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Behavior of a Bounded Non-Parametric H-Surface Near a Reentrant Corner

K. E. Lancaster and D. Siegel

Abstract. We investigate the manner in which a non-parametric surface $z = f(x, y)$ of prescribed mean curvature approaches its radial limits at a reentrant corner. We find, for example, that the solution $f(x, y)$ approaches a fixed value (an extreme value of its radial limits at the corner) as a Hölder continuous function with exponent $\frac{2}{3}$ as (x, y) approaches the reentrant corner non-tangentially from inside a distinguished half-space. We also mention an application of our results to a problem in the production of capacitors involving "dip-coating."

Keywords: *Minimal surfaces, H-surfaces, reentrant corners, dip-coating*

AMS subject classification: 35 J 67, 53 A 10, 35 J 65

1. Introduction

In this paper we consider first a bounded non-parametric surface $z = f(x, y)$ of prescribed mean curvature over a domain whose boundary has a reentrant corner P and which, when considered as a surface in $\mathbb{R}^3$, has a boundary branch point above the reentrant corner. In this case, it is known that there is a half-space from whose directions the radial limits of f at P are identical (i.e. Proposition 1). We will determine the manner in which $f(x, y)$ approaches this value as (x, y) approaches P in the vicinity of this half-space. We will also prove that if the prescribed mean curvature H is real-analytic, then "cusp solutions" do not occur and therefore the radial limits vary continuously with direction. We consider second a non-parametric minimal surface $z = f(x, y)$ over such a domain. In addition to the behavior of f from the vicinity of the half-space mentioned previously, we will determine the behavior of f near P from directions not in the half-space.

Throughout the paper we will let $H \in C^{1,\delta}(\mathbb{R}^3)$ for some $\delta \in (0, 1)$, Ω be a bounded Lipschitz domain in $\mathbb{R}^2$, and P be a (fixed) point on $\partial\Omega$. For convenience, we will assume $P = (0, 0)$. Define $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$ and $Nf = \nabla \cdot Tf$. We are interested in the following boundary value problems.

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Problem 1. Given a piecewise continuous function $\phi : \partial\Omega \to \mathbb{R}$, find a function $f \in C^2(\Omega) \cap C^0(\Omega \cup C)$ such that

$$Nf(x,y) = 2H(x,y,f(x,y)) \quad \text{for } (x,y) \in \Omega \tag{1.1}$$

$$f(x,y) = \phi(x,y) \quad \text{for } (x,y) \in C \tag{1.2}$$

where

$$C = \Big\{(x,y) \in \partial\Omega : \phi \text{ is continuous at } (x,y)\Big\}.$$

Problem 2. Given a piecewise continuous function $\gamma : \partial\Omega \to [0,\pi]$, find a function $f \in C^2(\Omega) \cap C^1(\Omega \cup C) \cap C^0(\overline{\Omega})$ such that

$$Nf(x,y) = 2H(x,y,f(x,y)) \quad \text{for } (x,y) \in \Omega \tag{1.3}$$

$$Tf(x,y) \cdot \nu(x,y) = \cos(\gamma(x,y)) \quad \text{for } (x,y) \in C \tag{1.4}$$

where

$$C = \Big\{(x,y) \in \partial\Omega : \gamma \text{ is continuous at } (x,y) \text{ and } \partial\Omega \text{ is of type } C^1 \text{ near } (x,y)\Big\}$$

and $\nu(x,y)$ is the exterior unit normal to Ω at $(x,y) \in C$.

These problems need not have a solution if appropriate boundary curvature conditions are not satisfied, as has been well illustrated (see, e.g., [3, 17]). However, if the boundary value problem has only one "bad" point, at P, either because $\partial\Omega$ is not smooth at P or the boundary data is discontinuous at P, there may exist a function $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\})$ which satisfies $Nf = 2H$ in Ω and satisfies the boundary condition at each point of $\partial\Omega \setminus \{P\}$. In some cases (see, i.e., [2, 10, 12, 14]) it has been shown that the radial limit of f at $P = (0,0)$ in the direction θ,

$$Rf(\theta) = \lim_{r \to 0+} f(r\cos\theta, r\sin\theta),$$

exists whenever $(r\cos\theta, r\sin\theta) \in \Omega$ for all sufficiently small $r > 0$ and $Rf(\theta)$ varies continuously with θ.

Unfortunately, in [2] (and in the concluding remark in [11]) the possibility that $Rf(\theta)$ might have jump discontinuities was not considered (see [12: Section 12] and [14]). On the other hand, no example is known, at least to the authors, which contradicts the conclusions of [2] and these results are known to be correct when H is constant (in a neighborhood of the z-axis).

One of the surprising conclusions obtained in, for example, [2, 10, 14] was the behavior of $Rf(\theta)$ as θ varies. For simplicity, let us assume $\partial\Omega \setminus \{P\}$ is of type C^1, $\partial\Omega$ has one-sided tangents at P, these tangents make angles $\theta = \alpha$ and $\theta = \beta$ with the x-axis, where $\alpha < \beta < \alpha + 2\pi$, and

$$\Big\{(r\cos\theta, r\sin\theta) : 0 < r < r(\theta) \text{ and } \alpha < \theta < \beta\Big\} \subset \Omega$$

for some $r(\theta) > 0$. We define $Rf(\alpha)$ to be the limiting value of f at P as P is approached along the portion of $\partial\Omega$ which is tangent to $\theta = \alpha$ and we define $Rf(\beta)$ similarly; when f satisfies (1.4), it is not clear that these limiting values need exist. We may summarize the current state of knowledge concerning the behavior of $Rf(\theta)$, including the possibility that Rf might have discontinuities, by the following

Proposition 1. *Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\})$, $f \notin C^0(\overline{\Omega})$, satisfy (1.1) and one of (1.2) or (1.4) on $\partial\Omega \setminus \{P\}$ (with $f \in C^1(\overline{\Omega} \setminus \{P\})$ if (1.4) is satisfied). Then either*

(i) *there exist α_0, β_0 with $\alpha \le \alpha_0 < \beta_0 \le \beta$ and there may exist a countable set $I \subset [\alpha_0, \beta_0]$ such that Rf exists on $[\alpha, \beta] \setminus I$, $Rf \in C^0([\alpha, \beta] \setminus I)$, and*

$$Rf \begin{cases} \text{is constant on } [\alpha, \alpha_0] \\ \text{is strictly monotonic on } [\alpha_0, \beta_0] \setminus I \\ \text{is constant on } [\beta_0, \beta] \end{cases}$$

or

(ii) *there exist $\alpha_0, \beta_0, \theta_1$ with $\alpha \le \alpha_0 < \theta_1 < \theta_1 + \pi < \beta_0 \le \beta$ and there may exist a countable set $I \subset [\alpha_0, \theta_1] \cup [\theta_1 + \pi, \beta_0]$ such that Rf exists on $[\alpha, \beta] \setminus I$, $Rf \in C^0([\alpha, \beta] \setminus I)$ and*

$$Rf \begin{cases} \text{is constant on } [\alpha, \alpha_0] \\ \text{is strictly increasing (decreasing) on } [\alpha_0, \theta_1] \setminus I \\ \text{is constant on } [\theta_1, \theta_1 + \pi] \\ \text{is strictly decreasing (resp. increasing) on } [\theta_1 + \pi, \beta_0] \setminus I \\ \text{is constant on } [\beta_0, \beta]. \end{cases}$$

If f satisfies (1.4) on $\partial\Omega \setminus \{P\}$, then $Rf(\alpha)$ and $Rf(\beta)$ both exist. In addition, if H is constant (on a neighborhood of the z-axis), $\phi \in C^0(\partial\Omega)$, and f satisfies (1.1) on $\partial\Omega \setminus \{P\}$ or if f satisfies (1.4) on $\partial\Omega \setminus \{P\}$ and either $H(0,0,\cdot)$ is strictly increasing or $H(x,y,z)$ depends only on z, is analytic, strictly decreasing, and unbounded from one side, then $I = \emptyset$.

Notice that in case (ii), we have a central "fan" $[\theta_1, \theta_1 + \pi]$ of directions in which the radial limits are all the same. This requires $\beta - \alpha > \pi$, of course, so that Ω has a reentrant corner at P. Let f be a solution of either of the boundary value problems which has radial limits for all $\theta \in [\alpha, \beta]$ and assume these limits behave as in (ii). Then the function f actually extends to be continuous on $\overline{\mathcal{H}}$, where $\mathcal{H}$ is the portion in Ω of the (open) half-space $\{(r\cos\theta, r\sin\theta) : r > 0 \text{ and } \theta_1 < \theta < \theta_1 + \pi\}$, when we define $f(0,0)$ to be $Rf(\theta_1)$.

Here we examine the manner in which $f(x,y)$ approaches its radial limits $Rf(\theta)$ as $(x,y) \to (0,0)$. We find, for example, that $f(x,y)$ approaches the value $Rf(\theta_1)$ as a Hölder continuous function with Hölder exponent $\frac{2}{3}$ independently of H and Ω, and boundary condition whenever case (ii) of Proposition 1 holds and (x,y) approaches P non-tangentially from inside $\Omega \cap \mathcal{H}$ (Theorem 1(v)) provided $H(0,0,Rf(\theta_1)) \neq 0$ or H is real-analytic (near the z-axis). We also find that Rf is continuous in neighborhoods of θ_1 and $\theta_1 + \pi$ in (α, β) (i.e. $\theta_1, \theta_1 + \pi \notin I$ in Proposition 1) and $Rf \in C^0([\alpha, \beta])$ (i.e. $I = \emptyset$) if H is real-analytic. We restrict our attention to case (ii) of Proposition 1 because it represents the more complicated situation; the behavior of f near P would be given by Theorem 1(iii) when Rf is monotonic on $[\alpha, \beta] \setminus I$.

If $H \equiv 0$, then a solution f of (1.1) is a non-parametric minimal surface and may be represented parametrically in terms of the Fourier coefficients of its boundary values. We examine this case in Theorem 2 and find, for example, that the location of the central "fan" of constant radial limits given in case (ii) of Proposition 1 is determined by the first few Fourier coefficients (i.e. (2.9)). While the determination of these Fourier coefficients depends on finding the "boundary correspondence" between the boundary of a parameter domain and the graph of ϕ, numerical algorithms based on this idea have been developed and implemented, such as [16] (developed under the supervision of Professor H. J. Wagner). The conclusions of Theorems 1 and 2 may have numerical applications in two ways. First, the formulas in Theorem 2, such as (2.9), may make programs such as [16] more general by removing the need for a symmetry assumption used to determine θ_1. Second, programs for finding non-parametric H-surfaces may be improved by making use of the a priori knowledge of the behavior of solutions of (1.1) near P. In particular, special finite elements near P or special modifications of other procedures might prove to be useful numerical tools.

2. Statement of main theorems

Before stating our first theorem, we require the following

Definition. For $(x,y) \in \Omega$, we define $\theta(x,y)$ to be the argument of $x+iy$ which satisfies $\alpha < \theta(x,y) < \beta$; that is, $x = r\cos\theta$ and $y = r\sin\theta$ with $r^2 = x^2+y^2$ and $\theta(x,y) \in (\alpha,\beta)$.

Theorem 1. *Assume $H \in C^{1,\delta}(\mathbb{R}^3)$ for some $\delta \in (0,1)$, Ω is a bounded Lipschitz domain in $\mathbb{R}^2$, $P = (0,0) \in \partial\Omega$, and $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\})$, $f \notin C^0(\overline{\Omega})$, satisfies* (1.1). *Suppose either*

(a) *there exists a piecewise continuous function ϕ defined on $\partial\Omega$ and continuous on $\partial\Omega \setminus \{P\}$ such that $f = \phi$ on $\partial\Omega \setminus \{P\}$*

or

(b) *$\partial\Omega \setminus \{P\}$ is of type C^1 and there exist $\epsilon > 0$ and $\gamma \in C^0(\partial\Omega \setminus \{P\})$ such that $\gamma(x,y) \in [\epsilon, \pi-\epsilon]$ for all $(x,y) \in \partial\Omega \setminus \{P\}$ and*

$$Tf(x,y) \cdot \nu(x,y) = \cos(\gamma(x,y)) \qquad \textit{for } (x,y) \in \partial\Omega \setminus \{P\}.$$

Suppose the graph of f over Ω has finite area, and, for some $M > 0$, $|f(x,y)| \le M$ for all $(x,y) \in \Omega$. Suppose also that Rf is not monotonic on $[\alpha,\beta]$ (i.e. case (ii) *of Proposition* 1 *holds), $\theta_1 \in (\alpha, \beta-\pi)$ is as indicated in case* (ii) *of Proposition* 1, *and either $H(x,y,z)$ is real-analytic in x,y,z or $H(0,0,Rf(\theta_1)) \ne 0$. Introduce new coordinates $(\bar{x},\bar{y})$ given by*

$$\bar{x} = -\cos(\theta_1)x - \sin(\theta_1)y \qquad \textit{and} \qquad \bar{y} = \sin(\theta_1)x - \cos(\theta_1)y \tag{2.1}$$

(as in Figure 1*). Denote f in these new coordinates by $\tilde{f}$, so that $\tilde{f}(\bar{x},\bar{y}) = f(x,y)$. Let*

$$\mathcal{H} = \left\{(r\cos\theta, r\sin\theta) \in \Omega : r > 0 \ \textit{ and } \ \theta_1 < \theta < \theta_1 + \pi\right\}$$

and $s = \operatorname{sgn}\big(Rf(\theta_1) - Rf(\theta_1 - \eta)\big)$ for any sufficiently small positive η.

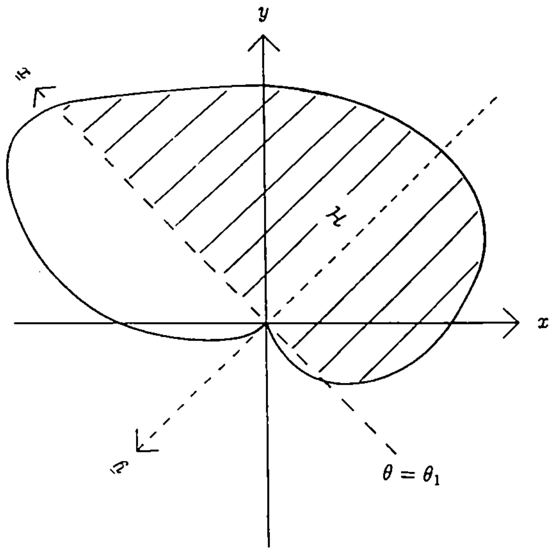

Figure 1: Original (i.e. x, y) and rotated (i.e. $\bar{x}, \bar{y}$) coordinates for the domain Ω with $\mathcal{H}$ shaded

Then:

(i) *The constant*

$$e = \left(\lim_{\bar{y} \uparrow 0} \frac{|\tilde{f}(0, \bar{y}) - Rf(\theta_1)|}{|\bar{y}|^{\frac{2}{3}}} \right)^{\frac{3}{2}}$$

is well defined and positive.

(ii) *For any closed C^1 domain D which satisfies $\mathcal{H} \cup \{P\} \subset D \subset \Omega \cup \{P\}$,*

$$f(x,y) = Rf(\theta_1) + sf^e(\bar{x}, \bar{y}) + R(x,y)$$

and

$$f^e(x,y) = \frac{1}{9e^2}\left(y(C(x,y))^{-1} + e(C(x,y))^2 \right)^2 - s(C(x,y))^2$$

where the graph of f^e is contained in the parametric surface $\{(2uv, e(3u^2v - v^3), u^2 - v^2) : v \geq 0\}$ (see Figure 2), A, B, C are given by

$$A(x,y) = y^2 + \sqrt{4e^4x^6 + y^4}$$

$$B(x,y) = 2^{\frac{1}{3}}\left(y^2 + \sqrt{4e^4x^6 + y^4}\right)^{\frac{2}{3}} - 2e^{\frac{4}{3}}x^2$$

$$C(x,y) = 2^{-\frac{4}{3}}e^{\frac{1}{3}}(B(x,y))^{\frac{3}{4}} - \frac{\sqrt{-4e^{\frac{1}{3}}yA(x,y) - (B(x,y))^{\frac{3}{2}}}}{4e^{\frac{2}{3}}(A(x,y))^{\frac{1}{6}}(B(x,y))^{\frac{1}{4}}},$$

the remainder $R(x,y) = o(f^e(x,y))$ *as* (x,y) *in* D *approaches* $(0,0)$, *and the remainder* $R(x,y) = O((x^2+y^2)^{\frac{2+\delta}{3}})$ *as* (x,y) *in* D *approaches* $(0,0)$ *if (in the* $(\bar{x},\bar{y})$ *coordinates)* $\partial D = \{(\bar{x}(t),\bar{y}(t)) : t \in \mathbb{R}\}$, $(\bar{x}(0),\bar{y}(0)) = (0,0)$, *and* $\bar{y}(t) = O(\bar{x}^2(t))$ *as* $t \to 0$.

(iii) *There exist a dense open subset* Λ *of* $(\alpha_0,\theta_1) \cup (\theta_1+\pi,\beta_0)$ *and a function* $g \in C^0(\Lambda)$ *such that*

$$f(x,y) = Rf(\theta(x,y)) + g(\theta(x,y))\bar{x} + O(x^2+y^2)$$

as (x,y) *in* $\mathcal{S}$ *approaches* $(0,0)$, *where* $\mathcal{S}$ *is a sector of the form*

$$\mathcal{S} = \Big\{(r\cos\theta, r\sin\theta) \in \Omega : r > 0 \;\; and \;\; \xi_1 \le \theta \le \xi_2\Big\}$$

with $[\xi_1,\xi_2] \subset \Lambda$.

(iv) *If* H *is real-analytic in a neighborhood of the z-axis, then* $Rf(\theta)$ *exists for all* $\theta \in [\alpha,\beta]$ *and* $Rf \in C^0([\alpha,\beta])$.

For certain types of approach to $(0,0)$ *in* (ii), *we obtain simpler formulas. In particular:*

(v) *If* $\mathcal{S} = \{(r\cos\theta, r\sin\theta) : \theta_1+\epsilon \le \theta \le \theta_1+\pi-\epsilon\}$ *for some* $\epsilon > 0$, *then*

$$f(x,y) = Rf(\theta_1) + se^{-\frac{2}{3}}|\bar{y}|^{\frac{2}{3}} + O\big(\sqrt{x^2+y^2}\big)$$

as (x,y) *in* $\mathcal{S}$ *approaches* $(0,0)$.

(vi) *As* $\bar{x} \to 0$,

$$\tilde{f}(\bar{x},0) = Rf(\theta_1) + \frac{2}{3}se^{-\frac{2}{3}}\bar{x} + O\big(|\bar{x}|^{1+\frac{\delta}{2}}\big).$$

We may also determine the behavior of $Rf(\theta)$ *as* θ *approaches* θ_1 *from below or* $\theta_1+\pi$ *from above. In fact:*

(vii) *As* $\theta \uparrow \theta_1$ *or* $\theta \downarrow \theta_1+\pi$,

$$Rf(\theta) = Rf(\theta_1) + \frac{4s}{9e^2}\tan^2(\theta-\theta_1) + O(|\tan(\theta-\theta_1)|^{2+\delta}). \tag{2.2}$$

Remark 1. As an illustration of these results, suppose $\theta_1 = -\pi$ (so $\bar{x} = x$ and $\bar{y} = y$), let Λ be as in (iii), and consider approaches in Ω to $(0,0)$ along rays. As $r \to 0+$,

$$f(r\cos\theta, r\sin\theta) = \begin{cases} Rf(\alpha) + o(1) & \text{if } \alpha < \theta \le \alpha_0 \\ Rf(\theta) \pm g(\theta)r^2\cos^2\theta + o(r^2) & \text{if } \alpha_0 < \theta < \theta_1 \ (\theta \in \Lambda) \\ Rf(\theta_1) \pm \frac{2}{3}e^{-\frac{2}{3}}r\cos\theta + o(r) & \text{if } \theta = \theta_1 \\ Rf(\theta_1) \pm e^{-\frac{2}{3}}(r\sin\theta)^{\frac{2}{3}} + o(r^{\frac{2}{3}}) & \text{if } \theta_1 < \theta < \theta_1+\pi \\ Rf(\theta_1) \pm \frac{2}{3}e^{-\frac{2}{3}}r\cos\theta + o(r) & \text{if } \theta = \theta_1+\pi \\ Rf(\theta) \pm g(\theta)r^2\cos^2\theta + o(r^2) & \text{if } \theta_1+\pi < \theta < \beta_0 \ (\theta \in \Lambda) \\ Rf(\beta) + o(1) & \text{if } \beta_0 < \theta \le \beta. \end{cases}$$

Remark 2. From conclusion (vi), we see that Rf is continuous at θ_1 and $\theta_1 + \pi$ and so, in the notation of Proposition 1, $\theta_1, \theta_1 + \pi \notin I$.

Remark 3. Notice that conclusion (vii) generalizes [11: pp. 654 - 655].

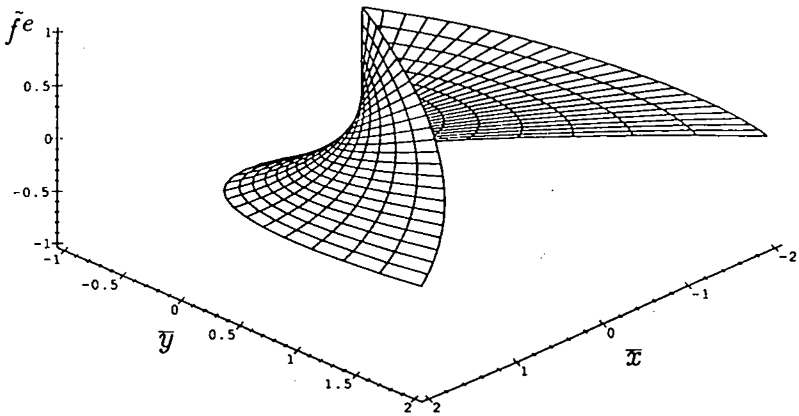

Figure 2: A portion of the parametric surface $\{(2uv, 3u^2v - v^3, u^2 - v^2) : v \geq 0\}$ near the origin

Suppose $H \equiv 0$ and Ω is a bounded Lipschitz domain in $\mathbb{R}^2$ which is locally convex at each point of $\partial\Omega \setminus \{P\}$ and can be written as

$$\Omega = \Big\{(r\cos\theta, r\sin\theta) : \ 0 < r < r(\theta) \ \text{ and } \ \alpha < \theta < \beta\Big\} \tag{2.3}$$

for some constants α and β with $0 < \beta - \alpha < 2\pi$ and some function r with $r(\theta) > 0$ for $\alpha < \theta < \beta$. Suppose also that ϕ is piecewise continuous on $\partial\Omega$. (We observe that if ϕ is continuous and $f \notin C^0(\overline{\Omega})$, then case (ii) of Proposition 1 automatically holds.) Let $f \in C^2(\Omega) \cap C^0(\overline{\Omega} \setminus \{P\})$ satisfy

$$\left.\begin{aligned} Nf &= 0 \quad \text{in } \Omega \\ f &= \phi \quad \text{on } \partial\Omega \setminus \{P\}. \end{aligned}\right\} \tag{2.4}$$

Assume $f \notin C^0(\overline{\Omega})$ and Rf is not monotonic on $[\alpha, \beta]$. Let $B = \{(u,v) : u^2 + v^2 < 1 \text{ and } v > 0\}$. Using the procedure in [10, 11] (also [2]) as indicated in the comments below preceeding the proof of Theorem 1 and an appropriate conformal map of the unit disk onto B, we find that there exists $X \in C^0(\overline{B} : \mathbb{R}^3) \cap C^2(B : \mathbb{R}^3)$,

$$X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big) \qquad \big((u,v) \in \overline{B}\big),$$

such that

$$\left.\begin{aligned} \Delta X &= \vec{0} \quad \text{in } B \\ X_u \cdot X_v &= 0 \quad \text{in } B \\ |X_u| &= |X_v| \quad \text{in } B \\ x(u,0) = y(u,0) &= 0 \quad \text{for } u \in [-1,1] \\ z(u,v) &= f(x(u,v),y(u,v)) \quad \text{for } (u,v) \in B \\ |X_u(u,v)| &\neq 0 \quad \text{iff } (u,v) \neq (0,0), \end{aligned}\right\} \tag{2.5}$$

and K is an orientation reversing homeomophism of B onto Ω, where $K : B \to \Omega$ is given by $K(u,v) = (x(u,v), y(u,v))$. Notice that $X(B)$ is the graph of f over Ω.

Let us denote by $a_n = a_n(\Omega,\phi)$ and $b_n = b_n(\Omega,\phi)$ the Fourier sine coefficients of $x(\cos t, \sin t)$ and $y(\cos t, \sin t)$, respectively, so that

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x(\cos\theta,\sin\theta)\sin(n\theta)\,d\theta \\ b_n &= \frac{2}{\pi}\int_0^\pi y(\cos\theta,\sin\theta)\sin(n\theta)\,d\theta \end{aligned} \qquad (n \geq 1). \tag{2.6}$$

Let $c_n = c_n(\Omega,\phi)$ denote the Fourier cosine coefficients of $z(\cos\theta, \sin\theta)$, so that

$$c_n = \frac{2}{\pi}\int_0^\pi z(\cos\theta,\sin\theta)\cos(n\theta)\,d\theta \qquad (n \geq 0). \tag{2.7}$$

Definition. Define θ to be a strictly increasing map from $(-1,1)$ to (α,β) such that

$$x_v(u,0) = |z_u(u,0)|\cos(\theta(u)) \qquad \text{and} \qquad y_v(u,0) = |z_u(u,0)|\sin(\theta(u))$$

with $\theta(0) = \lim_{u\to 0^+}\theta(u)$. Set $\alpha_0 = \lim_{u\to -1^+}\theta(u)$ and $\beta_0 = \lim_{u\to 1^-}\theta(u)$. (We observe that $\theta(0) = \theta_1 + \pi$ if θ_1 is given as in (2.9), $\tan\alpha_0 = \lim_{u\downarrow -1}\frac{y_v(u,0)}{x_v(u,0)}$ and $\tan\beta_0 = \lim_{u\uparrow 1}\frac{y_v(u,0)}{x_v(u,0)}$.)

Definition. Let u be the map from $[\alpha,\beta]$ to $[-1,1]$ satisfying the conditions $u \in C^0([\alpha,\beta])$ and $u(\theta(t)) = t$ for $t \in (-1,1)$; notice that $u \equiv -1$ on $[\alpha,\alpha_0]$ and $u \equiv 1$ on $[\beta_0,\beta]$.

Lemma 1.

(a) *For each* $u \in (-1,1)$ *with* $|X_u(u,0)| = |z_u(u,0)| \neq 0$, $\theta'(u) = \frac{d\theta}{du} > 0$.

(b) *Set* $L = \{\theta(u) : u \in (-1,1)$ *and* $z_u(u,0) \neq 0\}$. *Then* $u \in C^1(L)$ *and* $u'(\theta) = \frac{du}{d\theta} > 0$ *if* $\theta \in L$.

Notice that $L = (\alpha_0,\beta_0)$ or $L = (\alpha_0,\theta_1) \cup (\theta_1+\pi,\beta_0)$. We observe that the conclusions of Theorem 1 continue to hold (with $e = e_2 = -(\cos(\theta_1)a_2 + \sin(\theta_1)b_2)$ and $\Lambda = (\alpha_0,\theta_1) \cup (\theta_1+\pi,\beta_0)$.) In addition we can obtain the radial limits of f at P and the asymptotic behavior of f near P from the Fourier coefficients a_n, b_n, c_n as indicated in the following

Theorem 2. *Let Ω be given by (2.3), f be the solution of (2.4), and $X \in C^0(\overline{B} : \mathbb{R}^3) \cap C^2(B : \mathbb{R}^3)$ satisfy (2.5). Suppose $z_u(0,0) = 0$; hence $a_1 = b_1 = c_1 = 0$, $c_2^2 = a_2^2 + b_2^2 > 0$, Ω has a reentrant corner at P (i.e. $\beta - \alpha > \pi$), and the parametric minimal surface X has a boundary branch point at $X(0,0) = (0,0,c_0)$. Let $Rf(\alpha)$ and $Rf(\beta)$ denote the limits of ϕ at $(0,0)$ along $\partial\Omega$ from the appropriate directions. Then $Rf(\theta)$ exists for all $\theta \in (\alpha,\beta)$, $Rf \in C^0([\alpha,\beta])$, and*

$$Rf(\theta) = z(u(\theta),0) = c_0 + \sum_{n=2}^{\infty} c_n(u(\theta))^n \qquad (\theta \in [\alpha,\beta]). \tag{2.8}$$

Define $\theta_1 \in (\alpha, \beta - \pi)$ by

$$\sin(\theta_1)a_2 - \cos(\theta_1)b_2 = 0. \tag{2.9}$$

Then $u \equiv 0$ on $[\theta_1, \theta_1+\pi]$, $u \neq 0$ on $[\alpha,\theta_1)\cup(\theta_1+\pi,\beta]$, and when $\theta \in (\alpha_0,\theta_1)\cup(\theta_1+\pi,\beta_0)$, $u(\theta)$ satisfies

$$\sum_{n=2}^{\infty} n\big(\sin(\theta)a_n - \cos(\theta)b_n\big)(u(\theta))^{n-1} = 0, \tag{2.10}$$

and if $u \in (-1,0)\cup(0,1)$ satisfies (2.10), then $u = u(\theta)$. Further, if $u \in (-1,0)\cup(0,1)$,

$$\left(\sum_{n=2}^{\infty} nb_n u^{n-1}\right)\cos(\theta(u)) - \left(\sum_{n=2}^{\infty} na_n u^{n-1}\right)\sin(\theta(u)) = 0. \tag{2.11}$$

Set $s = \operatorname{sgn}(Rf(\theta_1) - Rf(\alpha))$ and define $g : [\alpha,\beta] \to \mathbb{R}$ by

$$g(\theta) = \frac{\sum_{n=2}^{\infty} c_n\left[nP(\theta)u(\theta) - \binom{n}{2}\right](u(\theta))^{n-2}}{\left(\sum_{n=2}^{\infty} n\big(\cos(\theta_1)a_n + \sin(\theta_1)b_n\big)u(\theta)^{n-1}\right)^2}$$

where

$$P(\theta) = \frac{\sum_{n=3}^{\infty}\binom{n}{3}\big(\cos(\theta)b_n - \sin(\theta)a_n\big)(u(\theta))^{n-3}}{\sum_{n=2}^{\infty} n(n-1)\big(\cos(\theta)b_n - \sin(\theta)a_n\big)(u(\theta))^{n-2}}.$$

Then as $(x,y) \to (0,0)$ with either

$$\liminf_{(x,y)\to(0,0)} \theta(x,y) > \alpha_0 \qquad \text{and} \qquad \limsup_{(x,y)\to(0,0)} \theta(x,y) < \theta_1$$

or

$$\liminf_{(x,y)\to(0,0)} \theta(x,y) > \theta_1 + \pi \qquad \text{and} \qquad \limsup_{(x,y)\to(0,0)} \theta(x,y) < \beta_0,$$

we have

$$f(x,y) = Rf(\theta(x,y)) + g(\theta(x,y))\,\overline{x}^2 + O((x^2+y^2)^2)$$

where $\overline{x} = -\cos(\theta_1)x - \sin(\theta_1)y$ and $\overline{y} = \sin(\theta_1)x - \cos(\theta_1)y$.

3. An application

As part of the process of manufacturing some capacitors, a well-known international firm applies a metallic coating to the bottom and a portion of the side of the capacitor using "dip-coating." One example of this part of the process consists of lowering the capacitor approximately 0.5 mm into a liquid metallic paste, letting it sit in the liquid for up to 20 seconds, removing it from the paste, turning it upside-down, and heating it until the coating dries. The manufacturer would like the coating of the side to have a uniform height, as in Figure 3, since otherwise precisely predicting the electrical properties of the device in advance might be difficult. However, the actual coating of a typical capacitor in the shape of a rectangular parallelpiped is "crescent shaped" as in Figure 4. If capillarity is primarily responsible for the shape of the coating, as seems to be the case, then our results can be applied to this problem, as illustrated in the following section.

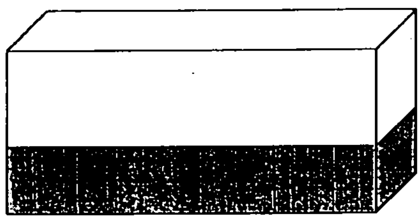

Figure 3: The desired coating of the side of the capacitor

Consider a constant contact angle $\gamma \in (0, \frac{\pi}{2})$ and a rectangle R with vertices $V = \{(0,0), (-2a,0), (0,-2b), (-2a,-2b)\}$ for some $a > 0$ and $b > 0$. Let C be a circle of radius $r_0 > \sqrt{a^2+b^2}$ centered at $(-a,-b)$, B be the disk of radius r_0 centered at $(-a,-b)$, and $T = B \times \{0\}$. Since the container which holds the metallic paste may not be too important, we will consider $S \cup T$ to be the container and assume the metallic fluid makes a contact angle of $\frac{\pi}{2}$ with the side of the container. The side of our capacitor is represented by $R \times [0,\infty)$. Let Ω_0 be the portion of the plane which is inside C and outside R and let $f \in C^2(\Omega_0) \cap C^1(\overline{\Omega_0} \setminus V)$ be the solution of

$$\left.\begin{aligned} Nf &= \kappa f + \lambda && \text{in } \Omega_0 \\ Tf \cdot \nu &= \cos\gamma && \text{on } R \setminus V \\ Tf \cdot \nu &= 0 && \text{on } C \end{aligned}\right\}$$

where $\kappa > 0$ and λ are appropriate constants. Then $f(x,y)$ represents the height of the liquid above the point (x,y) and the wetted portion of (half of) the side of the capacitor is the set

$$\Big\{(x,0,z) : -2a \le x \le 0,\ 0 \le z \le f(x,0)\Big\} \cup \Big\{(0,y,z) : -2b \le y \le 0,\ 0 \le z \le f(0,y)\Big\}.$$

If $a = b$ and $\gamma \in [\frac{\pi}{4}, \frac{\pi}{2})$, [14: Corollary 2] implies $f \in C^0(\overline{\Omega_0})$; we suspect f is continuous at each point of V even if $a \neq b$. On the other hand, there is (numerical and experimental) reason to suspect that f will be discontinuous on V if $\gamma \in (0, \frac{\pi}{4})$. If f is discontinuous at $(0,0) \in V$, then the results of Theorem 1 hold. For example, if $a = b$, then the radial limits of f at $(0,0)$ are constant on $[-\frac{\pi}{2}, -\frac{\pi}{2}+\gamma]$, $[-\frac{\pi}{4}, \frac{3\pi}{4}]$, and $[\pi-\gamma, \pi]$ and near $(0, 0, Rf(\frac{\pi}{4}))$ the graph of f is similar to Figure 2.

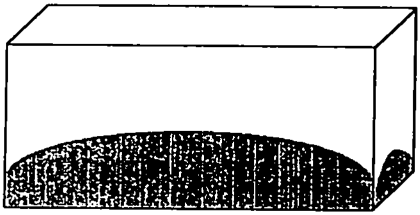

Figure 4: An approximation of the actual coating of the side of the capacitor

4. Proof of Theorem 1

Before beginning this proof, we wish to discuss briefly some results, specifically from [2, 6, 10, 11, 14] which we will use. In [2, 10, 11, 14] the graph $z = f(x,y)$ is represented parametrically in conformal coordinates. This representation is obtained as follows:

(a) For each $\epsilon > 0$, the portion of $z = f(x,y)$ outside the cylinder $C_\epsilon = \{(x,y,z) : x^2 + y^2 < \epsilon^2\}$ is represented as the image of a map Y_ϵ from the unit disk into $\mathbb{R}^3$ which is given in conformal coordinates and satisfies an appropriate three point condition.

(b) As ϵ approaches 0, the maps Y_ϵ are proven to converge to a map Y whose image is the closure of the graph of f and which satisfies other appropriate conditions (e.g. Y is conformal, of type C^2 inside the unit circle, of type C^0 on the closed unit circle, etc).

We note that the uniformization theorem is needed when $H \not\equiv 0$.

In [6], Robert Gulliver proved that minimizing surfaces of prescribed mean curvature do not have interior branch points. As one aspect of his investigation, he studied the behavior of prescribed mean curvature surfaces near branch points using, in part, modifications of the method of Hartman and Wintner [8]; we shall use the techniques in the proofs of Lemmas 2.1 and 7.3 and Corollary 7.1 of [6].

The proof of Theorem 1 will be given in six steps. Let

$$\Omega^{(\epsilon)} = \Big\{(r\cos\theta, r\sin\theta) \in \Omega : 0 < r < \epsilon\Big\}$$
$$\Gamma = \Big\{(x,y,f(x,y)) : (x,y) \in \partial\Omega \setminus \{P\}\Big\}$$
$$S_0 = \Big\{(x,y,f(x,y)) : (x,y) \in \Omega\Big\}.$$

We will use the unit half-disk

$$B = \Big\{(u,v) : u^2 + v^2 < 1 \text{ and } v > 0\Big\}$$

as our parameter domain and we will divide its boundary into two parts:

$$\partial'' B = \{(u,0) : -1 < u < 1\} \quad \text{and} \quad \partial' B = \{(u,v) : u^2 + v^2 = 1 \ \text{and} \ v \geq 0\}.$$

Step 1. *There is a parametric description of the surface* S_0

$$X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)^T \in C^2(B : \mathbb{R}^3)$$

which has the following seven properties:

(i) *X is a homeomorphism of B onto S_0.*

(ii) *X maps $\partial' B$ strictly monotonically onto $\overline{\Gamma}$.*

(iii) *X is conformal on B: $X_u \cdot X_v = 0, X_u^2 = X_v^2$ on B.*

(iv) $\Delta X := X_{uu} + X_{vv} = 2H(u,v) X_u \times X_v$ *where* $H(u,v) = H\big(x(u,v), y(u,v), z(u,v)\big)$.

(v) *$X \in C^0(\overline{B})$ and $x = y = 0$ on $\partial'' B$.*

(vi) *$X_u(u,v) = (0,0,0)$ if and only if $(u,v) = (0,0)$.*

(vii) *Writing $K(u,v) = (x(u,v), y(u,v)), K(\cos t, \sin t)$ moves clockwise about $\partial\Omega$ as t increases, $0 \leq t \leq \pi$ and K is orientation reversing on B.*

Proof. The existence of the map X follows as in [2] when $f = \phi$ on $\partial\Omega \setminus \{P\}$ and as in [14] when f satisfies (1.4) on $\partial\Omega \setminus \{P\}$ (see the comments preceeding the proof of the theorem) ∎

Step 2. *There is a C^2-extension of X, still denoted X, into a neighborhood W of $(0,0)$ such that, for some $a, b, \lambda \in \mathbb{R}$ with $a^2 + b^2 = \lambda^2 > 0$,*

$$X(u,v) = \big(2auv, 2buv, c_0 + \lambda(u^2 - v^2)\big)^T + \rho(w)$$

where $c_0 = Rf(\theta_1)$ and $D^k \rho(w) = o(|w|^{2-k})$ for $k = 0,1,2$ as $w = u + iv \to 0$ ($(u,v) \in W$). We may (and will) assume $\lambda > 0$.

Proof. From [9], we know that $X \in C^{1,\mu}(B \cup \partial'' B)$ for all $\mu \in (0,1)$. From Step 1(iv) we see that

$$\Delta x = 2H\big(x(u,v), y(u,v), z(u,v)\big)(y_u z_v - y_v z_u).$$

Let us denote the right-hand side by $k(u,v)$ and consider $x(u,v)$ as the solution of a linear equation (actually Poisson's equation). Let K be a compact subset of $B \cup \partial'' B$. Since $k(u,v)$ is in $C^{0,\delta}(K)$ and $x(u,0) = 0$, [5: Theorem 4.11 or Lemma 6.10] together with [4] implies $x \in C^{2,\delta}(K)$ (and so $x \in C^2(B \cup \partial'' B)$). Similarly, $y \in C^{2,\delta}(K)$ (and so $y \in C^2(B \cup \partial'' B)$). From the fact that X is conformal we see that $z(\cdot, 0) \in C^{2,\delta}(K \cap (-1,1))$; a similar argument to that above then shows that $z \in C^{2,\delta}(K)$. Thus $X \in C^{2,\delta}(K : \mathbb{R}^3)$ for each K which is a compact subset of $B \cup \partial'' B$; hence $X \in C^2(B \cup \partial'' B : \mathbb{R}^3)$. From [5: Theorem 6.19 and the remark following it] we see that $X \in C^{3,\delta}(K : \mathbb{R}^3)$ for each K as before.

Claim. *X can be extended to be of type C^2 on the closed disk $\overline{E}_\eta$ with $E_\eta = \{(u,v) : u^2+v^2 < \eta^2\}$ for all sufficiently small $\eta \in (0,1)$ such that in this disk X satisfies the system*

$$\triangle X = AX_u + BX_v \tag{4.1}$$

where A and B are matrices which are continuous on $\overline{E}_\eta$ and of C^1-type on the closed half-disks

$$\overline{E_\eta^+} = \Big\{(u,v) : mu^2 + v^2 \le \eta^2 \ \ and \ \ v \ge 0\Big\}$$

and

$$\overline{E_\eta^-} = \Big\{(u,v) : u^2 + v^2 \le \eta^2 \ \ and \ \ v \le 0\Big\}.$$

Assuming the claim is correct, the reasoning used to prove [6: Corollary 7.1] yields some $m \ge 1$ and some $\tau \in \mathbb{C}^3 \setminus \{0\}$ such that

$$X(u,v) = (0,0,c_0) + \mathrm{Re}\{\tau w^m\} + \rho(w)$$

where $D^k\rho(w) = o(|w|^{m-k})$ for $k = 0,1,2$ as $w = u+iv \to 0$. Since X is a "two-to-one" map of $\partial'' B$ into T, m must be even. Since X is one-to-one on B, m must equal two. Now $X_u(u,0) = (0,0,z_v(u,0))^T$ and $X_v(u,0) = (x_v(u,0), y_v(u,0), 0)^T$ for $-1 < u < 1$ and X is conformal on $B \cup \partial'' B$, so $\tau = (ia, ib, \lambda)^T$ where $a,b < 0$, we may introduce new coordinates $(x,y,\tilde{z})$ with $\tilde{z} = -z$ and so obtain $\lambda > 0$; we will assume this in the following. (Notice that this assumption implies $\operatorname{sgn} z_u(u,0) = \operatorname{sgn} u$. Regarding Step 2, also see [7].)

Proof of the Claim. Let us denote $H\big(x(u,v), y(u,v), z(u,v)\big)$ by $H(u,v)$. We first wish to extend X as a C^2-map on E_1 and then show that it satisfies a system of the form (4.1). Since we already know $X \in C^2(B \cup \partial'' B)$ and $x(u,0) = y(u,0) = 0$ for $u \in (-1,1)$ and it follows from the conformality of X that $z_v(u,0) = 0$ for $u \in (-1,1)$, we wish to extend $z(u,v)$ as an even function of v across $v = 0$ and extend $x(u,v)$ and $y(u,v)$ across $v = 0$ in a manner which makes the corresponding second derivatives of x and y from $v > 0$ and $v < 0$ agree at $v = 0$; notice that if $H(u,0) \ne 0$, then the odd extensions of x and y across $v = 0$ will not be of C^2-type at $(u,0)$. We extend X by defining, for $v < 0$,

$$\left.\begin{aligned} x(u,v) &= -x(u,-v) \\ &\quad - 2H(u,-v)\big(y_v(u,-v)z_u(u,-v) + y_u(u,-v)z_v(u,-v)\big)v^2 \\ y(u,v) &= -y(u,-v) \\ &\quad - 2H(u,-v)\big(x_v(u,-v)z_u(u,-v) + x_u(u,-v)z_v(u,-v)\big)v^2 \\ z(u,v) &= z(u,-v). \end{aligned}\right\} \tag{4.2}$$

Using the fact that $x(u,0) = y(u,0) = 0$, we see that X is of C^2-type on $E := E_1$. Differentiating $(4.2)_{1-2}$ gives

$$\begin{aligned} x_u(u,-v) &+ \big\{2\big[H(u,-v)z_v(u,-v)\big]_u v^2\big\} y_u(u,-v) \\ &+ \big\{2\big[H(u,-v)z_u(u,-v)\big]_u v^2\big\} y_v(u,-v) \\ &\qquad = -2x_u(u,v) - \big\{2H(u,-v)y_{uv}(u,-v)v^2\big\} z_u(u,v) \\ &\qquad\quad + \big\{2H(u,-v)y_{uu}(u,-v)v^2\big\} z_v(u,v) \end{aligned}$$

and

$$\begin{aligned} x_v(u,-v) &- \{2[H(u,-v)z_v(u,-v)v^2]_v\}y_u(u,-v) \\ &- \{2[H(u,-v)z_u(u,-v)v^2]_v\}y_v(u,-v) \\ &= -x_v(u,v) + \{2H(u,-v)y_{uv}(u,-v)v^2\}z_v(u,v) \\ &\quad - \{2H(u,-v)y_{vv}(u,-v)v^2\}z_u(u,v). \end{aligned}$$

Analogous equations hold for $y_u(u,-v)$ and $y_v(u,-v)$. This leads to a system of the form

$$(I+C)\begin{pmatrix} x_u(u,-v) \\ y_u(u,-v) \\ x_v(u,-v) \\ y_v(u,-v) \end{pmatrix} = \begin{pmatrix} -x_u(u,v) \\ -y_u(u,v) \\ x_v(u,v) \\ y_v(u,v) \end{pmatrix} + d \tag{4.3}$$

for $v < 0$, where C is of C^1-type on $\overline{E}_\eta^-$ and $C = 0$ for $v = 0$, $d_i = d_i' z_u + d_i'' z_v$ where d_i', d_i'' are of C^1-type on $\overline{E}_\eta^-$ and $d_i' = d_i'' = 0$ for $v = 0$. It follows from (4.3) that for $(u,v) \in \overline{E}_\eta^-$ with η sufficiently small

$$\left.\begin{aligned} x_u(u,-v) &= -x_u(u,v) + f_1 \\ y_u(u,-v) &= -y_u(u,v) + f_2 \\ x_v(u,-v) &= x_v(u,v) + f_3 \\ y_v(u,-v) &= y_v(u,v) + f_4 \end{aligned}\right\} \tag{4.4}$$

where

$$f_i = f_{i1}x_u(u,v) + f_{i2}y_u(u,v) + f_{i3}x_v(u,v) + f_{i4}y_v(u,v) + f_{i5}z_u(u,v) + f_{i6}z_v(u,v)$$

with $f_{ij} \in C^1$ on $\overline{E}_\eta^-$ and $f_{ij} = 0$ for $v = 0$. Additional functions with these same properties will be labeled f_i, respectively f_{ij}, with $i > 4$.

Taking the Laplacian of equations $(4.2)_{1-2}$ and using the formula

$$\begin{aligned} &\Delta\Big(y_v(u,-v)z_u(u,-v) + y_u(u,-v)z_v(u,-v)\Big) \\ &\quad = (\Delta y)_v z_u + y_v(\Delta z)_u + (\Delta y)_u z_v + y_u(\Delta z)_v + 2y_{uv}\Delta z + 2z_{uv}\Delta y \end{aligned}$$

(the right-hand side being evaluated at $(u,-v)$), the analogous formula which holds when y is replaced by x, and Step 1/(iv) to write Δx, Δy, and Δz in terms of first derivatives yields

$$\begin{aligned} &\Delta x(u,v) \\ &\quad = -\Delta x(u,-v) - 4H(u,-v)\Big(y_v(u,-v)z_u(u,-v) + y_u(u,-v)z_v(u,-v)\Big) + \dots \\ &\quad = -2H(u,-v)y_v(u,-v)z_u(u,-v) - 6H(u,-v)y_u(u,-v)z_v(u,-v) + \dots \\ &\quad = \{-2H(u,-v)y_v(u,-v)\}z_u(u,v) + f_7. \end{aligned}$$

To obtain the last equality, we have used (4.4) and the fact that $y_u(u,0) = 0$. Similarly, we have

$$\Delta y(u,v) = \{2H(u,-v)x_v(u,-v)\}z_u(u,v) + f_8.$$

In addition, we have

$$\begin{aligned}\Delta z(u,v) &= \Delta z(u,-v) \\ &= 2H(u,-v)\Big(x_u(u,-v)y_v(u,-v) - x_v(u,-v)y_u(u,-v)\Big) \\ &= f_9\end{aligned}$$

since $x_u(0,0) = y_u(0,0) = 0$.

We now define a matrix $A = (a_{ij})_{i,j=1}^3$ as follows:

$$\begin{aligned}
a_{11} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{71} & \text{for } v < 0 \end{cases} \\
a_{12} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{72} & \text{for } v < 0. \end{cases} \\
a_{13} &= \begin{cases} -2H(u,v)y_v(u,v) & \text{for } v \ge 0 \\ -2H(u,-v)y_v(u,v) + f_{75} & \text{for } v < 0 \end{cases} \\
a_{21} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{81} & \text{for } v < 0 \end{cases} \\
a_{22} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{82} & \text{for } v < 0 \end{cases} \\
a_{23} &= \begin{cases} 2H(u,v)x_v(u,v) & \text{for } v \ge 0 \\ 2H(u,-v)x_v(u,-v) + f_{85} & \text{for } v < 0 \end{cases} \\
a_{31} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{91} & \text{for } v < 0 \end{cases} \\
a_{32} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{92} & \text{for } v < 0 \end{cases} \\
a_{33} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{95} & \text{for } v < 0. \end{cases}
\end{aligned}$$

Further, we define a matrix $B = (b_{ij})_{i,j=1}^3$ by

$$\begin{aligned}
b_{11} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{73} & \text{for } v < 0 \end{cases} \\
b_{12} &= \begin{cases} 0 & \text{for } v \ge 0 \\ f_{74} & \text{for } v < 0 \end{cases} \\
b_{13} &= \begin{cases} 2H(u,v)y_u(u,v) & \text{for } v \ge 0 \\ f_{76} & \text{for } v < 0 \end{cases}
\end{aligned}$$

$$b_{21} = \begin{cases} 0 & \text{for } v \geq 0 \\ f_{83} & \text{for } v < 0 \end{cases}$$

$$b_{22} = \begin{cases} 0 & \text{for } v \geq 0 \\ f_{84} & \text{for } v < 0 \end{cases}$$

$$b_{23} = \begin{cases} -2H(u,v)x_u(u,v) & \text{for } v \geq 0 \\ f_{86} & \text{for } v < 0 \end{cases}$$

$$b_{31} = \begin{cases} 2H(u,v)y_u(u,v) & \text{for } v \geq 0 \\ f_{93} & \text{for } v < 0 \end{cases}$$

$$b_{32} = \begin{cases} 2H(u,v)x_u(u,v) & \text{for } v \geq 0 \\ f_{94} & \text{for } v < 0 \end{cases}$$

$$b_{33} = \begin{cases} 0 & \text{for } v \geq 0 \\ f_{96} & \text{for } v < 0 \end{cases}$$

With these choices for the matrices A and B, we see that (4.1) holds and so our claim is established ∎

Step 3. *Let us rotate the xy-plane through an angle of $\theta_1 + \pi$ and denote the new coordinates by $\overline{x}$ and $\overline{y}$, where $\overline{x} = -\cos(\theta_1)x - \sin(\theta_1)y$ and $\overline{y} = \sin(\theta_1)x - \cos(\theta_1)y$. We may write*

$$\Omega^{(\epsilon)} \cap \mathcal{H} = \left\{(\overline{x}, \overline{y}) : \overline{y} < 0 \;\; \text{and} \;\; \overline{x}^2 + \overline{y}^2 < \epsilon^2\right\}$$

where

$$\mathcal{H} = \left\{(r\cos\theta, r\sin\theta) : r > 0 \;\; \text{and} \;\; \theta_1 < \theta < \theta_1 + \pi\right\}$$

for $\epsilon > 0$ sufficiently small. Let us replace w by $\sqrt{\lambda}\,w$. Subsequently, let $\tilde{X}(u,v) = \left(\tilde{x}(u,v), \tilde{y}(u,v), z(u,v)\right)^T$ be given by

$$\begin{aligned} &\tilde{X}(u,v) = \\ &= \Big(-\cos(\theta_1)x(u,v) - \sin(\theta_1)y(u,v), \\ &\sin(\theta_1)x(u,v) - \cos(\theta_1)y(u,v), z(u,v)\Big)^T \end{aligned}$$

and define $\tilde{K}(u,v) = (\tilde{x}(u,v), \tilde{y}(u,v))$. Then for some $e, \xi \in \mathbb{R}$ with $e \neq 0$

$$\left.\begin{aligned} \tilde{x}(u,v) + iz(u,v) &= 2uv + i(c_0 + u^2 - v^2) + i\xi w^3 + \sigma(w) \\ \tilde{y}(u,v) &= e(3u^2v - v^3) + \tilde{\sigma}(w) \end{aligned}\right\} \tag{4.5}$$

where $\sigma(u) = \tilde{\sigma}(u) = 0$ for $-1 \leq u \leq 1$, $D^k\sigma(w) = O(|w|^{3+\delta-k})$ for $k = 0,1,2,3$ and $D^k\tilde{\sigma}(w) = O(|w|^{3+\delta-k})$ for $k = 0,1,2,3$ as $w = u + iv \to 0$ $((u,v) \in W)$.

Proof. We claim first that $\sin(\theta_1)a - \cos(\theta_1)b = 0$. Notice that the unit vector $\frac{X_v(u,0)}{|X_v(u,0)|}$ approaches $(\frac{a}{\lambda}, \frac{b}{\lambda}, 0)$ as $u \to 0+$ and approaches $(-\frac{a}{\lambda}, -\frac{b}{\lambda}, 0)$ as $u \to 0-$. Let $\theta_a \in (\alpha, \beta)$ satisfy $(-a, -b) = (\cos(\theta_a), \sin(\theta_a))$. Since $X(B)$ is a graph over the (x,y)-plane, the argument of the vector $(x_v(u,0), y_v(u,0))$ is greater than $\theta_a + \pi$ if $u > 0$ and less than θ_a if $u < 0$, where we require our argument function to vary continuously and have range $[\alpha, \beta]$. Using the general comparison principle and the fact that $z(u,v)$ approaches c_0 as $w = u + iv$ approaches 0, we see that $Rf(\theta) = c_0$ for all θ between θ_a and $\theta_a + \pi$. From our assumption about the behavior of Rf, this means $\theta_a = \theta_1$ and so $a = -\cos(\theta_1)$ and $b = -\sin(\theta_1)$. Our claim now follows.

From the proof of [6: Lemma 7.3] we see that there exists a complex number C such that $\tilde{y}(u,v) = \mathrm{Re}\{Cw^3\} + \tilde{\sigma}(w)$. Since $\tilde{y}(u,0) = 0$, $\mathrm{Re}\, C = 0$ and so $C = ie$ for some real e. Thus we have

$$\left.\begin{aligned} \tilde{x}(u,v) + iz(u,v) &= 2uv + i(c_0 + u^2 - v^2) + \sigma(w) \\ \tilde{y}(u,v) &= e(3u^2 v - v^3) + \tilde{\sigma}(w) \end{aligned}\right\} \tag{4.6}$$

where $\sigma(u) = \tilde{\sigma}(u) = 0$ for $-1 \le u \le 1$, $D^k\sigma(w) = o(|w|^{2-k})$ for $k = 0,1,2$ and $D^k\tilde{\sigma}(w) = o(|w|^{3-k})$ for $k = 0,1,2$ as $w = u + iv \to 0$ $((u,v) \in W)$. Since $X \in C^{3,\delta}(\overline{E}_d{}^+)$ for some $d > 0$, we may consider the third degree Taylor expansion $T(u,v)$ of X about $(0,0)$; the error term $X(u,v) - T(u,v)$ will be in $O(|w|^{3+\delta})$ as $\|(u,v)\| \to 0$. Let

$$T_{13}(u+iv) = \sum_{j=0}^{3} \left(\frac{\partial}{\partial u}\right)^j \left(\frac{\partial}{\partial v}\right)^{3-j} (\tilde{x} + iz)\Big|_{(0,0)} u^j v^{3-j}.$$

Since $\tilde{y}_w(0,0) = 0$, we find as in the proof of [6: Lemma 2.1] that T_{13} is analytic in $w = u + iv$. Thus

$$\left.\begin{aligned} \tilde{x}(u,v) &= 2uv + \xi(v^3 - 3u^2 v) + \sigma_1(w) \\ \tilde{y}(u,v) &= e(3u^2 v - v^3) + \sigma_2(w) \\ z(u,v) &= c_0 + u^2 - v^2 + \xi(u^3 - 3uv^2) + \sigma_3(w) \end{aligned}\right\} \tag{4.7}$$

where $D^k\sigma_j(w) = O(|w|^{3+\delta-k})$ for $k = 0,1,2,3$. Since $\tilde{x}(\cdot,0) \equiv \tilde{y}(\cdot,0) \equiv 0$, we see that $\sigma_j(w) = O(v|w|^{2+\delta})$ for $j = 1,2$.

We claim finally that $e \neq 0$. We will assume $e = 0$ and reach a contradiction. Let us suppose first $A = H(0,0,c_0) \neq 0$. From Step 1(iv), we see that

$$\Delta\tilde{y} = 2H(\tilde{x},\tilde{y},z)(\tilde{x}_v z_u - \tilde{x}_u z_v) = 8A(u^2 + v^2) + o(|w|^2) \tag{4.8}$$

and so $\tilde{y}$ must be of the form

$$\tilde{y}(u,v)b = d_4u^4 + d_3u^3v + d_2u^2v^2 + d_1uv^3 + d_0v^4 + o(|w|^4)$$

for some constants d_j. If we compute $\Delta\tilde{y}$, (4.8) implies $12d_4 + 2d_2 = 2d_2 + 12d_0 = 8A$ and $d_3 + d_1 = 0$. Thus $d_0 = d_4$ and, since $y(u,0) = 0$ implies $d_4 = 0$, $d_0 = 0$. Then we have

$$\tilde{y}(u,v) = d_3(u^3v - uv^3) + 4Au^2v^2 + o(|w|^4). \tag{4.9}$$

We claim that $d_3 = 0$. Suppose otherwise. Consider the map

$$g(t) = \tilde{y}(\rho\cos t, \rho\sin t)$$

where $\rho > 0$ is fixed and $0 \le t \le \pi$. For $\rho > 0$ small enough, (4.9) implies g has three changes of sign on $(0,\pi)$ and therefore $g(t)g(\pi - t) < 0$ when $0 < t < t_0$ if $t_0 > 0$ is small enough. Notice that $\tilde{y}(u,v) > 0$ if $u \neq 0$ and $v > 0$ is small enough, since $(\tilde{x}_v(u,0), \tilde{y}_v(u,0))$ points into the upper half-plane if $u \neq 0$. This means $g(t)g(\pi - t) > 0$ if $0 < t < \epsilon$ for $\epsilon > 0$ small enough, which yields a contradiction. Therefore, we see that $d_3 = 0$ and

$$\tilde{y}(u,v) = 4Au^2v^2 + o(|w|^4).$$

However, this implies $\tilde{y}(u,v) > 0$ near $(0,0)$ in B and this is impossible since $\tilde{y} < 0$ in $\Omega^{(\epsilon)} \cap \mathcal{H}$. Therefore, we have $e \neq 0$ if $A \neq 0$.

Suppose now that H is real-analytic, $H(0,0,c_0) = 0$, and $e = 0$. Then $\tilde{x}(u,v), \tilde{y}(u,v)$, and $z(u,v))$ are real-analytic on $B \cup \partial'' B$ and so extend to real-analytic functions in a neighborhood of $(0,0)$. Since properties (iii) and (iv) of Step 1 hold when $v \ge 0$, analyticity implies they continue to hold in this neighborhood. We may write

$$\tilde{y}(u,v) = \sum_{k=0}^{\infty}\sum_{l=1}^{\infty} a_{k,l}u^k v^l.$$

Let m be the total degree of the first non-zero term in this power series expansion of $\tilde{y}$ and let $\tilde{y}_1$ denote the terms of total degree m, so that $\tilde{y}(u,v) = \tilde{y}_1(u,v) + O(|w|^{m+1})$. Equation $(4.6)_2$ implies $m \ge 4$ and so

$$\tilde{y}_1(u,v) = \sum_{k=0}^{m-1}\sum_{l=1}^{m-k} a_{k,l}u^k v^l.$$

Since $H(0,0,z(0,0)) = 0$, we see (as in [6: Lemma 2.1]) that

$$\tilde{y}(u,v) = \tau \operatorname{Im}\{w^m\} + O(|w|^{m+1}) \tag{4.10}$$

for some $\tau > 0$. Let g be given by

$$g(\rho,t) = \tilde{y}(\rho\cos t, \rho\sin t).$$

Then the form of (4.10) implies $g(\rho,t) = \tau\rho^m \sin(mt) + o(\rho^m)$ as $\rho \to 0$. We may choose $\epsilon > 0$ small enough that, for each $k = 1,\dots,m$ and $0 < \rho \le \epsilon$, $\operatorname{sgn}(g(\rho,t)) = (-1)^k$ for all $t \in (\frac{(k-1)\pi}{m} + \delta, \frac{k\pi}{m} - \delta)$ where $\delta = \frac{\pi}{5m}$; this occurs since $\operatorname{sgn}(\tau\rho^m \sin(mt)) = (-1)^k$ if $\rho > 0$ and $t \in (\frac{(k-1)\pi}{m}, \frac{k\pi}{m})$ and, for $\rho > 0$ small enough, this term dominates g. A little thought will show that there is a closed Jordan curve $\sigma = \{\tilde{K}(u(s),v(s)) : 0 \le s \le 1\}$ in $\overline{\Omega}$ with the properties that $u^2(s) + v^2(s) \le \epsilon^2$, $(u(0),v(0)) = (\frac{\epsilon}{2},0)$, $(u(1),v(1)) = (-\frac{\epsilon}{2},0)$, there exists $0 < s_1 < 1$ such that $\tilde{x}(u(s),v(s)) > 0$ if $0 < s < s_1$ and $\tilde{x}(u(s),v(s)) < 0$ if $s_1 < s < 1$, and there exist s_2 and s_3 with $0 < s_2 < s_3 < 1$ such that

$$\tilde{y}(u(s),v(s)) \begin{cases} > 0 & \text{if } 0 < s < s_2 \\ < 0 & \text{if } s_2 < s < s_3 \\ > 0 & \text{if } s_3 < s < 1 \end{cases}.$$

Let $\rho(s) = \sqrt{u^2(s) + v^2(s)}$ and $t(s) \in [0, \pi]$ be the argument of $u(s) + iv(s)$. Then our earlier remarks yield

$$\operatorname{sgn}(g(\rho(s), t(s))) = \operatorname{sgn}(\tilde{y}(u(s), v(s))) = (-1)^k$$

if $t(s) \in (\frac{(k-1)\pi}{m} + \delta, \frac{k\pi}{m} - \delta)$ $(k = 1, \dots, m)$; thus $\tilde{y}(u(s), v(s))$ has at least $m - 1$ changes of sign as s varies from 0 to 1, since $t(0) = 0$ and $t(1) = \pi$. However, σ was constructed so that $\tilde{y}(u(s), v(s))$ has only two changes of sign as s varies from 0 to 1. This contradiction implies $e \neq 0$ ∎

Step 4. *If we write $\tilde{f}(\overline{x}, \overline{y}) = f(x, y)$, then*

$$\tilde{f}(\overline{x}, \overline{y}) = c_0 - e^{-\frac{2}{3}} \overline{y}^{\frac{2}{3}} + O(\sqrt{x^2 + y^2})$$

as $(\bar{x}, \bar{y})$ approaches $(0,0)$ non-tangentially from inside $\mathcal{H}$. Since $Rf(\theta_1) = c_0$, we see that conclusions (i) *and* (v) *of Theorem 1 hold.*

Proof. Let us use (4.5) to determine the preimage in B of the line $\tilde{y} = m\tilde{x}$. If $(u, v) \in B$ such that $\tilde{y}(u, v) = m\tilde{x}(u, v)$, then

$$(e + m\xi)(3u^2 v - v^3) - 2muv = o(v|w|^2).$$

If $q(w) \equiv (e + m\xi)(3u^2 - v^2) - 2mu = o(\|w\|^2)$, we have

$$3(e + m\xi)u^2 - 2mu - (v^2(e + m\xi) + q(w)) = 0.$$

Using the quadratic formula to solve for u when $e + m\xi \neq 0$ and taking the root $u_m(v)$ which approaches 0 as v approaches 0, we obtain

$$u_m(v) = \frac{m - \sqrt{m^2 + 3(e + m\xi)^2 v^2 - 3(e + m\xi)q(w)}}{3(e + m\xi)}.$$

If $m = \frac{\tilde{y}}{\tilde{x}}$ is bounded away from 0, we get $2mu_m(v) = (e + m\xi)v^2 + o(|v|^2)$. From (4.7) and the fact that $\tilde{x}(u_m(v), v) = \frac{1}{m}\tilde{y}(u_m(v), v)$ we obtain for $m \neq 0$

$$\left.\begin{aligned} \tilde{x}(u_m(v), v) &= -\frac{e}{m} v^3 + O(v^3) \\ \tilde{y}(u_m(v), v) &= -ev^3 + O(v^3) \\ z(u_m(v), v) &= c_0 - v^2 + O(v^3). \end{aligned}\right\}$$

If $(\overline{x}, \overline{y})$ approaches $(0,0)$ in such a way that the limiting values of $\overline{\theta}(\overline{x}, \overline{y})$ lie in $(-\pi, 0)$ (so $m = \tan(\overline{\theta}(\overline{x}, \overline{y}))$ is bounded away from 0), we obtain

$$\tilde{f}(\overline{x}, \overline{y}) = c_0 - e^{-\frac{2}{3}} \overline{y}^{\frac{2}{3}} + O(\sqrt{x^2 + y^2})$$

and Step 4 is proved ∎

Step 5. *Conclusions* (ii), (vi) *and* (vii) *of Theorem 1 hold.*

Proof. Let us define $u(x,y)$ and $v(x,y)$ for $(x,y) \in \Omega$ by the conditions that $(u(x,y), v(u,v)) \in B$ and

$$\left.\begin{array}{l} x = x(u(x,y), v(x,y)) \\ y = y(u(x,y), v(x,y)). \end{array}\right\}$$

Notice that if D is a closed C^1-domain with $\mathcal{H} \cup \{P\} \subset D \subset \Omega \cup \{P\}$ and if $(x,y) \in D$ approaches P, then $(u(x,y), v(x,y)) \to (0,0)$.

Notice that

$$(\bar{x}, \bar{y}) = (\tilde{x}(u(x,y), v(x,y)), \tilde{y}(u(x,y), v(x,y)))$$

where $(x,y) \in \Omega$ and (x,y) and $(\bar{x},\bar{y})$ are related as in Step 3. Let us write $\tilde{u}(\bar{x},\bar{y}) = u(x,y)$. The behavior of f as $(x,y) \in D$ approaches $(0,0)$ is given by the behavior of the parametic surface $X(u,v)$ as (u,v) approaches $(0,0)$; that is, by the behavior near $(0,0)$ of the parametric surface

$$X^e(u,v) = \left(2uv, e(3u^2v - v^3), c_0 + u^2 - v^2\right).$$

Now $(4.7)_1$ implies

$$\frac{\tilde{x}(u,v) - 2uv}{2v} = \frac{1}{2}\xi(v^2 - 3u^2) + O(|w|^{2+\delta}) = O(|w|^2)$$

and so

$$u = \frac{\tilde{x}(u,v)}{2v} + O(|w|^2) \qquad \text{as } |w|^2 = u^2 + v^2 \to 0. \tag{4.11}$$

Similarly, $(4.7)_2$ implies

$$\frac{\tilde{y}(u,v) - e(3u^2v - v^3)}{3ev} = O(|w|^{2+\delta})$$

and so

$$u^2 = \frac{\tilde{y}(u,v) + ev^3}{3ev} + O(|w|^{2+\delta}) \qquad \text{as } |w| \to 0. \tag{4.12}$$

Combining (4.11) and (4.12) yields

$$4ev^4 + 4\tilde{y}v - 3e\tilde{x}^2 = O(v^2|w|^{2+\delta}) \qquad \text{as } |w| \to 0 \tag{4.13}$$

where $u = \tilde{u}(\bar{x},\bar{y})$ and $v = \tilde{v}(\bar{x},\bar{y})$.

Let λ be implicitly defined as a function of t by the quartic equation (in λ)

$$4e\lambda^4 + 4\bar{y}\lambda - t = 0 \tag{4.14}$$

where we consider $\bar{x}$ and $\bar{y}$ to be fixed and choose λ to be the solution of (4.14) which satisfies $\lambda \geq 0$ and $\lambda = \tilde{v}(\bar{x},\bar{y})$ for $t = t_1 \equiv 4ev^4(\bar{x},\bar{y}) + 4\bar{y}\tilde{v}(\bar{x},\bar{y})$. Let $\nu(x,y)$ denote the value of λ when $t = t_0 \equiv 3e\bar{x}^2$, so that

$$\nu(\bar{x},\bar{y}) = \frac{e^{\frac{1}{3}}B(\bar{x},\bar{y}) - \sqrt{-4\bar{y}e^{\frac{1}{3}}A(\bar{x},\bar{y}) - (B(\bar{x},\bar{y}))^{\frac{3}{2}}}}{2^{\frac{4}{3}}e^{\frac{2}{3}}(A(\bar{x},\bar{y}))^{\frac{1}{6}}(B(\bar{x},\bar{y}))^{\frac{1}{4}}} \tag{4.15}$$

for $(x, y) \in D$. From (4.13) we have $t_1 - t_0 = O(v^2|w|^{2+\delta})$. Since

$$\frac{d\lambda}{dt} = \frac{1}{16ev^3 + 4\bar{y}} = O\left(\frac{1}{v|w|^2}\right)$$

we obtain

$$v(\tilde{x}(u,v), \tilde{y}(u,v)) = \nu(\bar{x}, \bar{y}) + O(v|w|^\delta). \tag{4.16}$$

Now from (4.11) and (4.16) we find

$$u(\tilde{x}(u,v), \tilde{y}(u,v)) = \frac{\bar{x}}{2\nu(\bar{x}, \bar{y})} + O(u|w|^\delta) \tag{4.17}$$

and so $(4.7)_4$ yields

$$z(u,v) = c_0 + \frac{\tilde{x}^2(u,v)}{4\nu^2(\tilde{x}(u,v), \tilde{y}(u,v))} - \nu^2(\tilde{x}(u,v), \tilde{y}(u,v)) + O(|w|^{2+\delta})$$

as $|w| \to 0$. Since $\tilde{f}(\bar{x}, \bar{y}) = z(\tilde{u}(\bar{x}, \bar{y}), \tilde{v}(\bar{x}, \bar{y}))$, the only remaining difficulty is writing the condition $O(|w(\bar{x}, \bar{y})|^{2+\delta})$ explicitly in terms of $\bar{x}$ and $\bar{y}$. Unfortunately, if we use (4.15) - (4.17) to find $|w|^{2+\delta}$ explicitly in terms of $\bar{x}$ and $\bar{y}$, we get a mess. (The reader is invited to try this using, for example, Maple V....good luck.) On the other hand, we know that $z(u,v) = c_0 + u^2 - v^2 + O(|w|^3)$ and so we certainly have

$$z(u,v) = c_0 + (u^2 - v^2)(1 + o(1)) \qquad \text{as } |w| \to 0.$$

This yields

$$f(x,y) = Rf(\theta_1) + sf^e(x,y)(1 + o(1)).$$

Hence we see that our remainder $R(x,y)$ is $o(f^e(x,y))$ as $(x,y) \in D$ approaches $(0,0)$.

Now suppose $\partial D = \{(\bar{x}(t), \bar{y}(t)) : t \in \mathbb{R}\}$ with $(\bar{x}(0), \bar{y}(0)) = (0,0)$ and $\bar{y}(t) = O(\bar{x}^2(t))$ as $t \to 0$. Then a straightforward calculation using (4.7) shows that if $u(t) = \tilde{u}(\bar{x}(t), \bar{y}(t))$ and $v(t) = \tilde{v}(\bar{x}(t), \bar{y}(t))$, then $v(t) = O(u(t))$ and $u(t) = O(v(t))$ as $t \to 0$. Using (4.7) we have

$$O(|w|^{2+\delta}) = O((x^2 + y^2)^{\frac{2+\delta}{3}}).$$

Actually, we have

$$O(|w|^{2+\delta}) = \begin{cases} O((x^2 + y^2)^{\frac{2+\delta}{2}}) & \text{when } v = O(u) \\ O((x^2 + y^2)^{\frac{2+\delta}{3}}) & \text{when } u = o(v). \end{cases}$$

The proof of conclusion (vi) of Theorem 1 follows from this discussion since $\tilde{y}(u,v) = 0$ if and only if $3u^2 = v^2 + O(|w|^{2+\delta})$ as $|w| \to 0$ and so $\tilde{u}(\bar{x}, 0) = O(\tilde{v}(\bar{x}, 0))$ and $\tilde{v}(\bar{x}, 0) = O(\tilde{u}(\bar{x}, 0))$ as $\bar{x} \to 0$.

Recall

$$Rf(\theta) = z(u(\theta), 0) = c_0 + (u(\theta))^2 + O(|u(\theta)|^3)$$

as $u(\theta) \to 0$. Notice that $u(\theta) \to 0$ if and only if $\theta \in (\alpha, \theta_1] \cup [\theta_1 + \pi, \beta)$ and $\tan(\theta - \theta_1) \to 0$. Since $\tilde{y}_v(u(\theta), 0) = \tan(\theta - \theta_1)\tilde{x}_v(u(\theta), 0)$, we have

$$3eu^2(\theta) + O(|u(\theta)|^{2+\delta}) = \tan(\theta - \theta_1)(2u(\theta) + O(|u(\theta)|^2))$$

or

$$u(\theta) = \frac{2}{3e}\tan(\theta - \theta_1) + O(|u(\theta)|^{1+\delta}) = \frac{2}{3e}\tan(\theta - \theta_1) + O(|\tan(\theta - \theta_1)|^{1+\delta})$$

as $\tan(\theta - \theta_1) \to 0$. Then (2.2) follows. This completes the proof of Step 5 ∎

Step 6. *Conclusions* (iii) *and* (iv) *of Theorem 1 hold.*

Proof. Let us write

$$\begin{aligned}
c(u) &= z(u,0)\\
a_1(u) &= \tilde{x}_v(u,0)\\
a_2(u) &= \tilde{y}_v(u,0)\\
b_1(u) &= \frac{1}{2}\tilde{x}_{vv}(u,0) = -H(0,0,z(u,0))a_2(u)c'(u)\\
b_2(u) &= \frac{1}{2}\tilde{y}_{vv}(u,0) = H(0,0,z(u,0))a_1(u)c'(u)\\
b_3(u) &= \frac{1}{2}z_{vv}(u,0).
\end{aligned}$$

Notice $\text{sgn}(a_1(u)) = \text{sgn}(u)$, $a_2(u) > 0$ if and only if $u \neq 0$, and $\text{sgn}(c'(u)) = \text{sgn}(u)$. Now

$$\left.\begin{aligned}
\tilde{x}(u,v) &= a_1(u)v + b_1(u)v^2 + O(v^3)\\
\tilde{y}(u,v) &= a_2(u)v + b_2(u)v^2 + O(v^3)\\
z(u,v) &= c(u) + b_3(u)v^2 + O(v^3)
\end{aligned}\right\}$$

and so

$$v = (a_1(u))^{-1}(\tilde{x} - b_1(u)v^2 + O(v^3)) = (a_2(u))^{-1}(\tilde{y} - b_2(u)v^2 + O(v^3)).$$

Hence

$$z(u,v) = e(u) + \frac{b_3(u)}{a_1^2(u)}\tilde{x}^2(u,v) + O(v^2|\tilde{x}(u,v)|)$$

and, when u is bounded away from 0,

$$z(u,v) = c(u) + \frac{b_3(u)}{a_1^2(u)}\tilde{x}^2(u,v)O + (|\tilde{x}(u,v)|^3).$$

Let u_1 represent an element of $(-1,0)\cup(0,1)$ and notice that the vector $(\tilde{x}_v(u_1,0), \tilde{y}_v(u_1,0))$ points in the direction $\bar{\theta}_1 \in (-\frac{3\pi}{2}, -\pi)\cup(0,\frac{\pi}{2})$ where

$$\frac{a_2(u_1)}{a_1(u_1)} = \frac{\tilde{y}_v(u_1,0)}{\tilde{x}_v(u_1,0)} = \tan\bar{\theta}_1.$$

Let us write $h = \frac{a_2}{a_1}$. Then Rf is continuous at θ_1 if and only if h is strictly increasing near u_1. If H is real-analytic near the z-axis, then $\tilde{X}(u,v)$ is real-analytic on a neighborhood of $\{(u,0) : -1 < u < 1\}$. This implies h is analytic on $(-1,1)$. Suppose $h(u) = \tan\theta_1$ for $u_1 \le u \le u_1 + \epsilon$, for some $\epsilon > 0$. Analyticity implies $h(u) = \tan\theta_1$ for all $u \in (-1,1)$ and so h is constant. However, (4.7) yield $h(u) = \frac{3\epsilon}{2}u + O(u^2)$ and so h cannot be constant. Therefore h is strictly increasing on $(-1,0)\cup(0,1)$ and so $Rf \in C^0([\alpha,\beta])$. This proves assertion (iv) of Theorem 1.

Let $J = \{u \in (-1,0) \cup (0,1) : h'(u) > 0\}$. For $u \in (-1,0) \cup (0,1)$, define $\theta(u) \in (\alpha_0, \theta_1) \cup (\theta_1 + \pi, \beta_0)$ by

$$\big(x_v(u,0), y_v(u,0)\big) = |z_u(u,0)|\big(\cos(\theta(u)), \sin(\theta(u))\big)$$

and $\tilde{\theta}(u) = \theta(u) - \theta_1 - \pi$; notice $\tan(\tilde{\theta}(u)) = h(u)$. Let $\Lambda = \{\theta(u) : u \in J\}$. Then Λ is a dense open subset of $(\alpha_0, \theta_1) \cup (\theta_1 + \pi, \beta_0)$ and $Rf \in C^0(\Lambda)$. Let $[\xi_1, \xi_2] \subset \Lambda$ and

$$\mathcal{S} = \Big\{(r\cos\theta, r\sin\theta) : \xi_1 \le \theta \le \xi_2 \text{ and } r > 0\Big\}.$$

Let $(x,y) \in \mathcal{S}$ and let $(\bar{x}, \bar{y})$ be related to (x,y) by (2.1) and define $\tilde{\theta}(\bar{x},\bar{y}) = \theta(x,y) - \theta_1 - \pi$. Then $u(\theta(x,y)) \in J$ and

$$h(u(\theta(x,y))) = \frac{\tilde{y}_v\big(\tilde{u}(\bar{x},\bar{y}), \tilde{v}(\bar{x},\bar{y})\big)}{\tilde{x}_v\big(\tilde{u}(\bar{x},\bar{y}), \tilde{v}(\bar{x},\bar{y})\big)}.$$

If we let $\bar{u}, \bar{v}$ and $\tilde{u}$ denote $\tilde{u}(\bar{x},\bar{y}), \tilde{v}(\bar{x},\bar{y})$ and $u(\tilde{\theta}(\bar{x},\bar{y}))$, respectively, we have

$$h(\tilde{u}) = h(\bar{u}) + H\big(0,0,z(\bar{u},0)\big)\frac{(c'(u))^3}{(a_1(\bar{u}))^2}\,\bar{v} + O(\bar{v}^2)$$

and so

$$\tilde{u} - \bar{u} = \frac{H(0,0,z(\bar{u},0))(c'(\bar{u}))^3}{h'(u)(a_1(\bar{u}))^2}\bar{v} + O(\bar{v}^2).$$

This implies

$$z(\bar{u},\bar{v}) = z(\tilde{u},0) + \frac{(c'(\bar{u}))^4 H(0,0,z(\tilde{u},0))}{h'(\tilde{u})(a_1(\tilde{u}))^2}\,\bar{v} + O(\bar{v}^2)$$

and

$$\tilde{f}(\bar{x},\bar{y}) = Rf(\tilde{\theta}(\bar{x},\bar{y})) + \bar{g}(\tilde{\theta}(\bar{x},\bar{y}))\bar{x} + O(\bar{x}^2)$$

where

$$\bar{g}(\theta) = \frac{(c'(\tilde{u}(\theta)))^4 H(0,0,R\tilde{f}(\theta))}{h'(\tilde{u}(\bar{\theta}))(a_1(\tilde{u}(\theta)))^3}$$

and $\tilde{u}(\bar{\theta}) = u(\bar{\theta} + \theta_1 + \pi)$ for $\bar{\theta} \in \{\theta - \theta_1 - \pi : \theta \in \Lambda\}$. This completes the proof of assertion (iii) of Theorem 1 ∎

5. Proof of Lemma 1

Suppose $u_0 \in (-1,1)$ and $|X_u(u_0,0)| \neq 0$, let $\theta_0 = \theta(u_0)$, and define

$$\tilde{X}(u,v) = \big(\tilde{x}(u,v), \tilde{y}(u,v), \tilde{z}(u,v)\big)$$

where

$$\left.\begin{aligned} \tilde{x}(u,v) &= \cos(\theta_0)x(u+u_0,v) + \sin(\theta_0)y(u+u_0,v) \\ \tilde{y}(u,v) &= -\sin(\theta_0)x(u+u_0,v) + \cos(\theta_0)y(u+u_0,v) \\ \tilde{z}(u,v) &= z(u+u_0,v). \end{aligned}\right\}$$

Then $\tilde{x}_v(0,0) = |z_u(0,0)| > 0$ and $\tilde{y}_v(0,0) = 0$. We may extend $\tilde{X}$ by reflection across the u-axis as a parametric minimal surface. If we continue to denote this extended minimal surface by $\tilde{X}$, then $\tilde{X}$ is a vector of harmonic functions and

$$\tilde{X}(u,-v) = \big(-\tilde{x}(u,v), -\tilde{y}(u,v), \tilde{z}(u,v)\big).$$

We may write

$$\tilde{X}(u,v) = \sum_{n=0}^{\infty} \mathrm{Re}\{A_n(u+iv)^n\}$$

and it is easy to see that $A_n = (-ia_n, -ib_n, c_n)$ for some real numbers a_n, b_n and c_n. Notice that $A_0 = (0,0,c_0)$ and $\tilde{y}_u(0,0) = \tilde{y}_v(0,0) = 0$, so

$$\tilde{x}(u,v) = \sum_{n=1}^{\infty} a_n \,\mathrm{Im}\left((u+iv)^n\right) = a_1 v + 2a_2 uv + \dots$$

$$\tilde{y}(u,v) = \sum_{n=2}^{\infty} b_n \,\mathrm{Im}\left((u+iv)^n\right) = 2b_2 uv + \dots.$$

Considering the sign pattern of $\tilde{x}(\rho\cos t, \rho\sin t)$ and $\tilde{y}(\rho\cos t, \rho\sin t)$ for small ρ as t varies from 0 to π, we see that $a_1 > 0$ and $b_2 > 0$ (e.g. the last part of the proof of Step 3 in the previous section). Since $\tilde{X}$ is conformal, we obtain

$$\tilde{z}(u,v) = c_0 + a_1 u + a_2(u^2 - v^2) + \dots.$$

If we define $\tilde{\theta}(u) = \theta(u) - \theta_0$, we see that

$$\tan(\tilde{\theta}(u)) = \frac{\tilde{y}_v(u,0)}{\tilde{x}_v(u,0)} \qquad \text{for } |u| < 1 - |u_0|$$

and so

$$\frac{d\theta}{du}(u_0) = \frac{d\tilde{\theta}}{du}(0) = \frac{\tilde{x}_v(0,0)\tilde{y}_{uv}(0,0) - \tilde{y}_v(0,0)\tilde{x}_{uv}(0,0)}{(x_v(0,0))^2} = \frac{2b_2}{a_1}$$

since $\sec^2(\tilde{\theta}(0)) = \sec^2(0) = 1$. This proves assertion (a) of Lemma 1.

To see that assertion (b) of Lemma 1 holds, we note that θ and u are inverse functions, θ is of C^1-type on $J = \{u \in (-1,1) : z_u(u,0) \neq 0\}$ (by the implicit function theorem), $\theta'(u) > 0$ on J, and $\theta \in L$ if and only if $u \in J$.

Remark 4. In the notation of Step 6 of the previous section, Lemma 1 means $h'(u) > 0$ for all $u \in L$.

6. Proof of Theorem 2

The proof of Theorem 2 will be given in six steps.

Step 1. *Define*

$$\Gamma_0 = \Big\{(x, y, \phi(x,y)) : (x,y) \in \partial\Omega\Big\} \quad \textit{and} \quad S_0 = \Big\{(x,y,f(x,y)) : (x,y) \in \Omega\Big\}.$$

Let $X \in C^2(B : \mathbb{R}^3) \cap C^0(\overline{B} : \mathbb{R}^3)$ *be the homeomorphism of* B *and* S_0 *with properties* (ii) - (vii) *of Step 1 of the proof of Theorem* 1*; here, of course,* $H \equiv 0$ *and the components of* X *are harmonic functions. Then we may extend* X *by reflection across the* u*-axis, so that* $x(u,-v) = -x(u,v)$, $y(u,-v) = -y(u,v)$, $z(u,-v) = z(u,v)$ *and* $X \in C^\omega(E)$ *where* $E = \{(u,v) : u^2+v^2 < 1\}$ *is the unit disk. If we let* a_n, b_n *and* c_n *be defined by* (2.6) *when* $n \geq 1$ *and* $c_0 = \int_0^\pi z(\cos t, \sin t)\, dt$*, then*

$$X(u,v) = \sum_{n=0}^{\infty} \mathrm{Re}\{A_n(u+iv)^n\} \tag{6.1}$$

for all $(u,v) \in B$*, where* $A_0 = (0,0,c_0)^T$ *and* $A_n = (-ia_n, -ib_n, c_n)^T$ *for* $n \geq 1$*. Also* $A_1 = (0,0,0)$ *and* $A_2 \neq (0,0,0)$.

Proof. The fact that X can be reflected across a line as a real-analytic parametrized surface is well known and, because of $(2.5)_{3-5}$ one can check that x and y reflect as odd functions of v while z reflects as an even function of v. Now

$$\sum_{n=0}^{\infty} a_n \sin(nt) \tag{6.2}$$

is the Fourier series expansion of $x^*(t) = x(\cos t, \sin t)$ since it is an odd function of t. Since x^* is continuous on $[0, 2\pi]$, standard results for Fourier series (see, e.g., [1: Subsection 38.10]) imply that (6.2) converges to x^* in $L^2([0,2\pi])$. Since $x^*(t)$ is the boundary value of the harmonic function $x(\rho\cos t, \rho \sin t)$ on $\rho = 1$, we see that

$$x(\rho\cos t, \rho\sin t) = \sum_{n=0}^{\infty} a_n \rho^n \sin(nt) \tag{6.3}$$

for $0 \leq \rho < 1$ and $0 \leq t \leq 2\pi$. A similar argument shows that

$$y(\rho\cos t, \rho\sin t) = \sum_{n=0}^{\infty} b_n \rho^n \sin(nt)$$

$$z(\rho\cos t, \rho\sin t) = \sum_{n=0}^{\infty} c_n \rho^n \cos(nt).$$

It is easy to see that each of these power series (in ρ) has radius of convergence ≥ 1 for each $t \in [0, 2\pi]$. Indeed, for each fixed $\rho_0 \in [0,1)$, $x(\rho_0 \cos t, \rho_0 \sin t)$, $y(\rho_0\cos t, \rho_0 \sin t)$ and $z(\rho_0 \cos t, \rho_0 \sin t)$ are smooth functions of t and so the Fourier series (6.3) and (6.4) converge for each $t \in [0,2\pi]$ (see, e.g., [1: Subsection 38.7]). This means that for each $t \in [0, 2\pi]$, the power series (6.3) and (6.4) converge when $\rho = \rho_0$ and so have radius of convergence $\geq \rho_0$.

Equation (6.1) then follows. From our hypothesis that $c_1 = 0$, we see that $A_1 = (0,0,0)$. As in Step 2 of the proof of Theorem 1, we obtain $a_2^2 + b_2^2 = c_2^2$ and $c_2 \neq 0$. Thus $A_2 \neq (0,0,0)$ ∎

Step 2. *$Rf(\theta)$ exists for each $\theta \in [\alpha,\beta]$ and Rf is a continuous function of θ. Define $\theta_1 \in (\alpha, \beta-\pi)$ by (2.9). Then $u(\theta) = 0$ and $Rf(\theta) = c_0$ for all $\theta \in [\theta_1, \theta_1+\pi]$. Further, (2.8) and (2.10) – (2.11) hold.*

Proof. From [10, 11] we see that $Rf(\theta)$ exists and behaves as indicated and $u(\theta)$ is a continuous function of θ. We notice that (2.8) holds because of the definition of $u(\theta)$ and $z(u,v)$. Also Step 3 of the proof of Theorem 1 implies $u(\theta) = 0$ and so $Rf(\theta) = z(0,0) = c_0$ for $\theta \in [\theta_1, \theta_1+\pi]$. We wish to show that (2.10) and (2.11) hold. Notice that

$$X_v(u,v) = \sum_{n=2}^{\infty} n\mathrm{Re}\{iA_n(u+iv)^{n-1}\} \tag{6.5}$$

for $u \in (-1,1)$. Now $u(\theta) \in (-1,0)$ if and only if $\theta \in (\alpha_0, \theta_1)$, and for such θ,

$$X_v(u(\theta),0) = z_u(u(\theta),0)(\cos\theta, \sin\theta, 0). \tag{6.6}$$

Similarly, $u(\theta) \in (0,1)$ if and only if $\theta \in (\theta_1+\pi, \beta_0)$ and (6.6) holds for these θ. From (6.5) and (6.6) we obtain the equations

$$\left.\begin{aligned} \sum_{n=2}^{\infty} na_n(u(\theta))^{n-1} &= z_u(u(\theta),0)\cos\theta \\ \sum_{n=2}^{\infty} nb_n(u(\theta))^{n-1} &= z_u(u(\theta),0)\sin\theta \end{aligned}\right\}$$

and (2.10) follows from solving each equation for z_u. Equation (2.11) follows in a similar manner. Suppose $u \in (-1,0) \cup (0,1)$ satisfies (2.10). Then (2.10) and (2.11) imply $\theta(u) = \theta$ and so $u = u(\theta)$ ∎

Step 3. *Let us define coordinates $(\overline{x}, \overline{y})$ by*

$$\left.\begin{aligned} \overline{x} &= -\cos(\theta_1)x - \sin(\theta_1)y \\ \overline{y} &= \sin(\theta_1)x - \cos(\theta_1)y \end{aligned}\right\}$$

and set $\Omega_1 = \{(\overline{x},\overline{y}) : (x,y) \in \Omega\}$. Set $e_n = -(\cos(\theta_1)a_n + \sin(\theta_1)b_n)$ and $f_n = \sin(\theta_1)a_n - \cos(\theta_1)b_n)$ for $n \geq 2$; $f_2 = 0$. Define $\tilde{X} : E \to \mathbb{R}^3$ by

$$\tilde{X}(u,v) = \left(\tilde{x}(u,v), \tilde{y}(u,v), z(u,v)\right)$$

where

$$\left.\begin{aligned} \tilde{x}(u,v) &= -\cos(\theta_1)x(u,v) - \sin(\theta_1)y(u,v) \\ \tilde{y}(u,v) &= \sin(\theta_1)x(u,v) - \cos(\theta_1)y(u,v). \end{aligned}\right\}$$

Let $\tilde{u}, \tilde{v} : \Omega_1 \to \mathbb{R}$ be defined by

$$\left.\begin{aligned} \tilde{x}(\tilde{u}(\overline{x},\overline{y}), \tilde{v}(\overline{x},\overline{y})) &= \overline{x} \\ \tilde{y}(\tilde{u}(\overline{x},\overline{y}), \tilde{v}(\overline{x},\overline{y})) &= \overline{y} \end{aligned}\right\}$$

so that $\tilde{u}(\overline{x},\overline{y}) = u(x,y)$ *and* $\tilde{v}(\overline{x},\overline{y}) = v(x,y)$. *For each* $(\overline{x},\overline{y}) \in \Omega_1$, *let us write* $\overline{u} = \tilde{u}(\overline{x},\overline{y})$ *and* $\overline{v} = \tilde{v}(\overline{x},\overline{y})$. *Then, as* $\overline{u} \to 0$ *or* $\overline{v} \to 0+$,

$$\left.\begin{aligned} \overline{x} &= M(\overline{u})\overline{u}\ \overline{v} + \overline{v}^3 k(\overline{v}) + O(\overline{u}\ \overline{v}^3) \\ \overline{y} &= N(\overline{u})\overline{u}^2\ \overline{v} + \overline{v}^3 l(\overline{v}) + O(\overline{u}\ \overline{v}^3) \end{aligned}\right\} \tag{6.7}$$

and $M(\overline{u}) > 0$ *and* $N(\overline{u}) > 0$ *for all* $\overline{u} \in (-1,1)$, *where*

$$\left.\begin{aligned} M(\overline{u}) &= \sum_{n=2}^{\infty} n e_n(\overline{u})^{n-2} \\ N(\overline{u}) &= \sum_{n=3}^{\infty} n f_n(\overline{u})^{n-2} \\ k(\overline{v}) &= \sum_{m=1}^{\infty} (-1)^m e_{2m+1}(\overline{v})^{2m-2} \\ l(\overline{v}) &= \sum_{m=1}^{\infty} (-1)^m f_{2m+1}(\overline{v})^{2m-2}. \end{aligned}\right\} \tag{6.8}$$

Proof. Notice that $(\tilde{x}_v(u,0), \tilde{y}_v(u,0))$ points into the first quadrant if $u > 0$ and its argument tends to 0 as $u \to 0+$. Since $\tilde{x}(u,v) = \sum_{n=2}^{\infty} e_n \mathrm{Im}\{(u+iv)^n\}$, $e_2 > 0$, and $\frac{\partial \tilde{x}}{\partial v}(u,0) \neq 0$ if $u \neq 0$, we see that $M(\bar{u}) > 0$. Now, as in Step 3 of the proof of Theorem 1, $f_3 \neq 0$ and, since $\frac{\partial \tilde{y}}{\partial v}(u,0) > 0$ if $u \neq 0$, we obtain $N(\bar{u}) > 0$. Now

$$\left.\begin{aligned} \overline{x} &= \sum_{n=2}^{\infty} e_n \mathrm{Im}\{ (\tilde{u}(\overline{x},\overline{y}) + i\tilde{v}(\overline{x},\overline{y}))^n \} \\ \overline{y} &= \sum_{n=3}^{\infty} f_n \mathrm{Im}\{ (\tilde{u}(\overline{x},\overline{y}) + i\tilde{v}(\overline{x},\overline{y}))^n \}. \end{aligned}\right\}$$

Then

$$\begin{aligned} \overline{x} &= \overline{v} \sum_{n=2}^{\infty} e_n \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} (-1)^k (\overline{u})^{n-(2k+1)} (\overline{v})^{2k} \\ &= M(\overline{u})\overline{u}\,\overline{v} + k(\overline{v})\overline{v}^3 + O(\overline{u}\,\overline{v}^3) \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \overline{y} &= \overline{v} \sum_{n=3}^{\infty} f_n \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} (-1)^k (\overline{u})^{n-(2k+1)} (\overline{v})^{2k} \\ &= N(\overline{u})\overline{u}^2\overline{v} + l(\overline{v})\overline{v}^3 + O(\overline{u}\,\overline{v}^3). \end{aligned} \tag{6.10}$$

Thus Step 3 is proved ∎

Step 4. *Define* $\tilde{\theta}(\bar{x},\bar{y}) = \theta(x,y) - \theta_1 - \pi$ *and* $\tilde{u}(\tilde{\theta}) = u(\tilde{\theta} + \theta_1 + \pi)$ (*i.e.* $\tilde{u}(\tilde{\theta}(\bar{x},\bar{y})) = u(\theta(x,y))$). *If* $(\bar{x},\bar{y}) \in \Omega_1$ *tends to* $(0,0)$ *in such a manner that*

$$0 < \liminf_{(\bar{x},\bar{y})\to(0,0)} \tilde{\theta}(\bar{x},\bar{y}) \leq \limsup_{(\bar{x},\bar{y})\to(0,0)} \tilde{\theta}(\bar{x},\bar{y}) < \beta_0 - \theta_1 - \pi,$$

then

$$\tilde{u}(\bar{x},\bar{y}) - \tilde{u}(\tilde{\theta}(\bar{x},\bar{y})) = p(\tilde{u}(\tilde{\theta}(\bar{x},\bar{y})))\,(\tilde{v}(\bar{x},\bar{y}))^2 + O((\tilde{v}(\bar{x},\bar{y}))^4). \tag{6.11}$$

Proof. Notice that

$$\liminf_{(\bar{x},\bar{y})\to(0,0)} \tilde{u}(\bar{x},\bar{y}) > 0 \qquad \text{and} \qquad \limsup_{(\bar{x},\bar{y})\to(0,0)} \tilde{u}(\bar{x},\bar{y}) < 1.$$

For $(\bar{x},\bar{y}) \in \Omega_1$, denote $\tilde{\theta}(\bar{x},\bar{y}), \tilde{u}(\bar{x},\bar{y})$ and $\tilde{v}(\bar{x},\bar{y})$ by $\bar{\theta}, \bar{u}$ and $\bar{v}$, respectively. Now (2.10) implies

$$\sum_{n=2}^{\infty} n\big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)\tilde{u}(\bar{\theta})^{n-1} = 0.$$

Since $\cos(\bar{\theta})\bar{y} - \sin(\bar{\theta})\bar{x} = 0$, (6.9) and (6.10) imply

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)\big((\bar{u})^{n-1} - (\tilde{u}(\bar{\theta}))^{n-1}\big)\bar{v} \\
&\quad + \sum_{n=3}^{\infty}\sum_{k=1}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} \big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)(\bar{u})^{n-(2k+1)}(\bar{v})^{2k+1} \\
&\quad = 0.
\end{aligned} \tag{6.12}$$

We notice that

$$\sum_{n=2}^{\infty} n\big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)(\bar{u})^{n-1} \neq 0$$

if $\bar{u} \neq \tilde{u}(\bar{\theta})$ from Step 2 and, from (2.10), if $\bar{u} \neq \tilde{u}(\bar{\theta})$,

$$\begin{aligned}
L(\bar{u},\bar{\theta}) &\equiv \sum_{n=2}^{\infty} n\big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)\sum_{l=0}^{n-2}(\bar{u})^{n-2-l}(\tilde{u}(\bar{\theta}))^l \\
&= \frac{1}{\bar{u} - \tilde{u}(\bar{\theta})}\sum_{n=2}^{\infty} n\big(\cos(\bar{\theta})f_n - \sin(\bar{\theta})e_n\big)\big((\bar{u})^{n-1} - (\tilde{u}(\bar{\theta})^{n-1})\big) \\
&\neq 0.
\end{aligned}$$

Notice that

$$(\bar{u} - \tilde{u}(\bar{\theta}))L(\bar{u},\bar{\theta}) = \cos(\bar{\theta})\big(\tilde{y}_v(\bar{u},0) - \tilde{y}_v(\tilde{u}(\bar{\theta}),0)\big) - \sin(\bar{\theta})\big(\tilde{x}_v(\bar{u},0) - \tilde{x}_v(\tilde{u}(\bar{\theta}),0)\big)$$

and so, differentating with respect to $\bar{u}$ and evaluating at $\tilde{u}(\bar{\theta})$, we obtain

$$\begin{aligned}
L(\tilde{u}(\bar{\theta}),\bar{\theta}) &= \cos(\bar{\theta})\tilde{y}_{uv}(\tilde{u}(\bar{\theta}),0) - \sin(\bar{\theta})\tilde{x}_{uv}(\tilde{u}(\bar{\theta}),0) \\
&= \frac{1}{|\tilde{z}_u(\tilde{u}(\bar{\theta}),0)|}\Big[\tilde{x}_v(\tilde{u}(\bar{\theta}),0)\tilde{y}_{uv}(\tilde{u}(\bar{\theta}),0) - \tilde{y}_v(\tilde{u}(\bar{\theta}),0)\tilde{x}_{uv}(\tilde{u}(\bar{\theta}),0)\Big].
\end{aligned}$$

On the other hand,

$$\tan(\bar\theta(\bar u)) = \frac{\tilde y_v(\bar u,0)}{\tilde x_v(\bar u,0)}$$

and so

$$\sec^2(\bar\theta(\bar u))\frac{d\bar\theta}{d\bar u} = \frac{\tilde x_v(\bar u,0)\tilde y_{uv}(\bar u,0) - \tilde y_v(\bar u,0)\tilde x_{uv}(\bar u,0)}{(\tilde x_v(\bar u,0))^2}.$$

Thus

$$L(\tilde u(\bar\theta),\bar\theta) = \sec^2(\bar\theta)\frac{(\tilde x_v(\tilde u(\bar\theta),0))^2}{|z_u(\tilde u(\bar\theta),0)|}\frac{d\bar\theta}{d\bar u}\bigg|_{\tilde u(\bar\theta)} = |z_u(\tilde u(\bar\theta),0)|\frac{d\bar\theta}{d\bar u} > 0.$$

Hence $L(\bar u,\bar\theta) > 0$ for $\bar u \in (-1,1)$. Now (6.12) implies

$$L(\bar u,\bar\theta)(\bar u - \tilde u(\bar\theta))\bar v = -Q(\bar u,\bar v,\bar\theta)$$

where

$$Q(\bar u,\bar v,\bar\theta) = \sum_{n=3}^{\infty}\sum_{k=1}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\big(\cos(\bar\theta)f_n - \sin(\bar\theta)e_n\big)(\bar u)^{n-(2k+1)}(\bar v)^{2k+1}.$$

Therefore, if

$$p(\bar u) = \frac{\sum_{n=2}^{\infty}\binom{n}{3}\big(\cos(\bar\theta)f_n - \sin(\bar\theta)e_n\big)(\bar u)^{n-3}}{L(\bar u,\bar\theta)}$$

we obtain

$$\bar u - \tilde u(\bar\theta) = p(\bar u)(\bar v)^2 + O((\bar v)^4).$$

Now

$$p(\bar u) = p(\tilde u(\bar\theta) + p(\bar u)\bar v^2 + O((\bar v)^4)) = p(\tilde u(\bar\theta)) + O(\bar v^2)$$

and so $\bar u - \tilde u(\bar\theta) = p(\tilde u(\bar\theta))\bar v^2 + O(\bar v^4)$. Thus Step 4 is proved ∎

Step 5. *Let $(x,y) \in \Omega$ tend to $(0,0)$ in such a manner that*

$$\liminf_{(x,y)\to(0,0)} \theta(x,y) > \theta_1 + \pi \qquad \textit{and} \qquad \limsup_{(x,y)\to(0,0)} \theta(x,y) < \beta_0.$$

Then

$$f(x,y) = Rf(\theta(x,y)) + \bar x^2 h(\theta(x,y)) + O(\bar x^3).$$

Proof. Notice that we are assuming

$$\liminf_{(x,y)\to(0,0)} u(x,y) > 0 \qquad \text{and} \qquad \limsup_{(x,y)\to(0,0)} u(x,y) < 1.$$

For $(x,y) \in \Omega$, denote $\tilde{u}(\bar{x},\bar{y}) = u(x,y)$, $\tilde{v}(\bar{x},\bar{y}) = v(x,y)$ and $\tilde{\theta}(\bar{x},\bar{y})$ by $\bar{u},\bar{v}$ and $\bar{\theta}$, respectively. Then

$$\begin{aligned} f(x,y) &= \tilde{f}(\bar{x},\bar{y}) \\ &= z(\bar{u},\bar{v}) \\ &= c_0 + \sum_{n=2}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \binom{n}{2k} c_n(\bar{u})^{n-2k}(\bar{v})^{2k} \\ &= c_0 + \sum_{n=2}^{\infty}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k \binom{n}{2k} c_n\big(\tilde{u}(\bar{\theta}) + p(\tilde{u}(\bar{\theta}))\bar{v}^2 + O(\bar{v}^4)\big)^{n-2k}(\bar{v})^{2k} \\ &= R\tilde{f}(\bar{\theta}) + (\bar{v})^2\sum_{n=2}^{\infty} c_n(\tilde{u}(\bar{\theta}))^{n-2}\left[n\tilde{u}(\bar{\theta})p(\tilde{u}(\bar{\theta})) - \binom{n}{2}\right] + O(\bar{v}^4). \end{aligned} \tag{6.13}$$

From $(6.8)_1$ and (6.11) we have

$$\begin{aligned} M(\bar{u}) &= M(\tilde{u}(\bar{\theta}) + p(\tilde{u}(\bar{\theta}))\bar{v}^2 + O(\bar{v}^4)) \\ &= M(\tilde{u}(\bar{\theta})) + \bar{v}^2\sum_{n=3}^{\infty} n(n-2)e_n(\tilde{u}(\bar{\theta}))^{n-3}p(\tilde{u}(\bar{\theta})) + O(\bar{v}^4) \\ &= M(\tilde{u}(\bar{\theta})) + \bar{v}^2 p(\tilde{u}(\bar{\theta}))M_1(\tilde{u}(\bar{\theta})) + O(\bar{v}^4) \end{aligned}$$

where

$$M_1(\bar{u}) = \sum_{n=3}^{\infty} n(n-2)e_n\bar{u}^{n-3}.$$

From $(6.7)_1$ and (6.11) we have

$$\bar{x} = M(\tilde{u}(\bar{\theta}))\tilde{u}(\bar{\theta})\bar{v} + M_1(\tilde{u}(\bar{\theta}))p(\tilde{u}(\bar{\theta}))\tilde{u}(\bar{\theta})\bar{v}^3 + O(\bar{v}^5)$$

and so

$$\bar{v} = \frac{\bar{x}}{M(\tilde{u}(\bar{\theta}))\tilde{u}(\bar{\theta})} + O(\bar{v}^2\bar{x}) = \bar{x}M(\tilde{u}(\bar{\theta}))\tilde{u}(\bar{\theta}) + O(\bar{x}^3).$$

Therefore, since $p(\bar{u}) = P(\theta)$, (6.13) implies

$$\begin{aligned} f(x,y) &= R\tilde{f}(\bar{\theta}) + \bar{x}^2\frac{\sum_{n=2}^{\infty} c_n(\tilde{u}(\bar{\theta}))^{n-2}\left[n\tilde{u}(\bar{\theta})p(\tilde{u}(\bar{\theta})) - \binom{n}{2}\right]}{\big(\tilde{u}(\bar{\theta})M(\theta u(\bar{\theta}))\big)^2} + O(\bar{x}^4) \\ &= Rf(\theta(x,y)) + \bar{x}^2 H(\theta(x,y)) + O(\bar{x}^4). \end{aligned}$$

Thus step 5 is proved ∎

Step 6. *The case in which $(x,y) \in \Omega$ tends to $(0,0)$ in such a manner that*

$$\liminf_{(x,y)\to(0,0)} \theta(x,y) > \alpha_0 \qquad \text{and} \qquad \limsup_{(x,y)\to(0,0)} \theta(x,y) < \theta_0$$

is essentially the same as Steps 4 *and* 5.

Remark 5. For numerical purposes, such as in [16], it would be useful to know that the series representations (6.3) and (6.4) converge on $0 \le \rho \le 1$ and $0 \le t \le 2\pi$. As an example, assume that $\partial\Omega \setminus \{P\}$ and ϕ are smooth. Then x^* is smooth on $(0,\pi) \cup (\pi, 2\pi)$ (see, e.g., [15: Subsection 349]) and hence (6.2) converges to $x^*(t)$ for each $t \in (0,\pi) \cup (\pi, 2\pi)$. Since $x^*(t) = 0$ and the series (6.2) converges to 0 when $t = 0$, $t = \pi$ or $t = 2\pi$, we see that (6.3) converges to $x(\rho\cos t, \rho\sin t)$ on $0 \le \rho \le 1$ and $0 \le t \le 2\pi$. A similar argument holds for y and z.

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