore, the integrands of the related operator applied to $w_m(t,z)$ are locally equi-continuous in z, provided f satisfies a Lipschitz condition with respect to w and $\partial_z w$.

Now regard any two points (t_1, z_1) and (t_2, z_2) in K_n with $t_2 \geq t_1$. Denote the image of w_n under the related operator by W_n. Then one has

$$\begin{aligned} &W_m(t_2,z_2) - W_m(t_1,z_1) \\ &\qquad = \big(W_m(t_2,z_2) - W_m(t_1,z_2)\big) + \big(W_m(t_1,z_2) - W_m(t_1,z_1)\big). \end{aligned}$$

The second difference on the right-hand side is locally equi-continuous in z because the integrand has this property as proved above. Moreover, the first difference is equi-continuous in t in view of the uniform boundedness of the integrand (note that the length of the integration interval equals $t_2 - t_1$). Summarizing these arguments, W_m turn out to be locally equi-continuous in (t,z).

Applying the Arzela-Ascoli theorem to $\{W_m\}_{m\geq 1}$ in K_1, there exists a subsequence which converges uniformly in K_1. Applying the Arzela-Ascoli theorem to the subsequence so obtained in K_2, we select a subsequence of the subsequence converging uniformly in K_2, and so on. The familiar diagonalization process yields a subsequence of $\{W_m\}_{m\geq 1}$ which converges locally uniformly. This completes the proof ∎

3. Approximations in $\mathcal{H}_*(\Omega,\eta)$ and $\mathcal{H}_{loc}(\Omega,\eta)$

Denote the related operator (3) by $\mathcal{A}$. As usual, successive approximations are defined by

$$w_{n+1} = \mathcal{A}w_n = \mathcal{A}^{n+1}w_0 \qquad (n \geq 0). \tag{7}$$

It is well-known that successive approximations may converge without the operator being contractive (for instance, in case of ordinary differential equations).

We shall, in the sequel, discuss the convergence of successive approximations both in $\mathcal{H}_*(\Omega,\eta)$ and $\mathcal{H}_{loc}(\Omega,\eta)$ under additional assumptions on the approximants. Observe that if the initial function w_0 belongs to $\mathcal{H}_*(\Omega,\eta)$, then all w_n belong to $\mathcal{H}_*(\Omega,\eta)$ as $\mathcal{A}$ maps $\mathcal{H}_*(\Omega,\eta)$ into itself. Notice, further, that the starting element of the successive approximations is not necessarily identical with the initial function (2).

Suppose, first, that the sequence $\{w_n\}_{n\geq 0}$ of successive approximations is bounded in $\mathcal{H}_*(\Omega,\eta)$. In view of the compactness of $\mathcal{A}$ (see Main Theorem) there exists a subsequence which is a Cauchy sequence in $\mathcal{H}_{loc}(\Omega,\eta)$. Assume, in addition, that the whole sequence is a Cauchy sequence in $\mathcal{H}_{loc}(\Omega,\eta)$. Then, on taking limits as $n\to\infty$ in (7) leads to $w_* = \mathcal{A}w_*$ for the limit function w_* showing that $w_*\in\mathcal{H}_{loc}(\Omega,\eta)$ is a solution of the initial value problem (1)-(2).

Suppose, finally, that $\{w_n\}_{m\geq 1}$ is a Cauchy sequence in $\mathcal{H}_*(\Omega,\eta)$. Since $\mathcal{A}$ is a bounded operator mapping $\mathcal{H}_*(\Omega,\eta)$ into itself, it follows on taking limits as $n\to\infty$, in (7) that $w_* = \mathcal{A}w_*$, i.e. the limit function $w_*\in\mathcal{H}_*(\Omega,\eta)$ is a solution of the initial value problem (1)-(2).

The behaviour of the approximations w_n $(n\geq 0)$ can thus be summarized as follows.

Corollary. *Let $\mathcal{H}_*(\Omega,\eta)$ and $\mathcal{H}_{loc}(\Omega,\eta)$ be defined as in Section 2. Let $\mathcal{A}$ be the operator related to the initial value problem (1) – (2) and defined by the right-hand side of (3). The successive approximations*

$$w_{n+1}(t,z) = \mathcal{A}w_n(t,z) = w_0(z) + \int_0^t f(\tau,z,w_n(\tau,z),\partial_z w_n(\tau,z))d\tau$$

behave in $\mathcal{H}_*(\Omega,\eta)$ and $\mathcal{H}_{loc}(\Omega,\eta)$ as follows:

1. *If $\{w_n\}_{n\geq 1}$ is a bounded sequence in $\mathcal{H}_*(\Omega,\eta)$, then some of its subsequences is a Cauchy sequence in $\mathcal{H}_{loc}(\Omega,\eta)$.*

2. *If $\{w_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}_{loc}(\Omega,\eta)$, then its limit solves the initial value problem (1) – (2) and belongs to $\mathcal{H}_{loc}(\Omega,\eta)$.*

3. *If $\{w_n\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{H}_*(\Omega,\eta)$, then its limit is a solution of the initial value problem (1) – (2) and belongs to $\mathcal{H}_*(\Omega,\eta)$.*

Remark. The following example shows that the limit function $w_*\in\mathcal{H}_*(\Omega,\eta)$, whereas successive approximations w_n fail to converge in $\mathcal{H}_*(\Omega,\eta)$. Consider again the initial value problem

$$\partial_t w = \partial_z w + \frac{1}{(1-z)^2},\quad w(0,z)\equiv 0$$

whose solution is given by (5). Here $\Omega = \{z : |z|<1\}, \eta = 1, 0\leq t<1-|z|$. The successive approximations are

$$w_n(t,z) = \frac{1-\left(\frac{t}{1-z}\right)^{n+1}}{1-z-t} - \frac{1}{1-z}\qquad (n\geq 0)$$

showing that $\|w_n\|_*\leq 3$ $(n\geq 0)$. Moreover,

$$|w_n(t,z)-w(t,z)|d(t,z) = \left|\frac{t}{1-z}\right|^{n+1}\cdot\frac{1-|z|-t}{|1-z-t|}\tag{8}$$

implying $\|w_n - w\|_* = 1$. The sequence $\{w_n\}_{n\geq 0}$ is, therefore, not convergent in $\mathcal{H}_*(\Omega,\eta)$. Relation (8) shows, however, that it is locally uniformly convergent in accordance with the corollary.

4. Concluding remark

W. Walter's approach to the Cauchy-Kovalevskaya problem can be modified in such a manner that initial value problems with *generalized analytic* initial functions can be solved as well. This has been done [3)] in [7] (cf. also [8]; for generalized analytic functions see L. Bers [2] and I. N. Vekua [9]). W. Walter's approach is also applicable to initial value problems in associated spaces: Suppose the initial function belongs to a space which is associated with the right-hand side of the differential equation (1). If the right-hand side of the differential equation (1) maps the associated space into itself and, further, the elements of the associated space permit a suitable interior estimate, the initial value problem can then also be reduced to a fixed-point problem for the related operator (see [3]). The considerations of the present paper can be carried out in these cases as well.

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[3)] Since the maximum modulus principle is not true for generalized analytic functions, in general, an analogous *-norm has to be defined via an exhaustion of Ω.