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A Non-Degeneracy Property for a Class of Degenerate Parabolic Equations

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Abstract. We deal with the initial and boundary value problem for the degenerate parabolic equation $u_t = \Delta\beta(u)$ in the cylinder $\Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is bounded, $\beta(0) = \beta'(0) = 0$, and $\beta' \geq 0$ (e.g., $\beta(u) = u|u|^{m-1}$ $(m > 1)$). We study the appearance of the free boundary, and prove under certain hypothesis on β that the free boundary has a finite speed of propagation, and is Hölder continuous. Further, we estimate the Lebesgue measure of the set where $u > 0$ is small and obtain the non-degeneracy property $\left|\{0 < \beta'(u(x,t)) < \epsilon\}\right| \leq c\epsilon^{\frac{1}{2}}$.

Keywords: *Free boundary problems, finite speed of propagation, porous medium equations*

AMS subject classification: Primary 35 K 65, secondary 35 R 35, 76 S 05

0. Introduction

Consider the initial and boundary value problem

$$\begin{aligned} u_t &= \Delta\beta(u) && \text{in } \Omega \times (0,T] \\ u(x,t) &= 0 && \text{on } \partial\Omega \times (0,T] \\ u(x,0) &= u_0(x) && \text{in } \Omega \end{aligned} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $T < +\infty$, β is a function with $\beta(0) = \beta'(0) = 0$ and $\beta' \geq 0$, and $u_0 \geq 0$. Written in divergence form $u_t = \operatorname{div}(\beta'(u)\nabla u)$ we see that (0.1) is a degenerate parabolic equation.

The model equation of this type is the porous medium equation

$$u_t = \Delta(u|u|^{m-1}) \qquad (m > 1). \tag{0.2}$$

Equation (0.2) has been the subject of intensive research, surveys can be found in [14, 16]. An interesting feature is the free boundary $\Gamma(t) = \partial \operatorname{supp} u(\cdot,t)$. Its behaviour in one dimension is studied in [2, 3, 6, 13], results in several dimensions are proven in [7, 9 - 11].

In detail the Cauchy problem in n dimensions is treated in [7] and the initial and boundary value problem in [11]. If $\operatorname{supp} u_0 \subset\subset \Omega$ and u_0 is not too flat near $\partial \operatorname{supp} u_0$,

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then there is no *waiting time* [13], that means the free boundary begins to move immediately. In detail $\Gamma(t)$ is strictly increasing for all $t > 0$ where $\Gamma(t) \cap \partial\Omega = \emptyset$. Further, the free boundary has a finite speed of propagation and $\Gamma(t)$ is Hölder continuous.

One consequence is the following *non-degeneracy property* [11]: Consider the set $\Omega_0 = \{(x,t) \in \Omega \times [0,T] : 0 < \beta'(u(x,t)) < \epsilon\}$ where $\beta(s) = s|s|^{m-1}$ $(m > 1)$. Then the Lebesgue measure $|\Omega_0|$ of Ω_0 satisfies

$$|\Omega_0| \le c\,\epsilon^{\frac{1}{2}}. \tag{0.3}$$

This estimate plays an important role in finite element analysis (see [11]). If Γ is sufficently smooth, the better result $|\Omega_0| \le c\,\epsilon$ can be shown, for example, if $\Omega \subset \mathbb{R}^1$ or if $\operatorname{supp} u(\cdot,t)$ is convex. The reason is that then the velocity of Γ is determined by the slope of u (see [7, 13]).

The aim of this paper is to prove the *non-degeneracy property* (0.3) in the case of general β. In Section 1 we state the assumptions on the data and the main result. In Section 2 we study the free boundary. We prove that under the hypotheses on β given in Section 1 the free boundary has a finite speed of propagation and is Hölder continuous. The proof depends in a crucial way on the *smoothing property* $u_t \ge -\frac{c}{t}u$. In Section 3 we prove the *non-degeneracy property*. There we will use suitable comparison functions.

1. Assumptions on the data and the main result

Let δ_0, δ_1, s_0 and c be positive constants and set $\Omega(t) = \operatorname{supp} u(\cdot,t)$ and $\Omega(0) = \operatorname{supp} u_0$. We need the following assumptions:

(H1) $u_0 \in L^\infty(\Omega)$ and $0 \le u_0(x) \le M$ $(x \in \Omega)$.

(H2) $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ is a connected open domain with Lipschitz boundary.

(H3) $\Omega(0) \subset\subset \Omega$, $\Omega(0)$ is a connected domain, and $\partial\Omega(0) \in C^2$.

(H4) $\beta'(u_0(x)) \ge c\big(\operatorname{dist}(x, \partial\Omega(0))\big)^{2-\delta}$ for $x \in \Omega(0)$ with $0 < \delta < 2$.

(H5) $\beta'(u(x,t)) \ge c\big(\operatorname{dist}(x, \partial\Omega(t))\big)^2$ $(t \in (0,\delta_0))$ if $\operatorname{dist}(x, \partial\Omega(t)) \le \delta_1$.

Furthermore, we suppose the following assumptions on β:

(A1) $\beta \in C^3(0, \|u_0\|_\infty)$.

(A2) $\beta(0) = \beta'(0) = 0$ and $\beta(s)$, $\beta'(s)$, $\beta''(s) > 0$ for all $s > 0$.

(A3) $\dfrac{\beta(s)\beta''(s)}{(\beta'(s))^2} \ge c > 0$ for all $s \in [0, \|u_0\|_\infty]$.

(A4) $0 < k_0 \le \dfrac{\beta'(s)\beta'''(s)}{(\beta''(s))^2} \le k_1 < 1$ $(s \in [0, s_0])$.

(A5) $\psi(\alpha s) \ge \alpha\,\psi(s)$ $(s \in [0, \beta(s_0)])$ where $\alpha \le 1$ and $\psi(s) = \beta'(\beta^{-1}(s))$.

(A6) $\beta(s) \ge s^m$ $(s \in [0, s_0))$ for some constant $m > 1$.

Remark. **i)** Assumption (A5) holds, if $\beta'(\beta^{-1}(s))$ is concave for all $s \in [0, \beta(s_0)]$.
ii) $\beta(s) = s|s|^{m-1}$ $(m > 1)$ satisfies assumptions (A1) - (A6).

In general, it is not to be expected that all solutions of problem (0.1) will be regular. But if $u_0 \in L^\infty(\Omega)$ and $T < +\infty$, then the existence of a unique weak solution u is known and it holds (see [15])

$$u \in L^\infty(0,T; L^\infty(\Omega)) \cap C(\Omega \times (0,T)) \qquad \text{and} \qquad \beta(u) \in L^2(0,T; H_0^1(\Omega)).$$

Further, $u_0 \geq 0$ implies $u(x,t) \geq 0$ for all $(x,t) \in \Omega \times [0,T]$.

The main object of the present paper is to prove a *non-degeneracy property* for wich we define the two sets

$$\Omega_0 = \Big\{(x,t) \in \Omega \times [0,T] : 0 < \beta'(u(x,t)) < \epsilon\Big\}$$

and

$$\Omega_0(t) = \Big\{x \in \Omega(t) : 0 < \beta'(u(x,t)) < \epsilon\Big\}$$

for $0 \leq t \leq T$.

Theorem 1.1. *For the Lebesgue measure $|\Omega_0|$ of Ω_0*

$$|\Omega_0| = |\cup_{0 \leq t \leq T} \Omega_0(t)| \leq c\,\epsilon^{\frac{1}{2}}$$

is satisfied.

2. The free boundary

Assumption (H3) implies that the support of u has a free boundary for some $t > 0$. We define the free boundaries

$$\Gamma = \cup_{0 < t \leq T} \partial\Omega(t) \setminus \partial\Omega \qquad \text{and} \qquad \Gamma(t) = \partial\Omega(t) \setminus \partial\Omega \quad (t > 0).$$

Lemma 2.8 below implies $\Omega(t_0) \subset \Omega(t_1)$ for $t_0 \leq t_1$. Then it follows as in [7] that if a vertical line segment $\sigma = \{(x_0,t) : t_0 < t < t_1\}$ satisfies $\sigma \subset \Gamma$, then $\{(x_0,t) : 0 < t < t_1\} \subset \Gamma$, and if Γ contains no vertical line segment, then Γ is strictly increasing in every point.

Further, hypothesis (H4) entails that there is no *waiting time*, that means $\overline{\Omega(0)} \cap \partial\Omega(t) = \emptyset$ for all $t > 0$. Hence the free boundary is strictly increasing for all $t > 0$ where $\Gamma(t) \cap \partial\Omega = \emptyset$. This result is due to [13], if $\beta(s) = s|s|^{m-1}$ $(m > 1)$ and $\Omega = \mathbb{R}^1$. The proof to equation (0.1) is similiar (one needs suitable comparison functions; see, for example, Section 3). Let us note that the conclusion fails, if we allow $\delta = 0$ in hypothesis (H4).

The set $\Omega(t)$ is open, thus let $\overline{\Omega(t)}$ denote its closure. In this section we generalize the ideas of [7, 11] in order to prove the two following theorems.

Theorem 2.1. *The following assertions are true.*

i) *Let $0 < \delta_0 \le t \le t+s \le T < +\infty$, $B \subset\subset \Omega$ a ball and U a $(cs^{1/\gamma})$-neighbourhood of $\Omega(t)$. Then there exist two constants c and $\gamma > 1$ independent of s and t such that $(U \cap B) \subset (\Omega(t+s) \cap B)$.*

ii) *Let $0 < \delta_0 \le t$ and $0 < \delta < 1$. Then $\overline{\Omega(t+\delta)}$ is contained in a $(c\delta^{1/2})$-neighbourhood of $\overline{\Omega(t)}$, where the constant c depends only on δ_0 and on the data.*

This theorem discribes the finite speed of propagation. In particular, let $t \ge \delta_0$, $z \in \Gamma(t_0)$, $\eta \in \mathbb{R}^n$, $|\eta| = 1$ and $g(s) = z + s\eta$ such that $g(s) \cap \overline{\Omega(t_0)} = \emptyset$ for all $s \in (0, \delta)$, for some $\delta > 0$. Further, let $\kappa_u(\eta, z, t_0)$ denote the velocity of $\Gamma(t_0)$ in the direction of η. Then by Theorem 2.1, there exists a constant c_1 independent of z, η and t_0 such that

$$\kappa_u(\eta, z, t_0) \ge c_1. \tag{2.1}$$

The free boundary Γ is strictly increasing. Thus for any $x \in \Omega \setminus \overline{\Omega(0)}$ there exists a unique point t_x such that $x \in \Gamma(t)$ if and only if $t = t_x$. Hence the free boundary is given by a function $t = G(x)$ $(x \in \Omega \setminus \overline{\Omega(0)})$ continuous in $\overline{\Omega} \setminus \overline{\Omega(0)}$.

Further, the proof of Theorem 2.1 yields the following property of G.

Theorem 2.2. *G is Hölder continuous on $\overline{\Omega} \setminus \overline{\Omega(0)}$ (with Hölder exponent γ) and uniformly Hölder continuous in any compact set $K \subset (\overline{\Omega} \setminus \overline{\Omega(0)})$.*

Corollary 2.3. *Let $t_1 < t_2$, $x_1 \in \Gamma(t_1)$ and $x_2 \in \Gamma(t_2)$ such that* $\operatorname{dist}(x_1, \Gamma(t_2)) = \operatorname{dist}(x_1, x_2)$. *If* $\operatorname{dist}(x_1, x_2)$ *is sufficently small, then*

$$dist(x_1, x_2) \le c\,|t_1 - t_2|^{\frac{1}{2}} \tag{2.2}$$

where the constant c is independent of x_1, x_2 and t_1, t_2.

An essential property of u is the *smoothing property* (2.3). A proof which uses semigroup theory can be found in [8]. Let us prove (2.3) using a comparison argument (see also [1, 5], if $\beta(s) = s|s|^{m-1}$ with $m > 1$). Hence we need the following comparison theorem [4].

Theorem 2.4. *Let $L(u) = u_t - \Delta\beta(u)$ and suppose (in the weak sense)*

1) *$L(u_1) \le 0$ and $L(u_2) \ge 0$ for all $(x,t) \in \Omega \times (t_0, t_1]$*

2) *$u_1(\cdot, t_0)$, $u_2(\cdot, t_0) \in L^2(\Omega)$ and $(u_1 - u_2)_t \in L^1(t_0, t_1; L^1(\Omega))$*

3) *$u_1(x, t_0) \le u_2(x, t_0)$ for all $x \in \Omega$*

4) *$u_1(x,t) \le u_2(x,t)$ for all $(x,t) \in \partial\Omega \times [t_0, t_1]$.*

Then

$$u_1(x,t) \le u_2(x,t)$$

for all $(x,t) \in \Omega \times [t_0, t_1]$.

Lemma 2.5. *Suppose assumptions* (A1) - (A3). *Then there exists a constant $k > 0$ such that*

$$u_t \ge -\frac{k}{t}u \tag{2.3}$$

for $t \in (0,T]$.

Proof. We will show

$$(\beta(u))_t \geq -\frac{k}{t}\beta(u) \tag{2.4}$$

for $t \in (0,T]$. Then the convexity of β entails

$$u_t \geq -\frac{k}{t}\,\frac{\beta(u)}{\beta'(u)} \geq -\frac{k}{t}\,u.$$

Consider the function $w = t\,(\beta(u))_t = t\,\beta'(u)\,u_t$. Since

$$w_t = \beta'(u)u_t + t\,\beta''(u)\,(u_t)^2 + t\,\beta'(u)\,(\Delta\beta(u))_t$$

we get

$$L(w) := w_t - \frac{w}{t} - \frac{\beta''(u)}{t(\beta'(u))^2}\,w^2 - \beta'(u)\Delta w = 0.$$

Further, $-k\beta(u)$ satisfies

$$L(-k\beta(u)) = \frac{k^2\beta(u)}{t}\left(\frac{1}{k} - \frac{\beta''(u)\,\beta(u)}{(\beta'(u))^2}\right)$$

and it follows $L(-k\beta(u)) \leq 0$ where $k = \frac{1}{c}$ with c given in assumption (A3). Assume u_0 to be smooth (otherwise one uses approximations (see, e.g., [15]). Then it holds $w_0(x) \equiv 0 \geq -k\,\beta(u_0(x))$ and $w(x,t) = 0 = -k\,\beta(u(x,t))$ for all $(x,t) \in \partial\Omega \times (0,T)$. Hence the comparison theorem yields $w \equiv t\,\beta'(u)u_t \geq -k\,\beta(u)$ ∎

Let

$$B(x,R) = \{y \in \mathbb{R}^n : |y-x| < R\}, \qquad B(R) = B(0,R), \qquad ⨍_\Omega f = \frac{1}{|\Omega|}\int_\Omega f.$$

Now we can establish two fundamental lemmas.

Lemma 2.6. *For arbitrary* $\delta_0 > 0$ *let* $t_0 \geq \delta_0$, $x_0 \in \Omega\backslash\overline{\Omega(t_0)}$, $R_0 := \operatorname{dist}(x_0, \partial\Omega(t_0)) < \operatorname{dist}(x_0, \partial\Omega)$ *and* $0 < R \leq R_0$. *There exist two constants* c *and* $\bar{c}$ *depending only on* δ_0, k_0, s_0, k, n *and* M *such that, for* $0 < \sigma < \bar{c}$,

$$\beta(u(x,t_0)) \equiv 0 \quad (x \in B(x_0,R)) \quad \text{and} \quad ⨍_{B(x_0,R)} \beta(u(x,t_0+\sigma))\,dx \leq c\,\frac{R^2}{\sigma}$$

implies

$$\beta(u(x,t_0+\sigma)) \equiv 0 \qquad (x \in B(x_0,\tfrac{R}{6})).$$

Corollary 2.7. *If* $\beta(u(x,t_0)) = 0$ *for all* $x \in B(x_0,R)$ *and if* $(x_0, t_0+\sigma)$ *belongs to* $\Gamma(t_0+\sigma)$, *then*

$$⨍_{B(x_0,R)} \beta(u(x,t_0+\sigma))\,dx > c\,\frac{R^2}{\sigma}.$$

Proof of Lemma 2.6. First we note that $\operatorname{dist}(x_0, \partial\Omega(t)) < \operatorname{dist}(x_0, \partial\Omega)$ entails the inclusion $B(x_0, R) \subset \Omega$. Define the function $v = \beta'(u)$. Some easy calculations show that v is a weak solution of the equation

$$v_t = v\Delta v + \left(1 - \frac{\beta'(u)\,\beta'''(u)}{(\beta''(u))^2}\right)|\nabla v|^2. \tag{2.5}$$

Let $\alpha = \frac{\sigma}{R^2}$ and

$$\tilde{u}(x,t) = \beta^{-1}\Big(\alpha\beta\big(u(x_0 + Rx, t_0 + \sigma t)\big)\Big).$$

By some direct calculations it follows that $\tilde{v} = \beta'(\tilde{u})$ is a weak solution of the equation

$$\tilde{v}_t = \alpha\,\psi\big(\alpha^{-1}\psi^{-1}(\tilde{v})\big)\Delta\tilde{v} + \alpha\,\frac{\beta'(u)}{\beta'(\tilde{u})}\left(1 - \frac{\beta'(\tilde{u})\,\beta'''(\tilde{u})}{(\beta''(\tilde{u}))^2}\right)|\nabla\tilde{v}|^2 \tag{2.6}$$

where $\psi = \beta'(\beta^{-1})$ (and $\alpha\,\psi\big(\alpha^{-1}\psi^{-1}(\tilde{v})\big) = \alpha\beta'(u)$). We distinguish two cases:

Case $\alpha \le 1$. The assumptions yield $\beta(\tilde{u}(x,0)) = 0$ for all $x \in B(1)$ and

$$\oint_{B(1)} \beta(\tilde{u}(x,1))\,dx = \alpha \oint_{B(1)} \beta\big(u(x_0 + Rx, t_0 + \sigma)\big)\,dx \le c \tag{2.7}$$

and from (2.3) it follows that

$$\Delta\beta(\tilde{u}) = \alpha\Delta\beta(u) \ge -\alpha\,\frac{k}{\delta_0}\,\sigma u \ge -\varepsilon_0 \tag{2.8}$$

where $\varepsilon_0 = \bar{c}\,\frac{kM}{\delta_0}$. Hence we get $\Delta\big(\beta(\tilde{u}) + \frac{\varepsilon_0}{2n}|x|^2\big) \ge 0$, thus $\beta(\tilde{u}) + \frac{\varepsilon_0}{2n}|x|^2$ is subharmonic. We obtain for $x \in B(\frac{1}{2})$

$$\begin{aligned}\beta(\tilde{u}(x,1)) + \frac{\varepsilon_0}{2n}|x|^2 &\le \oint_{B(x,\frac{1}{2})} \Big(\beta(\tilde{u}(\xi,1)) + \frac{\varepsilon_0}{2n}|\xi|^2\Big)\,d\xi \\ &\le 2^n \oint_{B(1)} \beta(\tilde{u}(\xi,1))\,d\xi + \frac{\varepsilon_0}{2n}.\end{aligned}$$

From (2.7) it follows that $\beta\big(\tilde{u}(x,1)\big) \le \big(2^n c + \frac{\varepsilon_0}{2n}\big)$ for all $x \in B(\frac{1}{2})$. Using (2.4), we get

$$(\beta(\tilde{u}))_t = \alpha\,(\beta(u))_t \ge -\alpha\,\frac{k\sigma}{\delta_0}\,\beta(u) \ge -\varepsilon_1\beta(\tilde{u}) \tag{2.9}$$

where $\varepsilon_1 = \bar{c}\,\frac{k}{\delta_0}$. We obtain $\beta(\tilde{u}(x,1)) \ge e^{-\varepsilon_1(1-t)}\beta(\tilde{u}(x,t))$, thus for all $x \in B(\frac{1}{2})$ and $t \in (0,1)$

$$\beta(\tilde{u}(x,t)) \le e^{\varepsilon_1}\left(2^n c + \frac{\varepsilon_0}{2n}\right) \quad \text{and} \quad \tilde{v}(x,t) \le \psi\left(e^{\varepsilon_1}\left(2^n c + \frac{\varepsilon_0}{2n}\right)\right). \tag{2.10}$$

Now we will apply the comparison theorem. We define the function

$$z(x,t) = \lambda\,(m_0 - 1)\left\{a^2 t + a\left(r - \frac{1}{3}\right)\right\}^+ \qquad (r = |x|,\ \lambda > 0)$$

where $m_0 > 1$ and $a = \frac{1}{6}$. Then $\operatorname{supp} z(x,t) = \{x \in \mathbb{R}^n : |x| \geq \frac{1}{3} - \frac{t}{6}\}$. If λ is sufficently small, then z satifies the inequality

$$z_t \geq \alpha\, \psi\big(\alpha^{-1}\psi^{-1}(z)\big)\Delta z + \frac{1}{m_0 - 1}\,|\nabla z|^2. \tag{2.11}$$

For it holds $\alpha\, \psi(\alpha^{-1}\psi^{-1}(z)) = 0$ if $z = 0$, next in $\operatorname{supp} z$ we have

$$z_t = \lambda(m_0 - 1)a^2, \qquad \Delta z = \lambda(m_0 - 1)(n-1)\frac{a}{r}, \qquad |\nabla z|^2 = \lambda^2(m_0-1)^2a^2$$

and assumption (A5) entails $\alpha\, \psi\big(\alpha^{-1}\psi^{-1}(z)\big) \leq \alpha\alpha^{-1}\psi(\psi^{-1}(z)) = z$, thus

$$a^2 \geq \left(\frac{\alpha\psi\big(\alpha^{-1}(\psi^{-1}(z))\big)}{\lambda a r}\,(n-1)a^2 + a^2\right)\lambda$$

if λ is sufficently small. Let $m_0 = \frac{2-k_0}{1-k_0}$ where k_0 is given in assumption (A4). Using (2.6), we get

$$\tilde{v}_t \leq \alpha\, \psi\big(\alpha^{-1}\psi^{-1}(\tilde{v})\big)\Delta\tilde{v} + \frac{1}{m_0-1}|\nabla\tilde{v}|^2, \tag{2.12}$$

for from assumption (A5) it follows that

$$\beta'(\tilde{u}) = \beta'\big(\beta^{-1}(\alpha\beta(u))\big) \geq \alpha\, \beta'\big(\beta^{-1}(\beta(u))\big) = \alpha\, \beta'(u).$$

The comparison theorem and (2.10) - (2.12) yield $\tilde{v}(x,t) \leq z(x,t)$ for all $(x,t) \in B(\frac{1}{2}) \times [0,1]$ if c and $\bar{c}$ are sufficently small. In particular we obtain $\tilde{v}(x,t) = z(x,t) = 0$ for all $(x,t) \in B(\frac{1}{6}) \times [0,1]$.

Case $\alpha > 1$. Now the assertion follows if we consider u instead of $\tilde{u}$ and use $R^2 < \sigma$. We get like above

$$v(x,t) \leq \psi\left(e^{\varepsilon_1}\left(2^n c + \frac{\varepsilon_0}{2n}\right)\right) \qquad \text{for all } (x,t) \in B(x_0, \frac{R}{2}) \times [t_0, t_0 + \sigma].$$

Next, we apply the comparison theorem in $B(x_0, \frac{R}{2}) \times [t_0, t_0 + \sigma]$ ∎

Lemma 2.8. *Let* $t_0 \geq \delta_0$, $x_0 \in \Omega \setminus \overline{\Omega(t_0)}$, $2R_0 = \operatorname{dist}(x_0, \partial\Omega)$, $0 < R < R_0$ *and* $0 < \sigma < \bar{c}$ *where* $\bar{c}$ *is sufficently small. If*

$$\oint_{B(x_0,R)} \beta(u(x,t_0))\, dx \geq \mu\frac{R^2}{\sigma},$$

then there exists a constant $\lambda > 0$ *independent of* σ, R, x_0 *and* t_0 *such that*

$$\beta\big(u(x_0, t_0 + \lambda\sigma)\big) > 0.$$

In particular λ *is small, if* μ *is large.*

Proof. We consider $\tilde{u}$ as above. The assumptions entail

$$\oint_{B(1)} \beta(\tilde{u}(x,0))\,dx = \alpha \oint_{B(1)} \beta\big(u(x_0 + Rx, t_0)\big)\,dx \geq \mu.$$

Using (2.9), we get $(\beta(\tilde{u}))_t \geq -\varepsilon_1 \beta(\tilde{u})$ where $\varepsilon_1 = \bar{c}\,\frac{k}{\delta_0}$. Consider

$$\varphi(t) = \int_{B(1)} \beta(\tilde{u}(x,t))\,dx.$$

It follows that $\varphi'(t) \geq -\varepsilon_1 \varphi(t)$, thus

$$\varphi(t) \geq e^{-\varepsilon_1 \lambda}\mu \qquad \text{for all } \ t \in [0, \lambda]. \tag{2.13}$$

If the assumption is not true, then

$$\beta(\tilde{u}(0,t)) = 0 \qquad \text{for all } \ t \in [0, \lambda] \tag{2.14}$$

and in particular $\beta(\tilde{u}(0,\lambda)) = 0$. Using m given in assumption (A6) we obtain as in [7] the existence of constants c_1, c_2, c_3 and δ such that

$$\int_0^t \varphi(s)\,ds \leq c_1 \int_0^t \beta(\tilde{u}(0,s))\,ds + c_2(\varepsilon_1)^{\tilde{\delta}} + c_3(\varphi(t))^{\frac{1}{m}} \tag{2.15}$$

and c_2, c_3 and δ depend only on m and n. Let $\lambda_0 := \frac{\lambda}{2}$ and $D(\lambda) := c_2(\varepsilon_1)^{\delta}(e^{\varepsilon_1 \lambda}\mu^{-1})^{\frac{1}{m}}$. Then from (2.13) - (2.15) it follows that

$$\int_0^t \varphi(s)\,ds \leq (c_3 + D(\lambda))(\varphi(t))^{\frac{1}{m}} \qquad \text{for all } \ t \in [\lambda_0, \lambda].$$

Now the function $\psi(t) = \int_0^t \varphi(s)\,ds$ satisfies

$$(\psi'(t))^{\frac{1}{m}} \geq B\,\psi(t) \qquad \text{for all } \ t \in (\lambda_0, \lambda]$$

where $B = (c_3 + D(\lambda))^{-1}$ and $\psi(\lambda_0) \geq \lambda_0 e^{-\varepsilon_1 \lambda}\mu$.

Next we compare ψ with the solution χ of the problem

$$\chi'(t) = (B\chi(t))^m \qquad (t \in (\lambda_0, \lambda])$$
$$\chi(\lambda_0) = \psi(\lambda_0).$$

We get $\psi(t) \geq \chi(t)$ for all $t \in [\lambda_0, \lambda]$. The function χ fulfills the equation

$$(m-1)(\chi(t))^{m-1} = (C - B^m t)^{-1}$$

where the constant C satisfies the equation $(m-1)(\psi(\lambda_0))^{m-1} = (C - B^m \lambda_0)^{-1}$.

It follows that $\chi(t) \to +\infty$ if $t \to \frac{C}{B^m}$, thus $\psi(t) \to +\infty$ if $\lambda \geq \frac{C}{B^m}$. This is a contradiction. Hence $\beta(\tilde{u}(0,\lambda)) > 0$ holds if

$$\lambda \geq \frac{C}{B^m} = \frac{\lambda}{2} + \frac{1}{(m-1)(\psi(\lambda_0))^{m-1} B^m}.$$

In particular this is true for small λ if μ is sufficently large ∎

The proofs of Theorem 2.1 and Theorem 2.2 follow now as in [7].

Remark. Another approach can be found in [12]. Instead of the *smoothing property* $u_t \geq -\frac{ku}{t}$ a generalized Harnack inequality and Moser iteration are used to study the free boundary of the porous medium equation with absorption

$$u_t = \Delta u^m - u^p \qquad \text{in } \mathbb{R}^n \times (0, \infty)$$

$(m > 1, p \geq 1)$.

3. The non-degeneracy property

In this section we will prove Theorem 1.1. We consider the set

$$\Omega_0(t) = \Big\{ x \in \Omega(t) : 0 < v(x,t) < \epsilon \Big\}$$

and define

$$\Gamma^*(t) = \Big\{ x \in \Gamma(t) : \operatorname{dist}\big(x, \mathbb{R}^n \setminus \overline{\Omega(t)}\big) > 0 \Big\}.$$

Let $\tau \geq \delta_0$ and $x \in \Omega_0(\tau)$. We distinguish the following three cases:

i) x near $\Gamma(\tau)$

ii) x near $\partial\Omega \cap \partial\Omega(\tau)$

iii) x near $\Gamma^*(t)$ for some $t \in (0, \tau)$

(the last case arises if there is a *close in*, that means a hole in the support disappears). We define for $\tau \geq \delta_0$ the following three sets:

$$M_3(\tau) = \Big\{ x \in \Omega(\tau) : \operatorname{dist}(x, \Gamma^*(t)) < \operatorname{dist}(x, \partial\Omega(\tau)) \text{ for some } t \in (0, \tau) \Big\}$$
$$M_2(\tau) = \Big\{ x \in \Omega(\tau) \setminus M_3(\tau) : \operatorname{dist}(x, \partial\Omega \cap \partial\Omega(\tau)) < \operatorname{dist}(x, \Gamma(\tau)) \Big\}$$
$$M_1(\tau) = \Big\{ x \in \Omega(\tau) : x \notin M_3(\tau) \cup M_2(\tau) \Big\}.$$

First we study the measure $|M_1(\tau)|$.

Propostion 3.1. *Let $t_0 \geq \delta_0$, $x_0 \in \Gamma(t_0)$, $x_s \in \Gamma(t_0 + s)$ and $dist(x_0, x_s) = dist(x_0, \Gamma(t_0 + s))$, further let $d_2(x_0) = dist\big(x_0, \partial\Omega \cup \big(\cup_{t_0 \leq t \leq T} \Gamma^*(t)\big)\big)$. There exist two constants $c_0 > 0$ and $d_1 > 0$ such that if $dist(x_0, x_s) \leq \min\{d_1, d_2(x_0)\}$, then*

$$v(x_0, t_0 + s) \geq c_0 \big(\operatorname{dist}(x_s, x_0)\big)^2 \tag{3.1}$$

where c_0 and d_1 are independent of x_0, t_0 and s.

Proof. Let $R := \operatorname{dist}(x_0, x_s)$, $B(x_s, R) \subset \Omega \setminus \Omega(t_0)$ and $\partial B(x_s, R) \cap \overline{\Omega(t_0)} = x_0$ (otherwise consider a suitable set $B'(x_s, R) \subset B(x_s, R)$ with $B'(x_s, R) \cap \overline{\Omega(t_0)} = x_0$).

Let $\eta(x_0,t_0)$ denote the inner normal to $\partial B(x_s,R)$ in x_0 and $|\eta|=1$. We define the function

$$g(x,t)=(m_0-1)\left\{\lambda a^2(t-t_0)+a(r-R)\right\}^+ \qquad (r=|x|) \tag{3.2}$$

where $\lambda>1$, $a>0$, $m_0=\frac{2-k_0}{1-k_0}$, $x\in B:=B(x_s,R)\cap\Omega$ and $t\in\left[t_0,t_0+\frac{R}{\lambda a}\right)$.

Let $t_1\in\left(t_0,t_0+\frac{R}{\lambda a}\right)$ and $|t_1-t_0|$ sufficently small, further let $a<c_1$ where c_1 is given in (2.1). This yields

i) $g_t\geq g\Delta g+(m_0-1)^{-1}|\nabla g|^2$ in $B\times(t_0,t_1]$

ii) $g(x,t_0)\equiv v(x,t_0)\equiv 0$ for all $x\in B$.

Let us now suppose that

iii) $g(x,t)\geq v(x,t)$ for all $x\in\partial B\times[t_0,t_1]$.

From (2.5) and assumption (A4) it follows that v is a weak solution of the inequality

$$v_t\leq v\Delta v+(m_0-1)^{-1}|\nabla v|^2.$$

Thus the comparison theorem entails $g(x,t)\geq v(x,t)$ for all $(x,t)\in B\times[t_0,t_1]$. Hence the velocity $\kappa_u(\eta,x_0,t_0)$ of $\Gamma(t_0)$ satisfies

$$\kappa_u(\eta,x_0,t_0)\leq\kappa_g(\eta,x_0,t_0)=\lambda a<c_1$$

if $|\lambda-1|$ is sufficently small. This contradicts (2.1). The continuity of v yields

$$v(x_0,t_0+s)>\sigma s \qquad \text{for all } s\in[0,\delta^*] \tag{3.3}$$

for some small δ^* and $\sigma=(m_0-1)\frac{c_1^2}{2}$.

Next we consider the function

$$p(x,s)=\begin{cases}\sigma s & \text{for } s\leq d_2(x)\\ \sigma d_2(x)-(s-d_2(x)) & \text{for } s>d_2(x).\end{cases}$$

For fixed $x\in\overline{\Omega\setminus\Omega(\delta_0)}$ there is a point t_x such that $x\in\partial\Omega(t_x)$ and $x\notin\overline{\Omega(t)}$ for all $t<t_x$. Let $s\geq 0$ and

$$F(s):=\min_{x\in\overline{\Omega\setminus\Omega(\delta_0)}}\big(v(x,t_x+s)-p(x,s)\big).$$

It holds $F(0)=0$, $F(s)>0$ for sufficently small s and F is continuous. Therefore there exists a $d_1>0$ such that $F(s)\geq 0$ for all $s\in[0,d_1]$. We conclude in view of (3.3) and (2.2) that $\operatorname{dist}(x_0,x_s)\leq\min\{d_1,d_2(x_0)\}$ implies

$$v(x_0,t_0+s)\geq\sigma\,|(t_0+s)-t_0|\geq c\left(\operatorname{dist}(x_0,x_s)\right)^2 \tag{3.4}$$

where c is independent of x_0, t_0 and s ∎

Now we construct some suitable comparison functions in order to estimate the measures $|M_2(\tau)|$ and $|M_3(\tau)|$.

Let $z,\ z'' \in \Omega$, $\eta = \frac{z''-z}{|z''-z|}$ parallel to the x_n-axis, $z''_n > z_n$, $z' = z + \lambda'\eta$ with $0 < \lambda' < \operatorname{dist}(z, z'')$ and $t_0 > \delta_0$. We define for $x \in \mathbb{R}^n \cap \{x_n \le z''_n\}$ and $t \ge t_0$

$$f(x,t) = (m_1 - 1)\left\{\alpha_1(t - t_0) + \alpha_2 l(x_n) - \alpha_3\big(d(x_1, \dots, x_{n-1})\big)^2\right\}^+ \tag{3.5}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$, l is a linear function with $l(z'_n) = 0$ and $l(z''_n) = |z'_n - z''_n|$, and

$$d(x_1, \dots, x_{n-1}) = \min_\lambda \sum_{i=1}^{n-1} |x_i - (z + \lambda\eta)_i|.$$

Let us consider the shape of f: It holds $f(z + \lambda\eta, t_0) = 0$ if $0 \le \lambda \le \lambda'$ and

$$\frac{d}{d\lambda} f(z + \lambda\eta, t_0) = (m_1 - 1)\,\alpha_2 =: \widetilde{\alpha}_2$$

if $\lambda' \le \lambda \le \operatorname{dist}(z, z'')$. Further, in any direction normal to η, the function f is decreasing and $\operatorname{supp} f(x, t') \subset \operatorname{supp} f(x, t'')$ for $t' < t''$.

Now put the point t_1 such that it holds $z \in \partial K$ where $K = \operatorname{supp} f(x, t_1)$ and suppose $K \subset \Omega$. Further, we define $S = \partial K \cap \{x_n = z''_n\}$.

Lemma 3.2. *Let $\delta_0 \le t_0 \le t_1$. For any z, z', z'', t_0 and c^* there exist $\alpha_1, \alpha_2, \alpha_3$ and t_1 such that*

$$f(z'', t_1) = c^* \tag{3.6}$$

$$f(x,t) \le c^* - \alpha_3\big(d(x_1, \dots, x_{n-1})\big)^2 \quad ((x,t) \in S \times [t_0, t_1]) \tag{3.7}$$

$$f_t \le f\Delta f + (m_1 - 1)^{-1}|\nabla f|^2. \tag{3.8}$$

In particular it holds

$$f(z + \lambda\eta, t_1) = \widetilde{\alpha}_2\lambda \equiv (m_1 - 1)\alpha_2\lambda$$

for $0 \le \lambda \le \operatorname{dist}(z, z'')$.

Proof. Set $h_0 = \frac{\operatorname{diam} S}{2}$, $h_1 = \operatorname{dist}(z, z'')$, $\operatorname{dist}(z, z') = \gamma h_1$ and $\operatorname{dist}(z', z'') = (1 - \gamma)h_1$ with $0 < \gamma < 1$, and $\alpha_1 = a\alpha_2^2$ with $0 < a < 1$. The definition of S yields $\alpha_3 := \frac{c^*}{(m_1 - 1)h_0^2}$. We require the following:

(i) $\alpha_1|t_1 - t_0| = \alpha_2\gamma h_1$

(ii) $\alpha_2(1 - \gamma)h_1 + \alpha_1|t_1 - t_0| = \frac{c^*}{m_1 - 1}$

(iii) $2\alpha_3 c^* = (1 - a)\alpha_2^2$.

Then (i) yields $z \in \partial \operatorname{supp} f(x, t_1)$, (ii) and (iii) entail (3.6) and (3.8) (note that in $\operatorname{supp} f(x,t)$ it holds $f_t = (m_1 - 1)\alpha_1$, $f\Delta f = -2\alpha_3(m_1 - 1)f$, $|\nabla f|^2 \ge (m_1 - 1)^2\alpha_2^2$ and $0 \le f \le c^*$). Now (i) requires

$$a = \frac{\gamma h_1}{\alpha_2|t_1 - t_0|}. \tag{3.9}$$

Inserting (i) into (ii), we obtain $\alpha_2 h_1 = \frac{c^*}{m_1 - 1}$. Noting the above definition of α_3, (iii) yields

$$h_0^2 = \frac{2(m_1 - 1)}{1 - a} h_1^2. \tag{3.10}$$

Hence the constant a is determined by (3.9) and h_0 is determined by (3.10). Finally note that $\frac{h_0}{h_1}$ must fulfil a special relation ∎

Now let us study the measure $|M_3(\tau)|$.

Proposition 3.3. *Let $\delta_0 \le t_0 < \tau$, $x \in M_3(\tau)$ and $\text{dist}(x, \Gamma^*(t_0)) \le \text{dist}(x, \Gamma^*(t))$ for all $t \in (0, \tau)$. Then for any sufficently small ϵ there exist two constants c_1 and c_2 such that*

$$v(x,t) \ge \epsilon \qquad \text{for all } \ t \in [t_0, T] \tag{3.11}$$

if $\text{dist}(x, \Gamma^*(t_0)) \ge c_1 \epsilon^{\frac{1}{2}}$ and if $\text{dist}(x, \partial\Omega) \ge c_2 \epsilon^{\frac{1}{2}}$. Further, c_1 and c_2 are independent of ϵ and t_0.

Proof. We fix a point $x^* \in M_3(t_0)$. Let $d := \text{dist}(x^*, \Gamma^*(t_0)) = \text{dist}(x^*, x_0)$ where $x_0 \in \Gamma^*(t_0)$. Then (3.1) provides two constants c_0 and d_1 such that

$$v(x^*, t_0) \ge c_0 d^2 \tag{3.12}$$

if $d \le d_1$. Now let us use f from (3.5) as a comparison function (all denotations are as above): We suppose $d \le c\epsilon^{\frac{1}{2}}$ where ϵ is sufficently small such that $3\epsilon^{\frac{1}{2}} \le \bar{c}_0 h_1$ where $\bar{c}_0 = \min\{1, c_0\}$. Further, put $h_1 = c_3 d$ ($c_3 < 1$ will be determined later), $\eta = \frac{x^* - x_0}{|x^* - x_0|}$ and

$$z = x_0 - \bar{c}_0^{-1}\epsilon^{\frac{1}{2}}\eta, \quad z' = x_0 + \bar{c}_0^{-1}\epsilon^{\frac{1}{2}}\eta, \quad z'' = x_0 + (c_3 d - \bar{c}_0^{-1}\epsilon^{\frac{1}{2}})\eta.$$

Now we apply Lemma 3.2. Let $c^* = \epsilon^{\frac{1}{2}} h_1$ where c^* is given in (3.6). This yields $\widetilde{\alpha}_2 = \epsilon^{\frac{1}{2}}$. Then the function f satisfies $f(z', t_0) = 0$ and $f(z'', t_0) \ge \epsilon$, and there exists a point t_1 such that

$$f(z, t_1) = 0, \qquad f(x_0, t_1) \ge \epsilon, \qquad f(z', t_1) \ge 2\epsilon, \qquad f(z'', t_1) \ge 3\epsilon.$$

We assume $t_1 = T$ (this is true if $\text{dist}(z', z'')$ is suitable or if severel functions f_i are considered one after another). Using $d \le c\epsilon^{\frac{1}{2}}$, we get $\text{supp} f(x,T) \subset \Omega$ if c_2 is suitable.

Now we are able to apply the comparison theorem. Below we will prove that

$$v(x,t) \ge f(x,t) \qquad \text{for all } \ (x,t) \in S \times [t_0, T]. \tag{3.13}$$

It follows

(i) $v(x,t) \ge f(x,t)$ for all $x \in \partial\,\text{supp} f(x,T) \times [t_0, T]$.

Inequalities (3.12) and (3.8) entail

(ii) $v(x, t_0) \ge f(x, t_0)$ for all $x \in \text{supp}\, f(x,T)$

(iii) $f_t \le f\Delta f + (m_1 - 1)^{-1}|\nabla f|^2$.

Put $m_1 = \frac{2-k_1}{1-k_1}$ where k_1 is given in assumption (A4). Then (2.5) yields

(iv) $v_t \geq v\Delta v + (m_1 - 1)^{-1}|\nabla v|^2$.

Thus by the comparison theorem we obtain $v(x,t) \geq f(x,t)$ for all $(x,t) \in \text{supp} f(x,T) \times [t_0, T]$. In particular it holds $v(z'', t) \geq \epsilon$ for all $t \in [t_0, T]$. This yields the assumption.

Finally we prove inequality (3.13). Note that c^* is defined by $c^* = \epsilon^{\frac{1}{2}} h_1$. Consider the *Barenblatt solution*

$$g(x,t) = m_1(t+\tau)^{-k(m_1-1)} \left\{ b^2 - \frac{k(m_1-1)}{2nm_1} \frac{|x-x^*|^2}{(t+\tau)^{2k/n}} \right\}^+$$

where $k = \left(m_1 - 1 + \frac{2}{n}\right)$. This function is a weak solution of the porous medium equation $g_t = g\Delta g + (m_1-1)^{-1}|\nabla g|$ (see, for example, [14]). Let $\text{supp}\, g(x,t_0) = B(x^*, r)$ where $r < d$ and c_3 (see above) are choosen such that $S \subset B(x^*, r)$. Now let $g(x^*, t_0) = c_4 d^2$ sufficently small. Then we obtain in view of (3.12) $g(x,t_0) \leq v(x,t_0)$ for all $x \in B(x^*, r)$. Next there exists a constant $c_5 > 1$ such that $\text{supp}\, g(x,t) \subset B(x^*, c_5 r)$ for all $t \in [t_0, T]$ and $B(x^*, c_5 r) \subset \Omega$ if c_2 is suitable and if $d \leq c\epsilon^{\frac{1}{2}}$. Thus the comparison theorem and (iv) entail $g(x,t) \leq v(x,t)$ for all $x \in B(x^*, c_5 r) \times [t_0, T]$. Further, it holds $g(z'', t) \geq c_6 d^2$ for all $t \in [t_0, T]$. Noting that $c^* = \epsilon^{\frac{1}{2}} h_1 = \epsilon^{\frac{1}{2}} c_3 d$, (3.7) yields $f(x,t) \leq g(x,t) \leq v(x,t)$ for all $x \in S \times [t_0, t_1]$ if $\epsilon^{\frac{1}{2}} \leq \min\left\{c_3^{-1} c_6 d, \bar{c}_0 c_3 \frac{d}{3}\right\}$ ∎

Proposition 3.4. *Let $t_0 \geq \delta_0$ and $x \in M_2(t_0)$. For any sufficently small ϵ there exists a constant c_1 such that*

$$v(x,t) \geq \epsilon \qquad \text{for all } \; t \in [t_0, T] \tag{3.14}$$

if $\text{dist}(x, \partial\Omega) \geq c_1 \epsilon^{\frac{1}{2}}$. *Further, c_1 is independent of ϵ and t_0.*

Proof. It follows like above by comparing f and v (here we use the fact that Ω has a Lipschitz boundary in order to choose a suitable constant c_1). Then $\text{supp}\, f(x,T) \subset \Omega$ and $f(x,t) \leq v(x,t)$ for all $(x,t) \in \text{supp}\, f(x,t) \times [t_0, T]$ ∎

Now the proof of the main theorem follows immediately.

Proof of Theorem 1.1. Hypotheses (H4) and (H5) entail $|\Omega_0(\tau)| \leq c\epsilon^{\frac{1}{2}}$ for $0 \leq \tau < \delta_0$ (near $\Gamma^*(\tau)$ proceed as above and use hypothesis (H5) instead of (3.12)). Next we consider $\tau \geq \delta_0$. Let $x \in \Omega(\tau)$. If ϵ is sufficently small, then by (3.1), (3.11) and (3.14) we obtain a constant c such that $v(x,t) \geq \epsilon$ if $\text{dist}\left(x, \partial\Omega(\tau) \cup \left(\cup_{\delta_0 \leq t \leq \tau} \Gamma^*(t)\right)\right) \geq c\epsilon^{\frac{1}{2}}$. Noting that $|\cup_{\delta_0 \leq t \leq \tau} \Gamma^*(t)| = 0$ we obtain the assumption ∎

Remark. Let $t_0 \geq \delta_0, x_0 \in \Gamma(t_0)$ and $x_s \in \Gamma(t_0 + s)$ such that $\text{dist}(x_0, x_s) = \text{dist}(x_0, \Gamma(t_0 + s))$. In order to prove (3.1), we have used (2.2):

$$\text{dist}(x_s, x_0) \leq c\,|(t_0 + s) - t_0|^{\frac{1}{\alpha}} \tag{3.15}$$

where $\alpha = 2$. But if this estimate holds for some $\alpha \in [1, 2]$, then the proof of Theorem 1.1 yields the better result $|\Omega_0| \leq c\epsilon^{\frac{1}{\alpha}}$. For example let us assume $\Omega \subset \mathbb{R}^1$. Then we can prove (3.15) for $\alpha = 1$ as in [13].

In general this is not to be expected: if (3.15) is satisfied for $\alpha = 1$, then the velocity of the free boundary has not only a lower bound (see (2.1)) but also an upper bound. For example this is impossible if there are holes in the support of u, even if Γ is smooth.

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