

Second Order Sufficient Optimality Conditions for a Nonlinear Elliptic Boundary Control Problem

E. Casas, F. Tröltzsch and A. Unger

Abstract. In this paper sufficient second order optimality conditions are established for optimal control problems governed by a linear elliptic equation with nonlinear boundary condition, where pointwise constraints on the control are given. The second order condition requires coercivity of the Lagrange function on a suitable subspace together with first order sufficient conditions on a certain set of strongly active points.

Keywords: *Optimal control, semilinear elliptic equations, second order conditions, sufficient optimality conditions*

AMS subject classification: Primary 49K20, secondary 49K27

1. Introduction

Optimal control problems governed by nonlinear elliptic partial differential equations have already been considered by many authors. We refer only to the papers of Casas [2, 3], to the book of Tiba [16], and to the references therein. Meanwhile, the existence of optimal controls and first order necessary optimality conditions of the maximum-principle type are well investigated. It is known that in the case of nonlinear equations the first order conditions are in general not sufficient for optimality. In this paper we are going to derive a second order sufficient optimality condition for a class of semilinear elliptic boundary control problems. For parabolic boundary control problems second order sufficient optimality conditions were established in papers by Goldberg and Tröltzsch [7, 8]. It is more or less obvious that these optimality conditions can be transferred by the same technique to elliptic problems. However, a comparison of the results in [7, 8] with second order optimality conditions for optimization problems in spaces of finite dimension reveals that the gap between the sufficient optimality conditions in [7, 8] and corresponding second order necessary optimality conditions is quite large. Taking into account the active set of optimal controls, this gap can be partially

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This research was partially supported by Deutsche Forschungsgemeinschaft (DFG), under Project number Tr 302/1-2.

closed. The known difficulty in the theory of sufficient optimality conditions for extremal problems in function spaces is the so-called two-norm discrepancy. This problem was resolved, for instance, by Ioffe [10] and Maurer [14].

In recent years further considerations have shown that some weaker sufficient optimality conditions can be established for abstract optimization problems in Banach spaces and for optimal control problems governed by ordinary differential equations. We refer, for instance, to Dontchev et al. [5] and Malanowski [13]. The main idea is to introduce a third norm taking into account the measure of positivity of some terms occurring in the variational inequality. We will use this idea to derive a second order sufficient optimality condition in the case of elliptic partial differential equations.

2. Formulation of the optimal control problem

We consider the *optimal control problem* to minimize

$$F(w, u) = \int_{\Omega} \varphi(w(x)) dx + \int_{\Gamma} \psi(u(x), w(x)) dS_x \quad (1)$$

subject to

$$\begin{aligned} -\Delta w(x) + w(x) &= 0 && \text{in } \Omega \\ \frac{\partial w}{\partial n}(x) &= b(w(x), u(x)) && \text{on } \Gamma \end{aligned} \quad (2)$$

and to the constraints on the control $u \in \mathcal{U}^{ad}$, where

$$\mathcal{U}^{ad} = \left\{ v \in L_{\infty}(\Gamma) \mid u_a \leq v(x) \leq u_b \text{ a.e. on } \Gamma \right\} \quad (3)$$

where $u_a, u_b \in \mathbb{R}$ with $u_a \leq u_b$. \mathcal{U}^{ad} is a non-empty, bounded, convex, and closed subset of $L_{\infty}(\Gamma)$. In this setting $\frac{\partial w}{\partial n}$ denotes the normal derivative (in the direction of the outward normal vector n on Γ), and dS_x is the Lebesgue surface measure defined on Γ .

The solution of the boundary value problem (2) is considered in the following weak sense.

Definition 1. A function $w \in W_2^1(\Omega)$ is said to be a *weak solution* of the boundary value problem (2), if for all $v \in W_2^1(\Omega)$ the equation

$$\int_{\Omega} \left(\sum_{i=1}^n \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} + wv \right) dx = \int_{\Gamma} b(w, u)v dS_x \quad (4)$$

is valid.

To make the optimal control problem (1) - (3) well defined we impose the following assumptions (A1) and (A2):

- (A1) The function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and all derivatives up to the second order are globally Lipschitz continuous. The functions $\psi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are

twice continuously differentiable with respect to both arguments. Furthermore, the function b is monotone decreasing with respect to the first argument and all partial derivatives of ψ and b up to the second order are globally Lipschitz continuous.

This property of the function b guarantees the existence of a unique weak solution of the boundary value problem (2) for each fixed $u \in \mathcal{U}^{ad}$ (cf. Kinderlehrer and Stampacchia [11]).

(A2) The set $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded domain with a sufficiently smooth boundary Γ such that the weak solution w of the boundary value problem (2) is an element of $W_p^1(\Omega)$ with some fixed $p > n$ and $W_p^1(\Omega)$ is continuously embedded into $C(\bar{\Omega})$. Moreover, we suppose that for all admissible controls $u_1, u_2 \in \mathcal{U}^{ad}$ and the corresponding solutions w_1 and w_2 of the boundary value problem (2) the estimate

$$\|w_1 - w_2\|_{W_p^1(\Omega)} \leq c_p \|u_1 - u_2\|_{L_p(\Gamma)} \tag{5}$$

is valid.

Note that assumption (A2) implies

$$\|w_1 - w_2\|_{C(\bar{\Omega})} \leq c \|w_1 - w_2\|_{W_p^1(\Omega)} \leq c' \|u_1 - u_2\|_{L_p(\Gamma)} \leq c_\infty \|u_1 - u_2\|_{L_\infty(\Gamma)}.$$

We have stated assumption (A2) in order to underline that our method can be transferred to more general elliptic equations, for instance including some elliptic operators with bounded and measurable coefficients.

Next, we shall show that assumption (A2) can be fulfilled for sufficiently regular domains Ω . For instance, suppose that Ω is a bounded domain with a C^1 -boundary Γ . Then we can use the following regularity result of Gröger.

Lemma 1 (see Gröger [9]). *The weak solution w of the linear boundary value problem*

$$\begin{aligned} -\Delta w + w &= 0 && \text{in } \Omega \\ \frac{\partial w}{\partial n} &= g && \text{on } \Gamma \end{aligned} \tag{6}$$

is an element of $W_{p'}^1(\Omega)$ ($1 < p' < \infty$) provided the function g is an element of $L_{p'}(\Gamma)$. Let w_1 and w_2 denote the solutions of the equation (6) corresponding to $g_1, g_2 \in L_{p'}(\Gamma)$, respectively. Then the estimate

$$\|w_1 - w_2\|_{W_{p'}^1(\Omega)} \leq c_L \|g_1 - g_2\|_{L_{p'}(\Gamma)}$$

is valid, where the constant c_L does not depend on the choice of g_1 and g_2 .

This lemma ensures the assumed regularity of the weak solution of problem (2). To verify estimate (5), we proceed in the following way.

A unique solution $w \in W_2^1(\Omega)$ of the state equation (2) exists due to the assumptions (A1) and (A2) on b and Ω . Furthermore, we have for the solutions w_1 and w_2 of the state equation (2) corresponding to admissible controls u_1 and u_2 from \mathcal{U}^{ad} , respectively, the following:

$$\begin{aligned} \|w_1 - w_2\|_{W_2^1(\Omega)}^2 &= \int_{\Omega} \left(\sum_{i=1}^n \left(\frac{\partial(w_1 - w_2)}{\partial x_i} \right)^2 + (w_1 - w_2)^2 \right) dx \\ &= \int_{\Gamma} \Gamma(b(w_1, u_1) - b(w_2, u_2))(w_1 - w_2) dS_x \\ &= \int_{\Gamma} \Gamma(b(w_1, u_1) - b(w_2, u_1))(w_1 - w_2) dS_x \\ &\quad + \int_{\Gamma} \Gamma(b(w_2, u_1) - b(w_2, u_2))(w_1 - w_2) dS_x. \end{aligned}$$

The first integrand is non-positive. Using the trace theorem and the Lipschitz property of b , we conclude finally

$$\|w_1 - w_2\|_{W_2^1(\Omega)} \leq c \|u_1 - u_2\|_{L_2(\Omega)}.$$

The trace theorem shows that the trace τw on Γ is in $W_2^{1/2}(\Gamma)$. This space is embedded into $L_{p^1}(\Gamma)$ with some $p^1 = 2 \frac{n-1}{n-2} > 2$ because of known embedding theorems. In the case $n = 2$ we can choose for instance $p^1 = 3$. Owing to the Lipschitz property of b and to the inclusion $u \in L_{\infty}(\Gamma)$, we can identify the function $b(w(\cdot), u(\cdot))$ with a function $g \in L_{p^1}(\Gamma)$. Lemma 1 implies that the solution w belongs to $W_{p^1}^1(\Omega)$. Moreover, Lemma 1 and the estimate of $\|w_1 - w_2\|$ in $W_2^1(\Omega)$ imply with a generic constant c

$$\begin{aligned} \|w_1 - w_2\|_{W_{p^1}^1(\Omega)} &\leq c \|b(w_1, u_1) - b(w_2, u_2)\|_{L_{p^1}(\Gamma)} \\ &\leq c (\|\tau w_1 - \tau w_2\|_{L_{p^1}(\Gamma)} + \|u_1 - u_2\|_{L_{p^1}(\Gamma)}) \\ &\leq c (c' \|w_1 - w_2\|_{W_2^1(\Omega)} + \|u_1 - u_2\|_{L_{p^1}(\Gamma)}) \\ &\leq c \|u_1 - u_2\|_{L_{p^1}(\Gamma)}. \end{aligned}$$

So we have proved property (A2), if $n = 2$. In the case $n > 2$ we repeat the considerations starting with p^1 instead of 2. In this way the solution w of the state equation (2) is seen to be an element of $W_{p^2}^1(\Omega)$ with some $p^2 > p^1$, and the stated estimate can be proved in $W_{p^2}^1(\Omega)$. Repeating this bootstrapping argument sufficiently often, we construct a sequence $\{p^i\}$, where $p^{i+1} > p^i$. It can be shown (see Eppler and Unger [6]) that it is possible to choose $p_0 = 2$ and $p^{i+1} = p^i + \frac{2}{n-2}$ for $i = 1, 2, \dots$. So we are sure to arrive at $p > n$ after a finite number of steps. The assumed estimate follows from the last estimation replacing the spaces $W_{p^i}^1(\Omega)$, $L_{p^i}(\Gamma)$ and $W_2^1(\Omega)$ by $W_{p^{i+1}}^1(\Omega)$, $L_{p^{i+1}}(\Gamma)$ and $W_{p^i}^1(\Omega)$ for $i = 1, 2, \dots$, respectively. As a further corollary we have that the norm of all feasible states u is uniformly bounded in $W_p^1(\Omega)$, since \mathcal{U}^{ad} is bounded.

If we assume in addition that the cost functional $F(w, u)$ is convex with respect to u and the function b in (2) is linear with respect to u , the existence of a solution of the

optimal control problem (1) - (3) follows by standard methods (cf., for instance, Eppler and Unger [6]). However, we shall not rely on this assumption. We just suppose that a fixed *reference control* u_0 is given, satisfying certain optimality conditions.

Let us introduce the *Lagrange function* $\mathcal{L} = \mathcal{L}(w, u, y)$ for our optimal control problem (1) - (3) in the following way:

$$\begin{aligned} \mathcal{L}(w, u, y) = & \int_{\Omega} \varphi(w) dx + \int_{\Gamma} \psi(w, u) dS_x \\ & - \int_{\Omega} \left(\sum_{i=1}^n \frac{\partial w}{\partial x_i} \frac{\partial y}{\partial x_i} + wy \right) dx + \int_{\Gamma} b(w, u)y dS_x. \end{aligned}$$

This function is well defined and twice continuously differentiable with respect to the space $(C(\bar{\Omega}) \cap W_2^1(\Omega)) \times L_{\infty}(\Gamma) \times (C(\bar{\Omega}) \cap W_2^1(\Omega))$.

Standard considerations apply to derive first order necessary optimality conditions at u_0 . They can be written in the form

$$\mathcal{L}_w(w_0, u_0, y_0) = 0 \quad (\text{adjoint equation})$$

and

$$\mathcal{L}_u(w_0, u_0, y_0)(u - u_0) \geq 0 \text{ for all } u \in \mathcal{U}^{ad} \quad (\text{variational inequality}).$$

Subscripts denote as usual associated partial derivatives. In detail, the adjoint equation for $y = y_0$ reads

$$\begin{aligned} -\Delta y + y &= \varphi_w(w_0) && \text{in } \Omega \\ \frac{\partial y}{\partial n} &= b_w(w_0, u_0)y + \psi_w(w_0, u_0) && \text{on } \Gamma. \end{aligned}$$

The variational inequality admits the form

$$\int_{\Gamma} (\psi_u(w_0, u_0) + b_u(w_0, u_0)y_0)(u - u_0)dS_x \geq 0 \quad \text{for all } u \in \mathcal{U}^{ad}.$$

To simplify the notation we denote in the sequel the pair (w, u) by v .

Definition 2. The pair $v = (w, u)$ is called *admissible*, if u belongs to \mathcal{U}^{ad} and w is the weak solution of the boundary value problem (2) corresponding to u .

Two norms of $v = (w, u)$ corresponding to the spaces

$$V_2 = W_2^1(\Omega) \times L_2(\Gamma) \quad \text{and} \quad V_{\infty} = W_p^1(\Omega) \times L_{\infty}(\Gamma)$$

are given by

$$\|v\|_2 = \left(\|w\|_{W_2^1(\Omega)}^2 + \|u\|_{L_2(\Gamma)}^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \|v\|_{\infty} = \max \left\{ \|w\|_{W_p^1(\Omega)}, \|u\|_{L_{\infty}(\Gamma)} \right\},$$

respectively.

The simplest second order sufficient optimality condition which can be transferred to our elliptic optimality problem (1) - (3) from the parabolic case can be formulated as follows:

Assume that for an admissible pair $v_0 = (w_0, u_0)$ the first order necessary optimality conditions are satisfied. Let y_0 be the associated adjoint state. Suppose the existence of an $\alpha > 0$ such that the second order derivative of the Lagrange function $\mathcal{L} = \mathcal{L}((w, u, y)$ with respect to w and u fulfils the estimate

$$\mathcal{L}_{vv}(v_0, y_0)[(h, \delta), (h, \delta)] \geq \alpha \|(h, \delta)\|_2^2 \tag{7}$$

for all (h, δ) , where $\delta = u - u_0$ with $u \in \mathcal{U}^{ad}$ and h is the weak solution of the corresponding linearized equation

$$\begin{aligned} -\Delta h + h &= 0 && \text{in } \Omega \\ \frac{\partial h}{\partial n} &= b_w(v_0)h + b_u(v_0)\delta && \text{on } \Gamma. \end{aligned}$$

In this condition, the second order derivative of the Lagrange function \mathcal{L} has to fulfil the estimate with respect to *all* admissible controls u . From the theory of optimization in spaces of finite dimensions weaker second order sufficient optimality conditions are known. More precisely, an estimate of type (7) has to be fulfilled only with respect to all u from some subset of the admissible set \mathcal{U}^{ad} . Using ideas of Dontchev et al. [5] and Malanowski [13] we will derive such a sufficient optimality condition in the next sections.

3. Motivation of a second order sufficient optimality condition

Let us consider the following mathematical programming problem:

(P) Minimize $f(x)$ subject to $x \in \mathbb{R}^k$ and $h(x) = 0$ as well as $g(x) \geq 0$

where f, h and g are smooth functions. For this problem the following second order sufficient optimality condition is known (cf., for instance, Collatz and Wetterling [4] or Spellucci [15]).

Theorem 1. *Let x^* be an admissible point of problem (P) satisfying the first order necessary optimality condition with associated Lagrange multipliers μ^* and λ^* . Then the point x^* is a strong local minimizer, if an $\alpha > 0$ exists such that*

$$z^T \left(\nabla^2 f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla^2 g_i(x^*) - \sum_{j=1}^k \mu_j^* \nabla^2 h_j(x^*) \right) z \geq \alpha z^T z \tag{8}$$

for all z satisfying the conditions

$$z^T \nabla h_i(x^*) = 0 \tag{9}$$

$$z^T \nabla g_i(x^*) \geq 0 \quad \text{if } g_i(x^*) = 0 \tag{10}$$

$$z^T \nabla g_i(x^*) = 0 \quad \text{if } \lambda_i^* > 0. \tag{11}$$

Comparing this sufficient optimality condition with the sufficient optimality condition (7) for the elliptic control problem (1) - (3) we observe:

Estimate (8) corresponds to estimate (7). Equation (9) can be viewed as a representation of the linearized partial differential equation in the sufficient optimality condition for the control problem (1) - (3). Inequality (10) means that $\tilde{x} = x^* + \alpha z$ is admissible with respect to the linearized inequality constraints for sufficiently small α . In the context of the control problem (1) - (3) this corresponds to restrictions on δ in the sufficient optimality condition for the elliptic control problem (1) - (3). However, there is no corresponding term for condition (11). It is this additional condition, which we shall add to the sufficient optimality condition.

Usually, necessary optimality conditions for optimal control problems have to be satisfied for all u from some control set \mathcal{U}^{ad} . To simplify the presentation, let this set be described by

$$\mathcal{U}^{ad} = \left\{ u \in L_\infty(\Gamma) \mid u(x) \geq 0 \text{ a.e. on } \Gamma \right\}.$$

Introducing formally a Lagrange multiplier function λ with respect to the inequality constraints, the variational inequality reads

$$\int_{\Gamma} (\psi_u(w_0, u_0) + b_u(w_0, u_0)y_0 - \lambda)(u - u_0) dS_x \geq 0$$

for all u from the whole space $L_\infty(\Gamma)$. Thus, the left factor under the integral sign has to vanish identically on Γ . In this way, we can identify λ with $\psi_u(w_0, u_0) + b_u(w_0, u_0)y_0$ in some sense. The formal multiplier function λ plays in this case the same role as the vector λ^* in Theorem 1. A suitable interpretation of condition (11) on z is now given in the case of the optimal control problem (1) - (3) by

$$u(x) - u_0(x) = 0 \quad \text{if } \psi_u(w_0(x), u_0(x)) + b_u(w_0(x), u_0(x))y_0(x) \geq \varepsilon > 0.$$

In the next section we will modify this intuitive additional condition and prove the sufficiency of a corresponding second order optimality condition.

4. Second order sufficient optimality condition

In what follows let $v_0 = (w_0, u_0)$ be an admissible reference pair for the optimal control problem (1) - (3). We do not assume that v_0 achieves the global minimum of this optimal control problem. However, we suppose that the first order necessary optimality conditions are fulfilled at v_0 with the associated adjoint state y_0 . This is the *optimality system*

$$\left. \begin{aligned}
 & \int_{\Gamma} (\psi_u(v_0) + b_u(v_0)y_0)(u - u_0) dS_x \geq 0 \text{ for all } u \in \mathcal{U}^{ad} \quad (\text{var. inequ.}) \\
 & -\Delta y_0 + y_0 = \varphi_w(v_0) \quad \text{in } \Omega \\
 & \frac{\partial y_0}{\partial n} = b_w(v_0)y_0 + \psi_w(v_0) \quad \text{on } \Gamma \quad (\text{adjoint equ.}) \\
 & -\Delta w_0 + w_0 = 0 \quad \text{in } \Omega \\
 & \frac{\partial w_0}{\partial n} = b(w_0, u_0) \quad \text{on } \Gamma \quad (\text{state equ.})
 \end{aligned} \right\} \quad (12)$$

In other words, $v_0 = (w_0, u_0)$ is a stationary pair, which need not be optimal in any sense. It is well known that the variational inequality holds if and only if

$$(\psi_u(v_0(x)) + b_u(v_0(x))y_0(x))(u - u_0(x)) \geq 0$$

almost everywhere on Γ for all $u \in [u_a, u_b]$.

Next, we define a set of positivity.

Definition 3. For $\epsilon > 0$ the set of positivity Γ_ϵ denotes the measurable subset of Γ such that

$$\left| \psi_u(v_0(x)) + b_u(v_0(x))y_0(x) \right| \geq \epsilon$$

holds for almost all $x \in \Gamma_\epsilon$.

Remark. Owing to the assumptions on Ω , the adjoint state y_0 is continuous on $\bar{\Omega}$. Thus, the function $\psi_u(v_0) + b_u(v_0)y_0$ is measurable on Γ , and the definition makes sense. It is possible that the set of positivity is of measure zero.

Corollary 1. *The estimate*

$$\int_{\Gamma_\epsilon} (\psi_u(v_0(x)) + b_u(v_0(x))y_0(x))(u(x) - u_0(x)) dS_x \geq \epsilon \|u - u_0\|_{L_1(\Gamma_\epsilon)}$$

holds true for all $u \in \mathcal{U}^{ad}$.

Before we start to discuss second order sufficient optimality conditions we derive some useful technical results. In all what follows, c denotes a generic constant. At first we remind of the continuity of the adjoint state y_0 . Therefore, the assumptions on the functions standing in the objective (1) and in the equations (2) allow us to prove the following statements.

Lemma 2. *The second order derivative of the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$ at (v_0, y_0) with respect to v fulfils the estimates*

$$\left| \mathcal{L}_{vv}(v_0, y_0)[(h_1, \delta_1), (h_2, \delta_2)] \right| \leq c_{\mathcal{L}} \|(h_1, \delta_1)\|_2 \|(h_2, \delta_2)\|_2 \tag{13}$$

and

$$\left| \mathcal{L}_{vv}(v_0, y_0)[(h, \delta), (h, \delta)] - \mathcal{L}_{vv}(v, y_0)[(h, \delta), (h, \delta)] \right| \leq C_{\mathcal{L}} \|v - v_0\|_{\infty} \|(h, \delta)\|_2^2$$

for all $(h, \delta), (h_i, \delta_i); v \in W_2^1(\Omega) \times L_2(\Gamma)$.

Proof. The second order derivative \mathcal{L}_{vv} of the Lagrange function \mathcal{L} is given by

$$\begin{aligned} & \mathcal{L}_{vv}(v_0, y_0)[(h_1, \delta_1), (h_2, \delta_2)] \\ &= \int_{\Gamma} \left(b_{ww}(v_0)h_1h_2 + b_{wu}(v_0)h_1\delta_2 + b_{uw}(v_0)\delta_1h_2 + b_{uu}(v_0)\delta_1\delta_2 \right) y_0 dS_x \\ &+ \int_{\Gamma} \left(\psi_{ww}(v_0)h_1h_2 + \psi_{wu}(v_0)h_1\delta_2 + \psi_{uw}(v_0)\delta_1h_2 + \psi_{uu}(v_0)\delta_1\delta_2 \right) dS_x \\ &+ \int_{\Omega} \varphi_{ww}(w_0)h_1h_2 dx. \end{aligned}$$

The control u_0 is bounded in the sense of $L_{\infty}(\Gamma)$, and the state w is bounded in the sense of $C(\bar{\Omega})$. Therefore, all second order derivatives under the integral sign are uniformly bounded in the sense of $L_{\infty}(\Gamma)$. The adjoint state y_0 is continuous on $\bar{\Omega}$. For that reason we are able to estimate

$$\begin{aligned} & \left| \mathcal{L}_{vv}(v_0, y_0)[(h_1, \delta_1), (h_2, \delta_2)] \right| \\ & \leq c \left| \int_{\Omega} h_1h_2 dx + \int_{\Gamma} \left(h_1h_2 + h_1\delta_2 + \delta_1h_2 + \delta_1\delta_2 \right) dS_x \right. \\ & \quad \left. + \int_{\Gamma} \left(h_1h_2 + h_1\delta_2 + \delta_1h_2 + \delta_1\delta_2 \right) dS_x \right| \\ & \leq c \|(h_1, \delta_1)\|_2 \|(h_2, \delta_2)\|_2. \end{aligned}$$

To prove the second estimate we easily find

$$\begin{aligned} & \mathcal{L}_{vv}(v_0, y_0)[(h, \delta), (h, \delta)] - \mathcal{L}_{vv}(v, y_0)[(h, \delta), (h, \delta)] \\ &= \int_{\Omega} (\varphi_{ww}(w_0) - \varphi_{ww}(w))h^2 dx + \int_{\Gamma} \left\{ (\psi_{ww}(v_0) - \psi_{ww}(v))h^2 \right. \\ & \quad \left. + 2(\psi_{wu}(v_0) - \psi_{wu}(v))h\delta + (\psi_{uu}(v_0) - \psi_{uu}(v))\delta^2 \right\} dS_x \\ &+ \int_{\Gamma} \left\{ (b_{ww}(v_0) - b_{ww}(v))h^2 \right. \\ & \quad \left. + 2(b_{wu}(v_0) - b_{wu}(v))h\delta + (b_{uu}(v_0) - b_{uu}(v))\delta^2 \right\} y_0 dS_x. \end{aligned}$$

The second order derivatives of φ , ψ and b are assumed to be Lipschitz continuous. Therefore,

$$\begin{aligned} & \left| \mathcal{L}_{vv}(v_0, y_0)[(h, \delta), (h, \delta)] - \mathcal{L}_{vv}(v, y_0)[(h, \delta), (h, \delta)] \right| \\ & \leq c \left(\int_{\Omega} |w_0 - w| h^2 dx + \int_{\Gamma} (|v_0 - v| h^2 + 2|v_0 - v| |h| |\delta| + |v_0 - v| \delta^2) dS_x \right. \\ & \quad \left. + \int_{\Gamma} (|v_0 - v| h^2 + 2|v_0 - v| |h| |\delta| + |v_0 - v| \delta^2) y_0 dS_x \right). \end{aligned}$$

Now the second estimate follows immediately ■

The next results are concerned with properties of remainder terms. We will use the following notations:

$$\begin{aligned} r_1^b(v_0, v) &= b(v) - b(v_0) - b_v(v_0)(v - v_0) \\ & \text{(first order remainder term of } b \text{ at } v_0) \end{aligned}$$

$$\begin{aligned} r_2^{\mathcal{L}}(v_0, y_0, v) &= \mathcal{L}(v, y_0) - \mathcal{L}(v_0, y_0) - \mathcal{L}_v(v - v_0) - \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\ & \text{(second order remainder term of } \mathcal{L} \text{ at } (v_0, y_0)) \end{aligned}$$

with $b_v(v_0)(v - v_0) = b_w(v_0)(w - w_0) - b_u(v_0)(u - u_0)$. In the sequel the sign $|l|$ denotes either the absolute value, if l is a real number, or the Euclidean length of l , if l is a vector.

Unfortunately, the Nemytskii operator b is not Fréchet differentiable from $L_2(\Gamma) \times L_2(\Gamma)$ to $L_2(\Gamma)$, since

$$\frac{\|r_1^b(v_0, v)\|_{L_2(\Gamma)}}{\|v - v_0\|_{L_2(\Gamma)}} \not\rightarrow 0$$

as $\|v - v_0\|_{L_2(\Gamma)} \rightarrow 0$. However, we have

Lemma 3. For the first order remainder term $r_1^b(v_0, v)$ of b at v_0 the estimate

$$\frac{\|r_1^b(v_0, v)\|_{L_2(\Gamma)}}{\|v - v_0\|_2} \leq c_r \|v - v_0\|_{\infty} \tag{14}$$

is valid for all $v \in W_p^1(\Omega) \times L_{\infty}(\Gamma)$.

Proof. We have by definition of r_1^b and the known mean value theorem in integral form:

$$\begin{aligned} \|r_1^b(v_0, v)\|_{L_2(\Gamma)}^2 &= \int_{\Gamma} (b(v) - b(v_0) - b_v(v_0)(v - v_0))^2 dS_x \\ &= \int_{\Gamma} \left(\int_0^1 (b_v(v_0 + s(v - v_0)) - b_v(v_0))(v - v_0) ds \right)^2 dS_x \\ &\leq \int_{\Gamma} \left(\int_0^1 c |s(v - v_0)| |v - v_0| ds \right)^2 dS_x \end{aligned}$$

(the last estimate follows from the Lipschitz property of $b_v = (b_w, b_u)$). Therefore,

$$\begin{aligned} \|r_1^b(v_0, v)\|_{L_2(\Gamma)}^2 &\leq c \int_{\Gamma} |v - v_0|^4 dS_x \\ &\leq c \|v - v_0\|_{C(\bar{\Omega}) \times L_{\infty}(\Gamma)}^2 \int_{\Gamma} |v - v_0|^2 dS_x \\ &\leq c \|v - v_0\|_{\infty}^2 \int_{\Gamma} |v - v_0|^2 dS_x. \end{aligned}$$

Note that $W_p^1(\Omega)$ is continuously embedded in $C(\bar{\Omega})$. So we end up with

$$\|r_1^b(v_0, v)\|_{L_2(\Gamma)}^2 \leq c \|v - v_0\|_{\infty}^2 \|v - v_0\|_2^2$$

and the lemma is proved ■

Lemma 4. For the second order remainder term $r_2^{\mathcal{L}}(v_0, y_0, v)$ of \mathcal{L} at (v_0, y_0) the estimate

$$\frac{|r_2^{\mathcal{L}}(v_0, y_0, v)|}{\|v - v_0\|_2^2} \leq c_r \|v - v_0\|_{\infty} \tag{15}$$

is valid for all $v \in W_p^1(\Omega) \times L_{\infty}(\Gamma)$.

Without limitation of generality we can use the same constant c_r in Lemma 3 and Lemma 4.

Proof of Lemma 4. We have

$$\begin{aligned} &r_2^{\mathcal{L}}(v_0, y_0, v) \\ &= \mathcal{L}(v, y_0) - \mathcal{L}(v_0, y_0) - \mathcal{L}_v(v_0, y_0)(v - v_0) - \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\ &= \int_0^1 \mathcal{L}_v(v_0 + s(v - v_0), y_0)(v - v_0) ds \\ &\quad - \mathcal{L}_v(v_0, y_0)(v - v_0) - \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\ &= \int_0^1 \left(\int_0^s \mathcal{L}_{vv}(v_0 + \tau(v - v_0), y_0)[v - v_0, v - v_0] d\tau + \mathcal{L}(v_0, y_0)(v - v_0) \right) ds \\ &\quad - \mathcal{L}_v(v_0, y_0)(v - v_0) - \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\ &= \int_0^1 \int_0^s \mathcal{L}_{vv}(v_0 + \tau(v - v_0), y_0)[v - v_0, v - v_0] d\tau ds \\ &\quad - \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \end{aligned}$$

$$= \int_0^1 \int_0^s (\mathcal{L}_{vv}(v_0 + \tau(v - v_0), y_0)[v - v_0, v - v_0] - \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0]) d\tau ds.$$

Invoking Lemma 2 we conclude

$$|r_2^{\mathcal{L}}(v_0, y_0, v)| \leq c \int_0^1 \int_0^s \tau \|v - v_0\|_{\infty} \|v - v_0\|_2^2 d\tau ds \leq c \|v - v_0\|_{\infty} \|v - v_0\|_2^2$$

and the assertion is proved ■

Now we are able to establish a strengthened second order sufficient optimality condition.

Theorem 2. *Suppose that the assumptions (A1) and (A2) are fulfilled. Let $v_0 = (w_0, u_0)$ be admissible for the optimal control problem (1) – (3) and fulfil the optimality system (12) with the associated adjoint state y_0 . Choose $\varepsilon > 0$ and the corresponding set of positivity Γ_{ε} . Suppose the existence of an $\alpha > 0$ such that*

$$\mathcal{L}_{vv}(v_0, y_0)[(h, z), (h, z)] \geq \alpha \|(h, z)\|_2^2 \tag{16}$$

holds for all $z = u - u_0$, where $u \in \mathcal{U}^{ad}$ and $u(x) = u_0(x)$ for almost all $x \in \Gamma_{\varepsilon}$, and h is the solution of the linearized equation

$$\begin{aligned} -\Delta h + h &= 0 && \text{in } \Omega \\ \frac{\partial h}{\partial n} &= b_w(v_0)h + b_u(v_0)z && \text{on } \Gamma. \end{aligned} \tag{17}$$

Then there exist $\beta > 0$ and $\rho > 0$ such that

$$F(v) \geq F(v_0) + \beta \|v - v_0\|_2^2$$

for all admissible pairs v with $\|v - v_0\|_{\infty} \leq \rho$.

Remark. The solution h of the linearized equation (17) exists as $b_w(v)$ is non-positive.

Proof of Theorem 2. a) Preparatory estimation of the cost functional $F = F(v)$. Let $v = (u, w)$ be an admissible pair. Suppose that v_0 fulfills the assumptions of the theorem. We have $F(v) = \mathcal{L}(v, y_0)$ since v is admissible. By means of a Taylor expansion of the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$ at v_0 with respect to v we get

$$\begin{aligned} F(v) &= \mathcal{L}(v, y_0) \\ &= \mathcal{L}(v_0, y_0) + \mathcal{L}_u(v_0, y_0)(u - u_0) + \mathcal{L}_w(v_0, y_0)(w - w_0) \\ &\quad + \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] + r_2^{\mathcal{L}}(v_0, y_0, v). \end{aligned}$$

The admissibility of v_0 yields $\mathcal{L}(v_0, y_0) = F(v_0)$. Moreover, the necessary optimality conditions are fulfilled at v_0 with adjoint state y_0 . Thus, the third term is equal to zero. On the other hand, the second term is non-negative due to the variational inequality in (12). However, we get even more. The definition of Γ_ϵ yields

$$\begin{aligned} \mathcal{L}_u(v_0, y_0)(u - u_0) &= \int_{\Gamma} (\psi_u(v_0) + b_u(v_0)y_0)(u - u_0) dS_x \\ &\geq \int_{\Gamma_\epsilon} (\psi_u(v_0) + b_u(v_0)y_0)(u - u_0) dS_x \\ &\geq \epsilon \|u - u_0\|_{L_1(\Gamma_\epsilon)}. \end{aligned}$$

Therefore, we have

$$F(v) \geq F(v_0) + \epsilon \|u - u_0\|_{L_1(\Gamma_\epsilon)} + r_2^{\mathcal{L}}(v_0, y_0, v) + \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0].$$

b) Splitting of the optimal control according to the set of positivity Γ_ϵ . Next, we discuss the second derivative of the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$. Here, we intend to exploit the estimate (16). To this aim, a new control \tilde{u} is introduced by

$$\tilde{u}(x) = \begin{cases} u_0(x) & \text{on } \Gamma_\epsilon \\ u(x) & \text{on } \Gamma \setminus \Gamma_\epsilon. \end{cases}$$

Let \tilde{w} be the state corresponding to the control \tilde{u} , i.e.

$$\begin{aligned} -\Delta \tilde{w} + \tilde{w} &= 0 && \text{in } \Omega \\ \frac{\partial \tilde{w}}{\partial n} &= b(\tilde{w}, \tilde{u}) && \text{on } \Gamma. \end{aligned}$$

Let \tilde{v} denote the pair (\tilde{w}, \tilde{u}) . Inserting these notations, we obtain by means of (13)

$$\begin{aligned} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] &= \mathcal{L}_{vv}(v_0, y_0)[v - \tilde{v} + \tilde{v} - v_0, v - \tilde{v} + \tilde{v} - v_0] \\ &= \mathcal{L}_{vv}(v_0, y_0)[v - \tilde{v}, v - \tilde{v}] \\ &\quad + \mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, \tilde{v} - v_0] + 2\mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, v - \tilde{v}] \\ &\geq \mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, \tilde{v} - v_0] - c\|\tilde{v} - v\|_2^2 - \tilde{c}\|\tilde{v} - v\|_2\|\tilde{v} - v_0\|_2. \end{aligned}$$

c) Linearization – linearized state w_t . Moreover, we introduce w_t as the solution of the linearized state equation at v_0 :

$$\begin{aligned} -\Delta w_t + w_t &= 0 \\ \frac{\partial w_t}{\partial n} &= b(v_0) + b_w(v_0)(w_t - w_0) + b_u(v_0)(\tilde{u} - u_0). \end{aligned}$$

Setting $v_t = (w_t, \tilde{u})$ and invoking (13), we continue by

$$\begin{aligned} & \mathcal{L}_{v\tilde{v}}(v_0, y_0)[\tilde{v} - v_0; \tilde{v} - v_0] \\ &= \mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_t + v_t - v_0, \tilde{v} - v_t + v_t - v_0] \\ &= \mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_t, \tilde{v} - v_t] \\ &\quad + \mathcal{L}_{vv}(v_0, y_0)[v_t - v_0, v_t - v_0] + 2\mathcal{L}_{vv}(v_0, y_0)[v_t - v_0, \tilde{v} - v_t] \\ &\geq \mathcal{L}_{vv}(v_0, y_0)[v_t - v_0, v_t - v_0] - c\|\tilde{v} - v_t\|_2^2 \\ &\quad - \bar{c}\|\tilde{v} - v_t\|_2\|v_t - v_0\|_2. \end{aligned}$$

For the first term of the last equation the estimate (16) applies. To handle the second and the third term we consider first the difference $\tilde{v} - v_t$. By definition we have

$$\|\tilde{v} - v_t\|_2 = \left(\|\tilde{w} - w_t\|_{W_2^1(\Omega)}^2 + \|\tilde{u} - \tilde{u}\|_{L_2(\Gamma)}^2 \right)^{1/2} = \|\tilde{w} - w_t\|_{W_2^1(\Omega)}$$

and $\tilde{w} - w_t$ fulfils the equations

$$\begin{aligned} 0 &= -\Delta(\tilde{w} - w_t) + (\tilde{w} - w_t) \\ \frac{\partial(\tilde{w} - w_t)}{\partial n} &= b(\tilde{w}, \tilde{u}) - b(w_0, u_0) - b_w(w_0, u_0)(w_t - w_0) - b_u(w_0, u_0)(\tilde{u} - u_0). \end{aligned}$$

For the right-hand side of the boundary condition we get

$$\begin{aligned} & b(\tilde{w}, \tilde{u}) - b(w_0, u_0) - b_w(w_0, u_0)(w_t - w_0) - b_u(w_0, u_0)(\tilde{u} - u_0) \\ &= b(\tilde{w}, \tilde{u}) - b(w_0, u_0) - b_w(w_0, u_0)(\tilde{w} - w_0) \\ &\quad - b_u(w_0, u_0)(\tilde{u} - u_0) - b_w(w_0, u_0)(\tilde{w} - w_t) \\ &= r_1^b(v_0, \tilde{v}) - b_w(w_0, u_0)(\tilde{w} - w_t). \end{aligned}$$

Thus the boundary condition is equivalent to

$$\frac{\partial(\tilde{w} - w_t)}{\partial n} + b_w(w_0, u_0)(\tilde{w} - w_t) = r_1^b(v_0, \tilde{v}).$$

Lemma 1 yields

$$\|\tilde{v} - v_t\|_2 = \|\tilde{w} - w_t\|_{W_2^1(\Omega)} \leq c\|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}, \tag{18}$$

the decisive estimate of the difference between state and linearized state.

d) Exploiting the second order condition (16) and the estimate (18). Using (16) and (18) we estimate

$$\begin{aligned}
 & \mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, \tilde{v} - v_0] \\
 & \geq \mathcal{L}_{vv}(v_0, y_0)[v_t - v_0, v_t - v_0] - c \|\tilde{v} - v_t\|_2^2 - \tilde{c} \|\tilde{v} - v_t\|_2 \|v_t - v_0\|_2 \\
 & \geq \alpha \|v_t - v_0\|_2^2 - c \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}^2 - \tilde{c} \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)} \|v_t - v_0\|_2 \\
 & \geq \alpha \|\tilde{v} - v_0\|_2^2 - 2\alpha \|v_t - \tilde{v}\|_2 \|\tilde{v} - v_0\|_2 - c \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}^2 \\
 & \quad - \tilde{c} \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)} \|v_t - \tilde{v}\|_2 - \tilde{c} \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)} \|\tilde{v} - v_0\|_2 \\
 & \geq \alpha \|\tilde{v} - v_0\|_2^2 - c \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}^2 - \tilde{c} \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)} \|\tilde{v} - v_0\|_2 \\
 & \geq \|\tilde{v} - v_0\|_2^2 \left(\alpha - \frac{c \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}^2}{\|\tilde{v} - v_0\|_2^2} - \frac{\tilde{c} \|r_1^b(v_0, \tilde{v})\|_{L_2(\Gamma)}}{\|\tilde{v} - v_0\|_2} \right).
 \end{aligned}$$

By (14),

$$\mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, \tilde{v} - v_0] \geq \|\tilde{v} - v_0\|_2^2 \left(\alpha - c \|\tilde{v} - v_0\|_\infty^2 - \tilde{c} \|\tilde{v} - v_0\|_\infty \right).$$

If v is sufficiently close to v_0 with respect to the V_∞ -norm, then $\|\tilde{v} - v_0\|_\infty$ is sufficiently close to zero, too. Therefore, the term in the brackets is greater or equal to $\tilde{\alpha} > 0$ for all admissible v such that $\|v - v_0\|_\infty \leq \rho$. In this way we conclude that

$$\mathcal{L}_{vv}(v_0, y_0)[\tilde{v} - v_0, \tilde{v} - v_0] \geq \tilde{\alpha} \|\tilde{v} - v_0\|_2^2$$

holds for all admissible v with $\|v - v_0\|_\infty \leq \rho$ and all corresponding pairs \tilde{v} .

e) Transformation back to the original quantity $v = (w, u)$. Further,

$$\begin{aligned}
 & \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\
 & \geq \tilde{\alpha} \|\tilde{v} - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|\tilde{v} - v_0\|_2 \\
 & \geq \tilde{\alpha} \|\tilde{v} - v + v - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|\tilde{v} - v + v - v_0\|_2 \\
 & \geq \tilde{\alpha} \|v - v_0\|_2^2 - 2\tilde{\alpha} \|v - v_0\|_2 \|\tilde{v} - v\|_2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|v - v_0\|_2 \\
 & \geq \tilde{\alpha} \|v - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|v - v_0\|_2.
 \end{aligned}$$

f) Final estimation using the first order sufficient optimality condition. Now we are

able to complete the estimation of the cost functional $F = F(v)$. We have

$$\begin{aligned} F(v) &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} + r_2^{\mathcal{L}}(v_0, y_0, v) \\ &\quad + \frac{1}{2} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \\ &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} + r_2^{\mathcal{L}}(v_0, y_0, v) \\ &\quad + \frac{1}{2} \left(\tilde{\alpha} \|v - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|v - v_0\|_2 \right) \\ &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} r_2^{\mathcal{L}}(v_0, y_0, v) \\ &\quad + \frac{1}{2} \tilde{\alpha} \|v - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|v - v_0\|_2. \end{aligned}$$

The next term under consideration is $\|\tilde{v} - v\|_2 \|v - v_0\|_2$. Youngs inequality implies

$$\|\tilde{v} - v\|_2 \|v - v_0\|_2 \leq \kappa^{-1} \|\tilde{v} - v\|_2^2 + \kappa \|v - v_0\|_2^2$$

for all $\kappa > 0$. Moreover, we have by construction $\|\tilde{v} - v\|_2 \leq c \|\tilde{u} - u\|_{L_2(\Gamma)}$, where the term on the right-hand side is uniformly bounded. Thus, we continue with

$$\begin{aligned} F(v) &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} + r_2^{\mathcal{L}}(v_0, y_0, v) \\ &\quad + \frac{1}{2} \tilde{\alpha} \|v - v_0\|_2^2 - c \|\tilde{v} - v\|_2^2 - \tilde{c} \|\tilde{v} - v\|_2 \|v - v_0\|_2 \\ &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} + r_2^{\mathcal{L}}(v_0, y_0, v) \\ &\quad + \frac{1}{2} \tilde{\alpha} \|v - v_0\|_2^2 - c \|\tilde{u} - u\|_{L_2(\Gamma)}^2 - \tilde{c} \left(\kappa^{-1} \|\tilde{u} - u\|_{L_2(\Gamma)}^2 + \kappa \|v - v_0\|_2^2 \right) \\ &\geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} - c \kappa^{-1} \|\tilde{u} - u\|_{L_2(\Gamma)}^2 \\ &\quad + \|v - v_0\|_2^2 \left(\frac{\tilde{\alpha}}{2} - \tilde{c} \kappa - \frac{|r_2^{\mathcal{L}}(v_0, y_0, v)|}{\|v - v_0\|_2^2} \right). \end{aligned}$$

By (15) we have that

$$|r_2^{\mathcal{L}}(v_0, y_0, v)| \leq c \|v - v_0\|_2^2 \|v - v_0\|_\infty.$$

Therefore, for $\|v - v_0\|_\infty \leq \rho_3 \leq \rho_2$ and κ sufficiently small, the term in the brackets is greater than or equal to $\alpha_1 > 0$ for all admissible v with $\|v - v_0\|_\infty \leq \rho_3$. Hence

$$F(v) \geq F(v_0) + \varepsilon \|u - u_0\|_{L_1(\Gamma_\varepsilon)} - c \kappa^{-1} \|\tilde{u} - u\|_{L_2(\Gamma)}^2 + \alpha_1 \|v - v_0\|_2^2.$$

Furthermore, \tilde{u} differs from u only on Γ_ε , where \tilde{u} is equal to u_0 provided that $\|v - v_0\|_\infty \leq \rho_3$ again. Therefore the norm $\|\tilde{u} - u\|_{L_2(\Gamma)}^2$ is equal to $\|u_0 - u\|_{L_2(\Gamma_\varepsilon)}^2 \leq \rho_3 \|u - u_0\|_{L_1(\Gamma_\varepsilon)}$. Now we take ρ_3 sufficiently small such that $\varepsilon - c(\kappa^{-1})\rho_3 > 0$. Then we are allowed to omit the terms with $\|u - u_0\|_{L_1(\Gamma_\varepsilon)}$ and arrive at

$$F(v) \geq F(v_0) + \alpha_1 \|v - v_0\|_2^2$$

for all admissible v with $\|v - v_0\|_\infty \leq \rho_3 = \rho$ ■

Re-formulating this result we arrive at

Theorem 3. *Suppose that the assumptions of Theorem 2 hold true. Then u_0 is a locally optimal control in the sense of $L_\infty(\Gamma)$.*

Proof. We choose $u \in \mathcal{U}^{ad}$ such that $\|u - u_0\|_{L_\infty(\Gamma)} \leq \min\{\rho/C, \rho\} \leq \rho$. By the estimate (5) we conclude for the state w corresponding to u

$$\|w - w_0\|_{W_p^1(\Omega)} \leq C \min\left\{\frac{\rho}{C}, \rho\right\} \leq \rho.$$

Therefore, we have $\|v - v_0\|_\infty \leq \rho$. Now the statement follows directly taking into account that $\|v - v_0\|_2^2 \geq \|u - u_0\|_{L_2(\Gamma)}^2$ ■

5. Remarks

The results of the preceding sections ensure local optimality of the control u_0 in a sufficiently small neighbourhood in $L_\infty(\Gamma)$. If jumps of u_0 cannot be excluded, all functions in this neighbourhood must have jumps at the same position. This is too strong for many applications. Aiming to weaken this, we consider now the optimal control problem (1) - (3) under stronger assumptions. We shall show that in this case the assumptions of Theorem 2 are sufficient for local optimality in the sense of $L_p(\Gamma)$, where $p < \infty$ is sufficiently large (cf. Section 2). To do so we suppose the following additional properties on ψ and b :

The functions ψ and b fulfil the assumptions (A1) and (A2) of Section 2 and have the form

$$b(w, u) = b_1(w) + b_2(w)u \tag{19}$$

$$\psi(w, u) = \psi_1(w) + \psi_2(w)u + \gamma u^2$$

where $\gamma \in L_\infty(\Gamma)$ with $\gamma(x) \geq 0$ almost everywhere on Γ . We assume that the functions b_1, b_2 and ψ_1, ψ_2 and all of their derivatives up to the second order are uniformly Lipschitz continuous.

The crucial point in the proof of Theorem 2 were the estimates of the first and second order remainder terms $r_1^b(v_0, v)$ and $r_2^{\mathcal{L}}(v_0, y_0, v)$ of b and the Lagrange function \mathcal{L} , respectively. In general, these estimates are valid only with respect to the norm in $W_p^1(\Omega) \times L_\infty(\Gamma)$. For that reason, Theorem 3 states only local optimality in the sense of $L_\infty(\Gamma)$. The additional assumptions (19) allow us to improve the estimates of the remainder terms. We set for convenience

$$\|v\|_p^p = \|(w, u)\|_p^p = \|w\|_{W_p^1(\Omega)}^p + \|u\|_{L_p(\Gamma)}^p.$$

Lemma 5. *The first order remainder term $r_1^b(v_0, v)$ of b fulfills the estimate*

$$\|r_1^b(v_0, v)\|_{L_2(\Gamma)} \leq c_{r,p} \|v - v_0\|_2 \|v - v_0\|_p$$

for all $v = (w, u) \in W_p^1(\Omega) \times L_\infty(\Gamma)$.

Proof. We have

$$\begin{aligned} \|r_1^b(v_0, v)\|_{L_2(\Gamma)}^2 &= \int_{\Gamma} \left(b(v) - b(v_0) - b_v(v_0)(v - v_0) \right)^2 dS_x \\ &= \int_{\Gamma} \left(\int_0^1 \left(b_v(v_0 + s(v - v_0)) - b_v(v_0) \right) (v - v_0) ds \right)^2 dS_x. \end{aligned} \tag{20}$$

Making use of the special structure of b we can rewrite the integrand at almost all $x \in \Gamma$ in the following way:

$$\begin{aligned} & \left| \left\{ b_v(v_0(x) + s(v(x) - v_0(x))) - b_v(v_0(x)) \right\} (v(x) - v_0(x)) \right| \\ &= \left| \left\{ b_{1w}(w_0(x) + s(w(x) - w_0(x))) - b_{1w}(w_0(x)) \right\} (w(x) - w_0(x)) \right. \\ & \quad + \left\{ b_{2w}(w_0(x) + s(w(x) - w_0(x))) \left(u_0(x) + s(u(x) - u_0(x)) \right) \right. \\ & \quad \left. \left. - b_{2w}(w_0(x)) u_0(x) \right\} (w(x) - w_0(x)) \right. \\ & \quad \left. + \left\{ b_2(w_0(x) + s(w(x) - w_0(x))) - b_2(w_0(x)) \right\} (u(x) - u_0(x)) \right|. \end{aligned}$$

The functions b_1 and b_2 and the corresponding derivatives are assumed to be Lipschitz continuous. Using $u_0 \in L_\infty(\Gamma)$, we get

$$\begin{aligned} & \left| b_v \left\{ v_0(x) + s(v(x) - v_0(x)) - b_v(v_0(x)) \right\} (v(x) - v_0(x)) \right| \\ & \leq c \left(\left| s(w(x) - w_0(x))(w(x) - w_0(x)) \right| + s \left| (u(x) - u_0(x))(w(x) - w_0(x)) \right| \right. \\ & \quad \left. \times |u_0(x)| \left| s(w(x) - w_0(x))(w(x) - w_0(x)) \right| \right) \\ & \leq c \left(\left| s(w(x) - w_0(x))(w(x) - w_0(x)) \right| + s \left| (u(x) - u_0(x))(w(x) - w_0(x)) \right| \right). \end{aligned}$$

Inserting this into (20) we find

$$\begin{aligned} \|r_1^b(v_0, v)\|_{L_2(\Gamma)}^2 &\leq c \int_{\Gamma} \left(\int_0^1 \left\{ \left| s(w(x) - w_0(x))(w(x) - w_0(x)) \right| \right. \right. \\ & \quad \left. \left. + s \left| (u(x) - u_0(x))(w(x) - w_0(x)) \right| \right\} ds \right)^2 dS_x \\ &\leq c \int_{\Gamma} \left\{ \left| (w(x) - w_0(x)) \right|^2 + \left| (u(x) - u_0(x))(w(x) - w_0(x)) \right| \right\}^2 dS_x \\ &\leq c \int_{\Gamma} \left\{ |w - w_0|^4 + |w - w_0|^3 |u - u_0| + |w - w_0|^2 |u - u_0|^2 \right\} dS_x \end{aligned}$$

$$\begin{aligned} &\leq c \|w - w_0\|_{C(\bar{\Omega})}^2 \int_{\Gamma} \left\{ |w - w_0|^2 + |w - w_0| |u - u_0| + |u - u_0|^2 \right\} dS_x \\ &\leq c \|w - w_0\|_{W_p^1(\Omega)}^2 \|v - v_0\|_2^2 \\ &\leq c \|v - v_0\|_p^2 \|v - v_0\|_2^2. \end{aligned}$$

The statement of the Lemma is now a simple conclusion ■

To prove estimates on the second order remainder term $r_2^{\mathcal{L}}(v_0, y_0, v)$ of the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$ we have used in the general case the Lipschitz argument of Lemma 2. This general result cannot be transferred to our special case replacing the ∞ -norm by the p -norm. However, we are able to show the following

Lemma 6. *For the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$ the estimate*

$$\begin{aligned} &\left| \mathcal{L}_{vv}(v_0 + \tau(v - v_0), y_0)[v - v_0, v - v_0] - \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \right| \\ &\qquad\qquad\qquad \leq c_{\mathcal{L},p} \|v - v_0\|_p \|v - v_0\|_2^2 \end{aligned}$$

holds for all admissible pairs $v \in W_p^1(\Omega) \times L_{\infty}(\Gamma)$ and $\tau \in (0, 1)$.

Proof. The form (19) of the functions ψ and b implies

$$\begin{aligned} \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] &= \int_{\Omega} \varphi_{ww}(w_0)(w - w_0)^2 dx \\ &\quad + \int_{\Gamma} \left\{ (\psi_{1ww}(w_0) + \psi_{2ww}(w_0)u_0)(w - w_0)^2 \right. \\ &\quad \left. + 2\psi_{2w}(w_0)(w - w_0)(u - u_0) + 2\gamma(u - u_0)^2 \right\} dS_x \\ &\quad + \int_{\Gamma} y_0 \left\{ b_{1ww}(w_0)(w - w_0)^2 + b_{2ww}(w_0)u_0(w - w_0)^2 \right. \\ &\quad \left. + 2b_{2wu}(w_0)(w - w_0)(u - u_0) \right\} dS_x. \end{aligned}$$

Analogously we have with $w_{\tau} = w_0 + \tau(w - w_0)$ and $u_{\tau} = u_0 + \tau(u - u_0)$

$$\begin{aligned} \mathcal{L}_{vv}(v_{\tau}, y_0)[v - v_0, v - v_0] &= \int_{\Omega} \varphi_{ww}(w_{\tau})(w - w_0)^2 dx \\ &\quad + \int_{\Gamma} \left\{ (\psi_{1ww}(w_{\tau}) + \psi_{2ww}(w_{\tau})u_{\tau})(w - w_0)^2 \right. \\ &\quad \left. + 2\psi_{2w}(w_{\tau})(w - w_0)(u - u_0) + 2\gamma(u - u_0)^2 \right\} dS_x \\ &\quad + \int_{\Gamma} y_0 \left\{ b_{1ww}(w_{\tau})(w - w_0)^2 + b_{2ww}(w_{\tau})u_{\tau}(w - w_0)^2 \right. \\ &\quad \left. + 2b_{2wu}(w_{\tau})(w - w_0)(u - u_0) \right\} dS_x. \end{aligned}$$

Using the Lipschitz continuity of the partial derivatives of φ , ψ and b , we can estimate

$$\begin{aligned}
 & \left| \mathcal{L}_{vv}(v_\tau, y_0)[v - v_0, v - v_0] - \mathcal{L}_{vv}(v_0, y_0)[v - v_0, v - v_0] \right| \\
 & \leq c \int_{\Gamma} \left\{ |w_\tau - w_0|(w - w_0)^2 + |u_\tau - u_0|(w - w_0)^2 \right. \\
 & \quad \left. + |w_\tau - w_0|(u - u_0)(w - w_0) \right\} dS_x + \int_{\Omega} |w_\tau - w_0|(w - w_0)^2 dx \\
 & \leq c \int_{\Gamma} \left\{ |w - w_0|(w - w_0)^2 + |u - u_0|(w - w_0)^2 \right. \\
 & \quad \left. + |w - w_0|(u - u_0)(w - w_0) \right\} dS_x + \int_{\Omega} |w - w_0|(w - w_0)^2 dx \\
 & \leq c \|w - w_0\|_{C(\bar{\Omega})} \left(\int_{\Gamma} \left\{ (w - w_0)^2 + |u - u_0| |w - w_0| \right. \right. \\
 & \quad \left. \left. + |(u - u_0)(w - w_0)| \right\} dS_x + \int_{\Omega} |w - w_0| |w - w_0| dx \right) \\
 & \leq c \|v - v_0\|_p \|v - v_0\|_2^2.
 \end{aligned}$$

In particular, the embedding result $\|w\|_{C(\bar{\Omega})} \leq c \|w\|_{W_p^1(\Omega)}$ was used. Therefore, the statement of the lemma holds true ■

The next statement is an immediate consequence.

Lemma 7. *The second order remainder term $r_2^{\mathcal{L}}(v_0, y_0, v)$ of the Lagrange function $\mathcal{L} = \mathcal{L}(v, y)$ at (v_0, y_0) satisfies the estimate*

$$|r_2^{\mathcal{L}}(v_0, y_0, v)| \leq C_{\mathcal{L}, p} \|v - v_0\|_p \|v - v_0\|_2^2$$

for all admissible $v \in W_p^1(\Omega) \times L_\infty(\Gamma)$.

Proof. The proof is along the lines of the proof of Lemma 4. We only have to use Lemma 6 instead of Lemma 2 for the Lipschitz estimate ■

Summarising up we arrive at

Theorem 4. *Let the general assumptions (A1), (A2), and (19) be fulfilled. Suppose further that the assumptions of Theorem 2 are satisfied. Then the reference control u_0 is locally optimal in the sense of $L_p(\Gamma)$.*

Proof. The proof of local optimality of u_0 in the sense of $L_\infty(\Gamma)$ was mainly based on $W_p^1(\Omega)$ -regularity of the state w_0 , on differentiability of the non-linearities in $C(\bar{\Omega}) \times L_\infty(\Gamma)$, and on the estimates of the remainder terms $r_1^{\mathcal{L}}(v_0, v)$ and $r_2^{\mathcal{L}}(v_0, y_0, v)$ of b and \mathcal{L} , respectively. The $W_p^1(\Omega)$ -regularity is preserved for controls $u \in L_p(\Gamma)$. Due to the strengthened assumptions on the appearance of u , the differentiability is guaranteed in $C(\bar{\Omega}) \times L_p(\Gamma)$, too. Therefore, to prove the statement of Theorem 4, we can proceed along the lines of the proofs of Theorem 2 and Theorem 3. The only difference is the use of estimates according to Lemma 5 and Lemma 7 instead of Lemma 3 and Lemma 4, respectively ■

Acknowledgement. The authors are very grateful to the referees for some helpful comments.

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Received 18.10.1995; in revised form 10.01.1996