Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 4, 779–787

On the Existence of Connecting Orbits

Yu Shu-Xiang

Abstract. Two existence criteria of orbits connecting a pair of critical points of planar differential equations are given.

Keywords: *Connecting orbits, parabolic sectors, exit points*

AMS subject classification: 34 C

1. Introduction

In this paper, we consider the differential system

$$\left.\begin{aligned}\frac{dx}{dt} &= X(x,y)\\ \frac{dy}{dt} &= Y(x,y)\end{aligned}\right\} \tag{1.1}$$

in the plane $\mathbb{R}^2$ where X and Y are continuous and assume that solutions of arbitrary initial value problems are unique. Let the vector field $V = (X, Y)$ define a flow $f(p,t)$ and let $p_1, p_2 \in \mathbb{R}^2$ be two isolated critical points of the system (1.1), i.e. $V(p_1) = V(p_2) = 0$.

Definition 1.1. If there is a point $p_0 \in \mathbb{R}^2$ such that

$$\lim_{t\to+\infty} f(p_0,t) = p_1 \quad \text{and} \quad \lim_{t\to-\infty} f(p_0,t) = p_2,$$

then $f(p_0, \mathbb{R})$ is called a *trajectory connecting* p_1 and p_2.

In some previous papers (see, e.g., [3, 4, 6]), generally it is assumed that one of two critical points p_1 and p_2 is a repeller or an attractor (about their definitions, see [3]). In the present paper we shall give some existence criteria for connecting orbits which contain no such assumption.

Yu Shu-Xiang: Academia Sinica, Institute of Mathematics, 100080 Beijing, China

ISSN 0232-2064 / $ 2.50

2. Definitions

Let ρ be a simple closed curve surrounding a critical point Q of the system (1.1). Then, a positive or negative *parabolic sector* of Q in ρ is defined as open subset D of the interior of ρ with boundary consisting of

(i) the critical point Q

(ii) the positive or negative semi-trajectory arcs $f(M_1, \mathbb{R}^+)$ and $f(M_2, \mathbb{R}^+)$ or $f(M_1, \mathbb{R}^-)$ and $f(M_2, \mathbb{R}^-)$, respectively, and

(iii) the oriented closed subarc ρ_{12} from M_1 to M_2

and such that when $t \to +\infty$ or $t \to -\infty$, then $f(M_i, t) \to Q$ $(i = 1, 2)$ and the closure of D contains no negative or positive semi-trajectory $f(M, \mathbb{R}^-)$ or $f(M, \mathbb{R}^+)$ which tends to Q as $t \to -\infty$ or $t \to +\infty$, respectively, where $M \in \rho_{12}$ (see [5: p. 163]).

Definition 2.1. A positive or negative parabolic sector D is said to be *regular* if the trajectory $\Gamma(M)$ through any point $M \in \rho_{12}$ is not tangent to ρ_{12} at M and $f(M, t) \to Q$ as $t \to +\infty$ or $t \to -\infty$, respectively.

Definition 2.2. Let D be a regular positive or negative parabolic sector. An endpoint $N_1 \in f(M_1, \mathbb{R})$ of a simple curve γ is called an *interior-side point* of γ with respect to D if γ is not tangent to the trajectory $f(M_1, R)$ at N_1 and if there is a neighbourhood $u(N_1)$ of N_1 on the curve γ such that either $u(N_1) \subset D$ or the positive or negative semi-trajectory originating from any point in $u(N_1)$ must intersect the closed subarc ρ_{12}.

For $N_2 \in f(M_2, \mathbb{R})$, an interior-side point of γ with respect to D can be defined similarly.

Let $T \subset \mathbb{R}^2$ be an open subset, $\overline{T}$ and ∂T its closure and boundary, respectively, and let $f(p, [a, b])$ denote the finite arc of the trajectory $f(p, \mathbb{R})$ corresponding to the interval $[a, b]$.

Definition 2.3 (see [5: p. 37]). A point $p_0 \in \partial T$ is called an *exit point* of T with respect to the system (1.1) if there exists an $\varepsilon > 0$ such that $f(p_0, (-\varepsilon, 0)) \subset T$. An exit point p_0 is called *strict* if there exists an $\varepsilon > 0$ such that $f(p_0, (0, \varepsilon)) \subset \mathbb{R}^2 \setminus \overline{T}$.

An *entrance point* (a *strict* entrance point) can be defined similarly.

In what follows, the set of exit points of T will be denoted by S_1 and the set of strict exit points by S_1^*.

Let

$$T_0 = \Big\{ p \in T \,\Big|\, f(p, t_1) \notin T \text{ for some } t_1 > 0 \Big\} \tag{2.1}$$

and $T_1 = T_0 \cup S_1$. We define the function $t_p : T_1 \to \mathbb{R}$ by

$$t_p = \sup \Big\{ t \geq 0 \,\Big|\, f(p, [0, t]) \subset T \Big\}. \tag{2.2}$$

It is easy to see that $f(p, [0, t_p]) \subset \overline{T}$ and $f(p, t_p) \in S_1$.

The following lemma holds (see also [2: p. 25]).

Lemma 2.1. *Let $T \subset \mathbb{R}^2$ be an open subset satisfying $S_1 = S_1^*$, i.e. all exit points of T are strict. Then the function t_p defined by* (2.2) *is continuous.*

Proof. By the condition $S_1 = S_1^*$ it follows that, for any given $p \in T_1$ and $\varepsilon > 0$,

$$f(p, [t_p, t_p + \varepsilon]) \not\subset T \cup S_1$$

and there is a point $t' \in (t_p, t_p + \varepsilon]$ such that $f(p, t') \notin \overline{T}$. Let V be a neighbourhood of $f(p, t')$ in $\mathbb{R}^2$ which is disjoint from $\overline{T}$, and let U be a neighbourhood of p in $\mathbb{R}^2$ such that $f(U, t') \subset V$ (by the continuity of $f(p, t)$). Then, for $p' \in U \cap T$, $f(p', t') \notin \overline{T}$. This implies $t_{p'} < t'$. Further, by $t' \in (t_p, t_p + \varepsilon]$ it follows that $t' \leq t_p + \varepsilon$. Therefore, $t_{p'} < t_p + \varepsilon$. This shows that t_p is upper semicontinuous.

Now let $p \in T_1$ and let $\varepsilon > 0$ be arbitrarily given. By the definition of t_p, it follows that $f(p, [0, t_p - \varepsilon]) \subset T$, hence $f(p, [0, t_p - \varepsilon]) \cap \partial T = \phi$. Therefore, for every $\tau \in [0, t_p - \varepsilon]$, there is a neighbourhood U_τ of $f(p, \tau)$ in $\mathbb{R}^2$ which is disjoint from ∂T. Since the trajectory arc $f(p, [0, t_p - \varepsilon])$ is compact, a finite number of the U_τ cover $f(p, [0, t_p - \varepsilon])$. Let $\tilde{U}$ be their union. Since $\tilde{U}$ is open, there is a neighbourhood $\tilde{V}$ of p in $\mathbb{R}^2$ such that $f(\tilde{V}, [0, t_p - \varepsilon]) \subset \tilde{U}$. Since $\tilde{U} \cap \partial T = \phi$, the inclusion $p' \in \tilde{V}$ implies $f(p', [0, t_p - \varepsilon]) \cap \partial T = \phi$. Thus $t_{p'} > t_p - \varepsilon$, and t_p is lower semicontinuous. This completes the proof ∎

Definition 2.4. If a simple closed curve C is the union of alternating non-closed whole trajectories and critical points, and if it is contained in the ω-limit set (or α-limit set) of some trajectory, then we say that C is a *singular closed trajectory.*

3. Results

In this section, we shall prove first the following theorem (see Figure 1).

Theorem 3.1. *Suppose the following:*

1. *Let p_1 and p_2 be two critical points of the system* (1.1) *and let one of them, say p_1, has a regular negative parabolic sector D_1 in some simple closed curve ρ_1.*

2. *Let M_1 and M_2 be two endpoints of the oriented closed subarc ρ_{12} of D_1.*

3. *Let $\widehat{N_1 N_2}$ be a simple curve connecting the points $N_1 \in f(M_1, \mathbb{R}^+)$ and $N_2 \in f(M_2, \mathbb{R}^+)$ such that each N_i $(i = 1, 2)$ is an interior-side point of $\widehat{N_1 N_2}$ with repsect to D_1.*

4. *Let B be the region enclosed by the segmental arcs $\widehat{M_1 M_2}$, $\widehat{N_1 N_2}$ and the trajectory arcs $\widehat{M_1 N_1}$, $\widehat{M_2 N_2}$, and let the following three conditions be satisfied:*

(i) *There is only one critical point p_2 in B.*

(ii) *There are no closed trajectories and singular closed trajectories in B.*

(iii) *All exit points of B lying on the curve $\widehat{N_1 N_2}$ are strict.*

Then there must be in $B \cup \overline{D}_1$ a trajectory connecting p_1 and p_2 (see Figure 1).

Proof. Since D_1 is a regular negative parabolic sector, it follows from Definition 2.1 that every point of the segmental arc $\widehat{M_1M_2}$ is a strict entrance point of B. The theorem proof proceeds by reduction to absurdity. Suppose there are no trajectories joining p_1 and p_2. Then, by the Poincaré-Bendixson theory of planar systems and by the conditions (i) and (ii) it follows that the positive semi-trajectory $f(p, \mathbb{R}^+)$ originating from any point $p \in \widehat{M_1M_2}$ must leave B from the point $p' \in \widehat{N_1N_2}$ for increasing time. It is easy to see that $p' \in S_1$ (where S_1 denotes the set of exit points of B). According to (2.2) we can define the function $t_p = \sup\{t \geq 0 | f(p, [0,t]) \subset B\}$ in the subset of $\overline{B}$. Then $p' = f(p, t_p) \in \widehat{N_1N_2}$ for $p \in \widehat{M_1M_2}$. Let

$$K = \left\{p' \in \widehat{N_1N_2} \,\middle|\, p' = f(p, t_p) \text{ for some } p \in \widehat{M_1M_2}\right\}.$$

Now we can prove that $K = \widehat{N_1N_2}$, i.e. every point of the arc $\widehat{N_1N_2}$ is in K. In fact, by the conditions of the theorem, each of the N_i $(i = 1,2)$ is an interior-side point of $\widehat{N_1N_2}$ with respect to D_1. This means (see Definition 2.2) that every point in some neighbourhood of N_i $(i = 1,2)$ on the arc $\widehat{N_1N_2}$ is in K. Therefore, if there is a point $p_0' \in \widehat{N_1N_2}$ such that p_0' is not in K, then K is a disconnected set on the arc $\widehat{N_1N_2}$.

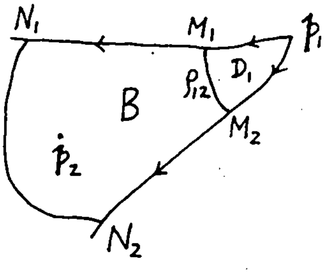

Figure 1

Since by Lemma 2.1 and by the continuity of the flow $f(p,t)$ it follows that $p' = f(p, t_p)$ is a continuous function of p, the arc $\widehat{M_1M_2}$ will be mapped into a continuous segmental arc in K. But this is impossible because K is disconnected and the images of M_1 and M_2 lie in two distinct connection components of K. Thus we have proved that $K = \widehat{N_1N_2}$, hence $\widehat{N_1N_2} \subset S_1$. Further, by the definition of K, we know that for every point $p_1' \in \widehat{N_1N_2}$ there must be a point $p_1 \in \widehat{M_1M_2}$ such that $p_1' = f(p_1, t_{p_1})$.

Consider now the mapping $p' = f(p, t_p)$ from $\widehat{M_1M_2}$ to $\widehat{N_1N_2}$. As stated above, it is surjective (onto). Further, if $f(p_1, t_{p_1}) = f(p_2, t_{p_2})$, then by the uniqueness of solutions it follows that $p_1 = p_2$. Hence this mapping is injective. Let

$$t_{p'} = \inf\left\{t \leq 0 \,\middle|\, f(p', [t, 0]) \subset B\right\}.$$

It is easy to see that $t_{p'}$ is defined for every point $p' \in \widehat{N_1N_2}$. Thus we get the inverse map $p = f(p', t_{p'})$. Using the same argument used in Lemma 2.1 we can prove that the inverse map is continuous. Hence $p' = f(p, t_p)$ is a homeomorphism from $\widehat{M_1 M_2}$ onto $\widehat{N_1 N_2}$. Therefore $\widehat{M_1 M_2}$ is mapped topologically onto $\widehat{N_1 N_2}$ by trajectories of (1.1), and B is filled by these trajectories. But this contradicts the fact that $p_2 \in B$. Hence Theorem 3.1 is proved ∎

Remark. In the case that there is a regular positive parabolic sector D, a theorem similar to Theorem 3.1 can be proved provided we make the change $t \to -t$ in the system (1.1).

Theorem 3.2. Suppose the following:

1. Let p_1 and p_2 be two critical points of the system (1.1) and let one of them, say p_1, have a positive parabolic sector D_1 with respect to some simple closed curve ρ_1.

2. Let M_1 and M_2 be two endpoints of the oriented closed subarc ρ_{12} of D_1, and let $f(M_i, t)$ $(i = 1, 2)$ are unbounded for $t \to -\infty$.

3. Let B be an unbounded sectorial region bounded by two unbounded curves $\widehat{p_1 M_i} \cup f(M_i, \mathbb{R}^-)$ $(i = 1, 2)$ with the same endpoint p_1 and containing D_1 in its interior, and let the following three conditions be satisfied in B:

(i) There is only one critical point p_2.

(ii) There are no closed trajectories and singular closed trajectories.

(iii) Every positive semi-trajectory of the system (1.1) is bounded.

Then there must be in B a trajectory connecting p_1 and p_2 (see Figure 2).

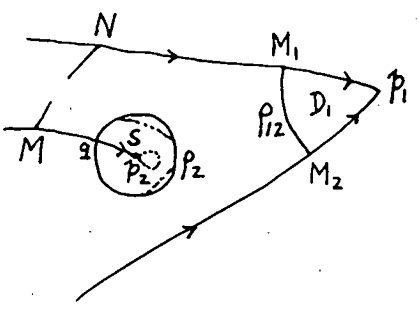

Figure 2

Proof. Consider the critical point p_2 and construct a circle ρ_2 of radius r with the centre p_2 such that $\rho_2 \cap \rho_{12} = \phi$. We distinguish the three cases

(I) there is at least one hyperbolic sector of p_2 in ρ_2

(II) there are no hyperbolic sectors of p_2, but it has at least an elliptic one in ρ_2

(III) there are no hyperbolic and elliptic sectors of p_2 in ρ_2

and consider them step by step.

Case (I): It is easy to see that there must be a point $N_1 \in \rho_2$ such that $f(N_1, t) \to p_2$ as $t \to -\infty$. By condition (iii), the positive semi-trajectory $f(N_1, \mathbb{R}^+)$ is bounded. Therefore, by the Poincaré-Bendixson theory of planar systems and condition (ii) we have $f(N_1, t) \to p_1$ as $t \to +\infty$. Hence Theorem 3.2 holds.

Case (II): By [5: p. 164], there is at most a finite number of elliptic sectors of p_2 in ρ_2. Since there are no hyperbolic sectors, the number of elliptic sectors is even and there must be at least one negative parabolic sector. Thus there must be a point $N \in \rho_2$ such that the trajectory $f(N, \mathbb{R})$ connects p_1 and p_2. Hence Theorem 3.2 holds.

Case (III): In this case, all sectors of p_2 in ρ_2 are parabolic. If there is one base solution (see [5: p. 162]) $\gamma(t)$ such that $\gamma(t) \to p_2$ as $t \to -\infty$, then the same argument used in case (I) implies $\gamma(t) \to p_1$ as $t \to +\infty$. Hence Theorem 3.2 holds.

Therefore, in what follows, assume that all base solutions are positive, i.e. they tend to p_2 as $t \to +\infty$. Consider a positive semi-trajectory $f(q, \mathbb{R}^+)$ which tends to p_2 as $t \to +\infty$, where $q \in \rho_2$. It is easy to see that $f(q,t)$ must be unbounded for $t \to -\infty$ because otherwise its α-limit set must contain critical points or closed orbits. But this contradicts conditions (i) and (ii). Therefore, there is a simple curve $\widehat{MN}$ connecting the point $M \in f(q, \mathbb{R}^-)$ and the point $N \in f(M_1, \mathbb{R}^-)$ or $N \in f(M_2, \mathbb{R}^-)$ such that $\widehat{MN} \cap \rho_2 = \phi$ and $\widehat{MN} \cap \overline{D}_1 = \phi$ (see Figure 2). By [5: p. 169], after deleting the hyperbolic part and the elliptic portion in every parabolic sector, one obtains a subregion S of the interior of ρ_2 such that the positive semi-trajectory through any point of S is interior to S for $t \geq 0$, and it tends to p_2 as $t \to +\infty$. Similary, after deleting the hyperbolic part and the elliptic portion from D_1, one obtains a subregion $D_1' \subset D_1$ and D_1' possesses properties similar to S.

From the continuous dependence of solutions on initial conditions it follows that the positive semi-trajectory originating from any point in a small neighbourhood of M on the curve $\widehat{MN}$ must enter S for increasing time, thus it tends to p_2 as $t \to +\infty$, while the positive semi-trajectory originating from any point in a small neighbourhood of N on the curve $\widehat{MN}$ must enter D_1' for increasing time, thus it tends to p_1 as $t \to +\infty$. Similarly, for any point $p \in \widehat{MN}$, if $f(p,t) \to p_2$ or $f(p,t) \to p_1$ as $t \to +\infty$, there is an open neighbourhood $\sigma(p)$ of p on the curve $\widehat{MN}$ such that for any point $\widehat{p} \in \sigma(p)$ the positive semi-trajectory $f(\widehat{p}, \mathbb{R}^+)$ must enter S or D_1' for increasing time, thus $f(\widehat{p},t) \to p_2$ or $f(\widehat{p},t) \to p_1$, respectively, as $t \to +\infty$. Therefore, there must be a point $Q \in \widehat{MN}$ such that $f(Q,t)$ tends to neither p_2 nor p_1 as $t \to +\infty$. By condition (iii), it follows that the ω-limit set of $f(Q,t)$ contains some closed trajectory or some critical point different from p_1 and p_2. But this contradicts the conditions (i) and (ii) ∎

4. An example

Consider the differential system

$$\left.\begin{aligned} \frac{dx}{dt} &= -\alpha\beta x^2 + \alpha xy \\ \frac{dy}{dt} &= y + \alpha y^2 - \alpha^2 x^2 \end{aligned}\right\} \tag{4.1}$$

in the plane $\mathbb{R}^2$ (see [1: p. 366]) where $\alpha > 0$ and $\beta > 0$ are constants. By [1: p. 367] we know that the critical point $O = (0,0)$ of the system (4.1) is a saddle node, i.e. a critical point whose canonical neighbourhood is the union of one negative parabolic sector and two hyperbolic sectors. The negative parabolic sector lies in the half-plane $x < 0$, its both boundary trajectories lie on the y-axis and all other trajectories of this sector tend to the origin O along the direction $\theta = \pi$ as $t \to -\infty$. The two hyperbolic sectors lie in the half-plane $x > 0$, and their common boundary trajectory tends to the origin O along the direction $\theta = 0$ as $t \to +\infty$. If let

$$\alpha - \beta^2 < 0, \tag{4.2}$$

then it is easy to show the following properties of the system (4.1):

(1) In addition to the origin O, the system (4.1) also have two critical points

$$O_1 = \left(0, -\frac{1}{\alpha}\right) \quad \text{and} \quad A = \left(\frac{\beta}{\alpha(\alpha-\beta^2)}, \frac{\beta^2}{\alpha(\alpha-\beta^2)}\right).$$

O_1 is a stable node and A is a saddle point. We shall prove that there must be a trajectory connecting O and A.

(2) Choose $M_2 = (0, -\frac{1}{3\alpha})$ and construct the straight line $y = -\frac{1}{3\alpha}$ through M_2. It intersects the straight line $y = \beta x$ at $Z = (-\frac{1}{3\alpha\beta}, -\frac{1}{3\alpha})$. Let $x_1 = -\frac{1}{3\alpha\beta} - \varepsilon$, where ε is assumed to be small enough ($0 < \varepsilon < \frac{1}{6\alpha\beta}$) and construct the straight line $x = x_1$ through $P = (x_1, -\frac{1}{3\alpha})$. It intersects the straight line $y = k_1$ at $Q = (x_1, k_1)$, where

$$\frac{-1+\sqrt{1+4\alpha^3 x_1^2}}{2\alpha} < k_1 < \frac{-1+\sqrt{1+\frac{\alpha}{\beta^2}}}{2\alpha}.$$

Let $M_1 = (0, k_1)$. Therefore, it is easy to see that

$$\frac{dy}{dt}\begin{cases} < 0 & \text{for all points on the line segment } M_2P \\ > 0 & \text{on the line segment } QM_1. \end{cases}$$

and

$$\frac{dx}{dt} < 0 \quad \text{on the line segment } PQ.$$

Thus the union $\gamma_1 = M_2P \cup PQUQM_1$ together with two negative semi-trajectories $\gamma^-(M_i)$ $(i = 1, 2)$ and the critical point O bound a regular negative parabolic sector D_1.

(3) Choose $N_2 = (0, -\frac{1}{2\alpha})$ and construct the straight line $y = -\frac{1}{2\alpha}$ through N_2. It intersects the straight line $y = \beta x$ at $Z_1 = (-\frac{1}{2\alpha\beta}, -\frac{1}{2\alpha})$. Let

$$R = \left(-\frac{1}{2\alpha\beta} + \varepsilon_1, -\frac{1}{2\alpha}\right)$$

where ε_1 is small enough ($0 < \varepsilon_1 < \frac{1}{2\alpha\beta}$). Clearly, $\frac{dy}{dt} < 0$ for all points on the line segment N_2R.

(4) We know from the second expression of (4.1) that $x = \pm\sqrt{\frac{k+\alpha k^2}{\alpha^2}}$ are the abscissa of those points on the straight line $y = k$ satisfying $\frac{dy}{dt} = 0$. Consider the straight line

$y = k' = \frac{\beta^2}{\alpha(\alpha-\beta^2)} - \varepsilon_2$ where ε_2 is small enough. Then, on the straight line $y = k'$ for $x < 0$, the abscissa satisfying $\frac{dy}{dt} = 0$ is as follows:

$$\tilde{x} = \frac{1}{\alpha(\alpha - \beta^2)} \cdot \sqrt{[\beta^2 - \varepsilon_2\alpha(\alpha - \beta^2)][1 - \varepsilon_2(\alpha - \beta^2)]}.$$

Moreover, the abscissa of the intersection point of two straight lines $y = k'$ and $y = \beta x$ is as follows:

$$x^* = \frac{\beta^2 - \alpha\varepsilon_2(\alpha - \beta^2)}{\alpha\beta(\alpha - \beta^2)}.$$

It is not difficult to check that $\tilde{x} < x^*$. Thus, we can take x_2 such that $\tilde{x} < x_2 < x^*$, and let $W = (x_2, k')$.

(5) Construct the straight line $x = -\frac{1}{2\alpha\beta} + \varepsilon_1$ through R. It intersects the straight line $y = k'$ at $L = (-\frac{1}{2\alpha\beta} + \varepsilon_1, k')$. Construct the straight line $x = x_2$ through the point W. It intersects the straight line $y = k_2$ at $E = (x_2, k_2)$ where it is assumed that

$$k_2 > \frac{-1 + \sqrt{1 + 4\alpha^3 x_2^2}}{2\alpha}.$$

Clearly, we have $k_2 > k_1$. Let $N_1 = (0, k_2)$. Then, it is easy to verify that

$$\frac{dy}{dt} \begin{cases} < 0 & \text{for all points on the line segment } N_2R \\ > 0 & \text{on the line segments } LW \text{ and } EN_1 \end{cases}$$

and

$$\frac{dx}{dt} \begin{cases} > 0 & \text{on the line segment } RL \\ < 0 & \text{on the line segment } WE. \end{cases}$$

Thus the union

$$\gamma_2 = N_2R \cup RL \cup LW \cup WE \cup EN_1$$

will serve as the simple curve $\widehat{N_1N_2}$ in Theorem 3.1. Consider the region B bounded by γ_i $(i = 1, 2)$ together with two line segments N_1M_1 and N_2M_2. Noting the fact that the y-axis is the union of trajectories of the system (4.1) and the properties of three critical points of (4.1), it is easy to see that there are no closed trajectories and singular closed trajectories in B. Further, it is clear that all exit points of B lying on γ_2 are strict. Therefore, all conditions of Theorem 3.1 are satisfied and it follows that there is in B a trajectory connecting O and A.

References

[1] Andronov, A. A., Leontovich, E. A., Gordon, I. I. and A. G. Maier: *Qualitative Theory of Second-Order Dynamic Systems.* New York: Wiley 1973.

[2] Conley, C.: *Isolated Invariant Sets and the Morse Index* (Conf. Board Math. Sci.: Vol. 38). Providence (R. I.): Amer. Math. Soc. 1978.

[3] Conley, C. C.: *Some aspects of the qualitative theory of differential equations.* Dyn. Systems 1 (1976), 1 - 12.

[4] Conley, C. C. and J. A. Smoller: *Viscosity matrices for two dimensional nonlinear hyperbolic systems.* Comm. Pure Appl. Math.23 (1970), 867 - 884.

[5] Hartman, P.: *Ordinary Differential Equations.* Boston: Birkhäuser Verlag 1982.

[6] Yu Shu-Xiang: *The existence of trajectories joining critical points.* J. Diff. Equ. 66 (1987), 230 - 242.

Received 02.01.1996