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Fibonacci Polynomials their Properties and Applications

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Abstract. The paper deals with polynomials characterized by coefficients determined by successive elements of the Fibonacci sequence. Basic properties and applications of the Fibonacci polynomials are demonstrated. The index of concentration of Fibonacci polynomials at k-th degree, locations of their zeros and optimization procedures for such polynomials are discussed. Illustrative examples are presented.

Keywords: *Fibonacci sequence, Fibonacci polynomials, recurrence relations, zeros of polynomials, stability of dynamical systems*

AMS subject classification: 43 C 05

1. Introduction

In this paper we deal with polynomials

$$p_n(x) = f_0x^0 + f_1x^1 + f_2x^2 + \dots + f_nx^n = \sum_{k=0}^{n} f_kx^k \qquad (n \geq 0) \tag{1}$$

and

$$q_n(z) = \frac{z^0}{f_0} + \frac{z^1}{f_1} + \frac{z^2}{f_2} + \dots + \frac{z^n}{f_n} = \sum_{k=0}^{n} \frac{1}{f_k} z^k \qquad (n \geq 0), \tag{2}$$

i.e. polynomials in $x \in \mathbb{C}$ and $z \in \mathbb{C}$ with coefficients determined by elements of the Fibonacci sequence

$$f_{k+2} = f_{k+1} + f_k \tag{3}$$

in which the first two values f_0 and f_1 are known. It is usualy assumed (see [2, 15, 16, 20]) that the absolute values of the subscripts in (3) are $|k| = 0, 1, 2, \dots$ and the initial values are $f_0 = 1$ and $f_1 = 1$. Table 1 gives the first few Fibonacci numbers generated by (3). In other publications (see [5, 6, 8, 11, 13, 19, 22]) it is assumed that the first terms of the Fibonacci sequence take the values $f_0 = 0$ and $f_1 = 1$. In the sequel we shall limit our attention to non-negative subscripts $k \geq 0$ only. The case of negative subscripts $k < 0$ can be treated analogously.

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Although the Fibonacci sequence has been studied extensively for some hundred years it still remains a fascinating area for exploration and there always seems to be some new aspects that can be revealed. Here, we explore some of the basic features of the Fibonacci sequence.

k	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
f_k	...	-5	-3	-2	-1	-1	0	1	1	2	3	5	8	13	...

Table 1. Successive elements of the Fibonacci sequence

It should be emphasized that polynomials play a central role not only in mathematics but also in many other domains of human activity to reveal phenomena in our environment and to design systems with desired properties (see [13 - 15]). The instantaneous state of many physical plants usually depends on several variables and is described by one or several state functions of one or several variables. These state variables, if sufficiently smooth, can be represented by polynomials, in some range and within some accuracy. Thus, the study of any plant, no matter how complicated it is, involves the study of polynomials.

In this paper we deal with fundamental concepts in the domain of newly created Fibonacci polynomials with the emphasis put on their properties and their possible application in various branches of mathematics and neighbouring disciplines. New concepts in the domain of the Fibonacci polynomials are presented. In particular, we discuss the index of concentration of the Fibonacci polynomials at degree k, and the location of zeros of the newly created polynomials are systematically studied. In this context we shall present new approaches allowing us to obtain a quantitative measure for the Fibonacci polynomials which result from the knowledge of qualitative ones. The fundamental notion and definition of concentration at low degree for polynomials are involved in the sequel. Illustrative examples are given along the presentation and sometimes examples will serve instead of formal proofs.

The paper is organized in the following way. We begin, in Section 2, by presenting some of the most important properties of Fibonacci polynomials with regard to their possible application. Section 3 is devoted to basic Fibonacci binomial identities established on the base of Fibonacci polynomials. Problems involving the need of applications of optimization approaches are included in Section 4. Conclusions and final remarks are presented in Section 5.

2. Basic properties of Fibonacci polynomials

In this section we shall demonstrate a set of the most useful properties of the Fibonacci polynomials which may not be commonly known.

2.1. Properties of polynomials (1). First, let us represent a general term of the Fibonacci sequence (3) by the Binet formula (see [18: Chapter IV/p. 52])

$$f_k = \frac{a^{k+1} - b^{k+1}}{a - b} \qquad (k \geq 0) \tag{4}$$

where

$$a = 1 - b = \frac{1+\sqrt{5}}{2}. \tag{5}$$

Substituting (4) into (1) and rearranging terms yields

$$p_n(x) = \frac{a}{a-b}\sum_{k=0}^{n}(ax)^k - \frac{b}{a-b}\sum_{k=0}^{n}(bx)^k \qquad (n \geq 0). \tag{6}$$

Taking into consideration the particular values of a and b, and assuming $ax \neq 1$ and $bx \neq 1$, we can use well known properties of the geometric progression and rewrite (6) in equivalent form as

$$p_n(x) = \frac{a}{a-b}\frac{(ax)^{n+1}-1}{ax-1} - \frac{b}{a-b}\frac{(bx)^{n+1}-1}{bx-1} \qquad (n \geq 0). \tag{7}$$

Since $a+b=1, a-b=\sqrt{5}$ and $ab=-1$ one gets

$$p_n(x) = \frac{1}{a-b}\frac{(a-b)-(a^{n+1}-b^{n+1})x^{n+2}-(a^{n+2}-b^{n+2})x^{n+1}}{-x^2-x+1} \qquad (n \geq 0). \tag{8}$$

Finally, taking into account (4) we obtain

$$p_n(x) = \frac{f_n x^{n+2} + f_{n+1}x^{n+1} - f_0}{x^2+x-1} \qquad (n \geq 0). \tag{9}$$

The above result indicates that any n-degree polynomial with coefficients determined by successive Fibonacci numbers can be represented as a ratio of two trinomials in x with appropriate degrees and coefficients from the set of Fibonacci numbers. It is worth mentioning that expression (9) can also be considered in inverse sense. This means that the polynomials $r_{n+2}(x) = f_n x^{n+2} + f_{n+1}x^{n+1} - 1$ $(n \geq 0)$ are divided without remainder by the polynomial $s_2(x) = x^2 + x - 1$.

The importance of (9) lies mainly in the fact that it can easily be used to prove many useful identities concerning the analysis of the set of Fibonacci numbers, in particular, when the variable x in $p_n = p_n(x)$ may take different numerical values and/or when n tends to infinity. To demonstrate these facts we shall consider some identities in the field of Fibonacci numbers.

At first, we take $x = 1$ and use (9) to express the sum of n successive Fibonacci numbers as

$$\sum_{k=0}^{n} f_k = p_n(x)|_{x=1} = f_{n+1} + f_n - f_0 = f_{n+2} - f_0 \qquad (n \geq 0). \tag{10}$$

Moreover, if we want to determine a partial sum of succesive Fibonacci numbers, then from (10) we obtain

$$\sum_{p=m}^{n} f_p = f_m + f_{m+1} + \ \dots\ + f_n = f_{n+2} - f_{m+1}. \tag{11}$$

Next, taking the first derivative with respect to x of both sides of (9) and using (1) we obtain

$$\begin{aligned} S_n(kf_k) &= \sum_{k=0}^{n} kf_k \\ &= \frac{d}{dx}p_n(x)\Big|_{x=1} \\ &= \left[\frac{d}{dx}\frac{f_n x^{n+2} + f_{n+1}x^{n+1} - f_0}{x^2+x-1}\right]\Big|_{x=1} \qquad (n \geq 0) \qquad (12) \\ &= (n+2)f_n + (n+1)f_{n+1} - 3(f_n + f_{n+1} - f_0) \\ &= nf_{n+2} - f_{n+3} + f_3. \end{aligned}$$

Therefore, to find $\sum_{k=0}^{n} kf_k$ we need to know the three Fibonacci numbers f_3, f_{n+2} and $-f_{n+3}$. In a similar way we can prove that

$$\begin{aligned} S_n^*(k^2 f_k) = \sum_{k=0}^{n} k^2 f_k &= \frac{d^2 p_n(x)}{dx^2}\Big|_{x=1} + S_n(kf_k) \\ &= (n+1)^2 f_{n+2} - (2n-1)f_{n+4} + 2f_{n+3} - 13f_0. \end{aligned} \qquad (13)$$

Other sums may be determined in similar manner.

Further, let us assume that $|x| < \frac{1}{a}$ and $n \to \infty$. Then from (9) we can determine the infinite Fibonacci polynomial. This is done by taking $\lim_{n\to\infty} p_n(x)$ in (1) and (9). This yields

$$p_\infty(x) = \lim_{n\to\infty} \sum_{k=0}^{n} (x^k f_k) = \lim_{n\to\infty} \frac{f_n x^{n+2} + f_{n+1}x^{n+1} - f_0}{x^2+x-1}. \qquad (14)$$

Observe that for $|x| < \frac{1}{a}$ we have $\lim_{n\to\infty} f_n x^n = 0$. By using this we obtain

$$p_\infty(x) = \frac{-f_0}{x^2+x-1} = \frac{1}{1-x-x^2}. \qquad (15)$$

Note that (15) generates successive terms of (1). Moreover, the above result can be used to evaluate $p_\infty(x)$ for various $|x| < \frac{1}{a}$. For instance, if $x_1 = \frac{1}{2}$ and $x_2 = -\frac{1}{2}$, then it can be easily checked that

$$\sum_{k=0}^{\infty} \frac{f_k}{2^k} = p_\infty\left(\frac{1}{2}\right) = 4 \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{f_k}{(-2)^k} = p_\infty\left(\frac{-1}{2}\right) = \frac{4}{5}.$$

Several other interesting cases of $p_n(x)$ for particular values of x and n can be investigated in similar manner.

2.2. Particular properties of polynomials (1). Let us now turn to the fundamental notion of the index of concentration at low degree $0 \leq k \leq n$ for a polynomial in x of

n-th degree. In the general case it is defined (see [1] or [17: Chapter II/p. 44]) by the expression

$$\delta_k(n) = \frac{\sum_{r=0}^{k} |d_r|}{\sum_{r=0}^{n} |d_r|} \tag{16$_a$}$$

where d_r $(0 \le r \le n)$ denotes the successive coefficients of the given polynomial

$$p_n(x) = d_0 + d_1 x + \ldots + d_k x^k + \ldots + d_n x^n. \tag{16$_b$}$$

Applying the above result to determine the index of concentration of the polynomial p_n at degree k we obtain

$$\delta_k(n) = \frac{\sum_{p=0}^{k} f_p}{\sum_{p=0}^{n} f_p}. \tag{16$_c$}$$

Relation $(16)_c$ is a measure for the relative importance of the terms of low degree inside the whole polynomial (1). This gives a possibility to determine the location of the zeros and the size of the polynomial in a given interval. On the other hand, polynomials (1) with different n do not have identical indices of concentration at the same degree k, but as we will see (due to the concentration property) the number of their zeros in any given disk remains uniformly bounded, independently of n.

Now, we want to determine the degree k of polynomial (1) for which $\delta_k(n) \le \frac{1}{2}$. This specific value can be used to determine the radius of an open disk centered at 0 and containing at most k zeros of p_n. Substituting the above value into (16) we get

$$f_{k+2} \le \frac{1}{2} f_{n+2} + \frac{1}{2}. \tag{17}$$

This implies that for sufficiently large n the relation

$$k \le n - 2 \tag{18}$$

holds. For instance, if $n = 3$, then for $k = 1$ we have from (16) that

$$\delta_1(3) = \frac{f_3 - 1}{f_5 - 1} = \frac{2}{7} < \frac{1}{2}.$$

But for $k = 2$ we obtain

$$\delta_2(3) = \frac{f_4 - 1}{f_5 - 1} = \frac{4}{7} > \frac{1}{2}.$$

To prove (18) we use (4) and take $n \ge 3$ to get

$$\delta_{n-2}(n) \simeq \left.\frac{a^{k+1}}{a^{n+1}}\right|_{k=n-2} = a^{-2} = \frac{3 - \sqrt{5}}{2} \simeq 0.381966\ldots. \tag{19}$$

Under the above conditions any polynomial (1) at degree k satysfying (18) has at most k roots in the open disk centered at 0 and radius $\rho(\delta, k)$ determined by the expression

$$\rho(\delta, k) = \left(\frac{1}{1-\delta}\right)^{\frac{1}{k+1}} - 1. \tag{20}$$

It has to be noted that for $\delta > \frac{1}{2}$ the estimation of the radius needs more complex analysis in a Hilbert space (see [20, 22]) and is beyond the scope of the present paper.

2.3. Singular cases of (9). Now we can consider two specific (singular) cases which appear when $x = \frac{1}{a} = -b$ or $x = \frac{1}{b} = -a$ are substituted into (9). Note that the identities

$$a^2 = a + 1 \qquad \text{and} \qquad b^2 = b + 1$$

hold. To consider the singular cases of (9) we can evaluate the ratio of the derivatives of the numerator and denominator of the right-hand side of (9). As a result, we find that

$$\begin{aligned} p_n\left(\frac{1}{a}\right) &= \left.\frac{\frac{d}{dx}[f_n x^{n+2} + f_{n+1}x^{n+1} - f_0]}{\frac{d}{dx}[x^2 + x - 1]}\right|_{x=\frac{1}{a}} \\ &= \left.\frac{(n+2)f_n x^{n+1} + (n+1)f_{n+1}x^n}{2x+1}\right|_{x=\frac{1}{a}} \end{aligned} \tag{21}$$

Again, making use of the Binet formula for f_n and f_{n+1}, substituting $x = \frac{1}{a}$ and rearranging terms we get

$$p_n\left(\frac{1}{a}\right) = \frac{(n+1)a}{a-b} - \frac{1}{(a-b)^2}\left(\left(\frac{b}{a}\right)^{n+1} - 1\right). \tag{22}$$

Next, consider $p_n(x)$ for $x = \frac{1}{b} = -a$. Again, we use the ratio of derivatives with repect to x to get

$$\begin{aligned} p_n\left(\frac{1}{b}\right) &= \left.\frac{(n+2)f_n x^{n+1} + (n+1)f_{n+1}x^n}{2x+1}\right|_{x=\frac{1}{b}} \\ &= -\left(\frac{(n+1)b}{a-b} + \frac{1}{(a-b)^2}\left(\left(\frac{a}{b}\right)^{n+1} - 1\right)\right). \end{aligned} \tag{23}$$

It is now evident that several other useful expressions based on (9) can be derived. We shall present some of them in the next subsection.

2.4. Properties of polynomials (2). Let us now turn to the study of basic properties of polynomials (2), i.e. polynomials with coefficients determined by the inverses of successive Fibonacci numbers.

Assuming a finite number of terms in (2) we can represent its right-hand side in a more compact form, namely

$$q_n(z) = \sum_{k=0}^{n} \frac{z^k}{f_k} = \frac{\sum_{p=0}^{n} {}^p f_n! z^p}{f_n!} \tag{24}$$

where

$$f_n! = f_0 f_1 f_2 \dots f_{n-1} f_n \tag{25}$$

and

$${}^p f_n! = \frac{f_n!}{f_p} \tag{26}$$

denote the Fibonacci factorials which are determined by the product of successive Fibonacci numbers and the partial Fibonacci factorials resulting from $f_n!$ by neglecting the f_p-th term. For example, for $n = 2, 3, 4$ we have

$$f_2! = f_0 f_1 f_2 = 2, \qquad f_3! = f_0 f_1 f_2 f_3 = 6, \qquad f_4! = f_0 f_1 f_2 f_3 f_4 = 30$$

and

$$\sum_{p=0}^{2} {}^p f_2! z^p = 2z^0 + 2z^1 + z^2$$

$$\sum_{p=0}^{3} {}^p f_3! z^p = 6z^0 + 6z^1 + 3z^2 + 2z^3$$

$$\sum_{p=0}^{4} {}^p f_4! z^p = 30z^0 + 30z^1 + 15z^2 + 10z^3 + 6z^5.$$

Finally, we obtain the corresponding polynomials (2) as

$$q_2(z) = \frac{2z^0 + 2z^1 + z^2}{2} = 1 + z + \frac{z^2}{2}$$

$$q_3(z) = \frac{6z^0 + 6z^1 + 3z^2 + 2z^3}{6} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3}$$

$$q_4(z) = \frac{30z^0 + 30z^1 + 15z^2 + 10z^3 + 6z^4}{30} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{5}.$$

It is also easily seen that the Fibonacci factorials fulfil the relations

$$f_n! = f_{n-1}! f_n \quad \text{and} \quad \sum_{p=0}^{n} {}^p f_n! = f_n \sum_{p=0}^{n-1} {}^p f_{n-1}! + f_{n-1}! \qquad (n \geq 0). \tag{27}$$

So, if we focus on a recursive process we can evaluate the Fibonacci factorials very effectively.

Next, substituting $z = 1$ into (24) we obtain the expression for the sum of inverses of n successive Fibonacci numbers

$$Q_n(1) = \sum_{k=0}^{n} \frac{1}{f_k} = \frac{\sum_{p=0}^{n} {}^p f_n!}{f_n!}. \tag{28}$$

We list the first few $Q_n(1)$ in the following table.

n	f_n	$f_n!$	$\sum_{p=0}^{n} {}^p f_n!$	$\sum_{k=0}^{n} \frac{1}{f_k} = Q_n(1)$
0	1	1	1	1.0
1	1	1	2	2.0
2	2	2	5	$5/2 = 2.5$
3	3	6	17	$17/6 \simeq 2.8333...$
4	5	30	91	$91/30 \simeq 3.03333...$
5	8	240	758	$379/120 \simeq 3.1583...$
6	13	3120	10094	$5047/1560 \simeq 3.2353...$
7	21	65520	215094	$107547/32760 \simeq 3.2829...$
⋮	⋮	⋮	⋮	⋮
∞	∞	∞	∞	$\simeq 3.3598856662....$

Table 2. Sum of inverses of the Fibonacci numbers

The above result can be useful in many of possible applications of Fibonacci polynomials and Fibonacci numbers. For instance, using (27) for $k \in (m, n)$ we obtain

$$\begin{aligned} \Delta_{m,n}(1) = \sum_{k=m}^{n} \frac{1}{f_k} &= \frac{\sum_{p=0}^{n} {}^{p}f_n!}{f_n!} - \frac{\sum_{p=0}^{m-1} {}^{p}f_{m-1}!}{f_{m-1}!} \\ &= \frac{\sum_{p=0}^{n} {}^{p}f_n! - \prod_{s=m}^{n} f_s \sum_{p=0}^{m-1} {}^{p}f_{m-1}!}{f_n!}. \end{aligned} \tag{29}$$

This result applied for three successive values $k = n-2, n-1, n$ gives the relation

$$\frac{1}{f_{n-2}} = \frac{f_{n-1} + f_n}{f_{n-1}f_n + (-1)^n} \tag{30}$$

from which we get

$$\frac{1}{f_{n-1}} + \frac{1}{f_n} = \frac{f_{n+1}}{f_{n-2}f_{n+1} - (-1)^n}. \tag{31}$$

Further, let us evaluate $Q_n(1)$ for $n \to \infty$. From (28) we get

$$Q_\infty(1) = \lim_{n\to\infty} \frac{\sum_{p=0}^{n} {}^{p}f_n!}{f_n!} \simeq 3.3598856662\ldots . \tag{32}$$

A proof of this result is not simple and can be performed by using the approach presented in [7]. For the sake of compactness of presentation it is omitted here.

2.5. Particular properties of polynomials (2). Let us now consider the index of concentration at low degrees of the polynomials (2) as $n \to \infty$. Using (32) it is easy to check that for $k = 0, 1, 2$ we have

$$\begin{aligned} \delta_0'(\infty) &= \lim_{n\to\infty} \frac{f_n!}{\sum_{p=0}^{n} {}^{p}f_n!} \simeq 0.2976291753\ldots \\ \delta_1'(\infty) &= 2 \cdot \lim_{n\to\infty} \frac{f_n!}{\sum_{p=0}^{n} {}^{p}f_n!} \simeq 0.5952583506\ldots \\ \delta_2'(\infty) &= 4 \cdot \lim_{n\to\infty} \frac{f_n!}{\sum_{p=0}^{n} {}^{p}f_n!} \simeq 1.1905167012\ldots . \end{aligned}$$

Thus the polynomials (2) are characterized by higher values of the indices of concentration at low degrees in comparison with that of polynomials (1). This is one of the important differences between the polynomials (1) and (2).

Applying (32) in circuit theory we get a simple expression for a parrallel connection of infinite number of resistors with resistances determined by successive Fibonacci numbers. Such type of electric circuits can be considered as an alternative structure for ladder networks composed of identical resistors (see [9, 16]).

Let us now examine other properties of the polynomials (2). Using (24) we can determine the index of concentration $\delta'_k(n)$ of the polynomial q_n at degree k as

$$\delta'_k(n) = \frac{\sum_{p=0}^{k} \frac{1}{f_p}}{\sum_{p=0}^{n} \frac{1}{f_p}} = \frac{f_n! \sum_{p=0}^{k} {}^p f_k!}{f_k! \sum_{p=0}^{n} {}^p f_n!}. \tag{33}$$

To transform (33) into a simpler form we can apply (25) and after performing a series of suitable manipulations on respective terms we get

$$\delta'_k(n) = 1 - \frac{\sum_{q=k+1}^{n} {}^q f_n!}{\sum_{p=0}^{n} {}^p f_n!}. \tag{34}$$

For instance, taking $n = 5$ and $k = 2$ we have

$$\delta'_2(5) = 1 - \frac{\sum_{q=3}^{5} {}^q f_5!}{\sum_{p=0}^{5} {}^p f_5!} = 1 - \frac{158}{758} \simeq 0.7915567\dots .$$

Observe that on the right-hand side of (34), both numerator and denominator depend on n and k, so for a given n there is a value of k at which $\delta'_k(n)$ is maximal. It is easily seen from (34) that this maximum equals 1 and that it is reached at $k = n$. The obtained result needs special attention for $|z| < 1$ and $n \to \infty$. Using (24) we obtain

$$\delta'_k(\infty) = \lim_{n\to\infty} \frac{\sum_{p=0}^{k} \frac{1}{f_p}}{\sum_{p=0}^{n} \frac{1}{f_p}} = \frac{\sum_{p=0}^{k} {}^p f_k!}{\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{f_k}} = \frac{\sum_{p=0}^{k} {}^p f_k!}{Q_\infty(1) f_k!}. \tag{35}$$

The above expression is useful for determining the influence of the coefficients of polynomials (2) on the index of concentration. For a given Fibonacci polynomial q_n the index of concentration $\eta'_{k,m}(n)$ for difference of m terms is defined as the ratio of the concentration indices at degree $k + m$ and k, respectively. Thus we can write

$$\eta'_{k,m}(n) = \frac{\delta'_{k+m}(n)}{\delta'_k(n)} = \frac{\sum_{p=0}^{k+m} {}^p f_{k+m}!}{\prod_{s=k+1}^{k+m} \sum_{p=0}^{k} {}^p f_k!} = 1 + \frac{\sum_{q=k+1}^{k+m} {}^q f_{k+m}!}{\sum_{p=0}^{k+m} {}^p f_k!}. \tag{36}$$

The concept of the index of concentration at fixed degree k of polynomials (1) and (2) is very important for the location of their zeros in the complex plane [21]. Details concerning this problem will be studied in the next section.

3. Zeros of the Fibonacci polynomials

One of the most important characteristics of the Fibonacci polynomials is the location of their zeros in the complex plane. Finding the location of the zeros is one of the major problems, and it is well known that if the polynomial's degree is at least 5, no exact algebraic solution can be given so, in general, the procedure has to be numerical. Here we shall present advantages offered in this domain.

Theorem 1. *All zeros of the polynomials* (1) *with* $n \in \mathbb{N}$ *lie in the annulus* $\frac{1}{2} \leq r \leq 1$ *where* r *denotes the radius of a disk centered at* 0 *in the complex plane.*

Proof. To prove the theorem we consider the real positive coefficients of the polynomials which are determined by the successive Fibonacci numbers. Using the result of Kakeya (see 10, 17]) we obtain the estimate for the absolute values of all zeros of the polynomials (1) as

$$\min\left(\frac{f_k}{f_{k+1}}\right) \leq |x| \leq \max\left(\frac{f_k}{f_{k+1}}\right) \qquad (0 \leq k \leq n-1). \tag{37}$$

Now we take into account (3) and find that

$$\min\left(\frac{f_k}{f_{k+1}}\right) = \left.\frac{f_k}{f_{k+1}}\right|_{k=1} = \frac{1}{2}$$

and

$$\max\left(\frac{f_k}{f_{k+1}}\right) = \left.\frac{f_k}{f_{k+1}}\right|_{k=0} = \frac{f_0}{f_1} = 1.$$

The first zeros $z_{n,k}$ of the polynomials (1) are given in the following table.

n	$x_{n,k}$			
1	-1.0			
2	$-0.25 + i0.6614$	$-0.25 - i0.6614$		
3	-0.7839	$0.0586 + i0.6495$	$0.0586 - i0.6495$	
4	$-0.5337 + i0.45831$	$-0.5337 - i0.45831$	$0.2337 + i0.5912$	$0.2337 - i0.5912$

Table 3. Zeros of polynomials (1)

The prof is completed by substituting the above estimates into (37) ∎

Looking at the above annulus in more detail we can find that the ratio of its maximal and minimal radii equals 2 and the area of the annulus is equal to $A_1 = \frac{3}{4}\pi \simeq 2.35619449\ldots$.

Applying a similar procedure with respect to polynomial (2) we can formulate the following theorem.

Theorem 2. *All zeros of the polynomials* (2) *with* $n \in \mathbb{N}$ *lie in the annulus* $1 \leq r \leq 2$ *where* r *is the radius of the disk centered at* 0 *in the complex plane.*

The proof can be performed in a manner similar to that in the previous case but for the sake of compactness it is omitted here. Note only that the ratio of the disk radii equals 2, but the area of the disk equals

$$A_2 = 3\pi \simeq 9.424777961\ldots .$$

Another interesting result can be derived from the above theorems. It can be easily verified that the difference of the areas of the two introduced disks equals

$$A_2 - A_1 = 3\pi - \frac{3}{4}\pi = \frac{9}{4}\pi = 3A_1 .$$

Thus the study of the locations of zeros of polynomials (1) and (2) in the complex plane leads to a new approach for determining the irrational number π. On the other hand, the sum of these disk areas is equal to

$$A_1 + A_2 = \frac{3}{4}\pi + 3\pi = \frac{15}{4}\pi.$$

Moreover, if the polynomials (1) and (2) represent discrete dynamical systems, then the location of their zeros provides useful information on the stability of such systems.

Tables 3 and 4 give the values of zeros of the polynomials (1) and (2) obtained by MATLAB procedures [12]. They agree well with the theorems above.

n	$q_{n,k}$			
1	-1.0			
2	$-1+i$	$-1-i$		
3	$-0.1732+i1.6033$	$-0.1732-i1.6033$	-1.1537	
4	$0.4091+i1.5141$	$0.4091-i1.5141$	$-1.2424+i0.5940$	$-1.2424-i0.5940$
5	$0.7439+i1.4355$	$0.7439-i1.4355$	$-0.9411+i1.2486$	$-0.9411-i1.2486$

Table 4. Zeros of polynomials (2)

Consider now the zeros of the polynomials (1) and (2) with positive real parts. To do this we first consider the family of polynomials (1) for different $n \geq 0$.

Theorem 3. *For every* $n > 2$, *the polynomial* p_n *in* (1) *has at least one zero with non-negative real part.*

Proof. To prove the above theorem we consider two polynomials (1) with degree n and $n+1$. Observe that using (1) and (3) we have

$$p_{n+1}(x) = p_n(x) + f_{n+1}x^{n+1} \qquad (n \geq 0).$$

Thus, the ratio of polynomials p_{n+1} and p_n can be represented in the form of the continued fraction

$$\begin{aligned}\tau_n(x) &= \frac{p_{n+1}(x)}{p_n(x)} \\ &= 1 + \frac{f_{n+1}x^{n+1}}{p_n(x)} \\ &= 1 + \cfrac{1}{\frac{f_n}{f_{n+1}x} + \frac{p_{n-1}(x)}{f_{n+1}x^{n+1}}} \\ &= 1 + \cfrac{1}{\frac{f_n}{f_{n+1}x} + \cfrac{1}{\frac{f_{n+1}x^2}{f_{n-1}} + \cfrac{1}{\frac{p_{n-2}(x)}{f_{n+1}x^{n+1}}}}} \\ &= 1 + \cfrac{1}{\frac{f_n}{f_{n+1}x} + \cfrac{1}{\frac{f_{n+1}x^2}{f_{n-1}} - \cfrac{1}{\frac{f_{n-1}^2}{f_{n+1}f_{n-2}x} - \cfrac{1}{\frac{f_{n+1}x^2}{f_{n-1}}} + \cdots}}}\end{aligned} \tag{38}$$

It can be easily verified that for $n > 2$ some of the partial coefficients in (38) take negative values. Moreover, the stated theorem follows immediately from the determinental Routh-Hurwitz stability criterion (see [10, 17]), since for $n > 2$ we have $D_1 = f_{n-1}, D_2 = (-1)^n$ and $D_3 = (-1)^n f_{n+1}$ showing that not all D_i can be positive, where D_i $(i = 1,2,3)$ denote the respective determinants. Thus, using results of [3, 10,17] we can state that not all zeros of (1) lie in the left half-plane and this completes the proof ∎

For instance, if $n = 3$, we have

$$\tau_3(x) = \left(1; \frac{3}{5x}; \frac{5x^2}{2}; -\frac{4}{5x}; \frac{5x^2}{2}; \frac{1}{5x}; -5x^2\right)$$

where the short notation has been used for the continued fraction.

Next, for the family of polynomials (2) with different degree $n \geq 0$ we have the following statement.

Theorem 4. *Some of the zeros of the polynomials* (2) *with* $n > 3$ *have non-negative real parts.*

The proof is similar to that above. For instance, if $n = 4$, then using the short notation for continued fraction $\tau_4' = \frac{q_5}{q_4}$ we obtain

$$\tau_4'(z) = \left(1; \frac{8}{5z}; \frac{3z^2}{8}; -\frac{16}{9z}; \frac{9z^2}{8}; \frac{2}{9z}; -\frac{9z^2}{8}; -\frac{16}{9z}; -\frac{3z^2}{8}\right).$$

This indicates that some zeros of the polynomial q_4 are located in the right half-plane.

Let us now focus on further details concerning the index of concentration of the polynomial (1) at low degree. Using the results presented above we are able to estimate the index of concentration $\delta_k(n)$ corresponding to the radius $\frac{1}{a} \leq r \leq 1$ of the disk centered at 0. Taking into account (12) we can formulate the following question: is it possible that polynomials (1) may have k zeros located in the open unit disk with index of concentration $\delta_k(n) \leq \frac{1}{2}$? That question is important for computations with Fibonacci polynomials on parallel processors. To answer it we consider the limit value $\delta_k(n) = \frac{1}{2}$, and using (20) we obtain

$$\rho\left(\frac{1}{2}, k\right) = \left(\frac{1}{1-\frac{1}{2}}\right)^{\frac{1}{k+1}} - 1 = 1 \qquad (0 \leq k \leq n). \tag{39}$$

Solving this equality gives $k = 0$. Thus, using (16) we can state that there exists any polynomial (1) with index of concentration $\delta \leq \frac{1}{2}$ which has some zeros located in the open unit disk centered at 0. It means that the location of the zeros of the polynomial (1) in the open unit disk corresponds to values of the index of concentration greater than $\frac{1}{2}$.

From (20) and (37) we obtain

$$\frac{1}{a} \leq \left(\frac{1}{1-\delta_k(n)}\right)^{\frac{1}{k+1}} - 1 \leq 1 \qquad (0 \leq k \leq n;\ n \geq 0). \tag{40}$$

Solving (40) and using (24) and (16) yield the estimate

$$1 - \frac{1}{a^{k+1}} \leq \frac{f_{k+2}-1}{f_{n+2}-1} \leq 1 - \frac{1}{2^{k+1}}. \tag{41}$$

Further, a relatively simple transformation of (41) leads to the solution

$$0 \leq k \leq n \tag{42}$$

which indicates that for each n and fixed value $\delta_k(n) \in (0,1)$ all zeros of the polynomial (1) are located in the ring with radius $\frac{1}{2} \leq r \leq 1$.

4. Optimizations with Fibonacci polynomials

The Fibonacci polynomials can be used to represent a given physical quantity, for example, voltage in a voltage divider (see [11, 13, 16]), or a fixed number as a sum of suitable components (see [4, 20]). Our particular interest here is the decomposition of a given value g into elements of a Fibonacci polynomial of n-th degree, i.e.

$$g = p_n(x) \tag{43}$$

where n and x are to be determined accordingly. To achieve this goal we take into account (9) and obtain

$$g = \frac{f_n x^{n+2} + f_{n+1} x^{n+1} - f_0}{x^2 + x - 1}. \tag{44}$$

Since n has to be taken from the set of integer numbers, there exists a remainder in (43). Let us denote this remainder as

$$\psi(x,n) = \frac{f_n x^{n+2} + f_{n+1}x^{n+1} - f_0 - (x^2+x-1)g}{x^2+x-1}. \tag{45}$$

Note that (45) depends on two variables: x and n. It means that in an ideal case we must have $\psi(x,n) = 0$ but in reality we are looking for the remainder $\psi(x,n)$ to be minimal for some values $n^* \simeq n$ and $x^* \simeq x$. That is an optimization problem. Thus we can write

$$\psi(x^*,n^*) = min_{n,x}\psi(x,n). \tag{46}$$

Following the standard minimization procedure we obtain two simultaneous equations

$$\frac{\partial\psi(x,n)}{\partial n} = 0 \qquad \text{and} \qquad \frac{\partial\psi(x,n)}{\partial x} = 0 \tag{47}$$

whose solutions yield the desired values of n and x. Performing necessary calculations we obtain

$$x^{n+2}\frac{df_n}{dn} + f_n x^{n+2}\ln x + x^{n+1}\frac{df_{n+1}}{dn} + f_{n+1}x^{n+1}\ln x = 0 \tag{48$_a$}$$

and

$$\begin{gathered}\Big[(n+2)f_n x^{n+1} + (n+1)f_{n+1}x^n - g(2x+1)\Big] \\ -\Big[f_n x^{n+2} + f_{n+1}x^{n+1} - (x^2+x-1)g\Big](2x+1) = 0.\end{gathered} \tag{48$_b$}$$

To determine the solution of (48) we need to use a special procedure due to the difficulties in the direct calculation of the derivative of the Fibonacci number f_n with respect to n. This special procedure is ased on a well known approximation of (4) by the expression

$$f_n \simeq \frac{a^{n+1}}{a-b} + O(n) \qquad (n \geq 2) \tag{49}$$

(see [8, 18, 19]) where $O(n)$ denotes a small term. Neglecting $O(n)$ in (49), we can represent succesive derivatives of f_n as

$$\frac{df_n}{dn} = \frac{a^{n+1}}{a-b}\ln a \qquad \text{and} \qquad \frac{df_{n+1}}{dn} = \frac{a^{n+2}}{a-b}\ln a. \tag{50}$$

Now, substituting (50) into equation $(48)_a$ gives

$$\frac{1}{a-b}\Big(x^{n+2}a^{n+1}\ln a + x^{n+2}a^{n+1}\ln x + x^{n+1}a^{n+2}\ln a + x^{n+1}a^{n+2}\ln x\Big) = 0. \tag{51}$$

It is clear that the condition $x > 0$ has to be satisfied, so we find that

$$(x+a)\ln(ax) = 0 \tag{52}$$

from which we obtain one component of the optimal solution

$$x^* = \frac{1}{a}. \tag{53}$$

Note that (53) determines the singular value for (9) and for this reason we need to take (22) to represent the given value g. Thus we can write

$$g \simeq \frac{(n+1)a}{a-b} - \frac{1}{(a-b)^2}\left(\left(\frac{b}{a}\right)^{n+1} - 1\right).$$

Solving this with respect to n gives the second component n^* of the desired optimal solution. For example, for $g = 3$, values of n^* and x^* for the optimal representation in terms of $p_n(x)$ are

$$n^* = 3 \qquad \text{and} \qquad x^* = \frac{1}{a} \simeq 0.6180339\ldots .$$

Finally, we can write

$$f_0 + \frac{f_1}{a} + \frac{f_2}{a^2} + \frac{f_3}{a^3} = 1 + \frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} \simeq 3.09017\ldots \simeq g$$

what is a good representation of the fixed value $g = 3$.

It has to be emphasized that the above problem can be also considered in terms of polynomial equations in two variables. Fixing one of them, preferably n, we can always find the unique positive value of x.

Our second problem concerning optimization with Fibonacci polynomials deals with a partition of a given value h into n parts in such a way that the first element is big as possible, while the other elements are smaller in ratios determined by the corresponding Fibonacci numbers. Denoting by h' the biggest part in such decompositions we have the relation

$$\Theta = \frac{h}{h'} = \sum_{k=0}^{n} \frac{z^k}{f_k}. \tag{54}$$

It is easily seen that the variable z appears as a control parameter. Such problems appear very often in practice, for example, in power systems where a number of loads are connected to a system but one of them is the most important and needs a fixed amount of electric energy supplied from a real source.

Thus we have two optimization problems. The first one appears when h and h' are fixed and we need to find the optimal solution of (54) with respect to n and z. The second problem appears when h and n are given and the optimal solution concerns h' and z. Observe that in the first case we can take $z = 1$ and then the problem is to determine n. Thus, using (28) we can write

$$\Theta = \frac{\sum_{p=0}^{n} {}^{p}f_p!}{f_n!}. \tag{55}$$

Note first that from (2), (3) and (32) we obtain $1 \le \Theta \le Q_\infty(1)$ so as $n \to \infty$ we have

$$h'_\infty = \frac{h}{Q_\infty(1)} = \min_n h'.$$

Thus for fixed $n < \infty$ we have

$$h' > h'_\infty.$$

The corresponding value of n is a solution of the equation

$$\frac{\Theta}{a-b}\prod_{p=0}^{n}(a^{k+1} - b^{k+1}) = \sum_{p=0}^{n}\prod_{k=0}^{n\ \prime}(a^{k+1} - b^{k+1}) \tag{56}$$

where superscript $'$ says that the term with index p has to be omitted.

Observe that this represents a complex nonlinear equation in n, and its solution can be found in a numerical process. In many practical calculations we can apply the result presented in Table 2 to solve (56) (equivalently (55)). For instance, let $h = 10$ and $h' = 4$. Then we have $\Theta = 2.5$ and from Table 2 we find directly that $n = 2$. Note that an alternative approach which takes the similar form as that with the polynomials (1) can be also applied.

The second problem of optimization with the polynomials (2) is much more complicated than the previous one. This problem is still under research.

5. Final discussion and concluding remarks

In this paper we have discussed new types of polynomials, $p_n = p_n(x)$ (see (1)) and $q_n = q_n(z)$ (see (2)), characterized by coefficients equal to the successive Fibonacci numbers or their inverses, respectively. A common feature of these polynomials is that all manipulations on their components are simple and can be easily implemented on a computer. It was proved that both polynomials lead to effective methods for establishing many of the Fibonacci identities.

The results obtained in this paper seem to be very useful in applications, e.g., in optimizations of electric networks, capacitors voltage dividers and other plants where more classical numerical treatments are difficult or at least expensive. Moreover, such characteristic properties of these new polynomials as the location of zeros on particular rings in the complex plane, their indices of concentration at low degree have been examined.

As an additional theoretical benefit, these new polynomials allowed an easy description of extremal terms in the product result, as well as the influence of a control variable when the partition of a given value into many smaller parts must be controlled appropriately. Our results also show that if we have some additional information about a given quantity or simple number we can, in special cases, get sharper bounds on its decompositions or partitions. This result can be considered as an alternative to solutions of such type problems in terms of polynomial equations in two variables.

Another area where interesting results may be obtained is that of index of concentration at low degree of products of polynomials p_n and q_m (both polynomials in $x \in \mathbb{R}$ and, in general case, of different degrees). Note that the polynomials p_n have large coefficients and the polynomials q_m are characterized by high concentrations at low degrees. But their product has a large coefficient. It is quite likely (but not proved yet) that the correct order of magnitude is of exponential type. The determination of the precise values of the estimates does not only involve computational accuracy but a better understanding of the problem itself.

It is worth mentioning that polynomials p_n and q_m have no common root (except for p_1 and q_1). Thus, following the Bezout identity (see [10: Chapter X/p. 200]), we obtain that there exist two other polynomials v_m and w_n such that $p_n v_m + q_m w_n = 1$. The problem of determining v_m and w_n is still open.

An interesting point related to the presented studies is that the polynomials p_n are solutions to the non-homogeneous recurrence

$$y_{n+1} - x y_n - x^2 y_{n-1} = 1 \qquad (n \geq 0) \tag{57}$$

with $y_{-1} = 0$ and $y_0 = 1$. To describe the solutions to this recurrence, we may use a general approach for solutions of second order difference equations with constant right-hand side term [8]. Thus under the condition

$$1 - x - x^2 \neq 0$$

we can solve (68) by superposition of the general and particular solutions. We obtain

$$y_n = A_1 \eta_1^n + A_2 \eta_2^n + \frac{1}{1 - x - x^2} \tag{58}$$

where η_1, η_2 and A_1, A_2 denote the roots of the corresponding characteristic equation and two arbitrary constants, respectively. Applying the whole procedure for determination of the solution corresponding to the given initial values we obtain

$$y_n = \frac{f_n x^{n+2} + f_{n+1} x^{n+1} - f_0}{x^2 + x - 1}. \tag{59}$$

Thus, comparing the right-hand sides of (9) and (58) we can state that

$$y_n = p_n(x). \tag{60}$$

It is now obvious that the polynomials p_n $(n \geq 0)$ are determined by the recurrence

$$p_{n+1}(x) - x p_n(x) - x^2 p_{n-1}(x) = 1 \qquad (n \geq 0)$$

with $p_{-1} = 0$ and $p_0 = 1$. In this way we have obtained an additional relation which can be considered as generating the Fibonacci numbers (successive coefficients of polynomial (60)).

The similar problem with polynomial (2) is still under investigation. It can be shown that the recurrence

$$q_{n+1}(z) = q_n(z) + \frac{z^{n+1}}{f_{n+1}} \qquad (n \geq 0)$$

holds true with $q_{-1} = 0$ and $q_0 = 1$. To get a more explicit version which could be free of the Fibonacci numbers some additional studies are needed.

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