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Fourier Multipliers between Weighted Anisotropic Function Spaces Part II. Besov-Triebel Spaces

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Abstract. We determine certain classes $\mathbf{M}(X^{s_0}_{p_0,q_0}(w_0), Y^{s_1}_{p_1,q_1}(w_1))$ of Fourier multipliers between weighted anisotropic Besov and Triebel spaces $X^{s_0}_{p_0,q_0}(w_0)$ and $Y^{s_1}_{p_1,q_1}(w_1)$ where $p_0 \leq 1$ and w_0, w_1 are weight functions of polynomial growth. To this end we refine a method based on discrete characterizations of function spaces which was introduced in Part I of the paper. Thus widely generalized versions of known results of Bui, Johnson and others are obtained in a unified way.

Keywords: *Fourier multipliers, weighted Besov and Triebel spaces, anisotropic spaces*

AMS subject classification: Primary 42 B 25, 46 E 35, secondary 46 E 39

1. Introduction

This is the continuation of the first part [6] of this work in which we considered Fourier multipliers between Besov spaces. Concerning definitions and notations we refer to this paper. Now we refine our methods to deal with the more general situation of Besov and Triebel spaces. This leads in particular to a generalization of the two theorems

$$\mathbf{M}(h_{p_0}, L_{p_1}) \rightleftarrows \mathcal{F}[B^{n(\frac{1}{p_0}-1)}_{p_1,\infty}] \qquad (0 < p_0 < 1 < p_1 < \infty)$$

and

$$\mathbf{M}(h_{p_0}, B^{s_1}_{p_1,q_1}) \rightleftarrows \mathcal{F}[B^{s}_{p_1,\infty}] \qquad \begin{cases} 0 < p_0 < 1 \leq p_1 < \infty \\ 1 \leq q_1 < \infty \\ s_1 \in \mathbb{R} \end{cases}$$

due to Johnson [11] and Bui [2]. These results are generalized to weighted anisotropic Besov-Triebel spaces and the ranges of the parameters involved are widely extended. Furthermore we sharpen these characterizations of multipliers by a couple of negative results.

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2. Triebel spaces

For $0 < p, q < \infty$, $s \in \mathbb{R}$ and $w \in W$ the (anisotropic inhomogeneous) *Triebel space* $F^s_{p,q}(\mathbb{R}^n; P, w)$ which is denoted by $F^s_{p,q}(w)$ for short contains all $f \in \mathcal{S}'$ (the space of tempered distributions) with finite quasinorm

$$\|f|F^s_{p,q}(w)\| = \left\| w \cdot \left\| \langle 2^{js}\mathcal{F}^{-1}[\phi_j \mathcal{F} f](\bullet)\rangle_{j\in\mathbf{N}} |\ell_q \right\| \right\|_p .$$

For remarks concerning the literature related to these spaces confer the references given in the first part [6] of the paper.

To extend our discrete methods to the $F^s_{p,q}(w)$ spaces we have to look for a discrete counterpart of $F^s_{p,q}(w)$. To this end we use the sequence space $f^s_{p,q}(\mathbb{R}^n; P, w)$ which is denoted by $f^s_{p,q}(w)$ for short ($0 < p, q \leq \infty$, $s \in \mathbb{R}$, $w \in W$) and which contains all complex sequences $\alpha = \langle \alpha^j_k \rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}$ with finite quasinorm

$$\|\alpha|f^s_{p,q}\| = \left\| \left\| \left\langle 2^{js} \sum_{k\in\mathbf{Z}^n} w^j_k |\alpha^j_k| \mathbf{1}^j_k(\bullet) \right\rangle_{j\in\mathbf{N}} \right| \ell_q \right\| \right\|_p$$

in which $w^j_k = w(A_{2^{-j}}k)$ and $\mathbf{1}^j_k$ denotes the characteristic function of the set

$$\Diamond^j_k = A_{2^{-j}}\left(k + \left[-\frac{1}{2}, \frac{1}{2}\right)^n\right).$$

These sets have measure $2^{-j\nu}$ and constitute a covering of $\mathbb{R}^n$ of pairwise disjoint elements for each fixed j. The connection with the Besov spaces is given by

$$\|f|B^s_{p,p}(w)\| = \|f|F^s_{p,p}(w)\| \qquad \text{and} \qquad \|\alpha|b^s_{p,p}(w)\| = \|\alpha|f^s_{p,p}(w)\|.$$

The unweighted spaces (i.e. $w \equiv 1$) are denoted by $F^s_{p,q}$ and $f^s_{p,q}$ as usual. In the sequel the symbols $X^s_{p,q}(w)$ and $Y^s_{p,q}(w)$ always denote Besov or Triebel spaces and the associated sequence spaces are denoted by $x^s_{p,q}(w)$ and $y^s_{p,q}(w)$, respectively.

The discrete characterization of $F^s_{p,q}(w)$ reads as follows.

Theorem 2.1 (Discrete characterization of Triebel spaces). *For $f \in \mathcal{S}'$ define the sequence* $\mathbf{se}f$ *by*

$$\mathbf{se}f = \left\langle (2\pi)^{-\frac{n}{2}} \mathcal{F}^{-1}[\phi_j \mathcal{F} f](A_{2^{-j}}k) \right\rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}.$$

For finite sequences $\alpha = \langle \alpha^j_k \rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}$ of complex numbers define the function $\mathbf{fu}\alpha$ *by*

$$\mathbf{fu}\,\alpha = \sum_{k\in\mathbf{Z}^n} \alpha^0_k \cdot (\mathcal{F}^{-1}\psi_0)(\bullet - k) + \sum_{j=1}^{\infty} \sum_{k\in\mathbf{Z}^n} \alpha^j_k \cdot (\mathcal{F}^{-1}\psi_1)(A_{2^j} \bullet - k).$$

Assume $0 < p, q < \infty$, $s \in \mathbb{R}$ and $w \in W$. Then the operators

$$\mathbf{se} : F^s_{p,q}(w) \to f^s_{p,q}(w) \qquad \textit{and} \qquad \mathbf{fu} : f^s_{p,q}(w) \to F^s_{p,q}(w)$$

are bounded (the unique extension of **fu** *to* $f^s_{p,q}(w)$ *is denoted by* **fu**, *too). Furthermore,* **fu** $\circ$ **se** $= id$ *on* $F^s_{p,q}(w)$ *and*

$$\|\mathbf{se} f | f^s_{p,q}(w)\| \sim \|f | F^s_{p,q}(w)\|$$

for all $f \in S'$.

Corollary 2.2 (Boundedness of linear operators). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w_0, w_1 \in W$. *A linear operator* $T : X^{s_0}_{p_0,q_0}(w_0) \to Y^{s_1}_{p_1,q_1}(w_1)$ *is bounded if and only if* **se** T **fu** $: x^{s_0}_{p_0,q_0}(w_0) \to y^{s_1}_{p_1,q_1}(w_1)$ *is bounded. The respective operator quasinorms are equivalent.*

3. Boundedness of matrix operators

Corresponding to the case of Besov spaces we start with the following lemma which can be proved by straightforward computations.

Lemma 3.1 (Boundedness of A and $A(w_0, w_1)$). *Assume* $0 < p_0, p_1, q_0, q_1 \le \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w_0, w_1 \in W$. *For a matrix* $A = \langle A^{j,l}_{k,m} \rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$ *define* $A(w_0, w_1)$ *by*

$$A(w_0, w_1) = \left\langle (w_1)^l_m A^{j,l}_{k,m} \frac{1}{(w_0)^j_k} \right\rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$$

with $(w_0)^l_m = w_0(A_{2^{-l}} m)$ *and* $(w_1)^j_k = w_1(A_{2^{-j}} k)$. *Then the relation*

$$\left\| A \middle| x^{s_0}_{p_0,q_0}(w_0), y^{s_1}_{p_1,q_1}(w_1) \right\| = \left\| A(w_0, w_1) \middle| x^{s_0}_{p_0,q_0}, y^{s_1}_{p_1,q_1} \right\|$$

holds for all A.

Now we prove a boundedness criterion for matrix operators analogous to the one for the case of Besov spaces. There we used a Sobolev-type embedding theorem for the sequence spaces $b^s_{p,q}(w)$ which was straightforward from the embedding of ℓ_p spaces. In the case of Triebel spaces the things are a little more complicated.

Lemma 3.2 (Sobolev-type embedding for $f^s_{p,q}$). *Assume* $0 < p_0, p_1, q_0, q_1 \le \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Then*

$$f^{s_0}_{p_0,q_0} \hookrightarrow f^{s_1}_{p_1,q_1} \qquad \left(s_0 - \tfrac{\nu}{p_0} = s_1 - \tfrac{\nu}{p_1}\right).$$

Proof. By the embedding of ℓ_p spaces it suffices to prove that $id : f^{s_0}_{p_0,\infty} \to f^{s_1}_{p_1,q_1}$ is bounded. This can be done by standard methods using an idea of Jawerth [9] (cf. [18: p. 129]) and has already be done in [4] ∎

Theorem 3.3 (Boundedness criterion for matrix operators). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *For a matrix* $A = \langle A^{j,l}_{k,m} \rangle^{j,l \in \mathbb{N}}_{k,m \in \mathbb{Z}^n}$ *put*

$$B(A; s_0, p_0, y^{s_1}_{p_1,q_1}) = \left\| \left\langle 2^{j(\frac{\nu}{p_0} - s_0)} \sup_{k \in \mathbb{Z}^n} \left\| \langle A^{j,l}_{k,m} \rangle^{l \in \mathbb{N}}_{m \in \mathbb{Z}^n} \middle| y^{s_1}_{p_1,q_1} \right\| \right\rangle_{j \in \mathbb{N}} \middle| \ell_\infty \right\|.$$

Then:

a) *The inequality*

$$B(A;s_0,p_0,y^{s_1}_{p_1,q_1}) \le \|A|x^{s_0}_{p_0,q_0},y^{s_1}_{p_1,q_1}\|$$

always holds.

b) *If* $p_0 < \min\{1,p_1,q_1\}$ *or* $\max\{p_0,q_0\} \le \min\{1,p_1,q_1\}$, *then the equivalence*

$$\|A|f^{s_0}_{p_0,q_0},y^{s_1}_{p_1,q_1}\| \sim B(A;s_0,p_0,y^{s_1}_{p_1,q_1})$$

holds for all A.

c) *If* $\max\{p_0,q_0\} \le \min\{1,p_1,q_1\}$, *then the equivalence*

$$\|A|b^{s_0}_{p_0,q_0},y^{s_1}_{p_1,q_1}\| \sim B(A;s_0,p_0,y^{s_1}_{p_1,q_1})$$

holds for all A.

Proof. Assertion a) can be proved as in [6] with the help of the sequences

$$(\varepsilon^j_k)^l_m = \begin{cases} 1 & \text{for } j=l \text{ and } k=m \\ 0 & \text{otherwise} \end{cases} \qquad (j,l\in\mathbb{N};\ k,m\in\mathbb{Z}^n).$$

To prove assertions b) and c) we refine the proof given in [6]. To this end we restrict ourselves to finite sequences and consider only the case of $y^{s_1}_{p_1,q_1} = f^{s_1}_{p_1,q_1}$. Put $r = \min\{1,p_1,q_1\}$. For $x \in \Diamond^l_{m_0}$ we obtain the estimate

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n}\left|\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}(\alpha^j_k\cdot A\varepsilon^j_k)^l_m\right|\mathbf{1}^l_m(x) &\le \left\|\langle(\alpha^j_k\cdot A\varepsilon^j_k)^l_{m_0}\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}|\ell_1\right\| \\ &\le \left\|\langle(\alpha^j_k\cdot A\varepsilon^j_k)^l_{m_0}\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}|\ell_r\right\| \\ &\le \left\|\left\langle\sum_{m\in\mathbb{Z}^n}|(\alpha^j_k\cdot A\varepsilon^j_k)^l_m|\mathbf{1}^l_m(x)\right\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}\middle|\ell_r\right\|\end{aligned}$$

for all α and all $x\in\mathbb{R}^n$. Applying the generalized Minkowski inequality twice we are lead to the relation

$$\begin{aligned}&\left\|A\left(\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^n}\alpha^j_k\varepsilon^j_k\right)\middle|f^{s_1}_{p_1,q_1}\right\| \\ &\le \left\|\left\|\left\langle 2^{ls_1}\left\|\left\langle\sum_{m\in\mathbb{Z}^n}|(\alpha^j_k\cdot A\varepsilon^j_k)^l_m|\mathbf{1}^l_m(\bullet)\right\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}\middle|\ell_r\right\|\right\rangle_{l\in\mathbb{N}}\middle|\ell_{q_1}\right\|\right\|_{p_1} \\ &= \left\|\left\|\left\langle 2^{ls_1}\left\|\left\langle\left(\sum_{m\in\mathbb{Z}^n}|(\alpha^j_k\cdot A\varepsilon^j_k)^l_m|\mathbf{1}^l_m(\bullet)\right)^r\right\rangle^{j\in\mathbb{N}}_{k\in\mathbb{Z}^n}\middle|\ell_1\right\|\right\rangle_{l\in\mathbb{N}}\middle|\ell_{\frac{q_1}{r}}\right\|\right\|^{\frac{1}{r}}_{\frac{p_1}{r}}\end{aligned}$$

$$\leq \left\| \left\langle \left\| \left\langle \left(2^{ls_1} \sum_{m\in\mathbf{Z}^n} |\alpha_k^j| \, |(A\varepsilon_k^j)_m^l| \mathbf{1}_m^l(\bullet) \right)^r \right\rangle_{l\in\mathbf{N}} \left| \ell_{\frac{q_1}{r}} \right\| \right\rangle_{k\in\mathbf{Z}^n}^{j\in\mathbf{N}} \left| \ell_1 \right\| \right\|_{\frac{p_1}{r}}^{\frac{1}{r}}$$

$$\leq \left\| \left\langle \left\| \left\| \left\langle \left(|\alpha_k^j| \, 2^{ls_1} \sum_{m\in\mathbf{Z}^n} |(A\varepsilon_k^j)_m^l| \mathbf{1}_m^l(\bullet) \right)^r \right\rangle_{l\in\mathbf{N}} \left| \ell_{\frac{q_1}{r}} \right\| \right\|_{\frac{p_1}{r}} \right\rangle_{k\in\mathbf{Z}^n}^{j\in\mathbf{N}} \left| \ell_1 \right\|^{\frac{1}{r}}$$

$$\leq \left\| \langle |\alpha_k^j| \cdot \| A\varepsilon_k^j | f_{p_1,q_1}^{s_1} \| \rangle_{k\in\mathbf{Z}^n}^{j\in\mathbf{N}} | \ell_r \right\|.$$

Now using the Sobolev-type embedding presented in Lemma 3.2 and the definition of $B(A; s_0, p_0, f_{p_1,q_1}^{s_1})$ we get

$$\begin{aligned}
&\left\| A \left(\sum_{j=0}^{\infty} \sum_{k\in\mathbf{Z}^n} \alpha_k^j \, \varepsilon_k^j \right) \Big| f_{p_1,q_1}^{s_1} \right\| \\
&\qquad \leq \sup_{j\in\mathbf{N}} \sup_{k\in\mathbf{Z}^n} \frac{\| A\varepsilon_k^j | f_{p_1,q_1}^{s_1} \|}{\| \varepsilon_k^j | x_{p_0,q_0}^{s_0} \|} \cdot \left\| \langle |\alpha_k^j| \cdot \| \varepsilon_k^j | x_{p_0,q_0}^{s_0} \| \rangle_{k\in\mathbf{Z}^n}^{j\in\mathbf{N}} | \ell_r \right\| \\
&\qquad = B(A; s_0, p_0, f_{p_1,q_1}^{s_1}) \cdot \| \alpha | b_{r,r}^{\sigma} \| \\
&\qquad \leq C \cdot B(A; s_0, p_0, f_{p_1,q_1}^{s_1}) \cdot \| \alpha | x_{p_0,q_0}^{s_0} \|
\end{aligned}$$

where $\sigma = s_0 + \nu\left(\frac{1}{r} - \frac{1}{p_0}\right)$. This completes the proof ∎

4. Fourier multipliers

For $M \in \mathcal{S}'$ the operator T_M is given by

$$T_M f = \mathcal{F}^{-1}[M\mathcal{F}f] \qquad (f \in \mathcal{S})$$

and the class of *Fourier multipliers* between the two spaces $X_{p_0,q_0}^{s_0}(w_0)$ and $Y_{p_1,q_1}^{s_1}(w_1)$ is defined by

$$\mathbf{M}(X_{p_0,q_0}^{s_0}(w_0), Y_{p_1,q_1}^{s_1}(w_1)) = \left\{ M \in \mathcal{S}' \,\middle|\, T_M : X_{p_0,q_0}^{s_0}(w_0) \to Y_{p_1,q_1}^{s_1}(w_1) \text{ bounded} \right\}$$

equipped with the quasinorm

$$\left\| M | \mathbf{M}(X_{p_0,q_0}^{s_0}(w_0), Y_{p_1,q_1}^{s_1}(w_1)) \right\| = \left\| T_M | X_{p_0,q_0}^{s_0}(w_0), Y_{p_1,q_1}^{s_1}(w_1) \right\|.$$

Concerning general results we have the following proposition.

Proposition 4.1 (Change of s). *Assume* $0 < p_0, p, q_0, q_1 < \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w_0, w_1 \in W$. *Then*

$$\mathbf{M}(X^{s_0}_{p_0,q_0}(w_0), Y^{s_1}_{p_1,q_1}(w_1)) \leftrightarrows \mathbf{M}(X^{s_1-s_0}_{p_0,q_0}(w_0), Y^{0}_{p_1,q_1}(w_1)).$$

Proof. It can be carried over from [6] since the mapping

$$I_\sigma : \mathbb{C}^{\mathbf{N}} \times \mathbb{Z}^n \to \mathbb{C}^{\mathbf{N}} \times \mathbb{Z}^n \qquad \text{with} \qquad I_\sigma \alpha = \langle 2^{j\sigma} \alpha^j_k \rangle^{j\in\mathbf{N}}_{k\in\mathbf{Z}^n}$$

is an isometric isomorphism $I_\sigma : f^s_{p,q}(w) \to f^{s-\sigma}_{p,q}(w)$ ∎

Now we are in a position to formulate and prove our main results.

Theorem 4.2 (Fourier multipliers between Triebel spaces). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Furthermore, assume that* $w_1 \in W_d$ *and that* $d \geq 0$ *and* $w_0 \in W$ *satisfy the condition* $\left\| \frac{(1+|\bullet|)^d}{w_0} \right\|_\infty < \infty$. *If either*

a) $p_0 < \min\{1, p_1, q_1\}$

or

b) $p_0 = \min\{1, p_1, q_1\}$ *and* $q_0 \leq \min\{p_1, q_1\}$,

then

$$\mathbf{M}(F^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)) \leftrightarrows \mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \qquad \left(\sigma = \nu\left(\tfrac{1}{p_0} - 1\right) + s_1 - s_0\right).$$

Proof. a) From the equivalences in Lemma 3.1, Theorem 3.3/b) and [6: Corollary 4.2] we obtain the relation

$$\begin{aligned} \big\|M\big|\mathbf{M}(F^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1))\big\| &\sim \big\|\widetilde{M}(w_0, w_1)\big| f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big\| \\ &\sim B\big(\widetilde{M}(w_0, w_1); s_0, p_0, f^{s_1}_{p_1,q_1}\big) \\ &\sim \big\|M\big|\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)]\big\|. \end{aligned}$$

b) Using [6: Corollary 4.2], Theorem 3.3/a) and Lemma 3.1 one gets the inequality

$$\begin{aligned} \big\|M\big|\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)]\big\| &\leq C_0 \cdot B\big(\widetilde{M}(w_0, w_1); s_0, p_0, f^{s_1}_{p_1,q_1}\big) \\ &\leq C_1 \cdot \big\|\widetilde{M}(w_0, w_1)\big| f^{s_0}_{p_0,q_0}, f^{s_1}_{p_1,q_1}\big\| \\ &\leq C_2 \cdot \big\|M\big|\mathbf{M}(F^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1))\big\| \end{aligned}$$

from which the embedding

$$\mathbf{M}(F^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)) \hookrightarrow \mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)]$$

follows. Because of $p_0 = \min\{1, p_1, q_1\}$ and $q_0 \leq \min\{p_1, q_1\}$ we have $\max\{p_0, q_0\} \leq \min\{p_1, q_1\}$. Thus we can apply the assertion [6: Theorem 5.3] from Part I of this paper to get

$$\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \hookrightarrow \mathbf{M}(B^{s_0}_{p_0,\max\{p_0,q_0\}}(w_0), B^{s_1}_{p_1,\min\{p_1,q_1\}}(w_1)).$$

Using the diagram

$$\begin{array}{ccc}
B^{s_0}_{p_0,\max\{p_0,q_0\}}(w_0) & \xrightarrow{\ T_M\ } & B^{s_1}_{p_1,\min\{p_1,q_1\}}(w_1) \\
id \Big\uparrow & & \Big\downarrow id \\
F^{s_0}_{p_0,q_0}(w_0) & \xrightarrow{\ T_M\ } & F^{s_1}_{p_1,q_1}(w_1)
\end{array}$$

we now conclude the desired embedding

$$\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \hookrightarrow \mathbf{M}\big(F^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)\big)$$

which completes the proof ∎

Remarks. The first result of this type goes back to Johnson [10: Theorem 5] who proved

$$\mathbf{M}(H_1, H_p) \rightleftarrows \mathcal{F}[\dot{B}^{0}_{p,\infty}(\mathbb{R}^n; I, 1)] \qquad (2 \le p < \infty).$$

Here the Hardy spaces H_p may be identified via $H_p \rightleftarrows \dot{F}^{0}_{p,2}(\mathbb{R}^n; I, 1)$ with homogeneous Triebel spaces. Johnson extended this result in [11: Theorem 8] to

$$\mathbf{M}(H_{p_0}, L_{p_1}) \rightleftarrows \mathcal{F}[\dot{B}^{n(\frac{1}{p_0}-1)}_{p_1,\infty}(\mathbb{R}^n; I, 1)] \qquad \begin{cases} 0 < p_0 < 1 \\ 1 < p_1 < \infty. \end{cases}$$

Here again L_p can be identified with a homogeneous Triebel space via $L_p \rightleftarrows \dot{F}^{0}_{p,2}(\mathbb{R}^n; I, 1)$ $(1 < p < \infty)$. Both results were carried over to the inhomogeneous case by Bui [2: Theorem 4] and read (using the Triebel space notation)

$$\mathbf{M}\big(F^{0}_{1,2}(\mathbb{R}^n; I, 1), F^{0}_{p,2}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}[B^{0}_{p,\infty}(\mathbb{R}^n; I, 1)] \qquad (2 \le p < \infty)$$

and

$$\mathbf{M}\big(F^{0}_{p_0,2}(\mathbb{R}^n; I, 1), F^{0}_{p_1,2}(\mathbb{R}^n; I, 1)\big) \rightleftarrows \mathcal{F}[B^{n(\frac{1}{p_0}-1)}_{p_1,\infty}(\mathbb{R}^n; I, 1)] \qquad \begin{cases} 0 < p_0 < 1 \\ 1 < p_1 < \infty. \end{cases}$$

They follow by choosing $q_0 = q_1 = 2$ und $s_0 = s_1 = 0$ in Theorem 4.2. The unweighted version of the first part of Theorem 4.2 was already proved in [4].

Theorem 4.3 (Multipliers between Triebel and Besov spaces). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Furthermore, assume that* $w_1 \in W_d$ *and that* $d \ge 0$ *and* $w_0 \in W$ *satisfy the condition* $\|\frac{(1+|\bullet|)^d}{w_0}\|_\infty < \infty$. *If either*

a) $p_0 < \min\{1, p_1, q_1\}$

or

b) $p_0 = \min\{1, p_1, q_1\}$ *und* $q_0 \le q_1$,

then

$$\mathbf{M}\big(F^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \rightleftarrows \mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \qquad \big(\sigma = \nu(\tfrac{1}{p_0} - 1) + s_1 - s_0\big).$$

Proof. The assertion can be proved in the same way as the previous theorem ∎

Remarks. In the case of homogeneous spaces Johnson [11: Theorem 7] proved the assertions

$$\mathbf{M}\big(H_{p_0}, \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n; I,1)\big) \rightleftarrows \mathcal{F}[\dot{B}^{s}_{p_1,\infty}(\mathbb{R}^n; I,1)] \qquad \begin{cases} 0 < p_0 < 1 \le p_1 < \infty \\ 1 \le q_1 < \infty \\ s_1 \in \mathbb{R} \end{cases}$$

where $s = n(\frac{1}{p_0} - 1) + s_1$ and

$$\mathbf{M}\big(H_1, \dot{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n; I,1)\big) \rightleftarrows \mathcal{F}[\dot{B}^{s_1}_{p_1,\infty}(\mathbb{R}^n; I,1)] \qquad \begin{cases} 1 \le p_1 < \infty \\ 2 \le q_1 < \infty \\ s_1 \in \mathbb{R} \end{cases}$$

in which the Hardy spaces may be identified with homogeneous Triebel spaces (cf. the previous remark). Bui [2: Theorem 4] carried over both these results to the case of inhomogeneous spaces. They read (using Triebel spaces in the notation instead of inhomogeneous Hardy spaces)

$$\mathbf{M}\big(F^0_{p_0,2}(\mathbb{R}^n; I,1), B^{s_1}_{p_1,q_1}(\mathbb{R}^n; I,1)\big) \rightleftarrows \mathcal{F}[B^{s}_{p_1,\infty}(\mathbb{R}^n; I,1)] \qquad \begin{cases} 0 < p_0 < 1 \\ 1 \le p_1, q_1 < \infty \\ s_1 \in \mathbb{R} \end{cases}$$

where as above $s = n(\frac{1}{p_0} - 1) + s_1$ and

$$\mathbf{M}\big(F^0_{1,2}(\mathbb{R}^n; I,1), B^{s_1}_{p_1,q_1}(\mathbb{R}^n; I,1)\big) \rightleftarrows \mathcal{F}[B^{s_1}_{p_1,\infty}(\mathbb{R}^n; I,1)] \qquad \begin{cases} 1 \le p_1 < \infty \\ 2 \le q_1 < \infty \\ s_1 \in \mathbb{R}. \end{cases}$$

These results can be obtained by choosing $q_0 = 2$ and $s_0 = 0$ in Theorem 4.3. The first part of Bui's theorem was already proved in [4] in the unweighted case.

Theorem 4.4 (Multipliers between Besov and Triebel spaces). *Assume* $0 < p_0, p_1, q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Furthermore, assume that* $w_1 \in W_d$ *and that* $d \ge 0$ *and* $w_0 \in W$ *satisfy the condition* $\|\frac{(1+|\bullet|)^d}{w_0}\|_\infty < \infty$. *If* $p_0 \le \min\{1, p_1\}$ *and* $q_0 \le \min\{p_1, q_1\}$, *then*

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)\big) \rightleftarrows \mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \qquad \big(\sigma = \nu(\tfrac{1}{p_0} - 1) + s_1 - s_0\big).$$

Proof. The embedding

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)]$$

is shown in the same way as in the proof of Theorem 4.2/b). For the converse embedding we use [6: Theorem 5.3] from Part I of this paper which states that

$$\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,\min\{p_1,q_1\}}(w_1)\big).$$

Using the embedding $B^{s_1}_{p_1,\min\{p_1,q_1\}}(w_1) \hookrightarrow F^{s_1}_{p_1,q_1}(w_1)$ we obtain the desired inclusion

$$\mathcal{F}[B^{\sigma}_{p_1,\infty}(w_1)] \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), F^{s_1}_{p_1,q_1}(w_1)\big)$$

and the proof is complete ∎

Remarks. Theorem 4.4 was proved in [4] in the unweighted case under the slightly stronger hypothesis $\max\{p_0, q_0\} \le \min\{1, p_1, q_1\}$.

5. Negative results

In this section we study the question under which conditions on the parameters of the function spaces the corresponding multiplier classes are empty. To this end we need the method of complex interpolation for Besov spaces and some duality arguments.

Theorem 5.1 (Complex interpolation of Besov spaces). *Assume* $1 \le p_0, p_1 \le \infty$, $1 \le q_0, q_1 < \infty$ *and* $s_0, s_1 \in \mathbb{R}$. *Furthermore, let* $w_0, w_1 \in W$ *and* $0 < \theta < 1$. *Then*

$$[B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)]_\theta \rightleftarrows B^s_{p,q}(w)$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, s = (1-\theta)s_0 + \theta s_1$ *and* $w = w_0^{1-\theta} \cdot w_1^\theta$.

For problems in connection with complex interpolation of weighted Besov-Triebel spaces confer the remarks given in [12: p. 321 ff.].

Proof of Theorem 5.1. We give a brief outline of the proof because the method is standard in interpolation theory (cf., e.g., Bergh and Löfström [1: Theorem 6.4.3]). First observe that for $1 \le u \le \infty$ and $v \in W$ the inequality

$$\|v \cdot \mathcal{F}^{-1}[\phi_j \mathcal{F} f]\|_u \le C \cdot \|v \cdot f\|_u$$

holds with a constant $C > 0$ independent of f. This follows from the definition of W. For $1 \le u \le \infty$, $1 \le r < \infty$, $t \in \mathbb{R}$ and $v \in W$ consider the space $\ell^t_r(L_u(v))$ of sequences of tempered distributions normed by

$$\big\|\langle g_j\rangle_{j\in\mathbb{N}} | \ell^t_r(L_u(v))\big\| = \big\|\langle 2^{jt}\|v g_j\|_u\rangle_{j\in\mathbb{N}} | \ell_r\big\|.$$

Similar as in the proof of [1: Theorem 6.4.3] one can show that the two mappings

$$\mathcal{I} : B^t_{u,r}(v) \to \ell^t_r(L_u(v)), \qquad \mathcal{I} f = \langle \mathcal{F}^{-1}[\phi_j \mathcal{F} f]\rangle_{j\in\mathbb{N}}$$

and

$$\mathcal{P} : \ell^t_r(L_u(v)) \to B^t_{u,r}(v), \qquad \mathcal{P}\langle g_j\rangle_{j\in\mathbb{N}} = \sum_{j=0}^{\infty} \mathcal{F}^{-1}[\eta_j \mathcal{F} g_j]$$

($\eta_j = \sum_{l=-1}^{1} \phi_{j+l}$) retract $B^t_{u,r}(v)$ to $\ell^t_r(L_u(v))$. The boundedness of $\mathcal{I}$ is straightforward and the above inequality is needed to show the boundedness of $\mathcal{P}$. The identity $\mathcal{P} \circ \mathcal{I} = id$ is obvious since η_j is identical 1 on the support of ϕ_j. Thus by [1: Theorem 6.4.2] it suffices to prove that

$$\big[\ell^{s_0}_{q_0}(L_{p_0}(w_0)), \ell^{s_1}_{q_1}(L_{p_1}(w_1))\big]_\theta \rightleftarrows \ell^s_q(L_p(w)).$$

To this end observe that the mapping $f \mapsto \tilde{f}$ given by

$$\tilde{f}(z) = \big\langle [2^{js_0} w_0]^{1-z} \cdot [2^{js_1} w_1]^z \cdot f(z)_j \big\rangle_{j\in\mathbb{N}}$$

is an isometric isomorphism

$$F\big[\ell^{s_0}_{q_0}(L_{p_0}(w_0)), \ell^{s_1}_{q_1}(L_{p_1}(w_1))\big] \to F\big[\ell_{q_0}(L_{p_0}), \ell_{q_1}(L_{p_1})\big].$$

From this we conclude

$$\Big\|\langle g_j\rangle_{j\in\mathbb{N}} \big| \big[\ell^{s_0}_{q_0}(L_{p_0}(w_0)), \ell^{s_1}_{q_1}(L_{p_1}(w_1))\big]_\theta\Big\| = \Big\|\langle 2^{js} w g_j\rangle_{j\in\mathbb{N}} \big| [\ell_{q_0}(L_{p_0}), \ell_{q_1}(L_{p_1})]_\theta\Big\|.$$

Combining [1: Theorem 5.1.2] and [1: Theorem 5.1.1] yields

$$[\ell_{q_0}(L_{p_0}), \ell_{q_1}(L_{p_1})]_\theta \rightleftarrows \ell_q\left([L_{p_0}, L_{p_1}]_\theta\right) \rightleftarrows \ell_q(L_p)$$

from which we infer the assertion ∎

Concerning duality a slight modification of the proof presented in [18: Subsection 2.11.2] results in the following theorem.

Theorem 5.2 (Dual spaces). *Assume* $1 < q < \infty$, $s \in \mathbb{R}$ *and* $w \in W$.

a) *If* $1 \leq p < \infty$, *then*

$$(B^s_{p,q}(w))' \rightleftarrows B^{-s}_{p',q'}(\tfrac{1}{w})$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$.

b) *If* $0 < p < 1$, *then*

$$(B^s_{p,q}(w))' \rightleftarrows B^{\sigma}_{\infty,q'}(\tfrac{1}{w}) \qquad \big(\sigma = \nu(\tfrac{1}{p} - 1) - s\big)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Now assume $w_0(x) = (1 + |x|)^{d_0}$ and $w_1(x) = (1 + |x|)^{d_1}$ with $d_0, d_1 \in \mathbb{R}$. Then one can use $w_0(x) = w_0(-x)$ and the corresponding formula for w_1 to show that the relation

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathbf{M}\big((B^{s_1}_{p_1,q_1}(w_1))', (B^{s_0}_{p_0,q_0}(w_0))'\big) \tag{1}$$

holds if the dual spaces are Besov spaces again. In the unweighted case this has been done in [4].

Concerning proofs of negative results we remark that we only have to consider the case of Besov spaces as long as the parameter q is not involved in the hypothesis (this will always be true in the sequel) because of the embedding

$$B^s_{p,\min\{p,q\}}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max\{p,q\}}(w) \qquad \begin{cases} 0 < p, q < \infty \\ s \in \mathbb{R},\ w \in W. \end{cases}$$

First we generalize a well known result of Hörmander [8]. Therefore we need the following lemma which can be proved by straightforward computations.

Lemma. *Assume* $s_0, s_1 \in \mathbb{R}$, $0 < q \leq \infty$ *and* $w(x) = (1 + |x|)^d$ *with* $d \geq 0$. *Then the embedding*

$$\mathcal{F}[B^{s_1 - s_0}_{1,\infty}(w)] \hookrightarrow \mathbf{M}(B^{s_0}_{2,q}, B^{s_1}_{2,q})$$

holds.

Theorem 5.4 (Trivial Fourier multipliers in the case of $p_0 > p_1$). *Assume* $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \leq \infty$, $s_0, s_1 \in \mathbb{R}$ *and* $w(x) = (1 + |x|)^d$ *with* $d \geq 0$. *Then*

$$\mathbf{M}\big(X^{s_0}_{p_0,q_0}(w), Y^{s_1}_{p_1,q_1}(w)\big) = \{0\} \qquad (p_0 > p_1).$$

In the case of $p_0 > 1$ *this is even true for all* $d \in \mathbb{R}$.

Proof. First we notice that the assertion for unweighted Besov spaces follows from a modification of Hörmander's argument. This situation was already considered in [4]. All other cases are reduced to this one.

Let $X^{s_0}_{p_0,q_0}(w) = B^{s_0}_{p_0,q_0}(w)$ and $Y^{s_1}_{p_1,q_1}(w) = B^{s_1}_{p_1,q_1}(w)$. From the diagram

$$\begin{array}{ccc} B^{s_0}_{p_0,q_0}(w) & \xrightarrow{\ T_M\ } & B^{s_1}_{p_1,q_1}(w) \\ \Big\uparrow id & & \Big\downarrow id \\ B^{s_0+1}_{p_0,2}(w) & \xrightarrow{\ T_M\ } & B^{s_1-1}_{p_1,2}(w) \end{array}$$

we infer

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{s_1}_{p_1,q_1}(w)\big) \hookrightarrow \mathbf{M}\big(B^{s_0+1}_{p_0,2}(w), B^{s_1-1}_{p_1,2}(w)\big) \tag{2}$$

(this argument also works in the case of different weight functions).

1. Assume $p_0 > 1$ and $M \in \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{s_1}_{p_1,q_1}(w)\big)$. Additionally we assume that $p_1 > 1$. The above embedding and the duality results in Theorem 5.2 and (1) show that

$$M \in \mathbf{M}\big(B^{s_0+1}_{p_0,2}(w), B^{s_1-1}_{p_1,2}(w)\big) \hookrightarrow \mathbf{M}\big(B^{-s_1+1}_{p_1',2}(\tfrac{1}{w}), B^{-s_0-1}_{p_0',2}(\tfrac{1}{w})\big).$$

Now apply complex interpolation

$$\begin{array}{lll} \Big[B^{s_0+1}_{p_0,2}(w)\ ,\ B^{-s_1+1}_{p_1',2}(\tfrac{1}{w})\Big]_{1/2} \rightleftarrows B^{(s_0-s_1+2)/2}_{r_0,2} & & \left(\tfrac{1}{r_0} = \tfrac{1}{2p_0} + \tfrac{1}{2p_1'}\right) \\ \Big\downarrow T_M \qquad\qquad \Big\downarrow T_M & & \\ \Big[B^{s_1-1}_{p_1,2}(w)\ ,\ B^{-s_0-1}_{p_0',2}(\tfrac{1}{w})\Big]_{1/2} \rightleftarrows B^{(s_1-s_0-2)/2}_{r_1,2} & & \left(\tfrac{1}{r_1} = \tfrac{1}{2p_1} + \tfrac{1}{2p_0'}\right) \end{array}$$

to obtain

$$M \in \mathbf{M}\big(B^{(s_0-s_1+2)/2}_{r_0,2}, B^{(s_1-s_0-2)/2}_{r_1,2}\big).$$

It suffices to show that $r_0 > r_1$ (notice that we are in an unweighted situation) and this follows from a simple computation.

In the case of $p_1 \le 1$ we first use a Sobolev-type embedding

$$B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{t}_{\frac{p_0+1}{2},q_1}(w) \qquad \left(t = s_1 + \nu\left(\tfrac{2}{p_0+1} - \tfrac{1}{p_1}\right)\right)$$

which proves

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{s_1}_{p_1,q_1}(w)\big) \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{t}_{(p_0+1)/2,q_1}(w)\big).$$

So we are again in the above situation and the theorem is proved in the case of $p_0 > 1$.

2. Now assume $p_0 \le 1$ and $M \in \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{s_1}_{p_1,q_1}(w)\big)$. Using a Sobolev-type embedding we get

$$B^{s_1-1}_{p_1,2}(w) \hookrightarrow B^{s-1}_{p_0,2} \qquad \left(s = s_1 + \nu\left(\tfrac{1}{p_0} - 1\right)\right)$$

and

$$\mathcal{F}[B^{\sigma}_{p_0,\infty}(w)] \hookrightarrow \mathcal{F}[B^{s-s_0-2}_{1,\infty}(w)] \qquad \Big(\sigma = \nu(\tfrac{1}{p_0}-1)+(s-1)-(s_0+1)\Big).$$

From [6: Theorem 5.2] and Lemma 5.3 we infer the embedding

$$\begin{aligned} &\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w), B^{s_1}_{p_1,q_1}(w)\big) \\ &\quad\hookrightarrow \mathbf{M}\big(B^{s_0+1}_{p_0,2}(w), B^{s_1-1}_{p_1,2}(w)\big) \hookrightarrow \mathbf{M}\big(B^{s_0+1}_{p_0,2}(w), B^{s-1}_{p_0,2}(w)\big) \\ &\quad\hookrightarrow \mathcal{F}[B^{\sigma}_{p_0,\infty}(w)] \hookrightarrow \mathcal{F}[B^{s-s_0-2}_{1,\infty}(w)] \hookrightarrow \mathbf{M}(B^{s_0+1}_{2,2}, B^{s-1}_{2,2}). \end{aligned}$$

We use complex interpolation

$$\begin{array}{ccc} \big[B^{s_0+1}_{p_0,2}(w)\ ,\ B^{s_0+1}_{2,2}\big]_\theta & \overset{\hookrightarrow}{\hookleftarrow} B^{s_0+1}_{t_0,2}(w_\theta) & \big(\tfrac{1}{t_0} = \tfrac{1-\theta}{p_0} + \tfrac{\theta}{2}\big) \\ \Big\downarrow T_M \qquad \Big\downarrow T_M & & \\ \big[B^{s_1-1}_{p_1,2}(w)\ ,\ B^{s-1}_{2,2}\big]_\theta & \overset{\hookrightarrow}{\hookleftarrow} B^{s_\theta}_{t_1,2}(w_\theta) & \big(\tfrac{1}{t_1} = \tfrac{1-\theta}{p_1} + \tfrac{\theta}{2}\big) \end{array}$$

to show

$$M \in \mathbf{M}\big(B^{s_0+1}_{t_0,2}(w_\theta), B^{s_\theta}_{t_1,2}(w_\theta)\big)$$

where $w_\theta(x) = (1+|x|)^{(1-\theta)d}$ and $s_\theta = (1-\theta)(s_1-1)+\theta(s-1)$. Now choose $0<\theta<1$ such that $t_0 > 1$. Because of $p_0 > p_1$ we have $t_0 > t_1$ and so we are in the situation considered in the first part of this proof ∎

Theorem 5.5 (Trivial Fourier multipliers in the case of $d_0 < d_1$). *Assume $0 < p_0, p_1 < \infty$, $0 < q_0, q_1 \le \infty$ and $s_0, s_1 \in \mathbb{R}$. If $w_0(x) = (1+|x|)^{d_0}$ and $w_1(x) = (1+|x|)^{d_1}$ with $d_0, d_1 \ge 0$, then*

$$\mathbf{M}\big(X^{s_0}_{p_0,q_0}(w_0), Y^{s_1}_{p_1,q_1}(w_1)\big) = \{0\} \qquad (d_0 < d_1).$$

Remark. When restricting to the weight functions

$$w_0(x) = (1+|x|)^{d_0} \qquad \text{and} \qquad w_1(x) = (1+|x|)^{d_1}$$

with $d_0, d_1 \ge 0$, we obtain a complete characterization of the determined classes of Fourier multipliers in the general case of $d_0, d_1 \ge 0$. In these characterizations given in the previous section the situation $d_0 \ge d_1$ is covered whereas the above theorem covers the remaining case of $d_0 < d_1$.

Proof of Theorem 5.5. We use the embedding (cf. formula (2))

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathbf{M}\big(B^{s_0+1}_{p_0,2}(w_0), B^{s_1-1}_{p_1,2}(w_1)\big).$$

Without loss of generality we may restrict to the case of $p_0 \le p_1$ since otherwise the assertion can be deduced from Theorem 5.4.

1. Assume $p_0 > 1$ and $M \in \mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1))$. From $B^{s_0+1}_{p_0,2}(w_1) \hookrightarrow B^{s_0+1}_{p_0,2}(w_0)$ and the duality results of Theorem 5.2 and (1) we conclude that

$$\begin{aligned}\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) &\hookrightarrow \mathbf{M}(B^{s_0+1}_{p_0,2}(w_0), B^{s_1-1}_{p_1,2}(w_1))\\ &\hookrightarrow \mathbf{M}(B^{s_0+1}_{p_0,2}(w_1), B^{s_1-1}_{p_1,2}(w_1))\\ &\hookrightarrow \mathbf{M}(B^{-s_1+1}_{p_1',2}(\tfrac{1}{w_1}), B^{-s_0-1}_{p_0',2}(\tfrac{1}{w_1})).\end{aligned}$$

Now apply complex interpolation

$$\begin{array}{ccccl} \left[B^{s_0+1}_{p_0,2}(w_0)\,,\; B^{-s_1+1}_{p_1',2}(\tfrac{1}{w_1})\right]_{\frac{1}{2}} & \begin{smallmatrix}\hookrightarrow\\ \hookleftarrow\end{smallmatrix} & B^{(s_0-s_1+2)/2}_{r_0,2}(w) & & (\tfrac{1}{r_0} = \tfrac{1}{2p_0} + \tfrac{1}{2p_1'}) \\ \Big\downarrow T_M \qquad\qquad \Big\downarrow T_M & & & & (w(x) = (1+|x|)^{\frac{d_0-d_1}{2}}) \\ \left[B^{s_1-1}_{p_1,2}(w_1)\,,\; B^{-s_0-1}_{p_0',2}(\tfrac{1}{w_1})\right]_{\frac{1}{2}} & \begin{smallmatrix}\hookrightarrow\\ \hookleftarrow\end{smallmatrix} & B^{(s_1-s_0-2)/2}_{r_1,2} & & (\tfrac{1}{r_1} = \tfrac{1}{2p_1} + \tfrac{1}{2p_0'}) \end{array}$$

to obtain

$$M \in \mathbf{M}(B^{(s_0-s_1+2)/2}_{r_0,2}(w), B^{(s_1-s_0-2)/2}_{r_1,2}).$$

From Theorem 3.3/a) and Lemma 3.1 we get

$$\begin{aligned} B(\tilde{M}(w,1);(s_0-s_1+2)/2, r_0, b^{(s_1-s_0-2)/2}_{r_1,2}) &\\ \le \big\|\widetilde{M}(w,1)\big|b^{(s_0-s_1+2)/2}_{r_0,2}, b^{(s_1-s_0-2)/2}_{r_1,2}\big\| &\\ \le C_0 \cdot \big\|M\big|\mathbf{M}(B^{(s_0-s_1+2)/2}_{r_0,2}(w), B^{(s_1-s_0-2)/2}_{r_1,2})\big\| &\\ < \infty. &\end{aligned}$$

Because of $d_0 < d_1$ we have $\|\frac{1}{w}\|_\infty = \infty$ and thus $M \equiv 0$ follows from [6: Corollary 4.2/b)]. This proves the theorem in the case of $p_0 > 1$.

2. Now assume $p_0 \le 1$. Additionally we assume that $p_1 > 1$. From the duality results of Theorem 5.2 and (1) we conclude that

$$\begin{aligned}\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) &\hookrightarrow \mathbf{M}(B^{s_0+1}_{p_0,2}(w_0), B^{s_1-1}_{p_1,2}(w_1))\\ &\hookrightarrow \mathbf{M}(B^{1-s_1}_{p_1',2}(\tfrac{1}{w_1}), B^{s}_{\infty,2}(\tfrac{1}{w_0}))\end{aligned}$$

where $s = -(s_0+1) + \nu(\frac{1}{p_0} - 1)$. Our next step is the proof of an embedding result. Hölder's inequality implies that

$$\big\|\tfrac{1}{\tilde{w}} \cdot \mathcal{F}^{-1}[\phi_j \mathcal{F} f]\big\|_p \le \big\|(1+|\bullet|)^{-\frac{n+1}{p}}\big\|_p \cdot \big\|\tfrac{1}{w_0} \cdot \mathcal{F}^{-1}[\phi_j \mathcal{F} f]\big\|_\infty$$

with $\tilde{w}(x) = (1+|x|)^{d_0 + \frac{n+1}{p}}$ holds for $1 < p < \infty$ and all $f \in S'$. This leads to $B^{s}_{\infty,2}(\frac{1}{w_0}) \hookrightarrow B^{s}_{p,2}(\frac{1}{\tilde{w}})$ which in connection with the duality results of Theorem 5.2 and (1) yields the embedding

$$\begin{aligned}\mathbf{M}(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)) &\hookrightarrow \mathbf{M}(B^{1-s_1}_{p_1',2}(\tfrac{1}{w_1}), B^{s}_{\infty,2}(\tfrac{1}{w_0}))\\ &\hookrightarrow \mathbf{M}(B^{1-s_1}_{p_1',2}(\tfrac{1}{w_1}), B^{s}_{p,2}(\tfrac{1}{\tilde{w}}))\\ &\hookrightarrow \mathbf{M}(B^{-s}_{p',2}(\tilde{w}), B^{s_1-1}_{p_1,2}(w_1)).\end{aligned}$$

Choose p so large that $d_0 + \frac{n+1}{p} < d_1$ and we are in the situation considered in the first part of this proof.

In the case of $p_1 \leq 1$ we first use a Sobolev-type embedding

$$B^{s_1}_{p_1,q_1}(w_1) \hookrightarrow B^{\sigma}_{2,q_1}(w_1) \qquad \left(\sigma = s_1 + \nu(\tfrac{1}{2} - \tfrac{1}{p_1})\right)$$

to obtain

$$\mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{s_1}_{p_1,q_1}(w_1)\big) \hookrightarrow \mathbf{M}\big(B^{s_0}_{p_0,q_0}(w_0), B^{\sigma}_{2,q_1}(w_1)\big).$$

Now we are again in the situation $p_1 > 1$ and the theorem is proved ∎

6. Proof of the discrete characterization

It remains to prove the discrete characterization of function spaces which was the basis of our considerations. Therefore we need a few preparations. We start with the anisotropic version

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|\{y : \varrho(y) \leq r\}|} \int_{\varrho(x-y)\leq r} |f(y)|\,dy \qquad (x \in \mathbb{R}^n)$$

of the *Hardy-Littlewood maximal function*. The following inequality due to Fefferman and Stein [7] in the isotropic case plays a basic role in the theory of function spaces.

Theorem 6.1 (Fefferman-Stein inequality). *Assume $1 < p,q < \infty$. Then there exists a constant $C > 0$ such that*

$$\big\| \, \|\langle \mathcal{M}f_j\rangle_{j\in\mathbb{N}}|\ell_q\| \, \big\|_p \leq C \cdot \big\| \, \|\langle f_j\rangle_{j\in\mathbb{N}}|\ell_q\| \, \big\|_p$$

holds for all sequences of functions $\langle f_j\rangle_{j\in\mathbb{N}} \in L_p(\ell_q)$.

According to Marschall [13: Subsection 1.1.4] this is a consequence of the extrapolation theory of Rubio de Francias [15] and the scalar case due to Calderón [3]. A different proof was proposed by Seeger [17].

Additionally we need a second maximal inequality due to Peetre [14] in the isotropic unweighted case. We will derive it from the following multiplier assertion.

Theorem 6.2 (Vector-valued Fourier multipliers). *Assume $0 < p,q < \infty$ and $\eta_j \in \mathcal{S}$ with $\operatorname{supp}\eta_j \subseteq \{\xi \in \mathbb{R}^n : \varrho^*(\xi) \leq 2^{j+2}\}$ for all $j \in \mathbb{N}$. Furthermore, let $w \in W$ and $N > \frac{\nu}{\min\{p,q\}}$. Then there exist constants $C, M > 0$ such that*

$$\left\| \left\| \left\langle \sup_{z\in\mathbb{R}^n} w(\cdot - z) \frac{|\mathcal{F}^{-1}[\eta_j \mathcal{F} f_j](\cdot - z)|}{[1 + 2^j \varrho(z)]^N} \right\rangle_{j\in\mathbb{N}} \middle| \ell_q \right\| \right\|_p \leq C \cdot A \cdot \big\| \, \|\langle w f_j\rangle_{j\in\mathbb{N}}|\ell_q\| \, \big\|_p$$

with

$$A = \sup_{j\in\mathbb{N}} \left\| (1 + |\cdot|)^M \cdot \mathcal{F}^{-1}[\eta_j(A^*_{2^j}\cdot)] \right\|_1$$

holds for all sequences $\langle f_j\rangle_{j\in\mathbb{N}}$ of tempered distributions.

Proof. First modify the inequality in [16: Theorem 1.4.2] with the help of the anisotropic Hardy-Littlewood maximal function to get

$$\sup_{z\in\mathbb{R}^n} w(x-z)\frac{|g(x-z)|}{[1+\varrho(z)]^N} \le C_0 \cdot \left(\mathcal{M}[|wg|^{\frac{\nu}{N}}](x)\right)^{\frac{N}{\nu}}$$

where the support of $\mathcal{F}g$ is contained in a given compact set. Then use anisotropic substitutions and the anisotropic Fefferman-Stein inequality in Theorem 6.1 to modify the proof of [16: Theorem 1.9.1] ∎

Now choose $\eta_j = \sum_{r=-1}^{1} \phi_{j+r}$ and $f_j = 2^{js}\mathcal{F}^{-1}[\phi_j \mathcal{F} f]$ to get the announced maximal inequality.

Corollary 6.3 (Peetre maximal inequality). *Assume* $0 < p, q < \infty$, $s \in \mathbb{R}$ *and* $w \in W$. *If* $N > \frac{\nu}{\min\{p,q\}}$, *then there exists a constant* $C > 0$ *such that*

$$\left\| \left\| \left\langle \sup_{z\in\mathbb{R}^n} w(\bullet - z)\frac{2^{js}|\mathcal{F}^{-1}[\phi_j \mathcal{F} f](\bullet - z)|}{[1+2^j\varrho(z)]^N} \right\rangle_{j\in\mathbb{N}} \middle| \ell_q \right\| \right\|_p \le C \cdot \|f|F^s_{p,q}(w)\|$$

holds for all $f \in \mathcal{S}'$.

To prove Theorem 2.1 we also need the following lemma.

Lemma 6.4. *With* $\alpha \in \mathbb{C}^{\mathbb{N}\times\mathbb{Z}^n}$ *we associate the sequence* $\langle [\alpha]_j \rangle_{j\in\mathbb{N}}$ *of functions defined by*

$$[\alpha]_j(x) = \sum_{k\in\mathbb{Z}^n} |\alpha^j_k| \cdot [1+\varrho(A_{2^j}x - k)]^{-L} \qquad (x \in \mathbb{R}^n).$$

Assume $0 < p, q < \infty$, $s \in \mathbb{R}$ *and* $w \in W$. *Then there exists a constant* $C > 0$ *such that*

$$\left\| w \cdot \|\langle 2^{js}[\alpha]_j(\bullet)\rangle_{j\in\mathbb{N}}|\ell_q\| \right\|_p \le C \cdot \|\alpha|f^s_{p,q}(w)\|$$

holds for all $\alpha \in f^s_{p,q}(w)$ *and a sufficiently large* $L > 0$.

Proof. Let $x \in \diamondsuit^j_{k_0}$. We decompose $\mathbb{Z}^n$ into the sets

$$K_r = \left\{ k \in \mathbb{Z}^n : 2^r - 1 \le \varrho(k_0 - k) < 2^{r+1} - 1 \right\} \qquad (r \in \mathbb{N}).$$

Define γ by

$$\gamma = \sup\left\{ \varrho(z) : z \in \left[-\frac{1}{2}, +\frac{1}{2}\right]^n \right\}. \tag{3}$$

For $y \in \diamondsuit^j_k$ we have $\varrho(k - A_{2^j}y) \le \gamma$ and $\varrho(A_{2^j}y - k) \le \gamma$. Let $k \in K_r$ and $x \in \diamondsuit^j_{k_0}$. Then

$$\begin{aligned} 2^r &\le \varrho(k_0 - k) + 1 \\ &\le C_0 \cdot \left(\varrho(k_0 - A_{2^j}x) + \varrho(A_{2^j}x - k) + 1\right) \\ &\le C_0 \cdot \left(1 + \gamma + \varrho(A_{2^j}x - k)\right) \\ &\le C_1 \cdot \left[1 + \varrho(A_{2^j}x - k)\right]. \end{aligned}$$

Furthermore we have the relation

$$w(x) \le C_2 \cdot w(A_{2^{-j}} k)\left[1+\varrho(x - A_{2^{-j}} k)\right]^d \le C_2 \cdot w_k^j \cdot \left[1+\varrho(A_{2^j} x - k)\right]^d$$

for a suitable $d \ge 0$. Choose $0 < t \le 1$ so that $1 < \frac{p}{t}, \frac{q}{t} < \infty$ to obtain

$$\begin{aligned} w(x)[\alpha]_j(x) &\le C_2 \cdot \sum_{k \in \mathbf{Z}^n} |\alpha_k^j| w_k^j \left[1+\varrho(A_{2^j} x - k)\right]^{-L+d} \\ &\le C_3 \cdot \sum_{r=0}^{\infty} \sum_{k \in K_r} 2^{(-L+d)r} |\alpha_k^j| w_k^j \\ &\le C_3 \cdot \sum_{r=0}^{\infty} 2^{(-L+d)r} \left(\sum_{k \in K_r} \left(|\alpha_k^j| w_k^j\right)^t \right)^{\frac{1}{t}}. \end{aligned}$$

The sum over K_r will now be estimated from above by a maximal function. We use the inequality

$$\begin{aligned} \varrho(x-y) &\le C_4 \cdot \left(\varrho(x - A_{2^{-j}} k_0) + \varrho(A_{2^{-j}} k_0 - A_{2^{-j}} k) + \varrho(A_{2^{-j}} k - y)\right) \\ &\le C_4 \cdot \left(\gamma 2^{-j} + (2^{r+1}-1)2^{-j} + \gamma 2^{-j}\right) \\ &\le C_5 \cdot 2^{r-j} \end{aligned}$$

which holds for all $y \in \diamondsuit_k^j$ $(k \in K_r)$ to get

$$\begin{aligned} \sum_{k \in K_r} (|\alpha_k^j| w_k^j)^t &= 2^{j\nu} \int_{\varrho(x-y) \le C_5 \cdot 2^{r-j}} \left[\sum_{k \in K_r} |\alpha_k^j| w_k^j \mathbf{1}_k^j(y) \right]^t dy \\ &\le C_6 \cdot 2^{r\nu} \mathcal{M} \left[\sum_{k \in \mathbf{Z}^n} |\alpha_k^j| w_k^j \mathbf{1}_k^j \right]^t (x) \end{aligned}$$

since $|\{y \in \mathbb{R}^n : \varrho(x-y) \le C_5 \cdot 2^{r-j}\}| = C_6 \cdot 2^{(r-j)\nu}$.

Now insert this in the above estimation of $w(x)[\alpha]_j(x)$ and take $L > 0$ sufficiently large to obtain

$$w(x)[\alpha]_j(x) \le C_7 \cdot \left(\mathcal{M} \left[\sum_{k \in \mathbf{Z}^n} |\alpha_k^j| w_k^j \mathbf{1}_k^j \right]^t (x) \right)^{\frac{1}{t}}.$$

Multiply this by 2^{js}, apply the quasinorms and the Fefferman-Stein inequality (Theorem 6.1) to get the estimate

$$\begin{aligned} &\left\| w \cdot \|\langle 2^{js}[\alpha]_j(\bullet)\rangle_{j \in \mathbf{N}} | \ell_q\| \right\|_p \\ &\le C_7 \cdot \left\| \left\| \left\langle 2^{js} \left(\mathcal{M} \left[\sum_{k \in \mathbf{Z}^n} |\alpha_k^j| w_k^j \mathbf{1}_k^j \right]^t (\bullet) \right)^{\frac{1}{t}} \right\rangle_{j \in \mathbf{N}} \middle| \ell_q \right\| \right\|_p \\ &\le C_7 \cdot \left\| \left\| \left\langle \mathcal{M} \left[2^{js} \sum_{k \in \mathbf{Z}^n} |\alpha_k^j| w_k^j \mathbf{1}_k^j \right]^t (\bullet) \right\rangle_{j \in \mathbf{N}} \middle| \ell_{\frac{q}{t}} \right\| \right\|_{\frac{p}{t}}^{\frac{1}{t}} \\ &\le C_8 \cdot \left\| \alpha | f_{p,q}^s(w) \right\| \end{aligned}$$

and the assertion is proved ∎

Proof of Theorem 2.1. We only consider the case of Triebel spaces because the assertions for Besov spaces can be deduced from this by real interpolation or by an interchange of the ℓ_q and L_p quasinorms. We need the following four steps to prove the assertion:

1. The operator **se** : $F^s_{p,q}(w) \to f^s_{p,q}(w)$ is bounded.
2. The operator **fu** : $f^s_{p,q}(w) \to F^s_{p,q}(w)$ is bounded.
3. We have $\mathbf{fu} \circ \mathbf{se} = id$ in $F^s_{p,q}(w)$.
4. $\|\mathbf{se}\bullet \mid f^s_{p,q}(w)\|$ is an equivalent quasinorm on $F^s_{p,q}(w)$.

Step 1. Let γ be defined as in the previous lemma (formula (3)). Since the sets $\diamondsuit^j_k$ are disjoint for fixed j we obtain from $\varrho(x - A_{2^{-j}}k) \le \gamma \cdot 2^{-j}$ $(x \in \diamondsuit^j_k)$ the relation

$$\begin{aligned} 2^{js} \sum_{k\in\mathbb{Z}^n} |(\mathbf{se}f)^j_k| w^j_k \mathbf{1}^j_k(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{k\in\mathbb{Z}^n} w(A_{2^{-j}}k) 2^{js} |\mathcal{F}^{-1}[\phi_j \mathcal{F} f](A_{2^{-j}}k)| \mathbf{1}^j_k(x) \\ &\le \frac{1}{(2\pi)^{\frac{n}{2}}} \sup_{\varrho(z)\le\gamma\cdot 2^{-j}} w(x-z) 2^{js} |\mathcal{F}^{-1}[\phi_j\mathcal{F}f](x-z)| \\ &\le C_0 \cdot \sup_{z\in\mathbb{R}^n} w(x-z) \frac{2^{js}|\mathcal{F}^{-1}[\phi_j\mathcal{F}f](x-z)|}{[1+2^j\varrho(z)]^N}. \end{aligned}$$

Choose $N > \frac{\nu}{\min\{p,q\}}$, apply the quasinorms and use the maximal inequality of Theorem 6.3 to get

$$\begin{aligned} \|\mathbf{se}f | f^s_{p,q}(w)\| &= \left\| \left\| \left\langle 2^{js} \sum_{k\in\mathbb{Z}^n} |(\mathbf{se}f)^j_k| w^j_k \mathbf{1}^j_k(\bullet) \right\rangle_{j\in\mathbb{N}} \middle| \ell_q \right\| \right\|_p \\ &\le C_0 \cdot \left\| \left\| \left\langle \sup_{z\in\mathbb{R}^n} w(\bullet - z) \frac{2^{js}|\mathcal{F}^{-1}[\phi_j\mathcal{F}f](\bullet - z)|}{[1+2^j\varrho(z)]^N} \right\rangle_{j\in\mathbb{N}} \middle| \ell_q \right\| \right\|_p \\ &\le C_1 \cdot \|f | F^s_{p,q}(w)\|. \end{aligned}$$

Step 2. Without loss of generality we may restrict ourselves to finite sequences since they are dense in $f^s_{p,q}(w)$. First we derive a pointwise estimate of

$$I_j(x) = |\mathcal{F}^{-1}[\phi_j \mathcal{F}(\mathbf{fu}\alpha)](x)| \qquad (j \in \mathbb{N}).$$

If $j \ge 2$, then the relation

$$\mathcal{F}^{-1}\left[\phi_j \mathcal{F}[(\mathcal{F}^{-1}\psi_1)(A_{2^l}\bullet - k)]\right](x) = \mathcal{F}^{-1}\left[\phi_1(A^*_{2^{l-j+1}}\bullet)\psi_1\right](A_{2^l}x - k)$$

holds for the terms appearing in $\mathbf{fu}\alpha$ and the right-hand side vanishs in the case of $|j-l| > 1$. From this we are lead to (put $l = j + r$)

$$\begin{aligned} I_j(x) &\le \sum_{l=1}^{\infty} \sum_{k\in\mathbb{Z}^n} |\alpha^l_k| \cdot \left|\mathcal{F}^{-1}\left[\phi_j\mathcal{F}[(\mathcal{F}^{-1}\psi_1)(A_{2^l}\bullet - k)]\right](x)\right| \\ &\le \sum_{r=-1}^{1} |\alpha^{j+r}_k| \cdot \left|\mathcal{F}^{-1}\left[\phi_1(A^*_{2^{r+1}}\bullet)\psi_1\right](A_{2^{j+r}}x - k)\right| \end{aligned}$$

and similar inequalities can be proved in the remaining cases. Because $\mathcal{F}^{-1}[\cdots]$ is a function from $\mathcal{S}$ we obtain

$$I_j(x) \le C_2 \cdot \sum_{r=-1}^{1} \sum_{k\in\mathbb{Z}^n} |\alpha_k^{j+r}| \cdot \left[1 + \varrho(A_{2^{j+r}} x - k)\right]^{-L} = C_2 \cdot \sum_{r=-1}^{1} [\alpha]_{j+r}(x)$$

for every $L > 0$ where C_2 does only depend on L. Now Lemma 6.4 shows that

$$\begin{aligned} \|\mathbf{fu}\alpha | F^s_{p,q}(w)\| &= \left\| w \cdot \|\langle 2^{js} I_j(\bullet)\rangle_{j\in\mathbb{N}} | \ell_q\| \right\|_p \\ &\le C_3 \cdot \left\| w \cdot \|\langle 2^{js} [\alpha]_j(\bullet)\rangle_{j\in\mathbb{N}} | \ell_q\| \right\|_p \\ &\le C_4 \cdot \|\alpha | f^s_{p,q}(w)\|. \end{aligned}$$

Step 3. Since $\mathcal{S}$ is dense in $F^s_{p,q}(w)$ we restrict ourselves to $f \in \mathcal{S}$. We apply the partition of unity

$$\phi_0(\xi)\psi_0(\xi) + \sum_{j=1}^{\infty} \phi_j(\xi)\psi_1(A^*_{2^{-j}}\xi) = 1$$

(which follows from the definitions of ψ_0 and ψ_1) and the defintion of $\mathcal{F}^{-1}$ to get

$$\begin{aligned} f(x) &= \mathcal{F}^{-1}[\phi_0\psi_0\mathcal{F}f](x) + \sum_{j=1}^{\infty} \mathcal{F}^{-1}\left[\phi_j\psi_1(A^*_{2^{-j}}\bullet)\mathcal{F}f\right](x) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi_0(\xi)\psi_0(\xi)\mathcal{F}f(\xi)e^{ix\cdot\xi}\,d\xi \\ &\quad + \sum_{j=1}^{\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi_j(\xi)\psi_1(A^*_{2^{-j}}\xi)\mathcal{F}f(\xi)e^{ix\cdot\xi}\,d\xi. \end{aligned}$$

The basic assumption $\{\xi \in \mathbb{R}^n : \varrho^*(\xi) \le 2\} \subseteq [-\pi,\pi]^n$ (cf. [5]) implies that

$$\begin{aligned} &\xi \in \operatorname{supp}\phi_j \implies \varrho^*(\xi) \le 2^{j+1} \implies A^*_{2^{-j}}\xi \in [-\pi,\pi]^n \\ &y \in \operatorname{supp}\psi_1 \implies \varrho^*(y) \le 2 \implies y \in [-\pi;\pi]^n \end{aligned}$$

and a Fourier series developement yields

$$\begin{aligned} &\psi_1(A^*_{2^{-j}}\xi)e^{ix\cdot\xi} \\ &\quad = \frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \int_{[-\pi,\pi]^n} \psi_1(y)\exp(ix\cdot A^*_{2^j}y)e^{-ik\cdot y}\,dy \cdot \exp(ik\cdot A^*_{2^{-j}}\xi) \\ &\quad = \sum_{k\in\mathbb{Z}^n} \frac{1}{(2\pi)^{\frac{n}{2}}}(\mathcal{F}^{-1}\psi_1)(A_{2^j}x - k)\cdot\exp(ik\cdot A^*_{2^{-j}}\xi) \end{aligned}$$

$(\xi \in \operatorname{supp}\phi_j)$ for $j \ge 1$ and

$$\psi_0(\xi)e^{ix\cdot\xi} = \sum_{k\in\mathbb{Z}^n} \frac{1}{(2\pi)^{\frac{n}{2}}}(\mathcal{F}^{-1}\psi_0)(x-k)\cdot\exp(ik\cdot\xi) \qquad (\xi \in \operatorname{supp}\phi_0).$$

Insert these two series in the above decomposition of f, interchange summation and integration and the assertion follows.

Step 4. We have

$$\begin{aligned}\|\mathrm{se}f|f^s_{p,q}(w)\| &\le C_1\cdot\|f|F^s_{p,q}(w)\|\\ &= C_1\cdot\|\mathbf{fu}(\mathrm{se}f)|F^s_{p,q}(w)\|\le C_1C_4\cdot\|\mathrm{se}f|f^s_{p,q}(w)\|\end{aligned}$$

which completes the proof ∎

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