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Heat Semi-Group and Function Spaces on Symmetric Spaces of Non-Compact Type

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Abstract. Besov-Triebel scales of function spaces defined on symmetric spaces of non-compact type are investigated. We prove an atomic decomposition theorem for the function spaces and give their characterization in terms of heat semigroup. In consequence we can describe the spectrum of the Laplace-Beltrami operator in these spaces and improve the generalized Riemann-Lebesgue lemma for the spherical Fourier transform.

Keywords: *Function spaces, symmetric spaces, heat semigroup, atoms*

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1. Preliminaries

We use standard notation and refer to [10, 12] for more details. Let $X = G/K$ be a Riemannian symmetric space of non-compact type. For convenience we will use the name Riemannian symmetric *manifold* instead of Riemannian symmetric space and reserve the word "space" for function spaces. The basis of harmonic analysis on X was settled around the sixties, mainly by Harish-Chandra and S. Helgason. The basic facts are the followings ones:

- One has a *Fourier transform* (or *Helgason-Fourier transform*)

$$\mathcal{H}f(\lambda, kM) = \int_G f(g) e^{-(\sqrt{-1}\lambda+\rho)H(g^{-1}k)} dg. \tag{1}$$

- One knows its behaviour on $L^2(X)$ (*Plancherel theorem*) and on $C_0^\infty(X)$ (*Paley-Wiener theorem*).
- One has an *inversion formula*

$$f(g) = \text{const} \int_{\mathfrak{a}^* \times K/M} \mathcal{H}f(\lambda, kM) e^{(\sqrt{-1}\lambda-\rho)H(g^{-1}k)} |c(\lambda)|^{-2} d\lambda dkM. \tag{2}$$

- The Laplacian Δ on X transforms under $\mathcal{H}$ as

$$\mathcal{H}\Delta f(\lambda, kM) = -(|\lambda|^2 + |\rho|^2)\mathcal{H}(f)(\lambda, kM). \tag{3}$$

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- For bi-K-invariant functions on G formulas (1) - (2) reduce to

$$\mathcal{H}f(\lambda) = \int_X f(x)\varphi_{-\lambda}(x)\,dx \tag{4}$$

and

$$f(x) = \text{const} \int_{\mathfrak{a}^*} \mathcal{H}f(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda \tag{5}$$

involving the *elementary spherical function*

$$\varphi_\lambda(gK) = \int_K e^{(\sqrt{-1}\lambda-\rho)H(gk)}dk \qquad (\lambda \in \mathfrak{a}_C^\star). \tag{6}$$

One knows also the behaviour of the Helgason-Fourier transform on more complicated objects, like L_p-Schwartz spaces $\mathcal{C}_p(X)$ $(0 < p \leq 2)$. The last are defined as

$$\mathcal{C}_p(X) = \left\{ f \in C^\infty(X) \,\middle|\, \begin{array}{l} D_1, D_2 \in U(\mathfrak{g}, r \geq 0) \\ \sup\limits_{\substack{k_1,k_2 \in K \\ H \in \mathfrak{a}}} \langle H\rangle^r \varphi_0^{-\frac{2}{p}}(e^H)|f(D_1 : k_1(e^H)k_2 : D_2)| < \infty \end{array} \right\}$$

where $f(D_1 : k_1(e^H)k_2 : D_2)$ denotes the natural action of $D_1, D_2 \in U(\mathfrak{g})$ (the universal enveloping algebra of $\mathfrak{g}$) on $f \in C^\infty(G)$ and $\langle H\rangle = (1 + |H|^2)^{\frac{1}{2}}$. For the description of the Fourier image $\mathcal{Z}_p(\mathfrak{a}^* \times B)$ of $\mathcal{C}_p(X)$ we refer to [2, 7]. For convenience we put $\mathcal{C}_p(X) = \mathcal{C}_2(X)$ if $p > 2$.

The best general references for Besov-Triebel scales on $\mathbb{R}^n$ are Triebel's books [24, 25]. In the second of them one can also find the definition of scales on complete Riemannian manifolds with bounded geometry via the uniform localization principle. Properties of the spaces as well as further references can be found there. In [17, 19] we have defined Besov-Triebel scales on symmetric spaces of non-compact type X for the Helgason-Fourier transform. In contrast to the Triebel method our approach is global (cf. also [15]). The definition is based on the construction of a continuous resolution of unity on $\mathfrak{a}^*$

$$\mathcal{H}k_{o,N} + \int_0^1 (\mathcal{H}k^N)^2(t\lambda)\frac{dt}{t} = 1 \tag{7}$$

where $k_{o,N}$ and k are bi-K-invariant test functions in X supported in the unit ball centered at the origin $o = eK$ of X and where $k^N = (-\Delta - |\rho|)^N k$ $(N \in \mathbb{N})$. We refer to [17, 18] for details. Using this resolution of unity we get the formula of Calderon type

$$f = f \star k_{o,N} + \int_0^1 f \star k_t^N \star k_t^N \frac{dt}{t} \qquad (f \in \mathcal{C}_p'(X),\ 1 \leq p \leq 2) \tag{8}$$

in which the convergence of the integral is understood in weak $\mathcal{C}_p'(X)$-sense (cf. [19]).

Definition 1. Let $s \in \mathbb{R}$, $N \in \mathbb{N}$ such that $2N > |s|$, $1 < p < \infty$ and $1 < q \leq \infty$ ($1 < p \leq \infty$ and $1 < q \leq \infty$ for Besov spaces). Then

$$\mathcal{F}^s_{p,q}(X) = \left\{ f \in \mathcal{C}_1' \;\middle|\; \begin{array}{l} \|f|\mathcal{F}^s_{p,q}(X)\|^{\{k^N\}} = \\ \|f \star k_{o,N}\|_p + \left\| \left(\int_0^1 t^{-sq} |f \star k_t^N(\cdot)|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p < \infty \end{array} \right\}$$

and

$$\mathcal{B}^s_{p,q}(X) = \left\{ f \in \mathcal{C}_1' \;\middle|\; \begin{array}{l} \|f|\mathcal{B}^s_{p,q}(X)\|^{\{k^N\}} = \\ \|f \star k_{0,N}\|_p + \left(\int_0^1 t^{-sq} \|f \star k_t^N | L_p(X)\|^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \end{array} \right\}$$

with usual modification if $q = \infty$.

Remark 1. Definition 1 is independent of the given resolution of unity. By H. Triebel [23], the spaces $\mathcal{F}^s_{p,q}(X)$ coincide with the spaces $F^s_{p,q}(X)$ defined on Riemannian manifolds with bounded geometry via uniform localization (cf. also [16, 17]). In [15, 19] the atomic decompositions of the above spaces is given.

2. Heat kernel on symmetric manifolds

The heat kernel on Riemannian manifolds was a subject of intensive study during the last decades (cf. [4, 6]). Let us recall basic facts. The heat semigroup $H_t = e^{t\Delta}$ $(t > 0)$ on $X = G/K$ is a positive symmetric diffusion semigroup satisfying the conservation property. It is realized by convolution on the right with heat kernel h_t being a positive bi-K-invariant Schwartz function on G with Fourier and Abel transforms

$$\mathcal{H}h_t(\lambda) = e^{-t(|\lambda|^2 + |\rho|^2)} \qquad \text{and} \qquad \mathcal{A}h_t(H) = \text{const} \cdot t^{-\frac{\alpha}{2}} e^{-|\rho|^2 t} e^{-\frac{|H|^2}{4t}},$$

respectively, where $\alpha = \dim \mathfrak{a}$, with $\mathfrak{a}$ the abelian subspace in the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. The function $\mathcal{H}h_t$ can be extended to an entire analytical function on $\mathfrak{a}^*_{\mathbb{C}}$ with polynomial growth in the tube $\mathcal{T}_1$. Thus $h_t \in \mathcal{C}_1(X)$ for every $t > 0$. We have good pointwise estimates for the heat kernel due to J.-Ph. Anker and others (cf. [3]). In our paper we use mainly estimates for $0 < t < t_o$, so we recall them here. For that we put

$$h_t^m(x) = \left(\frac{\partial}{\partial t} \right)^m h_t(x) \qquad (m \in \mathbb{N}_0). \tag{9}$$

Lemma 1. *Let $0 < t < t_o$ and $H \in \bar{\mathfrak{a}}_+$. Then there is a constant $C > 0$ depending on t_o such that the following assertions are true:*

1. *The inequality*

$$|h_t^m(e^H)| \leq C\, e^{-|\rho|^2 t - \rho(H) - \frac{|H|^2}{4t}} t^{\frac{n}{2} - 2m} \langle H \rangle^{n-\alpha} \sum_{l=0}^{m} t^l |H|^{2m-2l} \tag{10}$$

holds for every $m \in \mathbb{N}_0$.

2. *The inequality*

$$|\nabla^j h_t(e^H)| \le C\, t^{\frac{n}{2}-j} \langle H\rangle^{j+n-\alpha} e^{-|\rho|^2 t - \rho(H) - \frac{|H|^2}{4t}} \tag{11}$$

holds for any $j \in \mathbb{N}$.

3. *The inequality*

$$|h_t(e^H)| \ge C\, t^{-\frac{n}{2}} e^{-\frac{|H|^2}{4t}} \tag{12}$$

holds.

For the proof of inequality (10) we refer to [1, 3]. Inequality (11) is classical for the heat kernel on Riemannian manifolds if $|H|$ is bounded (see, e.g. [3: Formula (3.10)]). For $|H|$ large one can use the Flensted-Jensen reduction to the complex case (cf. [1]). The last inequality (12) follows from the Li-Yau-Harnack inequality [6].

We finish this section by proving a vector-valued local maximal inequality for the heat semigroup. We will denote by $M_o f$ the local Hardy-Littlewood maximal function, i.e. with supremum restricted to the balls of radius $0 < r \le 1$. The following lemma was proved in [19].

Lemma 2. *Let* $1 < p < \infty$, *let* $\{S_j\}_{j \ge 1}$ *be a family of subadditive operators defined in the space of locally integrable functions, let* h *be a real-valued non-negative bi-*K*-invariant function such that* $h(y^{-1} \cdot o) = h(y \cdot o)$ *for any* $y \in G$, *and let the convolution operator* $T(f) = f \star h$ *defined by* h *be of weak* 1-1 *type and of strong* p-p *type. We assume that there is a constant* $C > 0$ *independent of* j *such that the inequality*

$$|S_j f(x)| \le C\big((M_o|f|)(x) + T(|f|)(x)\big) \tag{13}$$

holds for a.e. $x \in X$. *Then the inequality*

$$\mathrm{vol}\left\{ x \in X \,\middle|\, \left(\sum_{j=1}^{\infty} |S_j f_j(x)|^q\right)^{\frac{1}{q}} > \lambda \right\} \le \frac{C_q}{\lambda} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \right\|_1 \tag{14}$$

holds for $1 < q < \infty$. *Moreover, if* $1 < q < \infty$, *then*

$$\left\| \left(\sum_{j=1}^{\infty} |S_j f_j|^q\right)^{\frac{1}{q}} \right\|_p \le C_{p,q} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \right\|_p \tag{15}$$

provided there is a constant $C > 0$ *independent of* j *such that*

$$\|S_j f\|_\infty \le C\, \|f\|_\infty. \tag{16}$$

Proposition 1. *Let* $1 < p < \infty$, $1 < q \le \infty$ *and*

$$H_t^m(f)(x) = t^m \big| |h_t^m| \star f(x) \big| \qquad (m \in \mathbb{N}_0). \tag{17}$$

Then the inequalities

$$\mathrm{vol}\left\{x \in X \,\middle|\, \left(\sum_{j=1}^{\infty}\left|\sup_{0<t\leq 1}\{H_t^m f_j(x)\}\right|^q\right)^{\frac{1}{q}} > \lambda\right\} \leq \frac{C_{q,m}}{\lambda}\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{\frac{1}{q}}\right\|_1 \tag{18}$$

and

$$\left\|\left(\sum_{j=1}^{\infty}\left|\sup_{0<t\leq 1}\{H_t^m f_j(\cdot)\}\right|^q\right)^{\frac{1}{q}}\right\|_p \leq C_{p,q,m}\left\|\left(\sum_{j=1}^{\infty}|f_j|^q\right)^{\frac{1}{q}}\right\|_p \tag{19}$$

hold for any number $m \in \mathbb{N}_0$ and any sequence $\{f_j\}_{j\geq 1}$ of locally integrable functions on X.

Proof. According to Lemma 1 it is sufficient to prove that there is an integrable function h on X such that the inequality

$$|H_t^m f(x)| \leq C\big((M_o|f|)(x) + h \star (|f|)(x)\big) \tag{20}$$

holds for a.e. $x \in X$. We divide $H^m f$ into the sum of two integrals

$$(H_t^m f)(x) = \int_{\Omega(o,1)} t^m |h_t^m(y)| f(y^{-1}x)\, dy + \int_{X\setminus\Omega(o,1)} t^m |h_t^m(y)| f(y^{-1}x)\, dy.$$

To estimate the first integral it is sufficient to use inequality (10) which in case $|H| \leq \sqrt{t}$ gives

$$|t^m h_t^m(e^H)| \leq C\, t^{-\frac{n}{2}}. \tag{21}$$

If $\sqrt{t} \leq |H| < 1$, then inequality (10) gives

$$|t^m h_t^m(e^H)| \leq C\, t^{-\frac{n}{2}}\left(\frac{|H|}{\sqrt{t}}\right)^{2m} e^{-\frac{1}{4}(\frac{|H|}{\sqrt{t}})^2} \leq C\, t^{-\frac{n}{2}}. \tag{22}$$

The last two estimates give

$$\left|\int_{\Omega(o,1)} t^m h_t^m(y) f(y^{-1}x)\, dy\right| \leq C\,(M_o|f|)(x).$$

On the other hand, for $|H| \geq 1$ inequality (10) implies

$$|t^m h_t^m e^H| \leq C\, e^{-\rho(H)} e^{-\frac{|H|^2}{8}} \langle H\rangle^{2m+n-\alpha}$$

and the expression on the right defines in usual way a bi-K-invariant integrable function on X. This proves the proposition ∎

Corollary 1. *Let $1 < p < \infty$ and $1 < q \leq \infty$. Then the inequalities*

$$\operatorname{vol}\left\{x \in X \,\middle|\, \left(\sum_{j=1}^{\infty} \left|\sup_{0<t\leq 1} \left\{t^m \left(\frac{d}{dt}\right)^m h_t \star f(x)\right\}\right|^q\right)^{\frac{1}{q}} > \lambda\right\} \leq \frac{C_{q,m}}{\lambda} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \right\|_1 \tag{23}$$

and

$$\left\| \left(\sum_{j=1}^{\infty} \left|\sup_{0<t\leq 1} \left\{t^m \left(\frac{d}{dt}\right)^m h_t \star f(x)\right\}\right|^q\right)^{\frac{1}{q}} \right\|_p \leq C_{p,q,m} \left\| \left(\sum_{j=1}^{\infty} |f_j|^q\right)^{\frac{1}{q}} \right\|_p \tag{24}$$

hold for any number $m \in \mathbb{N}_0$ and any sequence $\{f_j\}_{j\geq 1}$ of locally integrable functions on X.

3. Heat-extension norms and atomic decomposition

We start with the following standard observation, which is crucial for the paper. Let $f \in \mathcal{C}_2(X)$. The operator norm of $H_t : L_2 \mapsto L_\infty$ satisfies the estimate

$$\|H_t\|_{2,\infty} \sim t^{-\nu} e^{-|\rho|^2 t}$$

for every $t \in [1, \infty)$ where ν is a positive constant (cf. [5]). Using this estimate we get easily that

$$t^l \left(\frac{d}{dt}\right)^l h_t \star f(x) = t^l H_t \Delta^m f(x) \to 0$$

for every $x \in X$ if $t \to \infty$. Moreover, if $f \in \mathcal{C}_p(X)$ and $t \to 0$, then

$$t^l \left(\frac{d}{dt}\right)^l h_t \star f \to f \tag{25}$$

in $\mathcal{C}_p(X)$ (this may be checked for the Fourier image by direct calculations in $\mathcal{Z}_p(\mathfrak{a}^* \times B)$ and then the convergence follows by Eguchi's result [7]). Integrating by parts we get

$$\int_0^\infty t^k \left(\frac{d^k}{dt^k} H_t f\right) \frac{dt}{t} = (k-1) \int_0^\infty t^{k-2} \left(\frac{d^{k-2}}{dt^{k-2}} H_t f\right) \frac{dt}{t} = \ \dots \ = cf(x).$$

Thus

$$f(x) = C\left(h_{m,o} \star f + \int_0^1 t^k \frac{d^k}{dt^k} H_t f \frac{dt}{t}\right) \tag{26}$$

if $f \in C_p(X)$ where $h_{m,o} = \sum_{l=0}^{m-1} h_1^l$. Generalizing (25) we get for $t \to t_o \in (0,1]$ the convergence

$$t^l \left(\frac{d}{dt}\right)^l h_t \star f \to t_o^l \left(\frac{d}{dt}\right)^l h_{t_o} \star f$$

in $C_p(X)$. As a consequence (26) is true for every $f \in C_p'(X)$ if the integral convergence is understood in weak sense.

It will be convenient to introduce the following function spaces.

Definition 2. Let $s \in \mathbb{R}$, m a non-negative integer with $m > \frac{s}{2}$, $1 < p < \infty$ and $1 < q \leq \infty$. Then

$$\mathcal{F}_{p,q}^{s,m}(X) = \left\{ f \in C_p'(X) \;\middle|\; \begin{array}{l} \|f|F_{p,q}^{m,s}\| = \|f \star h_{0,m}\|_p + \\ \left\| \left(\int_0^1 t^{(m-\frac{s}{2})q} \left|\frac{d^m}{dt^m} H_t f(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}} \right\|_p < \infty \end{array} \right\}$$

and

$$\mathcal{B}_{p,q}^{s,m}(X) = \left\{ f \in C_p'(X) \;\middle|\; \begin{array}{l} \|f|B_{p,q}^{m,s}\| = \|f \star h_{0,m}\|_p + \\ \left(\int_0^1 t^{(m-\frac{s}{2})q} \left\|\frac{d^m}{dt^m} H_t f\right\|^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty \end{array} \right\}$$

with usual modification if $q = \infty$.

The above spaces are normed spaces. It will be proved that their definition is independent of m up to norm equivalence. Moreover, if $s > 0$, then the first term of the norms can be replaced by $\|f\|$.

For convenience we recall the definition and basic properties of atoms that we shall use.

Definition 3. Let $\Omega = \Omega(x,r)$ with $0 < r \leq 1$ be a geodesic ball in X, $s \in \mathbb{R}, 1 \leq p \leq \infty$, and let L and M be integers with

$$L \geq ([s]+1)_+ \qquad \text{and} \qquad M \geq \max([-s], -1) \tag{27}$$

where $(t)_+ = \max\{0, t\}$. A smooth function a is called an

a) *s-atom* centered in Ω if

$$\operatorname{supp} a \subset \Omega(x, 2r) \tag{28}$$

$$\sup_{x \in X} \{|(\Gamma^l a)(x)|\} \leq 1 \text{ for any } l \leq L. \tag{29}$$

b) (s,p)-*atom* centered in Ω if

$$\operatorname{supp} a \subset \Omega(x,2r) \tag{30}$$

$$\sup_{x\in X}\{|(\Gamma^l a)(x)|\} \le r^{s-2l-\frac{n}{p}} \text{ for any } l \le L. \tag{31}$$

$$D^\beta(\mathcal{H}a)(0,b) = 0 \text{ for any } |\beta| \le M \text{ and } b \in B = K/M. \tag{32}$$

If $M = -1$, then (32) means that no moment conditions are required.

The next lemma is a simple consequence of Definition 3 and the formula

$$A(g\cdot x, g(b)) = A(x,b) + A(g\cdot o, g(b)) \qquad (g\in G,\, b\in B,\, o = eK) \tag{33}$$

(cf. [13]).

Lemma 3 (cf. [15]). *Let a be an s-atom or (s,p)-atom centered at $\Omega(x,r)$. Then the function a_g $(g \in G)$ defined by $a_g(x) = a(g^{-1}x)$ is an s-atom or (s,p)-atom, respectively, centered at $\Omega(g\cdot x, r)$.*

The atomic decomposition with $p > 1$ requires a rigid control of the location of the support of the atom, therefore we need some coverings of the manifold X. Let $\{r_j\}_{j\ge 0}$ be a sequence of positive numbers decreasing to zero and let $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i=1}^\infty$ be a uniformly locally finite covering of X by balls of radius r_j. The sequence $\{\Omega_j\}_{j\ge 0}$ of coverings is called *uniformly locally finite* if there is a constant $C > 0$ such that for every $j \in \mathbb{N}$ any $x \in X$ is an element of at most C balls of the covering Ω_j.

Lemma 4 (cf. [15]). *Let X be a symmetric manifold of non-compact type. There is a uniformly locally finite sequence $\{\Omega_j\}_{j\ge 0}$ of coverings of X by geodesic balls $\Omega_j = \{\Omega(x_{j,i}, r_j)\}_{i\ge 0}$ of radius r_j. Moreover, if $l \in \mathbb{N}$ and $\Omega_{j,l} = \{\Omega(x_{j,i}, lr_j)\}_{i\ge 1}$, then the sequence $\{\Omega_{j,l}\}_{j\ge 0}$ is also uniformly locally finite.*

Let $\chi_{j,i}$ denote the characteristic function of the ball $\Omega(x_{j,i}, 2^{-j})$ and $\chi_{j,i}^{(p)} = 2^{\frac{jn}{p}}\chi_{j,i}$. Then $\|\chi_{j,i}^{(p)}\| \approx C$ for any $j,i \ge 0$.

Theorem 1. *Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ with $m > \frac{s}{2}$, $1 < p < \infty$ and $1 < q \le \infty$, L and M fixed integers satisfying (27). Let $\{\Omega_j\}_{j=0}^\infty$ with $\Omega_j = \{\Omega(x_{j,i}, 2^{-j})\}$ be a uniformly locally finite sequence of coverings of X.*

a) *Each f in $\mathcal{F}^{s,m}_{p,q}(X)$ or $B^{s,m}_{p,q}(X)$ can be decomposed as*

$$f = \sum_{i\in\mathbb{N}} s_i a_i + \sum_{j=0}^\infty \sum_{i\in\mathbb{N}} s_{j,i}\, a_{j,i} \qquad \big(\textit{convergence in } C_p'(X)\big) \tag{34}$$

where a_i is an s-atom related to the ball $\Omega(x_{1,i},1)$, $a_{j,i}$ is an (s,p)-atom related to the ball $\Omega(x_{j,i},2^{-j})$, s_i and $s_{j,i}$ are complex numbers with

$$\Big(\sum_{i\in\mathbb{N}} |s_i|^p\Big)^{\frac{1}{p}} + \left\| \Big(\sum_{j,i=0}^{\infty} \big(|s_{j,i}|\chi_{j,i}^{(p)}(\cdot)\big)^q\Big)^{\frac{1}{q}} \right\|_p < \infty \tag{35}$$

or

$$\left(\sum_{i\in\mathbb{N}} |s_i|^p\right)^{\frac{1}{p}} + \left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty} |s_{j,i}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty, \tag{36}$$

respectively.

b) *Conversely, suppose that $f \in C_p(X)$ can be represented as in* (34) *or* (35). *Then f is in $\mathcal{F}^{s,m}_{p,q}(X)$ or in $\mathcal{B}^{s,m}_{p,q}(X)$, respectively. Furthermore, the infimum of* (35) *with respect to all admissible representations (for fixed sequences of coverings and fixed integers L and M) is an equivalent norm in $\mathcal{F}^{s,m}_{p,q}(X)$ or $\mathcal{B}^{s,m}_{p,q}(X)$, respectively.*

We prove Theorem 1 in the last Section 5. The following corollary is an immediate consequence of Theorem 1 and the theorem in [19].

Corollary 2. *Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ with $m > \frac{s}{2}$, $1 < p < \infty$ and $1 < q \leq \infty$. Then*

$$\mathcal{F}^{m,s}_{p,q}(X) = \mathcal{F}^{s}_{p,q}(X) \qquad \text{and} \qquad \mathcal{B}^{m,s}_{p,q}(X) = \mathcal{B}^{s}_{p,q}(X)$$

in the sense of norm equivalence.

We have also the following discrete version of the norms.

Corollary 3. *Let $s \in \mathbb{R}$, $m \in \mathbb{N}_0$ with $m > \frac{s}{2}$, $1 < p < \infty$ and $1 < q \leq \infty$. Then*

$$\|f \star h_{0,m}\|_p + \left\| \left(\sum_{j=0}^{\infty} 2^{j(\frac{s}{2}-m)q} |h^m_{2^{-j}} f(\cdot)|^q\right)^{\frac{1}{q}} \right\|_p$$

and

$$\|f \star h_{0,m}\|_p + \left(\sum_{j=0}^{\infty} 2^{j(\frac{s}{2}-m)q} \|h^m_{2^{-j}} f\|^q\right)^{\frac{1}{q}}$$

are equivalent norms in $\mathcal{F}^s_{p,q}(X)$ and $\mathcal{B}^s_{p,q}(X)$, respectively.

The last corollary follows from Proposition 1 and (12) by standard calculations.

4. Some applications

The spectrum of the Laplacian Δ in $L_p(X)$ $(1 \leq p \leq \infty)$ was described precisely by M. Taylor in [20]. The L_p-spectrum of Δ is the "parabolic neighbourhood"

$$\mathcal{P}_p = \left\{ z^2 - |\rho|^2 : 0 \leq \operatorname{Re} z \leq \left|\frac{2}{p} - 1\right| |\rho| \quad (z \in \mathbb{C}\right\}$$

of the half line $(-\infty, -|\rho|^2]$. This is a consequence of the formula

$$\Delta\varphi_\lambda = -(\langle\lambda,\lambda\rangle + |\rho|^2)\varphi_\lambda$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form induced on $\mathfrak{a}^*_{\mathbb{C}}$ by the scalar product in $\mathfrak{a}^*$. We prove that the spectrum of Δ in $\mathcal{F}^s_{p,q}(X)$ and $\mathcal{B}^s_{p,q}(X)$ is exactly the same, so it is (s,q)-independent.

Lemma 5. *Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 < q \le \infty$. Then the following assertions are true:*

a) *$\varphi_\lambda \in \mathcal{F}^s_{p,q}(X)$ if and only if $\varphi_\lambda \in L_p(X)$ and*

$$\|\varphi_{\mu+i\nu}|\mathcal{F}^s_{p,q}(X)\| \sim C(q,m)\left(1 + \left(\langle\mu,\mu\rangle - \langle\nu,\nu\rangle + |\rho|^2\right)^{\frac{s}{2}}\right)\|\varphi_{\mu+i\nu}\|_p.$$

b) *$\varphi_\lambda \in \mathcal{B}^s_{p,q}(X)$ if and only if $\varphi_\lambda \in L_p(X)$ and*

$$\|\varphi_{\mu+i\nu}|\mathcal{B}^s_{p,q}(X)\| \sim C(q,m)\left(1 + \left(\langle\mu,\mu\rangle - \langle\nu,\nu\rangle + |\rho|^2\right)^{\frac{s}{2}}\right)\|\varphi_{\mu+i\nu}\|_p.$$

Proof. Since $h_t \star \varphi_\lambda = e^{-t(\langle\lambda,\lambda\rangle+|\rho|^2)}\varphi_\lambda$ we have

$$\left\|\left(\int_0^1 t^{(m-\frac{s}{2})q}\left|\frac{d^m}{dt^m}h_t \star \varphi_\lambda(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}}\right\|_p$$

$$\sim C\left(\langle\mu,\mu\rangle - \langle\nu,\nu\rangle + |\rho|^2\right)^{\frac{s}{2}}\left(\int_0^\infty t^{(m-\frac{s}{2})q}e^{-tq}\frac{dt}{t}\right)^{\frac{1}{q}}\|\varphi\|_\lambda.$$

if $\lambda = \mu + i\nu$. On the other hand

$$\varphi_\lambda \star h_{0,m} = \sum_{l=0}^{m-1}\varphi_\lambda \star h_1^l = \sum_{l=0}^{m-1}(-1)^l\left(\langle\lambda,\lambda\rangle + |\rho|^2\right)^l e^{-(\langle\lambda,\lambda\rangle+|\rho|^2)}\varphi_\lambda$$

and the proof is finished ∎

Theorem 2. *Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 < q \le \infty$. Then the spectrum of the Laplace operator Δ in $\mathcal{F}^s_{p,q}(X)$ and $\mathcal{B}^s_{p,q}(X)$ is the same as in $L_p(X)$, i.e. it coincides with the parabolic region $\mathcal{P}_p$.*

Proof. Let $z \in \mathbb{C}$ be in the resolvent set of Δ in $L_p(X)$. Then $(zI - \Delta)^{-1}$ can be represented by the convolution kernel k_z. It was proved by J. Ph. Anker [3] that the kernel is a C^∞-function outside the origin and an integrable function at the origin if z is out of the L^2-spectrum of Δ. Moreover, it is in $L_p(X)$ away from the origin when $\operatorname{Re} z > |\frac{2}{p} - 1||\rho|$.

If z is out of the parabolic region $\mathcal{P}_p$, then there is an $r \in (1,2)$ such that $\operatorname{Re} z > |\frac{2}{r} - 1||\rho| > |\frac{2}{p} - 1||\rho|$ and $r < p < r'$. So k_z is in $L_r(X)$ away from the origin. Now dividing k_z into two parts we get by the Minkowski inequality and the Kunze-Stein phenomenon that z is in the resolvent set of the Laplacian in $\mathcal{F}^s_{p,q}(X)$.

If $p > 2$, then it follows from the above lemma that the interior of the parabolic region is the point spectrum of Δ in $\mathcal{F}^s_{p,q}(X)$.

Let $1 < p < 2$. Using the Calderon formula it is easy to see that the space $\mathcal{F}^{-s}_{p',q'}(X)$ is contained in the dual space $\mathcal{F}^s_{p,q}(X)'$. So φ_λ defines a continuous functional on $\mathcal{F}^s_{p,q}(X)$ if $s_\lambda = -(\langle\lambda,\lambda\rangle + |\rho|^2)$ is in the interior of $\mathcal{P}_p$. We assume that $s_\lambda I - \Delta$ is invertible in

$\mathcal{F}^s_{p,q}(X)$. Let ψ be a smooth function belonging to $\mathcal{F}^s_{p,q}(X)$. Then $(s_\lambda I - \Delta)^{-1}\psi$ is a smooth element of $\mathcal{F}^s_{p,q}(X)$ and

$$(\varphi_\lambda, \psi) = \left(\varphi_\lambda, (s_\lambda I - \Delta)(s_\lambda I - \Delta)^{-1}\psi\right) = \left((s_\lambda I - \Delta)\varphi_\lambda, (s_\lambda I - \Delta)^{-1}\psi\right) = 0.$$

Thus φ_λ defines the zero functional which is impossible. So, s_λ is an element of the spectrum.

If $p = 2$, then it follows immediately from the lift property that the spectrum of the Laplacian in $\mathcal{F}^s_{2,q}(X)$ is independent on s. If z is an element of the L_2-spectrum, then one can find by the inversion formula a function $\psi \in \mathcal{C}(X)$ such that ψ is not an element of the domain of $(zI - \Delta)^{-1}$. If z was in the resolvent of Δ in $\mathcal{F}^s_{2,q}(X)$ for some negative s, then $(zI - \Delta)^{-1}\psi$ would be in $L_2(X)$, which is impossible. This proves the theorem ∎

Now we improve the generalized Riemann-Lebesgue lemma for the spherical Fourier transform. To formulate the statement we need the following notation.

Let $\mathcal{W}_1$ be the interior of the convex hull in $\mathfrak{a}^*$ of the images of ρ under the Weyl group W. For $\delta \in (0,1)$, we denote by $\mathcal{W}_\delta$ the dilate of $\mathcal{W}_1$ by δ. For $1 < p < 2$, let $\mathcal{T}_p$ denote the tube $\mathcal{T}_p = \mathfrak{a}^* + \sqrt{-1}\mathcal{W}_\delta$ over the polygon $\mathcal{W}_\delta$ with $\delta = \frac{2}{p} - 1$. Let Σ°_+ be the set of indivisible positive roots and let $d_\alpha = \dim \mathfrak{g}_\alpha + \dim \mathfrak{g}_{2\alpha}$ where $\mathfrak{g}_\alpha$ is the root space corresponding to $\alpha \in \Sigma^\circ_+$.

Theorem 3. *Let $1 < p < 2$, $\gamma(X) = \min_{\alpha\in\Sigma^\circ_+} d_\alpha$ and $\mathcal{T} = \mathfrak{a}^* + \sqrt{-1}\mathcal{W}$ a closed subtube of $\mathcal{T}_p$. Then the following assertions are true.*

1. *The spherical transform is a continuous mapping from $\mathcal{B}^{-\frac{\gamma(X)}{p'}}_{p,\infty}(X)$ into $L_\infty(\mathcal{T})$ and $\mathcal{H}f(\lambda)$ is for any $f \in \mathcal{B}^{-\frac{\gamma(X)}{p'}}_{p,\infty}(X)$ a holomorphic function inside $\mathcal{T}$.*

2. *If $s > -\frac{\gamma(X)}{p'}$, then in addition $\lim_{|\mu|\to\infty} |\mathcal{H}f(\mu + i\nu)| = 0$ for any $f \in \mathcal{B}^s_{p,\infty}(X)$ and $\nu \in \mathcal{W}$.*

Proof. Let $\Gamma(\lambda) = \prod_{\alpha\in\Sigma^\circ_+}(1 + |\langle\alpha,\lambda\rangle|)^{d_\alpha}$. Then for every closed subtube $\mathcal{T}$ of $\mathcal{T}_p$ there is a constant $C > 0$ such that $\|\varphi_\lambda\| \le C\,\Gamma(\lambda)^{-\frac{1}{p'}}$ (cf. [5]). So the above inequality and Lemma 5 gives

$$\|\varphi_{\mu+i\nu}|\mathcal{B}^s_{p,1}(X)\| \le C\left(1 + \left(\langle\mu,\mu\rangle - \langle\nu,\nu\rangle + |\rho|^2\right)^{\frac{s}{2}}\right)\Gamma(\mu + i\nu)^{-\frac{1}{p'}} \tag{37}$$

if $\mu + i\nu \in \mathcal{T}$. But the last assumption implies

$$\langle\mu,\mu\rangle - \langle\nu,\nu\rangle + |\rho|^2 \sim 1 + \langle\mu,\mu\rangle \tag{38}$$

$$1 + |\langle\alpha, \mu + \sqrt{-1}\,\nu\rangle| \sim 1 + |\langle\alpha,\mu\rangle|. \tag{39}$$

Moreover, Σ°_+ is a reduced root system (cf. [11: Lemma 3.2]). So we can define a W-invariant positive defined inner product $(\cdot,\cdot)$ such that $(\lambda,\gamma) = \sum_{\alpha\in\Sigma^\circ_+}(\lambda,\alpha)(\alpha,\gamma)$ for $\lambda,\gamma \in \mathfrak{a}^*$ (cf. [11: Chapter X/B7]). Thus

$$1 + \langle\mu,\mu\rangle \sim 1 + \sum_{\alpha\in\Sigma^\circ_+}(\alpha,\mu)^2 \le C \prod_{\alpha\in\Sigma^\circ_+}(1 + |\langle\alpha,\mu\rangle|)^2 \tag{40}$$

and

$$\|\varphi_{\mu+i\nu}|\mathcal{B}^s_{p,1}(X)\| \le C \prod_{\alpha\in\Sigma^o_+} (1+|\langle\alpha,\mu\rangle|)^{s-\frac{d_\alpha}{p}}. \tag{41}$$

Now the simple estimate

$$|\mathcal{H}f(\mu+\sqrt{-1}\nu)| \le \|f|\mathcal{B}^s_{p,\infty}(X)\|\,\|\varphi_{\mu+i\nu}|\mathcal{B}^{-s}_{p',1}(X)\|$$

ends the proof of the assertions.∎

Remark. It follows from elementary embeddings that Theorem 3 is true for every $\mathcal{B}^s_{p,q}(X)$ and every $\mathcal{F}^s_{p,q}(X)$. In particular it is true for Sobolev spaces $H^s_p(X)$ with $s \ge -\frac{\gamma(X)}{p'}$ with improves the result of M. Eguchi and K. Kumahara [8]. Theorem 3 is also strictly connected with [5: Theorem 2.1/Part 2].

The next corollary is an immediate consequence of Theorem 3 and [5: Theorem 2.1/Part 1].

Corollary 4. *Let $1 < p < 2$ and $r > p'$. Then there is a constant $C > 0$ such that*

$$\left(\int_{\mathfrak{a}^*} |\mathcal{H}f(\mu+\sqrt{-1}\nu)|^r |c(\mu)|^{-2} d\mu\right)^{\frac{1}{r}} \le C \left\| f \left| B^{\gamma(X)(\frac{1}{r}-\frac{1}{p'})}_{p,r}(X)\right.\right\|$$

for $f \in B^{\gamma(X)(\frac{1}{r}-\frac{1}{p'})}_{p,r}(X)$ and any $\nu \in \mathcal{W}$.

5. Proof of Theorem 1

We prove Theorem 1 for the $\mathcal{F}^s_{p,q}$-scale. For Besov spaces its proof is similar. We divide our proof into several steps. First we prove the theorem for $s > 0$. The case $s \le 0$ will be regarded in the last step of the proof.

Step 1. Let

$$f = \sum_{i\in\mathbb{N}} s_i a_i + \sum_{j=0}^{\infty}\sum_{i\in N} s_{j,i} a_{j,i}$$

with

$$\left(\sum_{i\in N} |s_i|^p\right)^{\frac{1}{p}} + \left\| \left(\sum_{j,i=0}^{\infty} \left(|s_{j,i}|\chi^{(p)}_{j,i}(\cdot)\right)^q \right)^{\frac{1}{q}} \right\|_p < \infty.$$

Then

$$\begin{aligned}
&\|f|\mathcal{F}^{s,m}_{p,q}\| \\
&\le \left\|\sum_{i\in\mathbb{N}} s_i a_i \star h_{0,m}\right\|_p + \left\| \left(\int_0^1 t^{(m-\frac{s}{2})q} \left| \sum_{i=0}^{\infty} s_i h^m_t \star a_j(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}} \right\|_p \\
&\quad + \left\| \sum_{j,i=0}^{\infty} s_{j,i}\, h_{0,m} \star a_{j,i}\right\|_p + \left\| \left(\int_0^1 t^{(m-\frac{s}{2})q} \left| \sum_{j,i=0}^{\infty} s_{j,i}\, h^m_t \star a_{j,i}(\cdot)\right|^q \frac{dt}{t}\right)^{\frac{1}{q}} \right\|_p.
\end{aligned}$$

We estimate every summand separately. The inequality

$$\left\| \sum_{i\in N} s_i a_i \star h_{0,m} \right\|_p \le C \left(\sum_{i\in N} |s_i|^p \right)^{\frac{1}{p}} \tag{42}$$

is obvious since the covering is uniformly locally finite and the functions a_j are uniformly bounded (cf. (28) and (29)). Let $r > 0$, $\tilde{\chi}_{j,i}$ the characteristic function of the ball $\Omega(x_{j,i}, r2^{-j})$ and $\tilde{\chi}^{(p)}_{j,i} = 2^{\frac{jn}{p}} \tilde{\chi}_{j,i}$. It should be clear that putting $\tilde{\chi}^{(p)}_{j,i}$ instead of $\chi^{(p)}_{j,i}$ in (35) we get equivalent norms. This observation and the definition of the atoms give us

$$\left\| \sum_{j,i=0}^{\infty} s_{j,i}\, h_{0,m} \star a_{j,i} \right\| \le \left\| \left(\sum_{j,i=0}^{\infty} |s_{j,i}|^q \tilde{\chi}^{(p)}_{j,i}(\cdot)^q \right)^{\frac{1}{q}} \right\|_p . \tag{43}$$

Let $J = \min\{L, m\}$. Then $J > \frac{s}{2}$ and

$$\left(\int_0^1 t^{(m-\frac{s}{2})q} \left| \sum_{i=0}^{\infty} s_i\, h_t^m \star a_i(x) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \le C \sup_{0<t\le 1} t^{m-J} \left| \sum_{i=0}^{\infty} s_i h_t^{m-J} \star \Delta^J a_i(x) \right| . \tag{44}$$

So

$$\left\| \left(\int_0^1 t^{(m-\frac{s}{2})q} \left| \sum_{i=0}^{\infty} s_i h_t^m \star a_i(\cdot) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p$$
$$\le C \left\| \sup_{0<t\le 1} t^{m-J} |h_t^{m-J}| \star \left| \sum_{i=0}^{\infty} s_i \Delta^J a_i(\cdot) \right| \right\|_p$$
$$\le C \left\| \sum_{i=0}^{\infty} |s_i \Delta^J a_i(\cdot)| \right\| \le C \left(\sum_{i=0}^{\infty} |s_i|^p \right)^{\frac{1}{p}} .$$

It remains to estimate the last summand which we divide into two parts. We have

$$\left(\int_0^1 t^{(m-\frac{s}{2})q} \left| \sum_{j,i=0}^{\infty} s_{j,i} h_t^m \star a_{j,i}(x) \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$\le \left(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-\frac{s}{2})q} \left(\sum_{j=0}^{[\frac{k}{2}]} \left| \sum_{i=0}^{\infty} s_{j,i}\, h_t^m \star a_{j,i}(x) \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$
$$+ \left(\sum_{k=0}^{\infty} \int_{2^{-k-1}}^{2^{-k}} t^{(m-\frac{s}{2})q} \left(\sum_{j=[\frac{k}{2}]}^{\infty} \left| \sum_{i=0}^{\infty} s_{j,i}\, h_t^m \star a_{j,i}(x) \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} .$$

If $j \le [\frac{k}{2}]$, then $(2j-k)(2J-s)$ is a non-positive number. Thus the first sum is less or equal to

$$\left(\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}\sqrt{2}^{(2j-k)(2J-s)}\sup_{0<t\le 1} t^{m-J}|h_t^m| \star \sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(x)|\right)^q\right)^{\frac{1}{q}}$$

$$\le C\left(\sum_{j=0}^{\infty}\left(\sup_{0<t\le 1} t^{m-J}|h_t^m| \star \sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(x)|\right)^q\right)^{\frac{1}{q}}.$$

Now the maximal inequality implies

$$\left\|\left(\sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^k} t^{(m-\frac{s}{2})q}\left(\sum_{j=0}^{[\frac{k}{2}]}\left|\sum_{i=0}^{\infty} s_{j,i}\, h_t^m \star a_{j,i}(\cdot)\right|\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{j=0}^{\infty}\sup_{0<t\le 1} t^{m-J}\left(|h_t^m| \star \sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|\right)^q\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|\right)^q\right)^{\frac{1}{q}}\right\|_p \le C\left\|\left(\sum_{i,j=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|^q\right)^{\frac{1}{q}}\right\|_p.$$

We estimate the second part. Now $k-2j\le 0$. We have

$$\left\|\left(\sum_{k=0}^{\infty}\int_{2^{-k-1}}^{2^k} t^{(m-\frac{s}{2})q}\left(\sum_{j=[\frac{k}{2}]}^{\infty}\left|\sum_{i=0}^{\infty} s_{j,i}\, h_t^m \star a_{j,i}(\cdot)\right|\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{k=0}^{\infty}\left(\sum_{j=[\frac{k}{2}]}^{\infty}\sqrt{2}^{(k-2j)s}\sup_{0<t\le 1} t^{m}|h_t^m| \star \sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|\right)^q\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{j=0}^{\infty}\left(\sup_{0<t\le 1} t^{m}|h_t^m| \star \sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|\right)^q\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|\right)^q\right)^{\frac{1}{q}}\right\|_p$$

$$\le C\left\|\left(\sum_{j,i=0}^{\infty}|s_{j,i}\tilde{\chi}_{j,i}^p(\cdot)|^q\right)^{\frac{1}{q}}\right\|_p.$$

Thus we proved the inequality

$$\|f|\mathcal{F}^{s,m}_{p,q}\| \leq C\left(\left(\sum_{i=0}^{\infty}|s_i|^p\right)^{\frac{1}{p}} + \left\|\left(\sum_{j,i}|s_{j,i}\chi^p_{j,i}(\cdot)|^q\right)^{\frac{1}{q}}\right\|_p\right). \tag{45}$$

Step 2. Now we decompose any distribution from $\mathcal{F}^{s,m}_{p,q}(X)$ into atoms. To do this we take a uniformly locally finite sequence $\{\Omega_j\}_{j\geq 1}$ of coverings of X with $r_j = \varepsilon 2^{-j}$, where ε is a fixed number such that $0 < \varepsilon < 1$. To deal with the decomposition we need some inequalities. The first one is the Harnack-Moser inequality for subsolutions of parabolic equations. We use the formulation for uniformly elliptic operators on Riemannian manifolds that is due to L. Saloff-Coste [14].

For the future use we need two constants $b > 0$ and $\delta > 0$. We choose these constants in such a way that the identities $b - \delta^2 = \frac{b}{16}$ and $b - \delta = \frac{b}{4}$ are satisfied. Such constants exist and both b and δ are greater than 1. Let $Q_{j,i} = (b4^{-j-1}, b4^{-j}) \times \Omega(x_{j,i}, 2^{-j})$. Then [14: Theorem 5.5] implies

$$\sup_{(t,x)\in Q_{j,i}} |h^m_t \star f(x)| \leq C\, 2^{jn} \int_{\Omega(x_{j,i},\delta 2^{-j})} \int_{b4^{-j-2}}^{b4^{-j}} |h^m_t \star f(x)| \frac{dt}{t}\, dx \tag{46}$$

where $C > 0$ is a constant depending on n, b and δ only.

For the reason that will be clear later on we assume that $\varepsilon b > 1$. Let $\{\psi_{j,i}\}$ be the smooth resolution of unity corresponding to the covering $\{\Omega(x_{j,i}, \varepsilon 2^{-j})\}$. We may assume that for every $m > 0$ there is a constant $b_m > 0$ such that the inequality

$$\left|\frac{\partial^{|\gamma|}}{\partial H^\gamma}\psi_{j,i} \circ \exp_{x_{j,i}}(H)\right| \leq b_m 2^{-j|\gamma|} \tag{47}$$

holds for every j and i, every $H \in T_{x_{j,i}}X$ and every multi-index γ such that $|\gamma| \leq m$. [27: Theorem III.1.5] implies that there is a constant $C > 0$ such that, for every $k < L$ and every $x \in [\frac{\varepsilon b}{2}, \varepsilon b] \times \Omega(o, \varepsilon)$,

$$|\nabla^k h^m_t \star f(x)| \leq C \int_{(\frac{b}{2}, b)\times\Omega(o,1)} |h^m_t \star f(y)|\, dy.$$

But G acts on X as a group of isometries, thus the above inequality is true for any unit geodesic ball of radius 1 with the same constant C. Now using the scaling method (cf. [26: Sections 7 and 8]) we can prove that the inequality

$$|\nabla|^k h^m_t \star f(x)| \leq C\, 2^{jk} \int_{Q_{j,i}} |h^m_t \star f(y)|\, dy \tag{48}$$

holds for any $x \in [\varepsilon b4^{-j-1}, \varepsilon b4^{-j}] \times \Omega(x_{j,i}, \varepsilon 2^{-j})$.

Step 3. Still assuming $s > 0$ we prove the converse inequality. We start with formula (26). Since $C_p(X)$ is dense in $C'_p(X)$ formula (26) if true for any $f \in C'_p(X)$

provided the convergence in (26) is understood in weak $C_p'(X)$-sense. For this part of proof it is convenient to change formula (26) a bit and to rewrite it in the form

$$f(x) = C\left(h_{m,0} \star f + \int_0^{\varepsilon b} t^k \left(\frac{d^k}{dt^k} H_t f\right) \frac{dt}{t}\right) \tag{49}$$

where b is the positive constant from Step 2. Let $\{\psi_{j,i}\}$ be the smooth resolution of unity described in the same step. Since $h_{m,0} = \sum_{l=0}^{m-1} h_{\varepsilon b}^l$ and $\varepsilon b > 1$ we can write

$$h_{m,0} \star f = \sum_{l=0}^{m-1} h_{\varepsilon b-1}^l \star h_1 \star f. \tag{50}$$

Let $\{E_i\}$ be a decomposition of X into a sum of disjoint sets such that $E_i \subset \Omega(x_i, \varepsilon)$. Let $GE_i = \pi^{-1}(E_i)$ with $\pi : G \mapsto X$ the natural projection. Using the above resolutions of unity and (49) - (50) we get the decomposition

$$\begin{aligned} f(x) &= C\left(h_{m,0} \star f + \int_0^{\varepsilon b} t^k \left(\frac{d^k}{dt^k} H_t f\right) \frac{dt}{t}\right) \\ &= C\left(h_{m,0} \star f + \sum_{j,i=0}^{\infty} \psi_{j,i} \int_{\varepsilon b4^{-j-1}}^{\varepsilon b2^{-j}} t^m h_t^m \star f \frac{dt}{t}\right) \\ &= C\left(\sum_{i=0}^{\infty} s_i a_i + \sum_{j,i=0}^{\infty} s_{j,i} a_{j,i}\right) \end{aligned}$$

of f where

$$a_{j,i}(x) = 2^{-j(m+2)} s_{j,i}^{-1} \psi_{j,i}(x) \int_{\varepsilon b4^{-j-1}}^{\varepsilon b4^{-j}} t^m h_t^m \star f(x) \frac{dt}{t} \tag{51}$$

$$a_i(x) = s_i^{-1} \int_{GE_i} f \star h_1(g) \left(\sum_{l=0}^{m-1} h_{\varepsilon b-1}^l (g^{-1}x)\right) dg \tag{52}$$

$$s_{j,i} = 2^{j(s-\frac{n}{p}-2m)} \sum_{l\in I_i} \sup_{x\in\Omega_{j,l}} |h_t \star \Delta^m f|(x) \tag{53}$$

$$s_i = \left(\int_{GE_i} |f \star h_1|^p(g)\, dg\right)^{\frac{1}{p}} \tag{54}$$

and

$$I_i = \left\{ l \in \mathbb{N} : \Omega(x_{j,l}, 2^{-j}) \cap \Omega(x_{j,i}, 2^{-j}) \neq \emptyset \right\}.$$

It follows from Step 2 that after suitable normalization $a_{j,i}$ are (s,p)-atoms (cf. (46) - (48)). The proof that after suitable normalization a_j are p-atoms is the same as in [15] and therefore it is omitted here.

Step 4. It should be clear that the expression

$$\|f\|_p + \left\| \left(\sum_{j=1}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-\frac{s}{2})q} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right|^q (\cdot) \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p$$

is an equivalent norm in $\mathcal{F}^{s,m}_{p,q}(X)$ if $s > 0$. We use that expression to estimates the atomic norm from above. Using the Fefferman-Stein maximal inequality [19] we get

$$\begin{aligned}
&\left\| \left(\sum_{j=1}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-\frac{s}{2})q} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right|^q (\cdot) \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p \\
&\geq C \left\| \left(\sum_{j=1}^{\infty} \left(\int_{b4^{-j-2}}^{b4^{-j}} t^{(m-\frac{s}{2})} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| (\cdot) \frac{dt}{t} \right)^q \right)^{\frac{1}{q}} \right\|_p \\
&\geq C \left\| \left(\sum_{j=1}^{\infty} M \left(\int_{b4^{-j-2}}^{b4^{-j}} t^{(m-\frac{s}{2})} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| \frac{dt}{t} \right) (\cdot)^q \right)^{\frac{1}{q}} \right\|_p \\
&\geq C \left\| \left(\sum_{j=1}^{\infty} 2^{jsq} 4^{-jmq} M \left(\int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| \frac{dt}{t} \right) (\cdot)^q \right)^{\frac{1}{q}} \right\|_p \\
&\geq C \left\| \left(\sum_{j=1}^{\infty} 2^{jq(s-\frac{n}{p})} 4^{-jmq} M \left(\int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| \frac{dt}{t} \right) (\cdot)^q \chi^{(p)}_{j,i}(\cdot)^q \right)^{\frac{1}{q}} \right\|_p .
\end{aligned}$$

But there is a constant $C > 0$ independent of j and i such that the inequalities

$$\begin{aligned}
&M \left(\int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| \frac{dt}{t} \right) (x) \\
&\qquad \geq C(\delta 2)^{nj} \left(\int_{\Omega(x_{i,l},\delta 2^{-j})} \int_{b4^{-j-2}}^{b4^{-j}} \left| \left(\frac{\partial}{\partial t}\right)^m f \star h_t \right| \frac{dt}{t} \right) (x) \\
&\qquad \geq C \sum_{l \in I_i} \sup_{x \in \Omega_{j,l}} |h_t \star \Delta^m f|(x)
\end{aligned}$$

hold for any $l \in I_i$. Therefore

$$\left\| \left(\sum_{j=1}^{\infty} \int_{b4^{-j-2}}^{b4^{-j}} t^{(m-\frac{s}{2})q} \left| \left(\frac{\partial}{\partial t}\right)^m \star h_t \right|^q (\cdot) \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_p \geq C \left\| \left(\sum_{j=1}^{\infty} |s_{j,i}|^q \chi_{j,i}^{(p)}(\cdot)^q \right)^{\frac{1}{q}} \right\|_p \tag{55}$$

Since the inequality $\left(\sum_i |s_i|^p\right)^{\frac{1}{p}} \leq C \|f\|_p$ is obvious Theorem 1 is proved for $s > 0$.

Step 5. Now we assume that $s \leq 0$. This case can be reduced to $1 < p \leq 2$. Moreover, Δ^{-1} maps the space $C_p(X)$ into the space $C_p(X)$. Thus Δ^{-1} can be extended to $C_p'(X)$. Let $f \in \mathcal{F}_{p,q}^{m,s}(X)$. It can be easily checked that if $2k > -s$, then the operator Δ^{-k} defines an isomorphism of $\mathcal{F}_{p,q}^{m,s}(X)$ onto $\mathcal{F}_{p,q}^{m+k,s+2k}(X)$ as well as an isomorphism of $\mathcal{F}_{p,q}^{s}(X)$ onto $\mathcal{F}_{p,q}^{s+2k}(X)$. So $f \in \mathcal{F}_{p,q}^{s}(X)$ and by [19: Theorem 1] this function can be represented as a sum of atoms. The same argument works in the opposite direction.

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