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On Linear Integro-Differential Equations with Integral Impulsive Conditions

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Abstract. Linear integro-differential equations with linear integral impulsive conditions are considered. Existence results and representations of solutions are obtained. Stability of these equations is investigated.

Keywords: *Integro-differential equations, impulsive conditions*

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1. Introduction

The application of solution representations is one of the basic methods in the theory of stability of functional-differential equations. For linear equations such representations have been constructed for a lot of equation classes including integro-differential equations (see [3 - 5]). Sometimes such representations are also called variation constant formulas. The present paper is aimed to obtain such formula for an integro-differential equation with integral impulsive conditions and to apply it in stability research. These impulsive conditions are natural for the equation considered since at any point the solution value is also determined by its prehistory.

Delay differential equations with the same integral impulsive conditions were studied in [1]. However, the approach used here is closer to the one in paper [2] on a delay equation with usual impulsive conditions $x(\tau_j) = B_j x(\tau_j - 0)$, where the solution value in the impulse point is defined only by its limit from the left and independent of prehistory. Not many publications are concerned with impulsive integro-differential equations (we name here [8 - 11]).

The paper is organized as follows. Sections 2 and 3 contain the statement of the problem and some auxiliary results. In particular, in Section 3 an existence theorem is presented and a fundamental function is explicitly estimated. Sections 4 and 5 contain the main results. In Section 4 a solution representation formula is obtained. In Section 5 a connection of various stability types such as asymptotic and exponential stability

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with properties of the fundamental matrix is established. In conclusion, an explicit stability result is presented.

2. Preliminaries

Let $\mathbb{R}^n$ be the space of n-dimensional column vectors $x = (x_1, x_2, \dots, x_n)$ with norm $\|x\| = \max_i |x_i|$, $\|\cdot\|$ the corresponding matrix norm and E_n the identity $(n \times n)$-matrix. Further, let $L[a,b]$ be the Lebesgue space with usual norm $\|x\|_{L[a,b]} = \int_a^b \|x(s)\|\, ds$, $L_\infty[a,b]$ the space of measurable essentially bounded functions $x : [a,b] \to \mathbb{R}^n$ with norm $\|x\|_{L_\infty[a,b]} = \text{vraisup}_{a \le s \le b}\|x(s)\|$ $(b \le \infty)$, $AC[a,b]$ the space of absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ with norm $\|x\|_{AC[a,b]} = \|x(a)\| + \|\dot{x}\|_{L[a,b]}$ and $\chi_\tau : [a,b] \to \mathbb{R}$ the characteristic function of the segment $[\tau, b]$, i.e. $\chi_\tau(t) = 0$ if $t < \tau$ and $\chi_\tau(t) = 1$ if $t \ge \tau$.

Let τ_i $(i \ge 0)$ with $a = \tau_0 < \tau_1 < \dots$ be fixed points. Denote by $PAC[a,b]$ the space of piecewise absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ with jumps at the points $\tau_i \in (a,b]$, i.e.

$$PAC[a,b] = \left\{ x : [a,b] \to \mathbb{R}^n \,\middle|\, x(t) = y(t) + \sum_{a < \tau_i \le b} \xi_i \chi_{\tau_i}(t) \text{ for } t \in [a,b] \right\}$$

with norm

$$\|x\|_{PAC[a,b]} = \|x(a)\| + \|\dot{x}\|_{L[a,b]} + \sum_{a < \tau_i \le b} \|\xi_i\|$$

where $y \in AC[a,b]$ and $\xi_i \in \mathbb{R}^n$. It is to be noted that a function $x \in PAC[a,b]$ is right continuous. In the sequel we consider $a \ge 0$ and $b \le \infty$. By $PAC[t_0, \infty)$ we mean the space of functions piecewise absolutely continuous in any finite interval $[\tau_i, \tau_{i+1}]$ with $\tau_i > t_0$.

If we consider functions $x \in PAC[a,b]$, we add to the set $\{\tau_i\}$ the point $t = a$. Denote for $x \in PAC[a,b]$

$$\Delta x(\tau_i) = x(\tau_i) - x(\tau_i - 0) \qquad \text{and} \qquad \Delta x(a) = x(a).$$

Lemma 1 (see [4]). *For any function $x \in PAC[a,b]$ the representation*

$$x(t) = \int_a^t \dot{x}(s)\, ds + \sum_{a \le \tau_i \le b} \Delta x(\tau_i) \chi_{\tau_i}(t) \qquad (t \in [a,b]) \tag{1}$$

holds.

In view of the above lemma $\|x\|_{PAC[a,b]} \ge \|x(s)\|$ for any $s \in [a,b]$. Relation (1) implies that $PAC[a,b]$ with $b < \infty$ is a Banach space.

3. Statement of the problem

Let $t_0 \geq 0$. Consider the linear delay impulsive integro-differential equation

$$\dot{x}(t) + A(t)x(t) + \int_0^t K(t,s)x(s)\,ds = r(t) \qquad (t \geq t_0,\ x(t) \in \mathbb{R}^n) \tag{2}$$

$$x(\xi) = \varphi(\xi) \qquad (\xi < t_0)$$

$$\Delta x(\tau_i) = \int_{t_0}^{\tau_i} b_i(s)\dot{x}(s)\,ds + \sum_{t_0 \leq \tau_j < \tau_i} \lambda_{ij}\Delta x(\tau_j) + \alpha_i \qquad (\tau_i > t_0) \tag{3}$$

under the following assumptions:

(a)$_1$ $0 = \tau_0 < \tau_1 < \ldots$ are fixed points with $\lim \tau_i = \infty$ as $i \to \infty$.

(a)$_2$ Columns of $A : [0,\infty) \to \mathbb{R}^{n\times n}$ and $r : [0,\infty) \to \mathbb{R}^n$ are integrable over any finite interval.

(a)$_3$ Columns of $K(t,s)$ are integrable over any finite square $[a,b] \times [a,b]$.

(a)$_4$ $b_i : [0,\tau_i] \to \mathbb{R}^{n\times n}$, the columns of b_i are measurable functions, essentially bounded in any finite interval.

(a)$_5$ $\lambda_{ij} \in \mathbb{R}^{n\times n}$.

(a)$_6$ $\varphi : [0,t_0) \to \mathbb{R}^n$ is a Borel measurable bounded function.

Let us introduce linear functionals $l_i : PAC[t_0,\infty) \to \mathbb{R}^n$ by

$$l_i(x) = \int_{t_0}^{\tau_i} b_i(s)\dot{x}(s)\,ds + \sum_{t_0 \leq \tau_j < \tau_i} \lambda_{ij}\Delta x(\tau_j) \qquad (\tau_i > t_0). \tag{4}$$

By the same symbol we will denote a matrix-valued functional acting in the space of matrix-valued functions with columns in $PAC[t_0,\infty)$ and defined by the same formula.

Definition. A function $x \in PAC[t_0,\infty)$ is said to be a *solution* of the impulsive differential equation (2)-(3) if equality (2) holds for almost all $t \in [t_0,\infty)$ and the impulsive conditions (3) hold.

Remarks. **1.** Usually (see [2: p. 927]), the conditions that define the magnitudes $\Delta x(\tau_i)$ of the solution jumps for a linear impulsive differential equation are of the type

$$\Delta x(\tau_i) = B_i\, x(\tau_i - 0) + \alpha_i.$$

If $b_i(s) \equiv B_i$ and $\lambda_{ij} = B_i$ $(j = 0,1,\ldots,i-1)$, then from (1) and (3) for any $x \in PAC[t_0,\infty)$ we obtain

$$\begin{aligned}
&\int_{t_0}^{\tau_i} b_i(s)\dot{x}(s)\,ds + \sum_{t_0 \leq \tau_j < \tau_i} \lambda_{ij}\Delta x(\tau_j) \\
&\qquad = B_i\left(x(\tau_i - 0) - \sum_{t_0 \leq \tau_j < \tau_i} \Delta x(\tau_j)\right) + B_i \sum_{t_0 \leq \tau_j < \tau_i} \Delta x(\tau_j) \\
&\qquad = B_i\, x(\tau_i - 0).
\end{aligned}$$

Therefore, the conditions (3) are of a more general type than the usual ones. For integro-differential systems it is natural to assume that the magnitude $\Delta x(\tau_i)$ of the jump of the solution depends, generally speaking, not only on the value $x(\tau_i - 0)$ but on the behavior of x on the interval $[t_0, \tau_i)$.

2. For an integro-differential equation an impulsive condition including both the function value and its integral also seems natural:

$$\Delta x(\tau_i) = B_i\, x(\tau_i - 0) + \int_{t_0}^{\tau_i} c_i(s)x(s)\, ds + \alpha_i \qquad (\tau_i > t_0). \tag{5}$$

Since by (1)

$$\begin{aligned} B_i x(\tau_i - 0) &+ \int_{t_0}^{\tau_i} c_i(s)x(s)\, ds \\ &= B_i\, x(\tau_i - 0) + \int_{t_0}^{\tau_i} c_i(s) \left(\int_{t_0}^{s} \dot{x}(\xi)\, d\xi + \sum_{t_0 \le \tau_j < \tau_i} \Delta x(\tau_j)\chi_{\tau_j}(s) \right) ds \\ &= B_i\, x(\tau_i - 0) + \int_{t_0}^{\tau_i} \dot{x}(s)\, ds \int_{s}^{\tau_i} c_i(\xi)\, d\xi + \sum_{t_0 \le \tau_j < \tau_i} \int_{t_0}^{\tau_j} c_i(s)\chi_{\tau_j}(s)\, ds\ \Delta x(\tau_j) \\ &= \int_{t_0}^{\tau_i} b_i(s)\dot{x}(s)\, ds + \sum_{t_0 \le \tau_j < \tau_i} \lambda_{ij}\Delta x(\tau_j) \end{aligned}$$

with

$$b_i(s) = B_i + \int_{s}^{\tau_i} c_i(\xi)\, d\xi \qquad \text{and} \qquad \lambda_{ij} = B_i + \int_{t_0}^{\tau_j} c_i(s)\chi_{\tau_j}(s)\, ds,$$

then (5) coincides with (3). Thus, (5) is a particular case of (3).

Definition. An impulsive integro-differential equation

$$\dot{x}(t) + A(t)\, x(t) + \int_{s}^{t} K(t,\xi)x(\xi)\, d\xi = 0 \qquad (t \ge s, x(t) \in \mathbb{R}^{n \times n}) \tag{6}$$

$$\Delta x(\tau_i) = \int_{s}^{\tau_i} b_i(\xi)\dot{x}(\xi)\, d\xi + \sum_{s \le \tau_j < \tau_i} \lambda_{ij}\Delta x(\tau_j) \qquad (\tau_i > s) \tag{7}$$

is said to be a homogeneous s-curtailed equation. The solution $X(\cdot, s)$ of this equation satisfying $X(s,s) = E_n$ is said to be a *fundamental function* of equation (2)-(3).

We assume $X(t,s) = 0$ if $t < s$.

Lemma 2. *Let assumptions* $(a)_1 - (a)_6$ *hold. Then for any* $t_0 \geq 0$ *the initial value problem* $(2) - (3)$, $x(t_0) = \alpha$ *has one and only one solution.*

Proof. Assume without loss of generality that $0 \leq t_0 < \tau_1$. In the interval $[t_0, \tau_1)$ the initial value problem for the integro-differential equation without impulses has one and only one solution $x(t)$ (see [4: p. 139] and [5: Theorem 10.3.9]) since it can be rewritten as

$$\dot{x}(t) + A(t)x(t) + \int_{t_0}^{t} K(t,s)x(s)\,ds = r(t) - \int_{0}^{t_0} K(t,s)\varphi(s)\,ds.$$

The value $x(\tau_1)$ is also defined uniquely by

$$x(\tau_1) = x(\tau_1 - 0) + \int_{t_0}^{\tau_1} b_1(s)\dot{x}(s)\,ds + \lambda_{10}x(t_0) + \alpha_1 = a.$$

Then, in the interval $[\tau_1, \tau_2)$ the function $x(t)$ is a solution of the initial value problem

$$\dot{x}(t) + A(t)x(t) + \int_{\tau_1}^{t} K(t,s)x(s)\,ds = r_1(t) \qquad (x(\tau_1) = a),$$

wherein

$$r_1(t) = r(t) - \int_{0}^{\tau_1} K(t,s)x(s)\,ds$$

is also defined uniquely and satisfies assumption $(a)_2$. Thus, the initial value problem has one and only one solution on $[t_0, \tau_2)$. Similarly, one obtains by induction that the problem has one and only one solution on the half-line $[t_0, \infty)$.

Remark. The homogeneous s-curtailed equation (6)-(7) is a particular case of (2) - (3), where $t_0 = s$ and $\varphi \equiv 0$.

Denote

$$\beta_i = \max\left\{\|b_i\|_{L_\infty[0,\tau_i)}, \lambda_{i0}, \lambda_{i1}, \ldots, \lambda_{ii-1}, 1\right\}.$$

The following auxiliary result gives an estimate of the fundamental matrix.

Lemma 3. *Suppose assumptions* $(a)_1 - (a)_5$ *are satisfied. Then, for the fundamental matrix* $X(t,s)$ *the estimate*

$$\|X(t,s)\| \leq \exp\left\{\prod_{s \leq \tau_i \leq t}(1+\beta_i)\int_{s}^{t}\left(\|A(\zeta)\| + \int_{\zeta}^{t}\|K(\xi,\zeta)\|d\xi\right)d\zeta\right\} \tag{8}$$

holds.

Proof. Denote $H(t,\zeta) = \int_\zeta^t \|K(\zeta,\xi)\|\,d\xi$ and by $x(t)$ a solution of the s-curtailed homogeneous problem (6) - (7). Let us estimate

$$y(t,s) = \|x\|_{PAC[s,t]} = \|x(s)\| + \int_s^t \|\dot{x}(\zeta)\|\,d\zeta + \sum_{s<\tau_j\le t} \|\Delta x(\tau_j)\|.$$

We recall $\|x(s)\| = 1$.

First, consider $\tau_{i-1} \le s < t < \tau_i$. Then,

$$\int_s^t \dot{x}(\zeta)\,d\zeta = -\int_s^t A(\zeta)x(\zeta)\,d\zeta - \int_s^t d\zeta \int_s^\zeta K(\zeta,\xi)x(\xi)\,d\xi,$$

and therefore

$$\begin{aligned} y(t,s) = \|x\|_{PAC[s,t]} &\le \|x(s)\| + \int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)\cdot \sup_{\xi\in[s,\zeta]} \|x(\xi)\|\,d\zeta \\ &\le 1 + \int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)y(\zeta,s)\,d\zeta. \end{aligned} \tag{9}$$

By the Gronwall-Bellman inequality

$$y(t,s) \le \exp\left\{\int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)\,d\zeta\right\}.$$

Next, let $\tau_{i-1} < s < \tau_i \le t < \tau_{i+1}$. Then, similarly,

$$\begin{aligned} y(t,s) &\le \int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)\cdot \sup_{\xi\in[s,\zeta]} \|x(\xi)\|\,d\zeta + 1 + \int_s^{\tau_i} \|b_i(\zeta)\|\,\|\dot{x}(\zeta)\|\,d\zeta \\ &\le \int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)y(\zeta,s)\,d\zeta + 1 + \beta_i\big(y(\tau_i-0,s)-1\big). \end{aligned}$$

Substituting into the last term $y(\tau_i - 0,s) - 1$ from (9), one obtains

$$\begin{aligned} y(t,s) &\le \int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)y(\zeta,s)\,d\zeta + 1 + \beta_i \int_s^{\tau_i} \big(\|A(\zeta)\| + H(t,\zeta)\big)y(\zeta,s)\,d\zeta \\ &\le 1 + (1+\beta_i)\int_s^t \big(\|A(\zeta)\| + H(t,\zeta)\big)y(\zeta,s)\,d\zeta. \end{aligned}$$

Thus by the Gronwall-Bellman inequality

$$y(t,s) \le \exp\left\{(1+\beta_i)\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) d\zeta\right\}.$$

Let $\tau_{i-1} < s < \tau_i < \tau_{i+1} < t < \tau_{i+2}$. Then,

$$\begin{aligned} y(t,s) \le &\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) \sup_{\xi\in[s,\zeta)} x(\zeta)\, d\zeta + 1 + \int_s^{\tau_i} \|b_i(\zeta)\|\,\|\dot{x}(\zeta)\|\, d\zeta \\ &+ \int_s^{\tau_{i+1}} \|b_{i+1}(\zeta)\|\,\|\dot{x}(\zeta)\|\, d\zeta + \|\lambda_{i+1\ i}\|\,\|\Delta x(\tau_i)\|. \end{aligned}$$

Similar to the previous argument one obtains that the first three terms do not exceed

$$1 + (1+\beta_i)\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) y(\zeta,s)\, d\zeta \tag{10}$$

and the last two terms do not exceed

$$\begin{aligned} &\beta_{i+1}\left(\int_s^{\tau_{i+1}} \|\dot{x}(\zeta)\|\, d\zeta + \|\Delta x(\tau_i)\|\right) \\ &\le \beta_{i+1}\left(y(\tau_{i+1}-0,s) - 1\right) \\ &\le \beta_{i+1}(1+\beta_i)\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) y(\zeta,s)\, d\zeta. \end{aligned} \tag{11}$$

From (10) and (11) one obtains by summation

$$y(t,s) \le 1 + (1+\beta_i)(1+\beta_{i+1})\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) y(\zeta,s)\, d\zeta.$$

By the Gronwall-Bellman inequality

$$y(t,s) \le \exp\left\{\prod_{j=i}^{i+1}(1+\beta_j)\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) d\zeta\right\}.$$

The induction step from

$$\tau_{i-1} \le s < \tau_i < \ldots < \tau_{j-1} < t \le \tau_j$$

to

$$\tau_{i-1} \le s < \tau_i < \ldots < \tau_j < t \le \tau_{j+1}$$

is similar. Hence,

$$\|X(t,s)\| \le y(t,s) \le \exp\left\{\prod_{s<\tau_i\le t}(1+\beta_i)\int_s^t \left(\|A(\zeta)\| + H(t,\zeta)\right) d\zeta\right\}$$

which concludes the proof ∎

It is to be emphasized that the proof gives the estimate (8) not only for $\|X(t,s)\|$, but also for

$$y(t,s) = \|X(t,s)\| + \int_s^t \|X'_\xi(\xi,s)\|\,d\xi + \sum_{s<\tau_j\le t} \|\Delta X(\tau_j,s)\|.$$

Thus, the following statement is valid.

Corollary. *Suppose the assumptions of Lemma 2 hold. Then,*

$$\int_s^t \|X'_\xi(\xi,s)\|\,d\xi \le \exp\left\{ \prod_{s\le\tau_i\le t}(1+\beta_i)\int_s^t \left(\|A(\zeta)\| + \int_\zeta^t \|K(\xi,\zeta)\|\,d\xi \right) d\zeta \right\}.$$

4. Solution representation

The main result of this section (Theorems 1 and 2) deals with the solution representation and extends results obtained in [4, 5] for non-impulsive equations.

Lemma 4. *Suppose assumptions $(a)_1 - (a)_6$ hold. Then, the solution of the initial value problem for the homogeneous equation (2) ($r(t) \equiv 0$, $\varphi = 0$), with non-homogeneous impulsive conditions (3), $x(t_0) = \alpha_0$ can be represented as*

$$x(t) = \sum_{\tau_k > t_0} X(t,\tau_k)\,\alpha_k + X(t,t_0)\,\alpha_0. \tag{12}$$

Proof. We perform the proof for $t_0 = 0$ (for $t_0 > 0$ the proof is similar). The function $x(t)$ defined by (12) satisfies the initial condition since $X(0,0) = E_n$ and $X(0,\tau_k) = 0$ ($k \ge 1$). Then, $x(t)$ also satisfies the homogeneous equation as a linear combination of functions $X(t,\tau_i)$ ($\tau_i < t$) satisfying the homogeneous equation.

Let us prove that $x(t)$ satisfies the impulsive conditions (3). For a fixed i the sum $x(\tau_i)$ in (12) contains only $i+1$ non-zero terms

$$X(\tau_i,0)\,\alpha_0,\ X(\tau_i,\tau_1)\,\alpha_1,\ \dots\ ,\ X(\tau_i,\tau_i)\,\alpha_i.$$

By the definition of the fundamental function $X(t,0), X(t,\tau_1), \dots, X(t,\tau_{i-1})$ satisfy the homogeneous i-th impulsive conditions

$$\begin{gathered} X(\tau_i,0) - X(\tau_i - 0, 0) = l_i(X(\cdot,0)) \\ X(\tau_i,\tau_1) - X(\tau_i - 0,\tau_1) = l_i(X(\cdot,\tau_1)) \\ \vdots \\ X(\tau_i,\tau_{i-1}) - X(\tau_i - 0,\tau_{i-1}) = l_i(X(\cdot,\tau_{i-1})) \end{gathered}$$

(we recall that the functionals l_i are defined by (4)), whereas the function $X(t,\tau_i)$ satisfies the non-homogeneous condition

$$X(\tau_i,\tau_i) - X(\tau_i - 0,\tau_i) = E_n - 0 = l_i(X(\cdot,\tau_i)) + E_n$$

since $X(t,\tau_i) = 0$ $(t < \tau_i)$. Thus,

$$\begin{aligned}
&x(\tau_i) - x(\tau_i - 0) \\
&= \sum_{k=0}^{i-1} \left(X(\tau_i,\tau_k) - X(\tau_i - 0,\tau_k)\right)\alpha_k + \left(X(\tau_i,\tau_i) - X(\tau_i - 0,\tau_i)\right)\alpha_i \\
&= \sum_{k=0}^{i-1} l_i(X(\cdot,\tau_k))\alpha_k + l_i(X(\cdot,\tau_i))\alpha_i + \alpha_i \\
&= l_i\left(\sum_{k=0}^{i} X(\cdot,\tau_k)\alpha_k\right) + \alpha_i \\
&= l_i(x) + \alpha_i
\end{aligned}$$

which completes the proof ∎

Remark. One can easily see that Lemma 4 is also valid for the homogeneous equation with impulsive conditions (5) since the above proof is based only on the linearity of impulsive conditions and on the properties of a fundamental function.

Consider the non-homogeneous equation (2) with homogeneous impulsive conditions (3) (i.e. $\alpha_i = 0$). The following statement gives the solution representation for this impulsive equation.

Lemma 5. *Suppose the assumptions* $(a)_1 - (a)_5$ *hold and* $X(t,s)$ *is the fundamental function of equation* $(2)-(3)$. *Then the solution* y *of the initial value problem* $(2)-(3)$, $y(t_0) = 0, \alpha_i = 0, \varphi(t) \equiv 0$ *can be represented as*

$$y(t) = \int_{t_0}^{t} X(t,s)r(s)\,ds + \sum_{t_0 \le \tau_i < t} X(t,\tau_i)\int_{t_0}^{\tau_i} b_i(s)r(s)\,ds. \tag{13}$$

Proof. We again perform the proof for $t_0 = 0$ without loss of generality. Denote

$$\beta_i = \int_0^{\tau_i} b_i(s)r(s)\,ds.$$

If y is defined by (13), then the three terms in the left-hand side of (2), where x is changed by y, are

$$\dot{y}(t) = r(t) + \int_0^t X_t'(t,s)r(s)\,ds + \sum_{i=1}^{\infty} X_t'(t,\tau_i)\beta_i \tag{14}$$

$$A(t)y(t) = \int_0^t A(t)X(t,s)r(s)\,ds + \sum_{i=1}^{\infty} A(t)X(t,\tau_i)\beta_i$$

$$\int_0^t K(t,s)y(s)\,ds = \int_0^t K(t,s)\left(\int_0^s X(s,\zeta)r(\zeta)\,d\zeta + \sum_{i=1}^{\infty} X(s,\tau_i)\beta_i\right)ds$$
$$= \int_0^t \left(\int_\zeta^t K(t,s)X(s,\zeta)\,ds\right) r(\zeta)\,d\zeta + \sum_{i=1}^{\infty}\int_{\tau_i}^t K(t,\zeta)X(\zeta,\tau_i)\,d\zeta\cdot\beta_i.$$

Hence,

$$\dot y(t) + A(t)y(t) + \int_0^t K(t,s)y(s)\,ds$$
$$= r(t) + \int_0^t \left(X_t'(t,s) + A(t)X(t,s) + \int_s^t K(t,\zeta)X(\zeta,s)\,d\zeta\right) r(s)\,ds$$
$$+ \sum_{i=1}^{\infty}\left(X_t'(t,\tau_i) + A(t)X(t,\tau_i) + \int_{\tau_i}^t K(t,\zeta)X(\zeta,\tau_i)\,d\zeta\right)\beta_i$$
$$\doteq r(t)$$

(the expressions in the brackets are equal to zero since $X(\cdot,s)$ for each s is a solution of the homogeneous equation (6)).

It remains to demonstrate that y satisfies the homogeneous impulsive conditions (3), namely

$$y(\tau_i) - y(\tau_i - 0) = l_i y = \int_0^{\tau_i} b_i(s)\dot y(s)\,ds + \sum_{j=0}^{i-1}\lambda_{ij}\Delta y(\tau_j). \tag{15}$$

First, let us prove that for any sequence $\{t_k\}$ tending to τ_i from the left the equalities

$$\lim_{t_k\to\tau_i-0}\int_0^{t_k} X(t_k,s)r(s)\,ds = \int_0^{\tau_i} X(\tau_i-0,s)r(s)\,ds$$
$$\lim_{t_k\to\tau_i-0}\int_0^{t_k} ds\int_0^{t_k} b_i(t)X_t'(t,s)\,dt = \int_0^{\tau_i} ds\int_0^{\tau_i} b_i(t)X_t'(t,s)\,dt$$

hold. By Lemma 3,

$$\|X(t_k,s)r(s)\| \le \exp\left\{\prod_{0<j<i}(1+B_j)\int_s^{\tau_i}\left(\|A(\zeta)\| + \int_\zeta^{\tau_i}\|K(\xi,\zeta)\|\,d\xi\right)d\zeta\right\}\cdot\|r(s)\|.$$

Therefore, the functions under the integral on the left-hand side of the first limit equality are uniformly bounded for $s \le \tau_i$. Thus, the Lebesgue convergence theorem yields the first equality. By assumption $(a)_4$ and the Corollary of Lemma 3, the functions $\int_0^{t_k} b_i(t)X_t'(t,s)\,dt$ are bounded for $s < t_k < \tau_i$, since

$$\left\| \int_0^{t_k} b_i(t)X_t'(t,s)\,dt \right\| \le \sup_{0\le t\le \tau_i} \|b_i(t)\| \int_0^{\tau_i} \|X_t'(t,s)\|\,dt.$$

Again, the Lebesgue convergence theorem leads to the second limit equality.

For each s the function $X(\cdot,s)$ satisfies the impulsive conditions

$$X(\tau_i,s) = X(\tau_i-0,s) + \int_0^{\tau_i} b_i(t)X_t'(t,s)\,dt + \sum_{s\le\tau_j<t} \lambda_{ij}\big(X(\tau_j,s) - X(\tau_j-0,s)\big).$$

Denote in (13) $(t_0 = 0)$

$$z_1(t) = \int_0^t X(t,s)r(s)\,ds \qquad \text{and} \qquad z_2(t) = \sum_{i=1}^{\infty} X(t,\tau_i) \int_0^{\tau_i} b_i(s)r(s)\,ds.$$

Then, $y(t) = z_1(t) + z_2(t)$ and

$$\dot z_1(t) = r(t) + \int_0^t X_t'(t,s)r(s)\,ds.$$

The latter equality and the above limit equalities yield

$$\begin{aligned}
z_1(\tau_i) - z_1(\tau_i-0) &= \int_0^{\tau_i} X(\tau_i,s)r(s)\,ds - \int_0^{\tau_i} X(\tau_i-0,s)r(s)\,ds \\
&= \int_0^{\tau_i}\left(\int_0^{\tau_i} b_i(t)X_t'(t,s)\,dt\right) r(s)\,ds \\
&\quad + \int_0^{\tau_i} \sum_{s\le\tau_j<\tau_i} \lambda_{ij}\big(X(\tau_j,s) - X(\tau_j-0,s)\big)r(s)\,ds \\
&= \int_0^{\tau_i} b_i(t)z_1'(t)\,dt - \int_0^{\tau_i} b_i(t)r(t)\,dt + \sum_{0\le\tau_j<\tau_i} \lambda_{ij}\Delta z_1(\tau_j) \\
&= l_i(z_1) - \int_0^{\tau_i} b_i(t)r(t)\,dt
\end{aligned}$$

since

$$\sum_{s\le\tau_j<\tau_i} \lambda_{ij}(X(\tau_j,s)-X(\tau_j-0,s)) = \sum_{0\le\tau_j<\tau_i} \lambda_{ij}(X(\tau_j,s)-X(\tau_j-0,s))$$

as $X(t,s)=0$ $(0\le t<s)$. By Lemma 4,

$$z_2(\tau_i)-z_2(\tau_i-0)=l_i(z_2)+\int_0^{\tau_i} b_i(t)r(t)\,dt.$$

Consequently,

$$\begin{aligned} y(\tau_i)-y(\tau_i-0) &= z_1(\tau_i)-z_1(\tau_i-0)+z_2(\tau_i)-z_2(\tau_i-0)\\ &= l_i(z_1)-\int_0^{\tau_i} b_i(t)r(t)\,dt+l_i(z_2)+\int_0^{\tau_i} b_i(t)r(t)\,dt\\ &= l_i(z_1)+l_i(z_2)=l_i(z_1+z_2)=l_i(y), \end{aligned}$$

i.e. y satisfies (15). The proof is complete ∎

Remark. After denoting

$$G(t,s)=X(t,s)+\sum_{\tau_i\ge s} \xi_{[s,\tau_i]}X(t,\tau_i)b_i(s) \tag{16}$$

where ξ_Ω is the characteristic function of the set Ω, (13) can be rewritten as

$$y(t)=\int_{t_0}^{t} G(t,s)r(s)\,ds \tag{17}$$

which gives a solution of the semi-homogeneous impulsive equation (2) - (3).

Lemma 6. *Suppose the assumptions $(a)_1-(a)_3$ hold and the columns of the matrices $c_i:[0,\tau_i]\to\mathbb{R}^{n\times n}$ are integrable over $[0,\tau_i]$. Then, the solution y of problem* (2),(5), *$y(t_0)=0$, $\varphi(t)\equiv 0$, $\alpha_i=0$ can be represented as*

$$y(t)=\int_{t_0}^{t} X(t,s)r(s)\,ds$$

where $X(t,s)$ is a fundamental function of (2),(5).

Proof. As demonstrated above the impulsive condition (5) can be rewritten as (3). Hence, by Lemma 2 the problem mentioned in the statement of the lemma has one and only one solution. Lemma 3 implies the estimate for the fundamental matrix $X(t,s)$ of the equation (2),(5). Similar to the proof of Lemma 5 y is shown to be a solution of

(2). Thus, it remains to prove that it also satisfies the impulsive conditions (5). By the definition $X(t,s)$ satisfies (5) as a function of the first argument

$$X(\tau_i, s) = (E + B_i)X(\tau_i - 0) + \int_0^{\tau_i} c_i(\tau)X(\tau, s)\,d\tau.$$

The application of Lemma 3 as in the proof of the previous lemma yields

$$\begin{aligned} y(\tau_i) &= \int_0^{\tau_i} X(\tau_i, s)r(s)\,ds \\ &= (E + B_i)\int_0^{\tau_i} X(\tau_i - 0, s)r(s)\,ds + \int_0^{\tau_i}\left(\int_0^{\tau_i} c_i(\tau)X(\tau, s)\,d\tau\right) r(s)\,ds \\ &= (E + B_i)y(\tau_i - 0) + \int_0^{\tau_i} c_i(\tau)\left(\int_0^{\tau_i} X(\tau, s)r(s)\,ds\right) d\tau \\ &= (E + B_i)y(\tau_i - 0) + \int_0^{\tau_i} c_i(\tau)\left(\int_0^{\tau} X(\tau, s)r(s)\,ds\right) d\tau \\ &= (E + B_i)y(\tau_i - 0) + \int_0^{\tau_i} c_i(\tau)y(\tau)\,d\tau. \end{aligned}$$

The latter equality is equivalent to (5), which completes the proof ∎

Lemma 2 and Lemmas 4 - 6 immediately imply the following results.

Theorem 1. *Let the assumptions $(a)_1 - (a)_6$ hold. Then, for any $\alpha_0 \in \mathbb{R}^n$ there exists one and only one solution of the initial value problem $(2) - (3)$, $x(t_0) = \alpha_0$ that can be represented as*

$$\begin{aligned} x(t) = &\int_{t_0}^{t} G(t,s)r(s)\,ds + \int_{t_0}^{t} G(t,s)\left(\int_0^{t_0} K(s,\xi)\varphi(\xi)\,d\xi\right) ds \\ &+ \sum_{\tau_j > t_0} X(t, \tau_j)\alpha_j + X(t, t_0)\alpha_0 \end{aligned} \tag{18}$$

where $G(t,s)$ is defined by (16).

Theorem 2. *Let the assumptions $(a)_1 - (a)_3$ hold and the columns of c_i be integrable on $[0, \tau_i]$. Then, for any $\alpha_0 \in \mathbb{R}^n$ there exists one and only one solution of problem $(2), (5), x(t_0) = \alpha_0$ which has the representation*

$$\begin{aligned} x(t) = &\int_{t_0}^{t} X(t,s)r(s)\,ds + \int_{t_0}^{t} X(t,s)\left(\int_0^{t_0} K(s,\xi)\varphi(\xi)\,d\xi\right) ds \\ &+ \sum_{\tau_j > t_0} X(t, \tau_j)\alpha_j + X(t, t_0)\alpha_0 \end{aligned}$$

where $X(t,s)$ *is a fundamental function of* (2), (5).

5. Stability

In this section Theorem 5 generalizes the stability test for ordinary differential equations with coefficients being integrable on the half-line.

Definition. Let x be any solution of the impulsive differential equation (2)-(3) with $r(t) \equiv 0$ and $\alpha_i = 0$. Equation (2)-(3) is said to be

(i) *stable* if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists a $\delta > 0$ not depending on t_0 such that $\|x(t_0)\| < \delta$ and $\mathrm{vraisup}_{t<t_0}\|\varphi(t)\| < \delta$ imply $\|x(t)\| < \varepsilon$ $(t \geq t_0)$;

(ii) *asymptotically stable* if, for each initial function φ, each t_0 and each initial value $x(t_0)$, $\lim_{t\to\infty}\|x(t)\| = 0$;

(iii) *exponentially stable* if for any $t_0 \geq 0$ there exist constants $N > 0$ and $\lambda > 0$ not depending on t_0 such that $\|x(t)\| \leq Ne^{-\lambda(t-t_0)}(\|x(t_0)\| + \mathrm{vraisup}_{t<t_0}\|\varphi(t)\|)$.

Lemma 7. *Let the assumptions* $(a)_1 - (a)_6$ *hold.*

(i) *If*

$$\operatorname*{vrai\,sup}_{t,s>0}\|X(t,s)\| < \infty$$

$$\operatorname*{vrai\,sup}_{t_0>0}\operatorname*{vrai\,sup}_{t>t_0}\int_{t_0}^{t}\|G(t,s)\|\int_{0}^{t_0}\|K(s,\tau)\|\,d\tau ds < \infty,$$

then equation $(2)-(3)$ *is stable.*

(ii) *If for every* $t_0 \geq 0$

$$\lim_{t\to\infty} X(t,t_0) = 0$$

$$\lim_{t\to\infty}\int_{t_0}^{t}\|G(t,s)\|\int_{0}^{t_0}\|K(s,\tau)\|\,d\tau ds = 0,$$

then equation $(2)-(3)$ *is asymptotically stable.*

(iii) *If there exist positive constants* N_1, N_2 *and* ν_1, ν_2 *such that*

$$\|X(t,t_0)\| \leq N_1 e^{-\nu_1(t-t_0)}$$

$$\int_{t_0}^{t}\|G(t,s)\|\int_{0}^{t_0}\|K(s,\tau)\|\,d\tau ds \leq N_2 e^{-\nu_2(t-t_0)},$$

then equation $(2)-(3)$ *is exponentially stable.*

Proof. If $f \equiv 0, \varphi \equiv 0$ and $\alpha_i = 0$, then by Theorem 1 the solution of equation (2) - (3) has the form

$$x(t) = X(t,t_0)\,x(t_0) + \int_{t_0}^{t} G(t,s)\,ds \int_{0}^{t_0} K(s,\xi)\varphi(\xi)\,d\xi.$$

Therefore,

$$\|x(t)\| \le \max\left\{ \|X(t,t_0)\|, \int_{t_0}^{t} \|G(t,s)\| \int_{0}^{t_0} \|K(s,\xi)\|\,d\xi ds \right\} \times \left(\|x(t_0)\| + \operatorname*{vrai\,sup}_{\xi<t_0} \|\varphi(\xi)\|\right).$$

Now, the stability definition immediately implies the statement of the lemma ∎

Additional constraints on the kernel K lead to more convenient stability tests.

Theorem 3. *Suppose the assumptions $(a)_1 - (a)_6$ hold and, in addition, there exist constants $M > 0$ and $\mu > 0$ such that $\|K(t,s)\| \le M\exp\{-\mu(t-s)\}$.*

(i) *If*

$$\operatorname*{vrai\,sup}_{t,s>0} \|X(t,s)\| < \infty$$
$$\operatorname*{vrai\,sup}_{t,s>0} \|G(t,s)\| < \infty,$$

then equation (2) – (3) is stable.

(ii) *If for every $t_0 \ge 0$*

$$\lim_{t\to\infty} \|X(t,t_0)\| = 0$$
$$\lim_{t\to\infty} \int_{t_0}^{t} \|G(t,s)\| e^{-\mu s} ds = 0,$$

then equation (2) – (3) is asymptotically stable.

(iii) *If there exist constants $N_1, N_2 > 0$ and $\nu_1, \nu_2 > 0$ such that*

$$\|X(t,s)\| \le N_1 e^{-\nu_1(t-s)}$$
$$\|G(t,s)\| \le N_2 e^{-\nu_2(t-s)},$$

then equation (2) – (3) is exponentially stable.

Proof. The proof is based on Lemma 7 and straightforward calculations ∎

Theorem 4. *Let the assumptions* $(a)_1 - (a)_6$ *hold and let there exist a constant* $\delta > 0$ *and a function* $k(t)$ *integrable over any finite interval* $[a, b]$ *such that* $K(t, s) = 0$ *if* $t - s > \delta$ *and* $\|K(t, s)\| \le k(t)$.

(i) *If*

$$\operatorname*{vrai\,sup}_{t,s>0} \|X(t,s)\| < \infty$$
$$\operatorname*{vrai\,sup}_{t,s>0} \|G(t,s)\| < \infty,$$

then equation $(2) - (3)$ *is stable.*

(ii) *If for each* $t_0 \ge 0$

$$\lim_{t\to\infty} \|X(t, t_0)\| = 0$$
$$\lim_{t\to\infty} \int_{t_0}^{t_0+\delta} \|G(t,s)\| k(s)\, ds = 0,$$

then equation $(2) - (3)$ *is asymptotically stable.*

(iii) *If there exist certain constants* $N_1, N_2 > 0$ *and* $\nu_1, \nu_2 > 0$ *such that*

$$\|X(t,s)\| \le N_1 e^{-\nu_1(t-s)}$$
$$\|G(t,s)\| \le N_2 e^{-\nu_2(t-s)},$$

and $\sup_{t>0} \int_t^{t+\delta} k(s)ds < \infty$, *then equation* $(2) - (3)$ *is exponentially stable.*

Proof. The assumptions of the theorem yield

$$\int_{t_0}^{t} \|G(t,s)\| \int_0^{t_0} \|K(s,\xi)\|\, d\xi ds = \int_{t_0}^{t_0+\delta} \|G(t,s)\| \int_{t_0-\delta}^{t_0} \|K(s,\xi)\|\, d\xi ds.$$

The assumption $\|K(t,s)\| \le k(t)$, Lemma 7 and the above equality immediately imply the statement of the theorem, which completes the proof ∎ .

Remarks. **1.** For equation (2),(5) in the assumptions of Lemma 7 and Theorems 3 and 4 the function $G(t, s)$ is to be replaced by the fundamental function $X(t, s)$. **2.** Such constraint on the kernel $K(t, s)$ as in Theorem 3 (exponential decay) occurs, for example, in elasticity problems [6]. The condition $K(t, s) = 0$ $(t - s > \delta)$ in Theorem 4 is an analogue of a bounded delay for delay equations.

The following theorem contains explicit stability results for equation (2),(5).

Theorem 5. *Suppose that for equation* (2),(5) *the assumptions* $(a)_1 - (a)_3$ *hold and the columns of* c_i *are integrable on* $[0, \tau_i]$. *In addition, let*

$$\int_0^\infty \|A(s)\|\, ds < \infty$$

$$\int_0^\infty \int_0^s \|K(s,\xi)\|\, d\xi ds < \infty$$

$$\sup_i \left\{ \int_0^{\tau_i} \|c_i(s)\|\, ds + \|E_n + B_i\| \right\} < 1.$$

Then, equation (2),(5) *is stable.*

Proof. We apply Lemma 7, wherein for equation (2),(5) $G(t,s) = X(t,s)$. So for the completeness of the proof it is enough to show the boundedness of the fundamental function on the half-line. In fact, if

$$\operatorname*{vrasup}_{t,s>0} \|X(t,s)\| \le M \qquad (M < \infty),$$

then the assumptions of the theorem yield that, for every $t_0 > 0$,

$$\begin{aligned} \int_{t_0}^t \|X(t,s)\| \int_0^{t_0} \|K(s,\xi)\|\, d\xi ds &\le M \int_{t_0}^t \int_0^s \|K(s,\xi)\|\, d\xi ds \\ &\le M \int_0^\infty \int_0^s \|K(s,\xi)\|\, d\xi ds < \infty. \end{aligned}$$

Consequently, by Lemma 7 equation (2),(5) is stable.

Now, let us prove that $X(t,s)$ is bounded. To this end consider the auxiliary equation

$$\dot{x}(t) = z(t) \tag{19}$$

with impulsive conditions (5). By Theorem 2, the solution $x(t)$ of problem (19), (5), $x(0) = 0, \alpha_i = 0$ is of the form

$$x(t) = \int_0^t X_0(t,s) z(s)\, ds$$

where $X_0(t,s)$ is a fundamental function of this equation. Thus, problem (2),(5), $x(0) = 0$, $\alpha_i = 0$ is equivalent to the equation

$$x(t) = \int_0^t X_0(t,s) r(s)\, ds - \int_0^t X_0(t,s) \left(A(s)x(s) + \int_0^s K(s,\xi) x(\xi)\, d\xi \right) ds. \tag{20}$$

The results of [1] imply $\operatorname{vraisup}_{t,s>0} \|X_0(t,s)\| \le 1$. Suppose $r \in L[0,\infty)$. Then, by (20) the solution of problem (2),(5), $x(0) = 0, \alpha_i = 0$ can be estimated as

$$\|x(t)\| \le \int_0^\infty \|r(s)\|\, ds + \int_0^t \left(\|A(s)\| + \int_s^t \|K(\xi,s)\|\, d\xi \right) \|x(s)\| ds.$$

The Gronwall-Bellman inequality implies

$$\begin{aligned} \|x(t)\| &\le \int_0^\infty \|r(s)\|\, ds \exp\left\{ \int_0^t \|A(s)\|\, ds + \int_0^t \int_0^s \|K(s,\xi)\|\, d\xi ds \right\} \\ &\le \int_0^\infty \|r(s)\|\, ds \exp\left\{ \int_0^\infty \|A(s)\|\, ds + \int_0^\infty \int_0^s \|K(s,\xi)\|\, d\xi ds \right\}. \end{aligned}$$

Therefore, if $r \in L[0,\infty)$, then the solution of the problem (2),(5), $x(0) = 0, \alpha_i = 0$ is in $L_\infty[0,\infty)$. On the other hand, by Theorem 2, the solution of this problem can be represented as

$$x(t) = \int_0^t X(t,s) r(s)\, ds.$$

The above argument yields that the integral operator with the kernel $X(t,s)$ acts from the space $L[0,\infty)$ into the space $L_\infty[0,\infty)$. Consequently (see [7: p. 191]), $\operatorname{vraisup}_{t,s>0} \|X(t,s)\| < \infty$. The latter inequality, as mentioned above, implies the stability of (2),(5). The proof of the theorem is complete ∎

References

[1] Anokhin, A., Berezansky, L. and E. Braverman: *Stability of linear delay impulsive equations.* Dyn. Systems Appl. 4 (1995), 173 - 189.

[2] Anokhin, A., Berezansky, L. and E. Braverman: *Exponential stability of linear delay impulsive differential equations.* J. Math. Anal. Appl. 193 (1995), 923 - 945.

[3] Burton, T. A.: *Volterra Integral and Differential Equations.* New York: Acad. Press 1983.

[4] Corduneanu, C.: *Integral Equations and Applications.* Cambridge: Univ. Press 1990.

[5] Gripenberg, G., Londen, S.-O. and O. Staffans: *Volterra Integral and Functional Equations.* Cambridge: Univ. Press 1990.

[6] Fabrizio, M. and A. Morro: *Mathematical Problems in Linear Viscoelasticity* (SIAM Studies in Appl. Math.: Vol. 12). Philadelphia: SIAM 1992.

[7] Kantorovich, L. V. and G. P. Akilov: *Functional Analysis.* Oxford: Pergamon Press 1982.

[8] Lakshmikantham, V., Bainov, D. D. and P. S. Simeonov: *Theory of Impulsive Differential Equations.* Singapore: World Scientific 1989.

[9] Rao, M. R. M. Sathanantham, S. and S. Sivasundaram: *Asymptotic behavior of solutions of impulsive integro-differential systems.* Appl. Math. Comput. 34 (1989), 195 - 221.

[10] Rao, M. R. M., Sanjay, K. S. S. Sivasundaram: *Stability of Volterra integro-differential equations with impulsive effect.* J. Math. Anal. Appl. 163 (1992), 47 - 59.

[11] Simeonov, P. S. and D. D. Bainov: *Stability of the solutions of a system of integro-differential equations with impulsive effect.* Math. Rep. Toyama Univ. 9 (1986), 1 - 24.

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