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Approximation of Solutions of Stochastic Differential Equations by Discontinuous Galerkin Methods

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Abstract. The generalized solution of a system of Stratonovich equations is approximated by a discontinuous Galerkin method. A piecewise polynomial approximation is introduced. The convergence and error estimates are proved. The solution of Galerkin equations can be approximated by the solution of a system of equations with an inhomogeneous random part and the simulation of a stochastic integral.

Keywords: *Approximation of solution of a stochastic differential equation, Stratonovich integral, Galerkin method*

AMS subject classification: Primary 60 H 20, secondary 60 H 30

1. Introduction

This paper is concerned with piecewise polynomial approximation of the solution of a system of ordinary stochastic differential equations in the sense of Stratonovich. Under certain conditions it is well known that a solution of a system of Stratonovich equations is a solution of a modified Ito system [3: p. 237]. The approximation of solutions of stochastic differential equations is studied by many authors. For example, a stochastic Taylor formula was developed and applied to the approximation of solutions of ordinary Ito equations (see [4: p. 163] and [5: p. 78]). Especially, stochastic variants of the Euler and Runge-Kutta methods are obtained by application of the stochastic Taylor formula. In [3: Theorem 7.2/p. 394] a Stratonovich equation is approximated by a sequence of stochastic differential equations with piecewise differentiable paths. Therefore the Stratonovich interpretation is often important for the applications.

Here we consider another method known in the deterministic case as completely discontinuous Galerkin method [1]. The investigations differ from the deterministic case since the paths of solutions of stochastic differential equations are not differentiable. Subsequently, the methods of stochastic analysis must be used. In Section 2 we interpret a system of Stratonovich equations on a fixed interval $[0, T]$ as system of stochastic

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variational equations and prove for the solution of the last an existence and uniqueness theorem and a regularity property (Theorem 2 and Theorem 3). The approximation is studied in Section 3. For that the interval $[0,T]$ is partitioned into N intervals $I_n = [t_{n-1}, t_n]$ by points t_n $(0 \le n \le N)$ with $0 = t_0 < t_1 < \dots < t_N = T$. On each interval I_n a random polynom (Galerkin approximation) is constructed by solution of random variational equations. If the partitions of the intervals are small enough, then a unique solution exists (Theorem 4). The convergence in mean square and error of approximation estimates are contained in Section 4. A computing possibility is given in Section 5.

2. Variational formulation

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\in[0,T]} \subset \mathcal{F}$, let $(w(t))_{t\in[0,T]}$ be an $\mathcal{F}_t$-adapted m-dimensional Wiener process and

$$\left.\begin{aligned} a_i &: [0,T] \times \mathbb{R}^d \to \mathbb{R} \\ \sigma_{ij} &: [0,T] \times \mathbb{R}^d \to \mathbb{R} \end{aligned}\right\} \qquad (i = 1, \dots, d;\ j = 1, \dots, m)$$

measurable functions with

$$|a_i(t,X)| + |\sigma_{ij}(t,X)| \le C(1 + |X|) \qquad (X \in \mathbb{R}^d,\ t \in [0,T]) \tag{2.1}$$

for some constant $C > 0$ and

$$|a_i(t,X) - a_i(t,Y)| \le D_1 |X - Y)| \qquad (X, Y \in \mathbb{R}^d,\ t \in [0,T]) \tag{2.2}$$

for some constant $D_1 > 0$. Assume that $\frac{\partial \sigma_{ij}(t,X)}{\partial X_k}$ exists and

$$\left| \frac{\partial \sigma_{ij}(t,X)}{\partial X_k} \right| \le K \qquad (i = 1, \dots, d;\ j = 1, \dots, m) \tag{2.3}$$

for some constant $K > 0$. Then the functions σ_{ij} are Lipschitz continuous over $\mathbb{R}^d$ with Lipschitz constant K. Define

$$\begin{aligned} a(t, X(t)) &= \big(a_i(t, X(t))\big)_{i=1,\dots,d} \\ \sigma(t, X(t)) &= \big(\sigma_{ij}(t, X(t))\big)_{\substack{i=1,\dots,d \\ j=1,\dots,m}} \end{aligned}$$

and let $X_0 : \Omega \to \mathbb{R}^d$ be $\mathcal{F}_0$-measurable.

Further we consider the system of Stratonovich equations

$$\left.\begin{aligned} dX(t) &= a(t, X(t))\, dt + \sigma(t, X(t)) \circ dw(t) \\ X(0) &= X_0 \end{aligned}\right\} \tag{2.4}$$

which is defined by

$$X(t) = X_0 + \int_0^t a(s, X(s))\, ds + \int_0^t \sigma(s, X(s)) \circ dw(s) \tag{2.5}$$

for all $t \in [0,T]$ with probability 1. The stochastic integral is defined in the sense of Stratonovich [3: p. 237]. From [3] we can deduce that (2.4) is equivalent to the modified Ito equation

$$\left.\begin{aligned} dX(t) &= b(t, X(t))\, dt + \sigma(t, X(t))\, dw(t) \\ X(0) &= X_0 \end{aligned}\right\} \tag{2.6}$$

where $b = b(t, X) \in \mathbb{R}^d$ with components

$$b_i(t, X) = a_i(t, X) + \frac{1}{2} \sum_{j=1}^{m} \sum_{k=1}^{d} \frac{\partial \sigma_{ij}(t, X)}{\partial X_k} \sigma_{kj}(t, X) \qquad (i = 1, \ldots, d),$$

equation (2.6) is defined as

$$X(t) = X_0 + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dw(s) \tag{2.6$'$}$$

$\mathcal{P}$-a.s. for all $t \in [0,T]$ and the stochastic integral is an Ito integral. Further assume

$$|b_i(t, X) - b_i(t, Y)| \leq D_2 |X - Y| \tag{2.7}$$

for fixed $D_2 > 0$.

The classical existence and uniqueness results are summarized in the next theorem (see, for example, [2: Satz 1/p. 38]).

Theorem 1. *Under the above assumptions equation* (2.6) *has a unique $\mathcal{F}_t$-adapted continuous $\mathbb{R}^d$-valued solution* $X(t) = \big(X_1(t), \ldots, X_d(t)\big)$ *with* $E \sup_{t \in [0,T]} |X_i(t)|^2 < \infty$ $(i = 1, \ldots, d)$.

Now we introduce a mesh-dependent variational formulation of equation (2.6). For a given number $N \in \mathbb{N}$ we introduce partitions

$$0 = t_0 < t_1 < \ \ldots \ < t_N = T$$

with

$$\max \big\{ t_{n+1} - t_n : \ n = 0, \ldots, N-1 \big\} =: h_N \to 0$$

as $N \to \infty$. Assume there is a constant $c > 0$ with $t_n - t_{n-1} \geq c\, h_N$ for all n. Let $H^1 = H^1(\Omega \times [0,T])$ denote the space of all $\mathcal{F}_t$-adapted random R^d-valued processes $(V(t))_{t \in [0,T]}$ where the paths have generalized derivations $(V'(t))_{t \in [0,T]}$ with

$$E \sup_{0 \leq t \leq T} |V'(t)|^2 < \infty.$$

That is, $(V'(t))_{t\in[0,T]}$ is an $\mathcal{F}_t$-adapted $\mathbb{R}^d$-valued process defined by

$$\int_0^T \langle V(t), \varphi'(t)\rangle dt = -\int_0^T \langle V'(t), \varphi\rangle dt$$

($\mathcal{P}$-a.s.) for all $\varphi \in C_0^\infty[0,T]$, where $\langle\cdot,\cdot\rangle$ is the scalar product over $\mathbb{R}^d$. Further we introduce the following function spaces:

- $L_n^2(\Omega)$: Space of all $\mathcal{F}_{t_n}$-measurable functions $U : \Omega \to \mathbb{R}^d$ with $E|U|^2 < \infty$ $(n = 0, ..., N)$.
- $\tilde{L}^2(\Omega \times [t_{n-1}, t_n])$: Space of all $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted processes $(Y(t))$ where $t \in [t_{n-1}, t_n]$ with $E\int_{t_{n-1}}^{t_n} |Y(t)|^2 dt < \infty$ $(n = 1, ..., N)$.
- $\tilde{Y}_N = \prod_{n=1}^{N} \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$.
- $\tilde{U}_N = \prod_{n=0}^{N} L_n^2(\Omega)$.
- $\tilde{V}_N = \prod_{n=1}^{N} H^1(\Omega \times [t_{n-1}, t_n])$.

The variational problem consists in finding of $(U_0, ..., U_N; Y_1, ..., Y_N) \in \tilde{U}_N \times \tilde{Y}_N$ such that

$$\left.\begin{aligned} U_0 &= X_0 \\ \langle U_n, V_n(t_n)\rangle &= \langle U_{n-1}, V_n(t_{n-1})\rangle \\ &\quad + \int_{t_{n-1}}^{t_n} \left(\langle Y_n(t), V_n'(t)\rangle + \langle b(t, Y_n(t)), V_n(t)\rangle\right) dt \\ &\quad + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t))\, dw(t)\rangle \end{aligned}\right\} \tag{2.8}$$

holds for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ and $n = 1, ..., N$. To solve problem (2.8) we apply the following

Lemma 1.

(1) *The function* $\Phi : H^1(\Omega \times [t_{n-1}, t_n]) \to \tilde{L}^2(\Omega \times [t_{n-1}, t_n]) \times L_n^2(\Omega)$ *defined by* $\Phi(V_n(\cdot)) = (-V_n'(\cdot), V_n(t_n))$ *is an isomorphism.*

(2) *Let* $(B_1(t))_{t\in[0,T]}$ *and* $(B_2(t))_{t\in[0,T]}$ *be* $\mathcal{F}_t$*-adapted stochastic processes with values in* $\mathbb{R}^d$ *and* $\mathbb{R}^{d\times m}$ *so that the Ito differential* $B_1(t)\,dt + B_2(t)\,dw(t)$ *exists and let*

$U_{n-1} \in L^2_{n-1}(\Omega)$ be given. Then there are $U_n \in L^2_n(\Omega)$ and $Y_n \in \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ with

$$\begin{aligned} &\langle U_n, V_n(t_n)\rangle - \langle U_{n-1}, V_n(t_{n-1})\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n'(t)\rangle dt \\ &\quad = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t)\, dw(t)\rangle \end{aligned} \tag{2.9}$$

$\mathcal{P}$-a.s. for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ $(n = 1, ..., N)$ and (U_n, Y_n) is with probability 1 unique.

Proof. Assertion (1) is obviously. Assertion (2): Applying the Ito formula to $\langle Y(t), V_n(t)\rangle$ where

$$\left.\begin{aligned} dY(t) &= B_1(t)\, dt + B_2(t)\, dw(t) \quad (t \in [t_{n-1}, t_n]) \\ Y(t_{n-1}) &= U_{n-1} \end{aligned}\right\}$$

and $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ we find

$$\begin{aligned} &\langle Y(t_n), V_n(t_n)\rangle - \langle U_{n-1}, V_n(t_{n-1})\rangle - \int_{t_{n-1}}^{t_n} \langle Y(t), V_n'(t)\rangle dt \\ &\quad = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t)\, dw(t)\rangle. \end{aligned}$$

Consequently, (U_n, Y_n) with $U_n = Y(t_n)$ and $Y_n = Y$ solve problem (2.9). It is easy to see with an indirect proof that (U_n, Y_n) is the unique solution of this problem with probability 1 ∎

Remark 1. Obviously, statement (2) in Lemma 1 holds also for the Stratonovich integral.

Theorem 2. *There is a unique solution $(U_0, ..., U_N; Y_1, ..., Y_N) \in \tilde{U}_N \times \tilde{Y}_N$ of problem (2.8) for sufficient small $h_n > 0$.*

Proof. On an interval $[t_{n-1}, t_n)$ suppose that an $\mathcal{F}_{t_{n-1}}$-measurable $\mathbb{R}^d$-valued variable U_{n-1} with $E|U_{n-1}|^2 < \infty$ is given. Then $(U_n, Y_n) \in L^2_n(\Omega) \times \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ has to be determined so that

$$\begin{aligned} &\langle U_n, V_n(t_n)\rangle - \langle U_{n-1}, V_n(t_{n-1})\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n'(t)\rangle dt \\ &\quad = \int_{t_{n-1}}^{t_n} \big\langle V_n(t), b(t, Y_n(t))\big\rangle dt + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \sigma(t, Y_n(t))\, dw(t)\big\rangle \end{aligned} \tag{2.10}$$

is valid for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$. We define recursively $(U_n^{i+1}, Y_n^{i+1}) \in L_n^2(\Omega) \times \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ $(i = 0, 1, \ldots)$ by the equations

$$\begin{aligned}\langle U_n^{i+1}, V_n(t_n)\rangle - \langle U_{n-1}, V_n(t_{n-1})\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n^{i+1}(t), V_n'(t)\rangle dt \\ = \int_{t_{n-1}}^{t_n} \big\langle V_n(t), b(t, Y_n^i(t))\big\rangle dt + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \sigma(t, Y_n^i(t))\, dw(t)\big\rangle\end{aligned} \tag{2.11}$$

where (2.11) holds for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ and $Y_n^0 \in \tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ is chosen arbitrarily. Lemma 1 shows that (U_n^{i+1}, Y_n^{i+1}) exists uniquely. We consider (2.11) for $i =: j$ and $i =: j-1$. Then we substract these equations and we obtain

$$\begin{aligned}\big\langle (U_n^{j+1} - U_n^j), V_n(t_n)\big\rangle - \int_{t_{n-1}}^{t_n} \big\langle Y_n^{j+1}(t) - Y_n^j(t), V_n'(t)\big\rangle dt \\ = \int_{t_{n-1}}^{t_n} \big\langle V_n(t), b(t, Y_n^j(t)) - b(t, Y_n^{j-1}(t))\big\rangle dt \\ + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \big(\sigma(t, Y_n^j(t)) - \sigma(t, Y_n^{j-1}(t))\big) dw(t)\big\rangle\end{aligned} \tag{2.12}$$

for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$. Obviously, the function

$$V_n(t) = \begin{cases} 0 & \text{for } t = t_{n-1} \\ \int_{t_{n-1}}^t (Y_n^j(s) - Y_n^{j+1}(s))ds & \text{for } t \in (t_{n-1}, t_n) \\ 0 & \text{for } t = t_n \end{cases} \tag{2.13}$$

is from $H^1(\Omega \times [t_{n-1}, t_n])$. If we choose the above process V_n, then we obtain from (2.12)

$$\begin{aligned}\int_{t_{n-1}}^{t_n} \big|Y_n^{j+1}(t) - Y_n^j(t)\big|^2 dt = \int_{t_{n-1}}^{t_n} \big\langle V_n(t), b(t, Y_n^j(t)) - b(t, Y_n^{j-1}(t))\big\rangle dt \\ + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \big(\sigma(t, Y_n^j(t)) - \sigma(t, Y_n^{j-1}(t))\big) dw(t)\big\rangle\end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
E\int_{t_{n-1}}^{t_n} & \left|Y_n^{j+1}(t) - Y_n^j(t)\right|^2 dt \\
&\le \left(E\int_{t_{n-1}}^{t_n} \left|b(t, Y_n^j(t)) - b(t, Y_n^{j-1}(t))\right|^2 dt\right)^{\frac{1}{2}} \left(E\int_{t_{n-1}}^{t_n} |V_n(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le D_2 \left(E\int_{t_{n-1}}^{t_n} \left|Y_n^j(t) - Y_n^{j-1}(t)\right|^2 dt\right)^{\frac{1}{2}} \left(E\int_{t_{n-1}}^{t_n} \left|\int_{t_{n-1}}^{t} (Y_n^j(s) - Y_n^{j+1}(s))ds\right|^2 dt\right)^{\frac{1}{2}} \\
&\le D_2 \left(E\int_{t_{n-1}}^{t_n} \left|Y_n^j(t) - Y_n^{j-1}(t)\right|^2 dt\right)^{\frac{1}{2}} h_N^{\frac{1}{2}} \left(E\int_{t_{n-1}}^{t_n} \left|Y_n^j(t) - Y_n^{j+1}(t)\right|^2 dt\right)^{\frac{1}{2}}
\end{aligned}
$$

where the properties of the Ito integral, the Lipschitz continuity of b and the Schwarz inequality in $L^2([t_{n-1}, t_n])$ were applied. Consequently, it follows

$$
\begin{aligned}
\left\|Y_n^{j+1}(\cdot) - Y_n^j(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])} &\le D_2 h_N^{\frac{1}{2}} \left\|Y_n^j(\cdot) - Y_n^{j-1}(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])} \\
&\;\;\vdots \\
&\le (D_2 h_N^{\frac{1}{2}})^j \left\|Y_n^1(\cdot) - Y_n^0(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])}
\end{aligned}
$$

and for $p > 0$

$$
\begin{aligned}
\Big\|Y_n^{j+p}(\cdot) & - Y_n^j(\cdot)\Big\|_{L^2(\Omega\times[t_{n-1},t_n])} \\
&\le \left\|Y_n^{j+p}(\cdot) - Y_n^{j+p-1}(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])} \\
&\quad + \left\|Y_n^{j+p-1}(\cdot) - Y_n^{j+p-2}(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])} \\
&\quad\;\vdots \\
&\quad + \left\|Y_N^{j+1}(\cdot) - Y_n^j(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])} \\
&\le \left[(D_2 h_N^{\frac{1}{2}})^{j+p-1} + (D_2 h_N^{\frac{1}{2}})^{j+p-2} + \ldots + (D_2 h_N^{\frac{1}{2}})^j\right] \\
&\quad \times \left\|Y_n^1(\cdot) - Y_n^0(\cdot)\right\|_{L^2(\Omega\times[t_{n-1},t_n])}.
\end{aligned}
$$

Therefore $\{Y_n^j(\cdot)\}_j$ is a Cauchy sequence in $\tilde{L}^2(\Omega \times [t_{n-1}, t_n])$, since the term [...] converges to 0 for sufficient small $h_N > 0$ as $j, p \to \infty$. Thus, the limit $Y_n(\cdot) = \lim_{j\to\infty} Y_n^j(\cdot)$

exists in $\tilde{L}^2(\Omega \times [t_{n-1}, t_n])$ and consequently,

$$\lim_{i\to\infty}\left[\int_{t_{n-1}}^{t_n} \langle Y_n^{i+1}(t), V_n'(t)\rangle\, dt + \int_{t_{n-1}}^{t_n} \langle V_n, b(t, Y_n^i(t))\rangle + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n^i(t))\, dw(t)\rangle\right]$$

$$= \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n'(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), b(t, Y_n(t))\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, Y_n(t))\, dw(t)\rangle$$

holds in probability. Then (2.11) shows if we choose $V_n \equiv c \in \mathbb{R}^d$ that there exists an $\mathcal{F}_{t_n}$-measurable function $U_n : \Omega \to \mathbb{R}^d$ with

$$\langle U_n, c\rangle - \langle U_{n-1}, c\rangle = \int_{t_{n-1}}^{t_n} \langle c, b(t, Y_n(t))\rangle dt + \int_{t_{n-1}}^{t_n} \langle c, \sigma(t, Y_n(t))\, dw(t)\rangle.$$

It follows from the properties of b and σ that $E|U_n|^2 < \infty$. It is clear that $(U_n, Y_n(\cdot))$ is the unique solution of (2.10). We proceed in this way to the next interval and so on in a finite number of steps. At the end we obtain the unique solution of problem (2.8) ∎

We can prove a regularity property of the solution of problem (2.8).

Theorem 3. *Let $(X_0; U_1, ..., U_N; Y_1, ..., Y_N)$ and $(X(t))$ the solutions of problems (2.8) and (2.6), respectively. Then $X(t_1) = U_1, ..., X(t_N) = U_N$ and $X(t) = Y_n(t)$ for $t \in [t_{n-1}, t_n)$ $(n = 1, ..., N)$.*

Proof. The solution of equation (2.6) also defines the solution of problem (2.8). This follows from the Ito formula:

$$\langle X(t_n), V_n(t_n)\rangle = \langle X(t_{n-1}, V_n(t_{n-1})\rangle + \int_{t_{n-1}}^{t_n} \langle X(t), V_n'(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle b(t, X(t)), V_n(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), \sigma(t, X(t))\, dw(t)\rangle$$

where $X(0) = X_0$. The statement of the theorem results from the uniqueness of the solution of problem (2.8) ∎

Lemma 2. *A solution of problem* (2.8) *is also a solution of the problem*

$$\left.\begin{aligned} U_0 &= X_0 \\ \langle U_n, V_n(t_n)\rangle &= \langle U_{n-1}, V_n(t_{n-1})\rangle \\ &\quad + \int_{t_{n-1}}^{t_n} \left[\langle Y_n(t), V_n'(t)\rangle + \langle a(t, Y_n(t)), V_n(t)\rangle\right] dt \\ &\quad + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \sigma(t, Y_n(t)) \circ dw(t)\big\rangle \end{aligned}\right\} \tag{2.15}$$

for all $V_n \in H^1(\Omega \times [t_{n-1}, t_n])$ $(n = 1, ..., N)$ *and conversely, where the stochastic integral is the Stratonovich integral.*

Proof. It is obviously ∎

3. Approximation

We want to approximate the solution of problem (2.8) by random polynoms. Let $P^k([t_{n-1}, t_n], \mathbb{R}^d)$ the space of all polynomials $P_1, ..., P_d$ of degree k. If their coefficients are from $L^2_{n-1}(\Omega)$, then we write $P^k_{n-1}([t_{n-1}, t_n], \mathbb{R}^d)$.

Lemma 3.

(1) *The function* $\Psi : P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d) \to P^k([t_{n-1}, t_n], \mathbb{R}^d) \times \mathbb{R}^d$ *defined by* $\Psi(V_n(\cdot)) = (-V_n'(\cdot), V_n(t_n))$ *is an isomorphism.*

(2) *Let* $(B_1(t))_{t\in[0,T]}$ *and* $(B_2(t))_{t\in[0,T]}$ *be* $\mathcal{F}_t$*-adapted continuous stochastic processes with values in* $\mathbb{R}^d$ *and* $\mathbb{R}^{d\times m}$ *so that the Ito differential* $B_1(t)\,dt + B_2(t)\,dw(t)$ *exists. Then there are* $U_n \in L^2_n(\Omega)$ *and* $Y_n \in P^k_{n-1}([t_{n-1}, t_n], \mathbb{R}^d)$ *with*

$$\langle U_n, V_n(t_n)\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n'(t)\rangle dt = \int_{t_{n-1}}^{t_n} \langle B_1(t), V_n(t)\rangle dt + \int_{t_{n-1}}^{t_n} \langle V_n(t), B_2(t)\, dw(t)\rangle$$

$\mathcal{P}$*-a.s. for all* $V_n \in P^{k+1}_{n-1}([t_{n-1}, t_n], \mathbb{R}^d)$ *and* (U_n, Y_n) *is with probability* 1 *unique.*

Proof. Assertion (1) is clear. Assertion (2): Let $t_{n-1} = s_0 < s_1 < \dots < s_r = t_n$ be a partition with

$$\lim_{r\to\infty} \max_{0<i\le r-1} (s_{i+1} - s_i) = 0.$$

If $V_n \in P^{k+1}([t_{n-1}, t_n], \mathbb{R}^d)$, then

$$\sum_{i=0}^{r-1} \big\langle B_1(s_i), V_n(s_i)\big\rangle (s_{i+1} - s_i) + \sum_{i=0}^{r-1} \big\langle V_n(s_i), B_2(s_i)\big(w(s_{i+1}) - w(s_i)\big)\big\rangle$$

defines for fixed $\omega \in \Omega$ a linear continuous functional ρ_r on the space of polynoms of degree $k+1$, and also a linear continuous functional on $P^k([t_{n-1},t_n],\mathbb{R}^d) \times R^d$ since $P^{k+1}([t_{n-1},t_n],\mathbb{R}^d)$ is isomorphic to this space. Then because of the definition of the isomorphism Ψ there are $U_n^r \in R^d$ and $Y_n^r(\cdot) \in P^k([t_{n-1},t_n],R^d)$ so that

$$\rho_r(V_n) = \langle U_n^r, V_n(t_n)\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n'(t)\rangle dt.$$

Subsequently we have

$$\begin{aligned} &\langle U_n^r, V_n(t_n)\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n'(t)\rangle dt \\ &= \sum_{i=0}^{r-1} \big\langle B_1(s_i), V_n(s_i)\big\rangle (s_{i+1}-s_i) + \sum_{i=0}^{r-1} \big\langle V_n(s_i), B_2(s_i)(w(s_{i+1})-w(s_i))\big\rangle \end{aligned} \tag{3.1}$$

for all $V_n \in \dot{P}^{k+1}([t_{n-1},t_n],\mathbb{R}^d)$ with probability 1 and for all r. If we choose especially $V_n \in P_{n-1}^{k+1}([t_{n-1},t_n],\mathbb{R}^d)$ so that $V_n(t_n)=0$, then

$$-E\int_{t_{n-1}}^{t_n} \langle Y_n^r(t), V_n'(t)\rangle dt = E\sum_{i=0}^{r-1} \langle B_1(s_i), V_n(s_i)\rangle (s_{i+1}-s_i)$$

defines a linear continuous functional on $P_{n-1}^k([t_{n-1},t_n],\mathbb{R}^d)$ where the values of the polynoms are zero in t_n. The space $P_{n-1}^k([t_{n-1},t_n],\mathbb{R}^d)$ is isomorph to the Hilbert space $L_{n-1}^2(\Omega;\mathbb{R}^d \times \ldots \times \mathbb{R}^d)$. Consequently Y_n^r is also from $P_{n-1}^k([t_{n-1},t_n],\mathbb{R}^d)$. Obviously, U_n^r is from $L_n^2(\Omega)$.

The left-hand side of (3.1) is convergent in mean square to the limit

$$\langle U_n, V_n(t_n)\rangle - \int_{t_{n-1}}^{t_n} \langle Y_n(t), V_n'(t)\rangle dt$$

since the right-hand side of (3.1) is convergent, namely to

$$\int_{t_{n-1}}^{t_n} \big\langle B_1(t), V_n(t)\big\rangle dt + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), B_2(t)\, dw(t)\big\rangle$$

in the mean square. The uniqueness of U_n and Y_n follows with an indirect proof ∎

We introduce the variational problem to find $(U_0, ..., U_N; Y_1, ..., Y_N)$ from $\tilde{U}_N \times \prod_{i=0}^{N-1} P_i^k([t_i, t_{i+1}], \mathbb{R}^d)$ so that

$$\left.\begin{aligned} U_0 &= X_0 \\ \langle U_n, V_n(t_n)\rangle &= \langle U_{n-1}, V_n(t_{n-1})\rangle \\ &\quad + \int_{t_{n-1}}^{t_n} \big(\langle Y_n(t), V_n'(t)\rangle + \langle b(t, Y_n(t)), V_n(t)\rangle\big)\, dt \\ &\quad + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \sigma(t, Y_n(t))\big\rangle dw(t) \end{aligned}\right\} \tag{3.2}$$

holds for all $V_n \in P_i^{k+1}([t_{n-1}, t_n], \mathbb{R}^d)$ and $n = 1, ..., N$.

Theorem 4. *Assume that the hypotheses of Theorem 2 are verified. Then the variational problem* (3.2) *has a unique solution for sufficiently small* $h_N > 0$.

The proof is like that for Theorem 2 if we apply Lemma 3 instead of Lemma 1. Hence it is omitted.

4. Error estimates

This section contains the theorem which establishes the convergence of U_n and $Y_n(\cdot)$ for $h_N \to 0$.

Theorem 5. *Let* X *and* $(U_0, \ldots, U_N; Y_1, \ldots, Y_N)$ *be the solutions of problems* (2.6) *and* (3.2) *where the assumptions* (2.1), (2.2), (2.3) *and* (2.7) *are fulfilled. Then*

(1) $\max_n (E|U_n - X(t_n)|^2)^{\frac{1}{2}} \le C h_N^{\frac{1}{2}}$

and

(2) $E \int_0^T \left| \sum_{n=1}^N Y_n(s)\chi_{[t_{n-1}, t_n]} - X(s) \right|^2 ds \le C_1 T h_N$

where C *and* C_1 *are positive constants.*

Proof. Assume X solves the stochastic equation (2.6). Then from equation (3.2) there follows

$$\begin{aligned} &\big\langle U_n - X(t_n), V_n(t_n)\big\rangle - \big\langle U_{n-1} - X(t_{n-1}), V_n(t_{n-1})\big\rangle \\ &\quad = \int_{t_{n-1}}^{t_n} \big(\big\langle Y_n(t) - X(t), V_n'(t)\big\rangle + \big\langle b(t, Y_n(t)) - b(t, X(t)), V_n(t)\big\rangle\big) dt \\ &\quad + \int_{t_{n-1}}^{t_n} \big\langle V_n(t), \sigma(t, Y_n(t)) - \sigma(t, X(t))\, dw(t)\big\rangle. \end{aligned} \tag{4.1}$$

If we choose $V_n(t) = (1,0,\ldots,0)$, $\ldots$, $V_n(t) = (0,\ldots,0,1)$, then we obtain equations for the components of $U_n - X(t_n)$. Then the following inequalities hold for the norms:

$$\begin{aligned}(E|U_n - X(t_n)|^2)^{\frac{1}{2}} &\le (E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} \\ &\quad + \left(E\left|\int_{t_{n-1}}^{t_n} (b(t,Y_n(t)) - b(t,X(t)))\,dt\right|^2\right)^{\frac{1}{2}} \\ &\quad + \left(E\left|\int_{t_{n-1}}^{t_n} (\sigma(t,Y_n(t)) - \sigma(t,X(t)))\,dw(t)\right|^2\right)^{\frac{1}{2}} \\ &\le (E(U_{n-1} - X(t_{n-1}))^2)^{\frac{1}{2}} \\ &\quad + D_2(1+2\sqrt{h_N})\left(\int_{t_{n-1}}^{t_n} E|Y_n(t) - X(t)|^2 ds\right)^{\frac{1}{2}}\end{aligned} \tag{4.2}$$

where the Lipschitz continuity of b and σ, the Schwarz inequality in $L^2([t_{n-1},t_n])$ and properties of the Ito integral were applied. Now introduce the L^2-projector $\mathcal{P}_L$ of $L^2([t_{n-1},t_n])$ onto $P^k([t_{n-1},t_n],\mathbb{R}^d)$ and an arbitrary polynomial $\bar{Y} \in P^k([t_{n-1},t_n],\mathbb{R}^d)$. If we substitute for V_n into (4.1) the solution of the problem

$$\left.\begin{aligned} \dot{V}_n(t) &= -\mathcal{P}_L(Y_n - \bar{Y}_n) \qquad (t \in [t_{n-1},t_n]) \\ V_n(t_{n-1}) &= 0 \end{aligned}\right\}$$

we obtain from equation (4.1) the estimate

$$\begin{aligned}&\left\langle U_n - X(t_n), -\int_{t_{n-1}}^{t_n} \mathcal{P}_L(Y_n - \bar{Y}_n)dt\right\rangle \\ &= -\int_{t_{n-1}}^{t_n} \langle Y_n - X(t), \mathcal{P}_L(Y_n - \bar{Y}_n)\rangle dt \\ &\quad - \int_{t_{n-1}}^{t} \left\langle b(t,Y_n) - b(t,X(t)), \int_{t_{n-1}}^{t} \mathcal{P}_L(Y_n - \bar{Y}_n)\,ds\right\rangle dt \\ &\quad - \int_{t_{n-1}}^{t_n} \left\langle \sigma(t,Y_n) - \sigma(t,X(t)), \int_{t_{n-1}}^{t} \mathcal{P}_L(Y_n - \bar{Y}_n)\,ds\,dw(t)\right\rangle.\end{aligned}$$

From the last equation we obtain with elementar transformations

$$\left(E\int_{t_{n-1}}^{t_n}|\mathcal{P}_L(Y_n-\bar{Y}_n|^2dt\right)^{\frac{1}{2}}$$
$$\le h_N^{\frac{1}{2}}(E|U_{n-1}-X(t_{n-1})|^2)^{\frac{1}{2}}$$
$$+\left[1+\sqrt{2}D_2+(2\sqrt{2}D_2+\sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right]$$
$$\times\left(E\int_{t_{n-1}}^{t_n}|\bar{Y}_n(t)-X(t)|^2dt\right)^{\frac{1}{2}} \tag{4.3}$$
$$+\left[\sqrt{2}D_2+(2\sqrt{2}D_2+\sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right]\left(E\int_{t_{n-1}}^{t_n}|Y_n(t)-\bar{Y}_n(t)|^2dt\right)^{\frac{1}{2}}$$

where properties of the projector and the Ito integral, the Lipschitz continuity of b, the Schwarz inequality in $L^2(\Omega\times[t_{n-1},t_n]\times[t_{n-1},t_n])$ and the triangle inequality were applied. The constants D_1 and D_2 are independent from h_N and n. Inequality (4.3) and the inequality $\beta_1|Y|^2\le|\langle\mathcal{P}_L(Y),Y(t_n)\rangle|\le\beta_2|Y|^2$ yield

$$\left[\beta_1-\sqrt{2}D_2-(2\sqrt{2}D_2+\sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right]\left(E\int_{t_{n-1}}^{t_n}|Y_n(t)-\bar{Y}_n(t)|^2dt\right)^{\frac{1}{2}}$$
$$\le h_N^{\frac{1}{2}}(E|U_{n-1}-X(t_{n-1})|^2)^{\frac{1}{2}} \tag{4.4}$$
$$+\left[1+\sqrt{2}D_2+(2\sqrt{2}D_2+\sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right]\left(E\int_{t_{n-1}}^{t_n}|\bar{Y}_n(t)-X(t)|^2dt\right)^{\frac{1}{2}}.$$

The last inequality and inequality (4.2) become

$$(E|U_n-X(t_n)|^2)^{\frac{1}{2}}$$
$$\le(E|U_{n-1}-X(t_{n-1})|^2)^{\frac{1}{2}}+qc\,h_N^{\frac{1}{2}}(E|U_{n-1}-X(t_{n-1})|^2)^{\frac{1}{2}}$$
$$+c\left[1+q\left\{1+\sqrt{2}D_2+(2\sqrt{2}D_2+\sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right\}\right] \tag{4.5}$$
$$\times\left(E\int_{t_{n-1}}^{t_n}|\bar{Y}_n(t)-X(t)|^2dt\right)^{\frac{1}{2}}$$

where c and q are constants which depend from the Lipschitz constant D_2 and T.

Through special selection of $\bar{Y}_n(\cdot)$ we obtain for $k_0 > 0$

$$\left(E\int_{t_{n-1}}^{t_n} |\bar{Y}_n(t) - X(t)|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_{t_{n-1}}^{t_n} E \sup_{t\in[t_{n-1},t_n]} |\bar{Y}_n(t) - X(t)|^2 dt\right)^{\frac{1}{2}} \leq k_0 h_N^{\frac{1}{2}}.$$

Then there follows

$$\begin{aligned}(E|U_n - X(t_n)|^2)^{\frac{1}{2}} &\leq \left[1 + cqh_N^{\frac{1}{2}}\right](E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} \\ &\quad + c\left[1 + q\left\{1 + h_N^{\frac{1}{2}}(2\sqrt{2}D_2 + \sqrt{2D_2h_n}) + \sqrt{2}D_2\right\}\right]k_0 h_N^{\frac{1}{2}}.\end{aligned} \tag{4.6}$$

Now we apply [1: Lemma A.2.2] of (4.6) with M = 1, $U_0 = 0$ and $\delta = 1$ and obtain

$$(E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq k_1 h_N^{\frac{1}{2}} \qquad (n \in \mathbb{N}).$$

This estimate substituted into the right side of (4.6) yields

$$\max_{1\leq n\leq N} (E|U_n - X(t_n)|^2)^{\frac{1}{2}} \leq C h_N^{\frac{1}{2}}$$

for all $n = 1, ..., N$ with some constant $C > 0$.

The second statement of Theorem 4 we obtain from (4.4) with the help of elementar transformations. Indeed, we have

$$\begin{aligned}\left(E\int_{t_{n-1}}^{t_n} |Y_n(t) - X(t)|^2 dt\right)^{\frac{1}{2}} &\leq c h_N^{\frac{1}{2}}(E|U_{n-1} - X(t_{n-1})|^2)^{\frac{1}{2}} \\ &\quad + c\left[1 + \sqrt{2}D_2 + (2\sqrt{2}D_2 + \sqrt{2D_2h_N})h_N^{\frac{1}{2}}\right] \\ &\quad \times \left(E\int_{t_{n-1}}^{t_n} |\bar{Y}_n(t) - X(t)|^2 dt\right)^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

Through special selection of $\bar{Y}_n$ defined as

$$\bar{Y}_n(t) = X(t_{n-2}) + \frac{1}{t_n - t_{n-2}}(X(t_{n-1}) - X(t_{n-2}))(t - t_{n-1})$$

we obtain

$$E\int_{t_{n-1}}^{t_n} |\bar{Y}_n(t) - X(t)|^2 dt \leq c_1 h_N^2$$

and then $E|X(t) - X(s)|^2 \le c_1|t-s|$ with some constant $c_1 > 0$. This equation, (4.6) and the first statement of Theorem 4 yield

$$E\int_{t_{n-1}}^{t_n} |Y_n(t) - X(t)|^2 dt \le c_2 h_N^2$$

and finally

$$E\int_0^T \left|\sum_{n=1}^N Y_n(s)\chi_{[t_{n-1},t_n]} - X(s)\right|^2 ds \le C_1 T h_N \qquad (C_1, T \in \mathbb{R}).$$

Thus the assertion is proved ∎

5. A Computing possibility

We now return to the problem of computing the solution of problem (3.2). We assume $d = 1$. It follows from (3.2) for $V_n = 1$ that

$$U_n = U_{n-1} + \int_{t_{n-1}}^{t_n} b(t, Y_n(t))\, dt + \int_{t_{n-1}}^{t_n} \sigma(t, Y_n(t))\, dw(t). \tag{5.1}$$

Let $\{\phi_0, \ldots, \phi_k\}$ be a base in $P^k([t_{n-1}, t_n], \mathbb{R}^1)$. Then Y_n has the representation

$$Y_n(t) = \sum_{j=0}^k Y_{nj}\phi_j(t)$$

with $Y_{nj} \in L^2_{n-1}(\Omega)$ and we have to determine Y_{nj} $(j = 1, \ldots, k)$. At first we calculate for given $\widehat{U}_{n-1} \in L^2_{n-1}(\Omega)$ (in the case $n = 1$ we have $\widehat{U}_0 = X_0$) random variables $\widehat{Y}_{nj} \in L^2_{n-1}(\Omega)$ with

$$\widehat{Y}_n(t) = \widehat{U}_{n-1} + \int_{t_{n-1}}^{t} b(s, \widehat{Y}_n(s))\, ds \tag{5.2}$$

where $\widehat{Y}_n(t) = \sum_{j=0}^k \widehat{Y}_{nj}\phi_j(t)$. That is, we have to solve for fixed $t \in [t_{n-1}, t_n]$ a (nonlinear) equation with random inhomogeneous part. Then we define $\widehat{U}_n$ as

$$\widehat{U}_n(t) = \widehat{U}_{n-1} + \int_{t_{n-1}}^{t} b(t, \widehat{Y}_n(t))\, dt + \int_{t_{n-1}}^{t} \sigma(t, \widehat{Y}_n(t))\, dw(t). \tag{5.3}$$

Obviously, $\left(\widehat{U}_n(t_n - 0), \widehat{Y}_n(\cdot)\right)$ is a solution of (5.1) and subsequently, it follows by the Ito formula that (U_n, Y_n) with

$$U_n = \widehat{U}_n(t_n - 0) \qquad \text{and} \qquad Y_n(t) = \widehat{Y}_n(t)$$

solves problem (3.2).

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