Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 3, 565–578

A Continuation Method for Weakly Condensing Operators

D. O'Regan

Abstract. We present a continuation result for weakly condensing operators between Banach spaces. There are given also a new fixed point result being in the spirit of Schauder's fixed point theorem and some applications to nonlinear operator equations.

Keywords: *Fixed points, continuation theory, measures of weak non-compactness*

AMS subject classification: 47 H 10

1. Introduction

This paper presents a general continuation theory for weakly condensing operators between Banach spaces. This extends some results of the author [16] which initially were motivated by the papers of Banas and Rivero [2], De Blasi [6] and Emmanuele [8]. We also present in Section 2 a new fixed point theorem for weakly condensing operators defined on a closed, bounded, convex subset of a separable, reflexive Banach space. This fixed point theorem is used in Section 3 to establish very general existence principles for nonlinear operator equations. In addition we show, in the case of second order boundary value problems, that the theory developed (in this paper) for weakly continuous operators can lead to the same conclusions as the theory of compact (strong) operators. In Section 4 we present a coincidence theory for weakly condensing operators..

For the remainder of this section we gather together some preliminaries that will be needed in the following sections. Let Ω_E be the bounded subsets of a Banach space E and let K^w be the family of all weakly compact subsets of E. Also let B be the unit ball of E. The DeBlasi [2, 6, 8] measure of weak non-compactness is the map $w : \Omega_E \to [0, \infty)$ defined by

$$w(X) = \inf \Big\{ t > 0 : \text{there exists } Y \in K^w \text{ with } X \subseteq Y + tB \Big\}$$

where $X \in \Omega_E$ (for other measures of weak non-compactness see [2]). For convenience we recall some properties of w. Let $S, T \in \Omega_E$. Then:

(i) $S \subseteq T$ implies $w(S) \le w(T)$.

(ii) $w(S) = 0$ if and only if $\overline{S^w} \in K^w$ where $\overline{S^w}$ is the weak closure of S in E.

D. O'Regan: Univ. Coll. Galway, Dept. Math., Galway, Ireland

ISSN 0232-2064 / \$ 2.50

(iii) $w(\overline{S^w}) = w(S)$.

(iv) $w(S \cup T) = \max\{w(S), w(T)\}$.

(v) $w(rS) = rw(S)$ for all $r > 0$.

(vi) $w(co(S)) = w(S)$.

(vii) $w(S + T) \leq w(S) + w(T)$.

Suppose $F : Y \subseteq E \to E$ maps bounded sets into bounded sets and is weakly continuous. We call F a

a) *w-Lipschitzian map* if there exists a constant $k \geq 0$ with $w(F(X)) \leq k\, w(X)$ for all bounded sets $X \subseteq Y$;

b) *w-condensing map* if F is w-Lipschitzian with $k = 1$ and $w(F(X)) < w(X)$ for all bounded sets X with $w(X) \neq 0$.

Theorem 1.1 (Emmanuele) [8]. *Let E be a Banach space, X a non-empty, bounded, closed, convex subset of E, and $F : X \to X$ a w-condensing map. Then F has a fixed point.*

Also the following results will be used in this paper.

Theorem 1.2 (see [14: p. 147]). *Every topological Hausdorff linear space is a Tychonoff space ($T_{3\frac{1}{2}}$).*

Theorem 1.3 (see [9: p. 124]). *If A is a compact subset of a Tychonoff space X, then for every closed set B disjoint from A there exists a continuous function $\mu : X \to [0,1]$ such that $\mu(x) = 1$ for $x \in A$ and $\mu(x) = 0$ for $x \in B$.*

Theorem 1.4 (see [13: p. 368]). *Let E be a Banach space whose dual space $E^\star$ is separable and let $A \subseteq E$ be weakly compact. Then there exists a weakly continuous retraction onto A.*

Theorem 1.5 (see [3: p. 126]). *A convex subset of a normed space is closed if and only if it is weakly closed.*

Theorem 1.6 (see [7: p. 425]). *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Theorem 1.7 (see [10: p. 65]). *In a Banach space, a bounded weakly closed set A is weakly compact if and only if for every weakly closed B which is disjoint from A, $d(A,B) = \inf\{\|a - b\| : a \in A,\, b \in B\} > 0$.*

Theorem 1.8 (Eberlein and Šmulian; see [7: p. 430]). *Suppose K is weakly closed in a Banach space E. Then the following assertions are equivalent:*

(i) *K is weakly compact.*

(ii) *K is weakly sequentially compact, i.e. any sequence in K has a subsequence which converges weakly.*

Theorem 1.9 (Krein and Šmulian; see [7: p. 434]). *If E is a Banach space and Q is a weakly compact subset of E, then $\overline{co}(Q)$ is weakly compact.*

2. Fixed point theory

In this section we present a continuation theory for w-condensing operators. The results were motivated by ideas in [12, 16]. Let E be a Banach space and let Q and C be closed, bounded, convex subsets of E with $Q \subseteq C$. Now let $X \subseteq Q$ and $A \subseteq X$ with A weakly closed in X and X weakly closed in Q.

Definition 2.1. We let $P_A(X,C)$ denote the set of all w-condensing mappings $F: X \to C$ such that F is fixed point free on A.

We call $N : X \times [0,1] \to C$ a *w-condensing mapping* if N is weakly continuous, $w(N(Y)) \leq w(\pi Y)$ for all sets $Y \subseteq X \times [0,1]$ and $w(N(Z)) < w(\pi Z)$ for all sets $Z \subseteq X \times [0,1]$ with $w(\pi Z) \neq 0$ where $\pi : X \times [0,1] \to X$ is the natural projection.

Definition 2.2. A map $F \in P_A(X,C)$ is *essential* if every map in $P_A(X,C)$ which agrees with F on A has a fixed point. Otherwise F is *inessential*, i.e. there exists a fixed point free map $G \in P_A(X,C)$ with $G = F$ on A.

Definition 2.3. Two maps $F, G \in P_A(X,C)$ are *homotopic* in $P_A(X,C)$ written $F \cong G$ in $P_A(X,C)$ (notice $\cong$ is an equivalence relation in $P_A(X,C)$) if there is a w-condensing mapping $N : X \times [0,1] \to C$ with $N_t(u) = N(u,t) : X \to C$ belonging to $P_A(X,C)$ for each $t \in [0,1]$ and $N_0 = F$ as well as $N_1 = G$.

Theorem 2.1. *Let C, X, A, Q, E be as above with $F \in P_A(X,C)$. Then the following assertions are equivalent:*

(i) *F is inessential.*

(ii) *There is a fixed point free $G \in P_A(X,C)$ such that $F \cong G$ in $P_A(X,C)$.*

Proof. We first show that assertion (i) implies assertion (ii). Let $G \in P_A(X,C)$ be a fixed point free map with $G = F$ on A. Define $N : X \times [0,1] \to C$ by

$$N(x,t) = tG(x) + (1-t)F(x).$$

Remark that (E,w), the space E endowed with the weak topology, is a locally convex linear topological space (in particular a Hausdorff space). Now $N : X \times [0,1] \to C$ is weakly continuous. To see this let $(x_\alpha, \lambda_\alpha)$ be a net in $X \times [0,1]$ with $x_\alpha \rightharpoonup x$ and $\lambda_\alpha \to \lambda$ where $\rightharpoonup$ denotes weak convergence. Then

$$N(x_\alpha, \lambda_\alpha) = \lambda_\alpha G(x_\alpha) + (1-\lambda_\alpha)F(x_\alpha) \rightharpoonup \lambda G(x) + (1-\lambda)F(x) = N(x,\lambda)$$

so N is Moore-Smith sequentially weakly continuous and consequently $N : X \times [0,1] \to C$ is weakly continuous. Also N is a w-condensing map. To see this let Y be a subset of $X \times [0,1]$ with $w(\pi Y) \neq 0$. Then

$$N(Y) \subseteq \operatorname{co}\big(G(\pi Y) \cup F(\pi Y)\big)$$

since if $(x,t) \in Y$, then $N(x,t) = tG(x) + (1-t)F(x) \subseteq \operatorname{co}\big(G(\pi Y) \cup F(\pi Y)\big)$. This together with the properties of w implies

$$\begin{aligned} w(N(Y)) &\leq w\big(\operatorname{co}\,(G(\pi Y) \cup F(\pi Y))\big) = w\big(G(\pi Y) \cup F(\pi Y)\big) \\ &= \max\big\{w(G(\pi Y)),\, w(F(\pi Y))\big\} < \max\big\{w(\pi Y), w(\pi Y)\big\} \\ &= w(\pi Y). \end{aligned}$$

Thus $N : X \times [0,1] \to C$ is a w-condensing map. Also since $F = G$ on A and G is fixed point free on A we have for $x \in A$ that

$$N_t(x) = tG(x) + (1-t)F(x) = G(x) \neq x,$$

so N_t is fixed point free on A for each $t \in [0,1]$. It remains to show $N_t \in P_A(X,C)$ for each $t \in [0,1]$. Fix $t \in [0,1]$ and let Z be a bounded subset of X with $w(Z) \neq 0$. Then

$$w(N_t(Z)) = w(N(Z \times \{t\})) < w(\pi(Z \times \{t\})) = w(Z)$$

since $\pi(Z \times \{t\}) = Z$. Thus for each $t \in [0,1]$ we have $N_t \in P_A(X,C)$. Finally $N_0 = F$ and $N_1 = G$ so $F \cong G$ in $P_A(X,C)$.

We next show that assertion (ii) implies assertion (i). Let $N : X \times [0,1] \to C$ be the w-condensing mapping from $G \in P_A(X,C)$ to F with $N_0 = G$ and $N_1 = F$. In particular N_t is fixed point free on A for each $t \in [0,1]$. Let

$$B = \Big\{ x \in X : x = N(x,t) \text{ for some } t \in [0,1] \Big\}.$$

If B is empty, then in particular $F = N_1$ has no fixed points and so F is inessential. So it remains to consider the case when B is non-empty. First note $B \cap A = \emptyset$. Also B is weakly closed. To see this let $x \in \overline{B^w}$. Then there is a Moore-Smith sequence (x_α) (i.e. a net) in B (i.e. $x_\alpha = N(x_\alpha, t_\alpha)$) which converges to x (i.e. $x_\alpha \rightharpoonup x$). Without loss of generality assume t_α converges to $t \in [0,1]$. Then since $N : X \times [0,1] \to C$ is weakly continuous it follows that $x = N(x,t)$ and thus B is weakly closed. Next we claim that B is weakly compact. If $w(B) \neq 0$, then since $B \subseteq N(B \times [0,1])$ we have

$$w(B) \leq w(N(B \times [0,1])) < w(\pi(B \times [0,1])) = w(B)$$

since $\pi(B \times [0,1]) = B$. This contradiction implies B is weakly compact.

Now $E = (E,w)$, the space E endowed with the weak topology, is a locally convex Hausdorff topological space. Thus E (and hence X, with subspace topology) is a Tychonoff space by Theorem 1.2. This together with theorem 1.3 implies that there is a continuous (weakly) function $\mu : X \to [0,1]$ with $\mu(A) = 1$ and $\mu(B) = 0$. Define $J : X \to C$ by $J(x) = N(x, \mu(x))$. Now J is weakly continuous. To see this let (x_α) be a net in X with $x_\alpha \rightharpoonup x$. Then $\mu(x_\alpha) \rightharpoonup \mu(x)$ and so $\mu(x_\alpha) \to \mu(x)$ since strong and weak convergence are the same in finite-dimensional spaces. Consequently since $N : X \times [0,1] \to C$ is weakly continuous we have that $J(x_\alpha) = N(x_\alpha, \mu(x_\alpha)) \rightharpoonup N(x, \mu(x)) = J(x)$ so J is weakly continuous. We <u>claim</u> that $J : X \to C$ is a w-condensing fixed point free map with $J = F$ on A. If this is true, then $J \in P_A(X,C)$ and J is a fixed point free map which agrees with F on A. Consequently F is inessential and we are finished.

It remains to prove the claim. J is fixed point free since $J(x) = x$ means $N(x,\mu(x)) = x$ which implies $x \in B$ and so $\mu(x) = 0$ (i.e. $N(x,0) = x$), which is a contradiction since $N(x,0) = G(x)$ is fixed point free. To see that $J = F$ on A notice if $x \in A$, then $\mu(x) = 1$ and so $J(x) = N(x,\mu(x)) = N(x,1) = F(x)$. It remains to show J is a w-condensing map. Let Ω be a bounded subset of X with $w(\Omega) \neq 0$. Now $\Omega^\star = \{ (x,\mu(x)) : x \in \Omega \} \subseteq X \times [0,1]$. Then since $J(\Omega) = N(\Omega^\star)$ and $\pi(\Omega^\star) = \Omega$ we have

$$w(J(\Omega)) = w(N(\Omega^\star)) < w(\pi(\Omega^\star)) = w(\Omega).$$

Thus J is a w-condensing map ∎

Theorem 2.2. *Let C, X, A, Q, E be as above. Suppose F and G are two maps in $P_A(X,C)$ such that $F \cong G$ in $P_A(X,C)$. Then F is essential if and only if G is essential.*

Proof. If F is inessential, then Theorem 2.1 guarantees a fixed point free map $T \in P_A(X,C)$ with $F \cong T$ in $P_A(X,C)$. Thus $G \cong T$ in $P_A(X,C)$ and so G is inessential by Theorem 2.1. Symmetry will now imply that F is inessential if and only if G is inessential ∎

Theorem 2.3. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition let U be a weakly open subset of Q with $u_0 \in U$. Then the constant map $F(\overline{U^w}) = u_0$ is essential in $P_{\partial_Q U}(\overline{U^w}, C)$, where $\overline{U^w}$ denotes the weak closure of U in Q and $\partial_Q U$ the weak boundary of U in Q.*

Proof. Let $G : \overline{U^w} \to C$ be any w-condensing map with $G|_{\partial_Q U} = F|_{\partial_Q U} = u_0$. Define

$$I(x) = \begin{cases} G(x) & \text{for } x \in \overline{U^w} \\ u_0 & \text{for } x \in C/\overline{U^w}. \end{cases}$$

It is easy to see that $I : C \to C$ is weakly continuous. In fact $I : C \to C$ is a w-condensing map. To see this let Ω be a bounded subset of C with $w(\Omega) \neq 0$. Then since $I(\Omega) \subseteq G(\Omega \cap \overline{U^w}) \cup \{0\}$ we have

$$w(I(\Omega)) \leq w\big(G(\Omega \cap \overline{U^w}) \cup \{0\}\big) \leq w\big(G(\Omega \cap \overline{U^w})\big) \leq w(G(\Omega)) < w(\Omega).$$

Thus $I : C \to C$ is a w-condensing map. Emmanuele's fixed point theorem (Theorem 1.1) implies that I has a fixed point, say, y in C. In addition since $I(x) = u_0 \in U$ for $x \in C/\overline{U^w}$ we have $y \in \overline{U^w}$. Thus $y = I(y) = G(y)$ and since $G|_{\partial_Q U} = u_0$ we have $y \in U$. Hence G has a fixed point $y \in U$ so F is essential ∎

Theorems 2.2 and 2.3 immediately yields the following nonlinear alternative of Leray-Schauder type which was proved in [16] by different methods.

Theorem 2.4. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition let U be a weakly open subset of Q, with $p \in U$ and $F : \overline{U^w} \to C$ a w-condensing map. Then either*

(A1) *F has a fixed point*

or

(A2) *there is a point $u \in \partial_Q U$ and $\lambda \in (0,1)$ with $u = \lambda F u + (1-\lambda)p$.*

Proof. We assume $F|_{\partial_Q U}$ is fixed point free for otherwise $(A1)$ is satisfied. Let $G : \overline{U^w} \to C$ be the constant map $u \mapsto p$ and consider the homotopy $N : \overline{U^w} \times [0,1] \to C$ joining G and F given by $N(u,t) = tF(u) + (1-t)p$. Now N is a w-condensing map since if Ω is a subset of $\overline{U^w} \times [0,1]$ with $w(\pi\,\Omega) \neq 0$, then

$$N(\Omega) \subseteq \operatorname{co}(F(\pi\,\Omega) \cup \{p\})$$

implies

$$w(N(\Omega)) \leq w\big(\operatorname{co}(F(\pi\,\Omega) \cup \{p\})\big) = w\big(F(\pi\,\Omega) \cup \{p\}\big) = w(F(\pi\,\Omega)) < w(\pi\,\Omega).$$

Now either N_t is fixed point free on $\partial_Q U$ for each $t \in [0,1]$ or it is not. If N_t is fixed point free on $\partial_Q U$ for each $t \in [0,1]$, then Theorems 2.2 and 2.3 imply that F must have a fixed point so $(A1)$ occurs. If N_t is not fixed point free on $\partial_Q U$ for each $t \in [0,1]$, then there exists $u \in \partial_Q U$ with $u = \lambda Fu + (1-\lambda)p$ for some λ with $0 \le \lambda \le 1$. Now $\lambda \neq 0$ since $p \in U$ and $\lambda \neq 1$ since $F|_{\partial_Q U}$ was assumed to be fixed point free. Hence $(A2)$ occurs ∎

Next we present a new fixed point result for weakly continuous maps in separable, reflexive Banach spaces.

Theorem 2.5. *Let $E = (E, \|\cdot\|)$ be a separable and reflexive Banach space, let Q and C be closed, bounded, convex subsets of E with $Q \subseteq C$ and $0 \in Q$ and let $F : Q \to C$ be a weakly continuous map. In addition suppose the following:*

(H) *For any $\Omega_\epsilon = \{x \in E : d(x,Q) \le \epsilon\}$ $(\epsilon > 0)$, if $\{(x_j,\lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial_{\Omega_\epsilon} Q$ and $\lambda_j \to \lambda$, and if $x = \lambda F(x)$ with $0 \le \lambda < 1$, then $\lambda_j F(x_j) \in Q$ for j sufficiently large where $d(x,y) = \|x-y\|$, $(x,y \in E)$ and $\partial_{\Omega_\epsilon} Q$ is the weak boundary of Q relative to Ω_ϵ (i.e. $\overline{\Omega_\epsilon/Q}^w \cap Q$).*

Then F has a fixed point in Q.

Remark. A special case of assumption (H) is the following condition which is all we need for the applications in Section 3:

(H)* If $\{(x_j,\lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x$ and $\lambda_j \to \lambda$ and if $x = \lambda F(x)$, $0 \le \lambda < 1$, then $\lambda_j F(x_j) \in Q$ for j sufficiently large.

Proof of Theorem 2.5. Let $r : E \to Q$ be the weakly continuous retraction guaranteed by Theorem 1.4 (notice (i) that E^* is separable since E is separable and reflexive and (ii) that Q is weakly compact by Theorems 1.5 and 1.6). Consider

$$B = \{x \in E : x = Fr(x)\}.$$

We claim $B \neq \emptyset$. To see this look at rF. Notice $rF : Q \to Q$ is weakly continuous. Also rF is a w-condensing map. To see this first notice $F(Q)$ is relatively weakly compact by Theorem 1.6 since $F(Q) \subseteq C$ and C is bounded. This together with the fact that $r : E \to Q$ is weakly continuous implies $r(\overline{F(Q)}^w)$ is a weakly compact set so $r(F(Q))$ is relatively weakly compact. Thus $w(r(F(Q))) = 0$ so $rF : Q \to Q$ is a w-condensing map. Emmanuele's fixed point theorem (Theorem 1.1) implies that there exists $y \in Q$ with $y = rF(y)$. Let $z = F(y)$ and we have

$$Fr(z) = Fr(F(y)) = F(y) = z,$$

so $z \in B$ and $B \neq \emptyset$. In addition the weak continuity of Fr implies that B is weakly closed. Also B is weakly compact since $B \subseteq Fr(B) \subseteq F(Q)$ and so $w(B) \le w(F(Q)) = 0$.

We now show $B \cap Q \neq \emptyset$. To do this we argue by contradiction. Suppose $B \cap Q = \emptyset$. Then since Q is a weakly closed set (since Q is closed and convex) and B is a weakly compact set we have from Theorem 1.7 that

$$d(B,Q) = \inf\{\|x-y\| : x \in B, y \in Q\} > 0.$$

Thus there exists $\epsilon > 0$ with $\Omega_\epsilon \cap B = \emptyset$ where $\Omega_\epsilon = \{x \in E : d(x,Q) \le \epsilon\}$ and Ω_ϵ is weakly compact from Theorem 1.6 (since Ω_ϵ is a closed, convex, bounded subset of E). Also the weak topology on Ω_ϵ is metrizable [3: p. 136]; let $d^\star$ denote the metric. For $i \in \mathbb{N}$ let

$$U_i = \left\{x \in \Omega_\epsilon : d^\star(x,Q) < \frac{\epsilon}{i}\right\}.$$

Fix $i \in \mathbb{N}$. Now U_i is $d^\star$-open in Ω_ϵ so U_i is weakly open in Ω_ϵ. Also

$$\overline{U_i^w} = \overline{U_i^{d^\star}} = \left\{x \in \Omega_\epsilon : d^\star(x,Q) \le \frac{\epsilon}{i}\right\} \quad \text{and} \quad \partial_{\Omega_\epsilon} U_i = \left\{x \in \Omega_\epsilon : d^\star(x,Q) = \frac{\epsilon}{i}\right\}.$$

Now Theorem 2.4 (with Fr and $p = 0$) implies (since $\Omega_\epsilon \cap B = \emptyset$) that there exists $y_i \in \partial_{\Omega_\epsilon} U_i$ and $\lambda_i \in (0,1)$ with $y_i = \lambda_i Fr(y_i)$ (notice that $Fr : \overline{U_i^w} \to C$ where $\overline{U_i^w}$ is the weak closure of U_i in Ω_ϵ is a w-condensing map since $F(Q)$ is relatively weakly compact so $w(Fr(\overline{U_i^w})) = 0$). Consequently for each $j \in \mathbb{N}$ there exists $(y_j, \lambda_j) \in \partial_{\Omega_\epsilon} U_j \times (0,1)$ with $y_j = \lambda_j\, Fr(y_j)$. Notice in particular since $y_j \in \partial_{\Omega_\epsilon} U_j$ that

$$\lambda_j\, Fr(y_j) \notin Q \quad \text{for} \quad j \in \{1,2,\ldots\}. \tag{2.1}$$

We now claim that

$$D = \left\{x \in E : x = \lambda Fr(x) \text{ for some } \lambda \in [0,1]\right\}$$

is weakly compact. Clearly D is weakly closed since $Fr : E \to C$ is weakly continuous. Also since $D \subseteq \operatorname{co}(F(Q) \cup \{0\})$ we have $w(D) \le w(\operatorname{co}(F(Q) \cup \{0\})) = w(F(Q)) = 0$. Thus D is weakly compact (so weakly sequentially compact by the Eberlein-Šmulian theorem (Theorem 1.8)). This together with $d^\star(y_j, Q) = \frac{\epsilon}{j}$ and $|\lambda_j| \le 1$ ($j \in \mathbb{N}$) implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star$ and $y_j \rightharpoonup y^\star \in \partial_{\Omega_\epsilon} Q$; also $y_j = \lambda_j Fr(y_j)$ so $y^\star = \lambda^\star Fr(y^\star) = \lambda^\star F(y^\star)$. If $\lambda^\star = 1$, then $y^\star = F(y^\star)$ which contradicts $B \cap Q = \emptyset$. Hence we may assume $0 \le \lambda^\star < 1$. But in this case assumption (H) with $x_j = r(y_j) \in Q$, $x = y^\star = r(y^\star) \in \partial_{\Omega_\epsilon} Q$ implies $\lambda_j Fr(y_j) \in Q$ for j sufficiently large. This contradicts (2.1). Thus $B \cap Q \ne \emptyset$, so there exists $x \in Q$ with $x = Fr(x)$, i.e. $x = F(x)$ ∎

3. Applications to operator equations

In this section we present existence principles for nonlinear operators. We motivate our study by first considering the second order boundary value problem

$$\left.\begin{array}{l} y'' + f(t,y,y') = 0 \quad \text{a.e. on } [0,1] \\ y(0) = y(1) = 0 \end{array}\right\} \tag{3.1}$$

where $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a L^p-Carathéodory function with $p > 1$. By the last we mean the following:

(a) $t \mapsto f(t,u,v)$ is measurable for all $(u,v) \in \mathbb{R}^2$.

(b) $(u, v) \mapsto f(t, u, v)$ is continuous for a.e. $t \in [0, 1]$.

(c) For any $r > 0$ there exists $h_r \in L^p[0, 1]$ such that $|f(t, u, v)| \leq h_r(t)$ for a.e. $t \in [0, 1]$ and all $|u| \leq r$ and $|v| \leq r$.

By a solution to problem (3.1) we mean a function $y \in W^{2,p}[0, 1]$ (i.e. $y' \in AC[0, 1]$ and $y'' \in L^p[0, 1]$) which satisfies the differential equation in (3.1) a.e. and $y(0) = y(1) = 0$. We now combine the fixed point theory of Section 2 together with some ideas in Corduneanu [4] to obtain an existence principle for the boundary value problem (3.1).

For this define first operators

$$H_1, H_2 : L^p[0, 1] \to C[0, 1] \subseteq L^p[0, 1]$$

by

$$H_1 u(t) = \int_0^1 G(t, s) u(s)\, ds \qquad \text{and} \qquad H_2 u(t) = \int_0^1 G_t(t, s) u(s)\, ds$$

where

$$G(t, s) = \begin{cases} (t-1)s & \text{for } 0 \leq s \leq t \leq 1 \\ (s-1)t & \text{for } 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to check that solving problem (3.1) is equivalent to finding a solution $u \in L^p[0, 1]$ to the equation

$$u = -f\big(t, H_1(u), H_2(u)\big). \tag{3.2}$$

Remark that if u is a solution of equation (3.2), then $y(t) = \int_0^1 G(t, s) u(s)\, ds$ is a solution of problem (3.1) whereas if w is a solution of problem (3.1), then $v = w''$ is a solution of equation (3.2).

Further on, define an operator $N : L^p[0, 1] \to L^p[0, 1]$ by

$$Nu(t) = -f\big(t, H_1(u(t)), H_2(u(t))\big). \tag{3.3}$$

Consequently solving equation (3.1) is equivalent to finding a fixed point $u \in L^p[0, 1]$ to

$$u = N u. \tag{3.4}$$

Let Q be a bounded, closed, convex subset of $L^p[0, 1]$. We claim that $N : Q \to L^p[0, 1]$ is weakly continuous (we need only check N is weakly sequentially continuous [5: p. 93]). Suppose $y_n, w \in Q$ with $y_n \rightharpoonup w$ in $L^p[0, 1]$ (i.e. $\int_0^1 y_n\, g\, dt \to \int_0^1 w\, g\, dt$ for all $g \in L^q[0, 1]$ where $\frac{1}{p} + \frac{1}{q} = 1$). We must show $Ny_n \rightharpoonup Nw$ in $L^p[0, 1]$. Let $g \in L^q[0, 1]$. Notice

$$\begin{aligned} &\left| \int_0^1 (Ny_n - Nw) g\, dt \right| \\ &= \left| \int_0^1 \Big[f\big(t, H_1(y_n), H_2(y_n)\big) - f\big(t, H_1(w), H_2(w)\big) \Big] g\, dt \right| \\ &\leq \left(\int_0^1 \Big| f\big(t, H_1(y_n), H_2(y_n)\big) - f\big(t, H_1(w), H_2(w)\big) \Big|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |g|^q\, dt \right)^{\frac{1}{q}}. \end{aligned}$$

If we show

$$\int_0^1 \left|f(t, H_1(y_n), H_2(y_n)) - f(t, H_1(w), H_2(w))\right|^p dt \to 0 \quad \text{as } y_n \rightharpoonup w, \tag{3.5}$$

then $\int_0^1 (Ny_n) g\, dt \to \int_0^1 (Nw) g\, dt$. As a result $Ny_n \rightharpoonup Nw$ in $L^p[0,1]$ and so $N : Q \to L^p[0,1]$ is weakly continuous. It remains to prove (3.5). First we show, for each $t \in [0,1]$, that

$$y_n \rightharpoonup w \quad \Longrightarrow \quad H_i(y_n(t)) \to H_i(w(t)) \ \ (i = 1,2). \tag{3.6}$$

We prove (3.6) with $i = 1$ (the case $i = 2$ is similar). Fix $t \in [0,1]$. Then

$$\left|H_1(y_n(t)) - H_1(w(t))\right| = \left|\int_0^1 G(t,s)[y_n(s) - w(s)]\, ds\right| \to 0 \quad \text{as } y_n \rightharpoonup w$$

since $G(t,\cdot)$ is in $L^q[0,1]$ for fixed $t \in [0,1]$. Now (3.6) together with the fact that f is a L^p-Carathéodory function gives

$$\left.\begin{array}{c} y_n \rightharpoonup w \quad \text{implies} \\ f(t, H_1(y_n(t)), H_2(y_n(t))) \to f(t, H_1(w(t)), H_2(w(t))) \text{ a.e. on } [0,1]. \end{array}\right\} \tag{3.7}$$

Also for $u \in Q$ we have

$$\begin{aligned} |H_1 u(t)| &= \left|\int_0^1 G(t,s) u(s)\, ds\right| \\ &\le \left(\int_0^1 |u|^p\, dt\right)^{\frac{1}{p}} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q\, ds\right)^{\frac{1}{q}} \quad \text{for all } t \in [0,1]. \end{aligned}$$

Since $Q \subseteq L^p[0,1]$ is bounded there exists $r > 0$ with

$$|H_i u(t)| \le r \qquad \text{for all } t \in [0,1] \text{ and } u \in Q \ (i = 1,2). \tag{3.8}$$

Now (3.5) follows immediately from (3.7), (3.8), the fact that f is a L^p-Carathéodory function, and the Lebesgue dominated convergence theorem (see [7: p. 151]) ∎

Theorem 3.1. *Let f be a L^p-Carathéodory function with $p > 1$. In addition suppose there is a constant M_0, independent of λ, with*

$$\|y''\|_{L^p} = \left(\int_0^1 |y''|^p\, dt\right)^{\frac{1}{p}} \le M_0$$

for any solution y *to the problem*

$$\left.\begin{aligned} y'' + \lambda f(t,y,y') = 0 \quad &a.e. \ on \ [0,1] \\ y(0) = y(1) = 0 \end{aligned}\right\} \tag{3.9$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then problem* (3.1) *has at least one solution.*

Remark. Theorem 3.1 could be proved using the theory of compact (strong) operators (see [15: Chapter 3]). However here we will supply a proof based on Theorem 2.5. This has the added advantage of automatically yielding a new and very general existence principle for nonlinear operator equations.

Proof of Theorem 3.1. Let

$$Q = \{u \in L^p[0,1] : \|u\|_{L^p} \le M_0 + 1\}$$

and notice Q is a closed, bounded, convex subset of $L^p[0,1]$. Solving problem $(3.9)_\lambda$ is equivalent to finding a fixed point $u \in L^p[0,1]$ of

$$u = \lambda N u \tag{3.10$_\lambda$}$$

where N is as defined in (3.3). Notice if y is a solution of problem $(3.9)_\lambda$, then $u = y''$ is a solution of equation $(3.10)_\lambda$ whereas if w a solution of equation $(3.10)_\lambda$, then $v(t) = \int_0^1 G(t,s)w(s)\,ds$ is a solution of problem $(3.9)_\lambda$.

We know $N : Q \to L^p[0,1]$ is weakly continuous. Also since $N(Q)$ is relatively weakly compact, then the Krein-Šmulian theorem (Theorem 1.9) implies that $C = \overline{co}(\overline{N(Q)^w})$ is a closed, bounded, convex subset of $L^p[0,1]$. The result follows immediately from Theorem 2.5 once we show that condition (H)* is satisfied. Take a sequence $\{(x_j,\lambda_j)\}_{j=1}^\infty$ in $Q \times [0,1]$ with $\lambda_j \to \lambda$ and $x_j \rightharpoonup x$ with $x = \lambda N(x)$ for $0 \le \lambda < 1$. We need to show $\lambda_j N(x_j) \in Q$ for j sufficiently large.

First notice since $x_j \rightharpoonup x$ we have (as in (3.5)) that

$$\begin{aligned} &\int_0^1 |N(x_j(t)) - N(x(t))|^p\,dt \\ &\qquad = \int_0^1 \Big| f\big(t, H_1(x_j(t)), H_2(x_j(t))\big) - f\big(t, H_1(x(t)), H_2(x(t))\big)\Big|^p\,dt \\ &\qquad \to 0. \end{aligned}$$

Then given $\epsilon > 0$ (say $\epsilon < \frac{1}{3}$) there exists $j_0 \in \mathbb{N}$ with

$$\|N(x_j)\|_{L^p} \le \|N(x)\|_{L^p} + \epsilon \qquad (j \ge j_0). \tag{3.11}$$

Also $x = \lambda N(x)$ together with the fact that $\|u\|_{L^p} \le M_0$ for any solution u to equation $(3.10)_\lambda$ implies

$$\|\lambda N(x)\|_{L^p} \le M_0. \tag{3.12}$$

Consequently (3.11), (3.12) and $\lambda_j \to \lambda$ implies that there exists $j_1 \geq j_0$ with

$$\|\lambda_j N(x_j)\|_{L^p} \leq M_0 + \frac{1}{2} \qquad (j \geq j_1).$$

Thus $\lambda_j N(x_j) \in Q$ for j sufficiently large so condition (H)* is satisfied. Theorem 2.5 now guarantees that equation $(3.10)_1$ has a solution (and consequently problem (3.1) has a solution) ∎

Essentially the same reasoning as in Theorem 3.1 now establishes immediately a general existence principle for the operator equation

$$u = T(u) \tag{3.13}$$

where $T : L^p([0,1], \mathbb{R}^n) \to L^p([0,1], \mathbb{R}^n)$ with $p > 1$.

Theorem 3.2. *Let Q be a closed, bounded, convex subset of $L^p([0,1], \mathbb{R}^n)$ with $p > 1$. Also assume $T : Q \to L^p([0,1], \mathbb{R}^n)$ is weakly continuous and*

$$\{x_j\}_{j=1}^{\infty} \subset Q,\, x_j \rightharpoonup x \quad \Longrightarrow \quad \int_0^1 |T(x_j(t)) - T(x(t))|^p \, dt \to 0. \tag{3.14}$$

In addition suppose there is a constant M_0, independent of λ, with $\|u\|_{L^p} \leq M_0$ for any solution u to the equation

$$u = \lambda T(u) \tag*{$(3.15)_\lambda$}$$

for each $\lambda \in (0,1)$. Then equation (3.13) has at least one solution in Q.

Remarks. **(i)** In Theorem 3.2 to show $T : Q \to L^p([0,1], \mathbb{R}^n)$ is weakly continuous we need only show that $T : Q \to L^p([0,1], \mathbb{R}^n)$ is weakly sequentially continuous which follows immediately from [5: p. 93]. **(ii)** In Theorem 3.2, $L^p([0,1], \mathbb{R}^n)$ may be replaced by $L^p([0,1], B)$ where B is a separable and reflexive Banach space (notice $L^p([0,1], B)$ is a separable and reflexive Banach space [7]).

4. Coincidence theory

A coincidence theory is developed for w-condensing maps. Let E be a Banach space and let Q and C be closed, bounded, convex subsets of E with $Q \subseteq C$. Also $X \subseteq Q$ and A is a weakly compact subset of X, X weakly closed in Q. We let $L : X \to C$ be a weakly continuous operator.

Definition 4.1. We let $P_A(X, C; L)$ denote the set of all w-condensing mappings $F : X \to C$ such that $L - F$ is zero free on A.

Definition 4.2. A map $F \in P_A(X, C; L)$ is

(i) *L-essential* if for every $G \in P_A(X, C; L)$ which agrees with F on A we have that $L - G$ has a zero in X;

(ii) *L-inessential* if there exists a $G \in P_A(X,C;L)$ which agrees with F on A and $L-G$ is zero free on X.

Definition 4.3. Two mappings $F,G \in P_A(X,C;L)$ are *homotopic* in $P_A(X,C;L)$ written $F \cong G$ in $P_A(X,C;L)$ if there is a w-condensing mapping $N : X \times [0,1] \to C$ with $N_t(u) = N(u,t) : X \to C$ belonging to $P_A(X,C;L)$ for each $t \in [0,1]$ and $N_0 = F$ as well as $N_1 = G$.

Theorem 4.1. *Let C, X, A, Q, E, L be as above with $F \in P_A(X,C;L)$. Then the following assertions are equivalent:*

(i) *F is L-inessential.*

(ii) *There is a $G \in P_A(X,C;L)$ with $F \cong G$ in $P_A(X,C;L)$ and with $L-G$ zero free on X.*

Proof. We first show that assertion (i) implies assertion (ii). Let $G \in P_A(X,C;L)$ with $G = F$ on A and $L-G$ zero free on X. Define $N : X \times [0,1] \to C$ by

$$N(x,t) = tG(x) + (1-t)F(x).$$

As in Theorem 2.1, $N : X \times [0,1] \to C$ is a w-condensing map. Also since $F = G$ on A and $L-G$ is zero point free on A we have for $x \in A$ that

$$L(x) - N_t(x) = L(x) - [tG(x) + (1-t)F(x)] = L(x) - G(x) \neq 0$$

so $L - N_t$ is zero free on A for each $t \in [0,1]$. Then (as in Theorem 2.1) $N_t \in P_A(X,C;L)$ for each $t \in [0,1]$. Finally $N_0 = F$ and $N_1 = G$ so $F \cong G$ in $P_A(X,C;L)$.

Next we show that assertion (ii) implies assertion (i). Let $N : X \times [0,1] \to C$ be a w-condensing mapping from $G \in P_A(X,C;L)$ to F with $N_0 = G$ and $N_1 = F$. In particular $L - N_t$ is zero free on A for each $t \in [0,1]$. Let

$$B = \Big\{ x \in X : L(x) = N(x,t) \text{ for some } t \in [0,1] \Big\}.$$

If $B = \emptyset$, then F is L-inessential. So assume $B \neq \emptyset$. As in Theorem 2.1 (since L is weakly continuous) we have that B is weakly closed. Also A is a weakly compact subset of X. Then there exists a continuous (weakly) function $\mu : X \to [0,1]$ with $\mu(A) = 1$ and $\mu(B) = 0$. Define $J : X \to C$ by $J(x) = N(x,\mu(x))$. As in Theorem 2.1, $J : X \to C$ is a w-condensing map with $J = F$ on A. Also $L-J$ is zero free since $L(x) - J(x) = 0$ means $L(x) = N(x,\mu(x))$ which implies $x \in B$ and so $\mu(x) = 0$ (i.e. $L(x) = N(x,0)$), which is a contradiction since $L(x) - N(x,0) = L(x) - G(x)$ is zero free. Thus $J \in P_A(X,C;L)$ and so F is L-inessential ∎

Remark. We can remove the assumption that A is a weakly compact subset of X provided extra conditions are put on L. For example suppose $A \subseteq X$ with A weakly closed in X and X weakly closed in Q and suppose all maps considered F, G, N, N_t (in Definitions 4.1 - 4.3) are w-compact (w-condensing with $k = 0$, i.e. $T : Y \subseteq E \to E$ is a w-compact map if T maps bounded sets into bounded sets, is weakly continuous and $w(T(Z)) = 0$ for all bounded sets $Z \subseteq Y$). Assume $L : X \to C$ is a weakly continuous operator with

$$\Omega \subset C \text{ weakly compact} \quad \Longrightarrow \quad L^{-1}(\Omega) \text{ relatively weakly compact} \tag{4.1}$$

holding where L^{-1} denotes the inverse image. Then assertions (i) and (ii) in Theorem 4.1 are equivalent. The same reasoning as in Theorem 4.1 establishes this. There is only one place where the arguement is different – we need to guarantee the existence of a weakly continuous $\mu : X \to [0,1]$ with $\mu(A) = 1$ and $\mu(B) = 0$. This is immediate once we show B is weakly compact. Since $L(B) \subseteq N(B \times [0,1])$ we have, since $N : X \times [0,1] \to C$ is a w-compact map, that $L(B)$ is relatively weakly compact. Also since $B \subseteq L^{-1}(L(B)) \subseteq L^{-1}(\overline{L(B)^w})$ we have that B is weakly compact.

Essentially the same reasoning as in Theorem 2.2 establishes the next result.

Theorem 4.2. *Let C, X, A, Q, E, L be as above. Suppose F and G are two maps in $P_A(X, C; L)$ such that $F \cong G$ in $P_A(X, C; L)$. Then F is L-essential if and only if G is L-essential.*

We also have the following nonlinear alternative of Leray-Schauder type.

Theorem 4.3. *Let Q and C be closed, bounded, convex subsets of a Banach space E with $Q \subseteq C$. In addition let U be a weakly open subset of Q and $L : \overline{U^w} \to C$ a weakly continuous map. Also suppose $\partial_Q U$ is a weakly compact subset of $\overline{U^w}$ and $G \in P_{\partial_Q U}(\overline{U^w}, C; L)$ is L-essential. Then every w-condensing map $F : \overline{U^w} \to C$ has at least one of the properties*

(A1) *$L(x) = F(x)$ for some $x \in \overline{U^w}$*

or

(A2) *there exist $u \in \partial_Q U$ and $\lambda \in (0,1)$ with $L(u) = \lambda F(u) + (1-\lambda)G(u)$.*

Proof. Assume $L - F|_{\partial_Q U}$ is zero free and consider the homotopy $N : \overline{U^w} \times [0,1] \to C$ joining F and G given by

$$N(u,t) = tF(u) + (1-t)G(u).$$

As in Theorem 2.4, $N : \overline{U^w} \times [0,1] \to C$ is a w-condensing map. Now either $L - N_t$ is zero free on $\partial_Q U$ for each $t \in [0,1]$ or it is not. If $L - N_t$ is zero free on $\partial_Q U$ for each $t \in [0,1]$, then Theorem 4.2 implies that $L - F$ has a zero in U so property (A1) occurs. If $L - N_t$ is not zero free on $\partial_Q U$ for each $t \in [0,1]$, then there exist $u \in \partial_Q U$ and $\lambda \in [0,1]$ with $L(u) - [\lambda F(u) + (1-\lambda)G(u)] = 0$. Now $\lambda \neq 0$ since $G \in P_{\partial_Q U}(\overline{U^w}, C; L)$ (in particular $L - G$ is zero free on $\partial_Q U$) and $\lambda \neq 1$ since $L - F|_{\partial_Q U}$ was assumed to be zero free. Hence property $(A2)$ occurs ∎

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Received 25.10.1995