Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 15 (1996), No. 4, 917–933

On a Class of Nonlinear Neumann Problems of Parabolic Type: Blow–Up of Solutions

M. A. Pozio and A. Tesei

Abstract. We investigate large time behaviour of solutions for a class of nonlinear Neumann parabolic problems of indefinite type, possibly degenerate. Depending on the features of the problem, several parameters play a role to establish global boundedness or finite time blow-up of solutions. The occurrence of either situation is related with the existence of stationary solutions. Proofs make extensive use of monotonicity methods.

Keywords: *Nonlinear Neumann parabolic equations, reaction terms of indefinite type, blow-up of solutions, existence and non-existence of stationary solutions*

AMS subject classification: 35 K 20, 35 K 55, 35 K 65, 35 B 05, 35 B 40

1. Introduction

In this paper we study large time behaviour of non-negative solutions of the parabolic problem

$$\left.\begin{aligned} \partial_t u &= \Delta u^m + h(x)u^m + a(x)u^p && \text{in } (0,T)\times\Omega \\ \frac{\partial u^m}{\partial n} &= 0 && \text{in } (0,T)\times\partial\Omega \\ u &= u_0 && \text{in } \{0\}\times\Omega \end{aligned}\right\} \tag{1.1}$$

where $m > 1$, $p > 1$, $0 < T \leq +\infty$ and $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) is a bounded connected domain with smooth boundary $\partial\Omega$. The functions a and h are Hölder continuous and a is non-identically zero in $\overline{\Omega}$. The initial value u_0 is continuous and non-negative in Ω.

In the case $h = 0$ problem (1.1) was suggested as a mathematical model for the evolution of a population which lives in an inhomogeneous habitat (see [17, 19] and references therein). This is the reason why we are interested only in non-negative solutions, which we will call solutions for simplicity.

For $p = m$ or, equivalently, $a = 0$, the behaviour of solutions can be easily described; we omit here this case. In the general case we consider three classes of problems, depending on h, a and Ω. For two of them a complete picture of the behaviour of solutions is given – namely, we obtain necessary and sufficient conditions for finite time

M. A. Pozio: Università di Roma "La Sapienza", Dip. di Matem. "G. Castelnuovo", P.le A. Moro 2, I – 00185 Roma, Italy; e-mail: pozio@mat.uniroma1.it
A. Tesei: Università di Roma "La Sapienza", Dip. di Matem. "G. Castelnuovo", P.le A. Moro 2, I – 00185 Roma, Italy; e-mail: tesei@vaxiac.iac.rm.cnr.it

ISSN 0232-2064 / $ 2.50

blow-up of solutions with positive initial values. Moreover, we prove that the opposite conditions are necessary and sufficient for existence of stationary solutions positive on a suitable set. Such conditions were first given for the stationary case in [6] if $p < m$ and $h = 0$, and more recently in [8, 9] if $m < p$. Related problems were studied in [1, 4, 5, 20], yet no finite time blow-up was obtained. As a special case we improve a result given in [4]. In fact, in [4] the unboundedness of solutions was proved under suitable assumptions; under the same assumptions we prove here that such solutions blow-up in finite time.

The case $m = 1$, as well as the case of Dirichlet homogeneous boundary conditions, can be investigated with the same methods; we do not consider them for brevity. However, we use a result in [14] where finite time blow-up was proved for Dirichlet boundary conditions in the case $m = 1$, $h = 0$ and $a = c > 0$, c a constant (see also [18]).

The results concerning blow-up, as well as the existence of non-trivial stationary solutions, depend both on the assumption

$$\Omega_+ = \{x \in \Omega : a(x) > 0\} \neq \emptyset \tag{1.2}$$

and on the sign of the quantities μ_0 and A which are now to be defined. Indeed, denote by μ_0 the first eigenvalue of the problem

$$\left.\begin{aligned} -\Delta\varphi - h(x)\varphi &= \mu\varphi \qquad \text{in } \Omega \\ \frac{\partial\varphi}{\partial n} &= 0 \qquad \text{on } \partial\Omega \end{aligned}\right\} \tag{1.3}$$

and by φ_0 the associated eigenfunction ($\|\varphi_0\|_\infty = 1$, $\varphi_0 > 0$ in Ω). Define also

$$A = \int_\Omega a(x)\varphi_0^{\frac{p}{m}+1}(x)\,dx. \tag{1.4}$$

First consider the case $p < m$. Then the predominant behaviour for large values of u is given by the diffusion term; hence we expect boundedness of u if $\mu_0 > 0$, and unboundedness if $\mu_0 < 0$. If $\mu_0 = 0$, the behaviour should depend on the function a; actually, we prove that the dependence is through the sign of A. Moreover, finite time blow-up proves to occur when we expect unboundedness.

If we exclude that $\mu_0 = 0 = A$, one of the following three cases will occur, which characterize the behaviour of solutions both for $p < m$ and $m < p$:

(B) Either $\mu_0 > 0$, or $A < 0$ and $\mu_0 = 0$.

(U) $\mu_0 \leq 0$, $A \geq 0$ and $\mu_0^2 + A^2 > 0$.

(I) $\mu_0 < 0$ and $A < 0$.

In cases (B) and (U) we expect boundedness and unboundedness, respectively, while the case (I) is of indeterminacy.

The following theorem holds.

Theorem 1.1. *Let $p < m$, $\Omega_+ \neq \emptyset$ and $\mu_0^2 + A^2 > 0$. Assume that condition* (I) *does not hold. Then the following statements are equivalent:*

(i) *Condition* (B) *holds.*

(ii) *There exist non-trivial stationary solutions of problem* (1.1) *positive on Ω_+.*

(iii) *For any initial value u_0 the solution of problem* (1.1) *is global and uniformly bounded in* $[0, +\infty)$.

Moreover, the following statements are equivalent:

(j) *Condition* (U) *holds.*

(jj) *There does not exist any stationary solution of problem* (1.1) *positive on Ω_+.*

(jjj) *For any initial value $u_0 > 0$ in $\overline{\Omega}$ the solution of problem* (1.1) *blows up in finite time.*

Remark 1.1. The requirement of positivity in Ω_+ for non-trivial stationary solutions of problem (1.1) is needed, since for $p < m$ such solutions may vanish in some subset of Ω. In fact, in some examples no solution of problem (1.1) positive on Ω_+ exists, although solutions vanishing on some connected component of Ω_+ do (see [6] and [2, 7, 10, 15, 21]). On the other hand, if $m \leq p$, stationary solutions are strictly positive in $\overline{\Omega}$ by the maximum principle.

Remark 1.2. The case $\Omega_+ = \emptyset$ is not considered in Theorem 1.1. In this case, if moreover $\mu_0 \geq 0$, the diffusion and the reaction terms have the same effect. Hence only the trivial stationary solution $u = 0$ of problem (1.1) exists; moreover, it is globally asymptotically stable, namely

$$\|u(t)\|_\infty \longrightarrow 0 \quad \text{as } t \to +\infty$$

for any initial value u_0.

Some partial results relative to the case of condition (I) are given in the following theorem and in the subsequent remark.

Theorem 1.2. *Let $\Omega_+ \neq \emptyset$ and condition* (I) *hold. Then in both cases $p < m$ and $m < p$ the solution of problem* (1.1) *blows up in finite time for suitable initial values.*

Remark 1.3. Proposition 3.6 and Example 3.1 show that a stationary solution positive in Ω_+ may exist or not for problem (1.1) satisfying condition (I).

Next we consider the case $m < p$. If $N \geq 3$, the following restriction on p will be used:

(C) $p < m\,\dfrac{N+2}{N-2}$ if $N \geq 3$.

We have the following result.

Theorem 1.3. *Let $m < p$ and $\Omega_+ \neq \emptyset$. Then for solutions of problem* (1.1) *finite time blow-up occurs for suitable initial values u_0. Moreover, let condition* (C) *hold, $\mu_0^2 + A^2 > 0$ and condition* (I) *not hold. Then the following statements are equivalent:*

(i) *Condition* (B) *holds.*

(ii) *There exist non-trivial stationary solutions of problem* (1.1).

(iii) *There exist globally bounded solutions of problem* (1.1) *for suitable initial values $u_0 \neq 0$.*

Moreover, the following statements are equivalent:

(j) *Condition* (U) *holds.*

(jj) *Non-trivial stationary solutions of problem* (1.1) *do not exist.*

(jjj) *For any initial value $u_0 \neq 0$ the solution of problem* (1.1) *blows up in finite time.*

Remark 1.4. As before, the case $\Omega_+ = \emptyset$ is not considered in Theorem 1.3; the results in Remark 1.2 hold also for $m < p$, if $\mu_0 \geq 0$. For $\mu_0 < 0$, see Propositions 3.4 and 3.5.

Conditions on initial data, which imply blow-up, are given in Lemma 4.4.

In Section 2 we state some definitions and some known existence and comparison results of problem (1.1). In Section 3 we give results, which describe existence or non-existence of stationary solutions and blow-up phenomena of problem (1.1), depending on the data of the problem. Theorems 1.1 - 1.3 are consequences of such results. Proofs are given in Section 4.

Acknowledgement. *The authors wish to thank Professor Catherine Bandle for useful suggestions.*

2. Mathematical framework

For any $\tau > 0$ let us define $Q_\tau = (0, \tau] \times \Omega$ and $\Sigma_\tau = (0, \tau] \times \partial\Omega$. We also set $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$. Let $u_0 \in L^\infty(\Omega)$. By a *solution* of problem (1.1) in $[0, \tau]$ we mean a function $u \in C([0, \tau]; L^1(\Omega)) \cap L^\infty(Q_\tau)$ such that

$$\int_\Omega u(t)\chi(t) - \iint_{Q_t} \{u\chi_t + u^m \Delta\chi\} = \int_\Omega u_0\chi(0) + \iint_{Q_t} \{hu^m + au^p\}\chi \qquad (2.1)$$

for all $0 \leq \chi \in C^2(\overline{Q}_t)$ with $\frac{\partial\chi}{\partial n} = 0$ on Σ_t and any $t \in [0, \tau]$. A *global* solution of problem (1.1) is a solution in $[0, \tau]$ for any $\tau > 0$. The notation $u = u(t; u_0)$ $(t \geq 0)$ will be used to stress the dependence on the initial value u_0. Moreover, *supersolutions* of problem (1.1) are defined replacing the symbol "=" by "≥" in equality (2.1); similarly for *subsolutions.*

Concerning existence, uniqueness and non-negativity of solutions of problem (1.1) the following holds (see [2, 12]).

Lemma 2.1. *For any $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ a.e. in Ω, there is a $\tau > 0$ (depending only on $\|u_0\|_\infty$) such that there exists a unique non-negative solution of problem (1.1) in $[0,\tau]$ which is continuous in $(0,\tau] \times \overline{\Omega}$. Either the solution is global, or there is a maximal existence interval $[0,T)$ $(0 < T < \infty)$ such that $\|u\|_{L^\infty(Q_\tau)} \to +\infty$ as $\tau \to T^-$.*

If the maximal time of existence T is finite, it is referred to as the *blow-up time* of the solution. By the above lemma we can assume without loss of generality that the initial value u_0 is continuous. Observe that the statement of non-negativity in Lemma 2.1 follows by the following more general statement (see [2, 11]).

Proposition 2.1 (Comparison result).

(i) *If u is a subsolution and v a supersolution of problem (1.1) and $0 \leq u(0) \leq v(0)$ in $\overline{\Omega}$, then $u(t) \leq v(t)$ in the existence interval of v. In particular, this is true if u and/or v are solutions.*

(ii) *If $\overline{u}_0$ is a stationary supersolution of problem (1.1), then $u(t;\overline{u}_0)$ is non-increasing in t, exists in $[0,+\infty)$ and $u^* = \lim_{t\to+\infty} u(t;\overline{u}_0)$ is a stationary solution of problem (1.1). If $\underline{u}_0$ is a stationary subsolution of problem (1.1), then $u(t;\underline{u}_0)$ is non-decreasing in t and, if it exists in $[0,+\infty)$ and is uniformly bounded from above, then $u_* = \lim_{t\to+\infty} u(t;\underline{u}_0)$ is a stationary solution of problem (1.1).*

It is well known that the solution of problem (1.1) is classical in open regions where it is positive [16]. Moreover, for any continuous initial value $u_0 \geq 0$ with $u_0 \not\equiv 0$ in Ω the solution $u(t;u_0)$ is positive in any region of positivity of u_0 [15].

Let us also mention the following result.

Lemma 2.2. *Let $m \leq p$ and $u_0 \geq 0$ with $u_0 \not\equiv 0$. Then either the solution of problem (1.1) blows up in finite time, or there exists $t_0 \geq 0$ such that it is strictly positive in $(t_0,+\infty) \times \Omega$. In the first case u can vanish somewhere at the blow-up time.*

Proof. Observe that the solution of problem (1.1) lies above the solution of the corresponding initial-boundary value problem with homogeneous Dirichlet boundary conditions. For the latter there exists $t_0 \geq 0$ such that for any $t \geq t_0$ the solution is positive in Ω and lies above a smooth function with strictly negative outward derivative at the boundary (see [10]). Then by comparison with a suitable subsolution (see [3: Section 3B]) the solution of problem (1.1) is positive in $\overline{\Omega}$ for any $t > t_0$, unless it blows up in finite time. This proves the claim ∎

An explicit example where blow-up occurs before the solution becomes positive in the whole of Ω is given in [13] in the case $m = p$. Finally, remark that a stationary solution u of problem (1.1) solves the stationary problem

$$\left.\begin{aligned} -\Delta u^m - h(x)u^m &= a(x)u^p && \text{in } \Omega \\ \frac{\partial u^m}{\partial n} &= 0 && \text{in } \partial\Omega, \end{aligned}\right\} \tag{2.2}$$

i.e., by (2.1), $u \in L^\infty(\Omega)$ and satisfies

$$-\int_\Omega u^m \Delta\chi = \int_\Omega \{hu^m + au^p\}\chi \tag{2.3}$$

for any $0 \leq \chi \in C^2(\overline{\Omega})$ with $\frac{\partial\chi}{\partial n} = 0$ on $\partial\Omega$. In view of Hölder continuity of the functions h and a, the function u is a classical solution of problem (2.2).

3. Auxiliary results

The proofs of Theorems 1.1 - 1.3 follow from the results of this section. We first consider the case $p < m$: then non-negative stationary solutions need not be positive in Ω (see Remark 1.1).

The following result concerning stationary solutions is stated without proof, since it is a simple extension of results in [6] (see also [8, 9]).

Proposition 3.1. *Assume $p < m$.*

(i) *Let $\mu_0 > 0$. Then non-trivial stationary solutions of problem* (1.1) *exist if and only if the set Ω_+ is non-empty.*

(ii) *Let $\mu_0 = 0$. Then conditions $\Omega_+ \neq \emptyset$ and $A < 0$ are sufficient for the existence of non-trivial stationary solutions of problem* (1.1)*; moreover, $\Omega_+ \neq \emptyset$ and*

$$\int_{\{u>0\}} a\varphi_0^{\frac{p}{m}+1} < 0 \tag{3.1}$$

are necessary conditions for the existence of a non-trivial stationary solution u of problem (1.1).

(iii) *Let $\mu_0 < 0$. Then condition* (3.1) *is necessary for the existence of a non-trivial stationary solution u of problem* (1.1).

Remark 3.1. In Theorem 1.1 stationary solutions of problem (1.1) positive in Ω_+ are considered. For any such solution u the necessary condition (3.1) implies $A < 0$. Then Theorem 1.1 follows by Proposition 3.1 as for the stationary results.

In the next section we prove the following result concerning the evolutionary problem (1.1).

Proposition 3.2. *Assume $p < m$.*

(i) *Let $\mu_0 > 0$. Then for any initial value u_0 the solution of problem* (1.1) *is global and uniformly bounded in* $[0, +\infty)$.

(ii) *Let $\mu_0 = 0$. If $A < 0$, the same conclusion as in case* (i) *holds true. If $A > 0$, the solution $u(t; u_0)$ blows up in finite time for any initial value $u_0 > 0$ in $\overline{\Omega}$.*

(iii) *Let $\mu_0 < 0$. If $A \geq 0$, then for any initial value $u_0 > 0$ in $\overline{\Omega}$ the solution of problem* (1.1) *blows up in finite time. If $A < 0$, then the solution blows up in finite time for suitable initial values u_0.*

Observe that a typical feature of the case $p < m$ is the existence of free boundary stationary solutions (see [6, 20, 21]). Hence, if $u_0 = 0$ in some connected component of the set Ω_+, the corresponding solution of problem (1.1) can converge to some steady state solution, which identically vanishes in the same component [7, 21]. This shows that the condition $u_0 > 0$ in $\overline{\Omega}$ in Proposition 3.2/(ii) cannot in general be omitted. However, the following holds.

Proposition 3.3. *Assume $h = 0$ and $A > 0$. Then for any initial value $u_0 \neq 0$ on each connected component of Ω_+, the solution $u(t; u_0)$ of problem* (1.1) *blows up in finite time.*

Remark 3.2. Under the assumptions of Proposition 3.3 we have $\mu_0 = 0$ and $A > 0$. Thus by Proposition 3.2/(ii) we get finite time blow-up only for solutions with strictly positive initial values. Hence Proposition 3.3 improves Proposition 3.2/(ii), at least if $h = 0$. We conjecture that the condition $h = 0$ can be removed if $\mu_0 = 0$.

The stationary problem for $m < p$ was considered in [8, 9], where the following results were proved.

Proposition 3.4. *Let $m < p$.*

(i) *If $\mu_0 > 0$, then condition $\Omega_+ \neq \emptyset$ is necessary for the existence of non-trivial stationary solutions of problem* (1.1). *It is also sufficient if condition* (C) *is satisfied.*

(ii) *If $\mu_0 = 0$, then conditions $\Omega_+ \neq \emptyset$ and $A < 0$ are necessary and, if condition* (C) *holds, also sufficient for the existence of non-trivial stationary solutions of problem* (1.1).

(iii) *If $\mu_0 < 0$, then condition $A < 0$ is necessary for the existence of positive solutions of problem* (1.1). *Moreover, if $a < 0$ in $\overline{\Omega}$, then there exists a positive solution of problem* (1.1).

Concerning the evolutionary problem we have the following result (see the next section).

Proposition 3.5. *Let $m < p$. Then if $\Omega_+ \neq \emptyset$, for solutions of problem*(1.1) *finite time blow-up occurs for suitable initial values u_0. Moreover, the following holds:*

(i) *Let $\mu_0 > 0$. If the set Ω_+ is non-empty and condition* (C) *holds, then the solution $u(t; u_0)$ of problem* (1.1) *is global for suitable initial values u_0.*

(ii) *Let $\mu_0 = 0$. If $\Omega_+ \neq \emptyset$, $A < 0$ and condition* (C) *holds, then the solution of problem* (1.1) *is global for suitable initial values u_0. If $A > 0$, then for any initial value $u_0 \neq 0$ the solution of problem* (1.1) *blows up in finite time.*

(iii) *Let $\mu_0 < 0$. If $A \geq 0$, then for any initial value $u_0 \neq 0$ the solution of problem* (1.1) *blows up in finite time. If $a < 0$ in $\overline{\Omega}$, then the solution of problem* (1.1) *is global for suitable initial values u_0.*

The proofs of Theorems 1.1 - 1.3 follow from the propositions above. To complete the description of case (I) given in Remark 1.3, we need the following additional result (see [9: Theorems 3 and 6]).

Proposition 3.6. *Let $m < p$, $\Omega_+ \neq \emptyset$ and $A < 0$. Assume $h = h_\tau$ defined by $h_\tau(x) = q(x) + \tau$ is such that for $\tau = 0$ we have $\mu_0 = 0$, hence $\mu_0 = -\tau$ for any τ. Then there exists a $\tau^* > 0$ such that problem* (1.1) *has a non-trivial stationary solution for any $\tau < \tau^*$, while no such solution exists if $\tau > \tau^*$.*

Proposition 3.6 is concerned only with the case $m < p$. However, the non-existence result in Proposition 3.6 was proved in [9] without using this condition; hence it is true also for $p < m$. Concerning existence for $p < m$, when condition (I) holds, let us mention the following example.

Example 3.1. Take Ω, $a = a_0$ and $h = h_0$ such that $\mu_0 = 0$, $\Omega_+ \neq \emptyset$ and $A = A_0 < 0$. Then there exists a non-trivial stationary solution u_* of problem (1.1) (Proposition 3.1). Take $\gamma > 0$ so small to have $a_0(x) - \gamma u_*^{m-p}(x) > 0$ for some $x \in \Omega$. Then u_* is a non-trivial stationary solution of problem (1.1) with

$$a(x) = a_0(x) - \gamma u_*^{m-p}(x) \qquad \text{and} \qquad h(x) = h_0(x) + \gamma.$$

Indeed,

$$\Delta u_*^m + h u_*^m + a u_*^p = -a_0(x) u_*^p + \gamma u_*^m + (a_0(x) - \gamma u_*^{m-p}) u_*^p = 0.$$

Moreover, the new problem satisfies condition (I) since $\mu_0 = -\gamma < 0$ and $A < A_0 < 0$. Observe that in this example it is not relevant whether $p < m$ or $m < p$. Indeed, by Proposition 3.4 u^* exists also for $m < p$ and a is well defined since $u_* > 0$ in $\overline{\Omega}$, by the maximum principle.

4. Proofs

We first prove a lemma, which is the parabolic counterpart of an identity proved in [9] for the stationary case.

Lemma 4.1. *Let $p > 1$. If $u_0 > 0$ in $\overline{\Omega}$ and $T \leq +\infty$ is the maximal existence time of the solution of problem (1.1), then for any $\tau \in (0, T)$ the identity*

$$\begin{aligned} &\frac{p}{m} \iint_{Q_\tau} \left| m \varphi_0^{\frac{p+m}{2m}} u^{\frac{m-p-2}{2}} \nabla u - \varphi_0^{\frac{p-m}{2m}} u^{\frac{m-p}{2}} \nabla \varphi_0 \right|^2 \\ &- \mu_0 \iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1} u^{m-p} + \tau \int_\Omega a \varphi_0^{\frac{p}{m}+1} \\ &= \frac{1}{p-1} \int_\Omega \varphi_0^{\frac{p}{m}+1} u_0^{-(p-1)} - \frac{1}{p-1} \int_\Omega \varphi_0^{\frac{p}{m}+1} u^{-(p-1)}(\tau) \end{aligned} \tag{4.1}$$

holds.

Proof. By [15] the solution $u(t; u_0)$ of problem (1.1) is strictly positive in $\overline{\Omega}$ for any $t \in [0, T)$; hence problem (1.1) is satisfied pointwise, not only in weak sense. Also (1.3) for $\mu = \mu_0$ and $\varphi = \varphi_0$ is satisfied pointwise. Then for any $(t, x) \in [0, T) \times \Omega$ we can multiply the first equations in (1.3) and (1.1) by $\varphi_0^{\frac{p}{m}} u^{m-p}$ and $\varphi_0^{\frac{p}{m}+1} u^{-p}$, respectively, and integrate in Q_τ, for any given $\tau \in (0, T)$. Integrating by parts in space we obtain

$$\iint_{Q_\tau} \langle \nabla(\varphi_0^{\frac{p}{m}} u^{m-p}), \nabla \varphi_0 \rangle - \iint_{Q_\tau} h \varphi_0^{\frac{p}{m}+1} u^{m-p} = \mu_0 \iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1} u^{m-p}$$

and

$$\begin{aligned} \iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1} u^{-p} u_t = &- \iint_{Q_\tau} \langle \nabla \varphi_0^{\frac{p}{m}+1} u^{-p}, \nabla u^m \rangle \\ &+ \iint_{Q_\tau} h \varphi_0^{\frac{p}{m}+1} u^{m-p} + \iint_{Q_\tau} a \varphi_0^{\frac{p}{m}+1}, \end{aligned}$$

respectively. From the above equalities we obtain easily

$$\begin{aligned}\frac{1}{p-1}\int_\Omega \varphi_0^{\frac{p}{m}+1}u_0^{-(p-1)} - \frac{1}{p-1}\int_\Omega \varphi_0^{\frac{p}{m}+1}u^{-(p-1)}(\tau) &= pm\iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1}u^{-(p+1)+m-1}|\nabla u|^2 \\ &\quad - (m+p)\iint_{Q_\tau} \varphi_0^{\frac{p}{m}}u^{m-p-1}\langle\nabla\varphi_0,\nabla u\rangle \\ &\quad + (m-p)\iint_{Q_\tau} \varphi_0^{\frac{p}{m}}u^{m-p-1}\langle\nabla u,\nabla\varphi_0\rangle \\ &\quad + \frac{p}{m}\iint_{Q_\tau} \varphi_0^{\frac{p}{m}-1}u^{m-p}|\nabla\varphi_0|^2 \\ &\quad - \mu_0\iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1}u^{m-p} + \tau\int_\Omega a\varphi_0^{\frac{p}{m}+1}.\end{aligned}$$

Hence equality (4.1) follows ∎

Proof of Proposition 3.2. Assertion (i): There exists a $k_0 > 0$ such that $k\varphi_0^{\frac{1}{m}}$ is a stationary supersolution of problem (1.1) for any $k \geq k_0$. Hence the first claim follows by the comparison results in Proposition 2.1.

Assertion (ii): Let $A < 0$. Arbitrarily large stationary supersolutions of problem (1.1) can be constructed in this case following the same path as in [6]. Hence the first claim follows by Proposition 2.1/(ii). Now let $A > 0$ and $u_0 > 0$ in $\overline{\Omega}$. Then Lemma 4.1 applies and equality (4.1) with $\mu_0 = 0$ holds. This implies

$$\frac{1}{p-1}\int_\Omega \varphi_0^{\frac{p}{m}+1}u_0^{-(p-1)} \geq \tau\int_\Omega a\varphi_0^{\frac{p}{m}+1} = \tau A.$$

It follows that

$$T \leq \frac{1}{(p-1)A}\left(\int_\Omega \varphi_0^{\frac{p}{m}+1}u_0^{-(p-1)}\right) < +\infty.$$

This proves the claim.

Assertion (iii): If $A > 0$, the proof follows as in (ii) since $-\mu_0 > 0$. Let $A = 0$. Then the initial assumption $a \neq 0$ implies $\Omega_+ \neq \emptyset$. Since $u_0 > 0$ in $\overline{\Omega}$, there exists a non-trivial stationary subsolution $\underline{u}$ of problem (1.1) such that $u_0 \geq \underline{u}$ in $\overline{\Omega}$ (see [21: Lemma 3]). By the comparison results in Proposition 2.1 we obtain

$$u(t;u_0) \geq \underline{u} \qquad \text{for any } t \in [0,T). \tag{4.2}$$

Moreover, Lemma 4.1 applies. Hence, by equality (4.1) and inequality (4.2), for any $\tau \in (0,T)$ we obtain

$$\frac{1}{p-1}\int_\Omega \varphi_0^{\frac{p}{m}+1}u_0^{-(p-1)} \geq |\mu_0|\iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1}u^{m-p} \geq |\mu_0|\tau\int_\Omega \varphi_0^{\frac{p}{m}+1}\underline{u}^{m-p}.$$

This implies

$$T \le \frac{1}{(p-1)|\mu_0|}\left(\int_\Omega \varphi_0^{\frac{p}{m}+1} u_0^{-(p-1)}\right)\left(\int_\Omega \varphi_0^{\frac{p}{m}+1}\underline{u}^{m-p}\right)^{-1} < +\infty.$$

Then the claim in the case $A = 0$ follows.

Now let $A < 0$. We can prove the result by constructing subsolutions of problem (1.1) which blow-up in finite time. In fact, let us define

$$\underline{u}(t) = \rho(t)\varphi_0^{\frac{1}{m}} \qquad (t \in [0, T_0)) \tag{4.3}$$

where

$$\rho(t) = \rho_0\left(1 - \frac{t}{T_0}\right)^{\frac{-1}{p-1}} \tag{4.4}$$

$$\rho_0 \ge \left(2\|a\|_\infty |\mu_0|^{-1}(\min_{\overline{\Omega}}\varphi_0)^{-\frac{m-1}{m}}\right)^{\frac{1}{m-p}} \tag{4.5}$$

and

$$T_0 = \left((p-1)\|a\|_\infty \rho_0^{p-1}\right)^{-1}. \tag{4.6}$$

Since $p < m$, for any $(t,x) \in [0,T_0) \times \Omega$ we have

$$\begin{aligned}
&\underline{u}_t - (\Delta + h)\underline{u}^m - a\underline{u}^p \\
&= \varphi_0^{\frac{1}{m}}\left[\rho' + \rho^p\left(\rho^{m-p}\varphi_0^{\frac{m-1}{m}}\mu_0 - a\varphi_0^{\frac{p-1}{m}}\right)\right] \\
&\le \varphi_0^{\frac{1}{m}}\left[\rho' + \rho^p\left(-\rho^{m-p}(\min_{\overline{\Omega}}\varphi_0)^{\frac{m-1}{m}}|\mu_0| + \|a\|_\infty\right)\right] \\
&\le \varphi_0^{\frac{1}{m}}\left[\rho' - \|a\|_\infty \rho^p\right] \\
&= 0.
\end{aligned}$$

Then by the comparison results in Proposition 2.1 the solution $u(t;u_0)$ of problem (1.1) blows up in finite time for any initial value $u_0 \ge \underline{u}(0)$ in $\overline{\Omega}$ ∎

In the proof of Proposition 3.3 the following results will be used.

Lemma 4.2 *Let $p < m$. If the initial value $u_0 \not\equiv 0$ on each connected component of $\Omega_+ \neq \emptyset$, then there exists a non-trivial stationary subsolution $\underline{u}_0$ of problem* (1.1) *such that $\underline{u}_0 \le u_0$ and $\underline{u}_0 \not\equiv 0$ on each connected component of Ω_+.*

Proof. If $h = 0$, the proof follows by [21: Lemma 3]. However, the same proof applies to the case $h \neq 0$.

Lemma 4.3. *Let $p < m$ and $\Omega_1 \subseteq \Omega_+$ be a set with a finite number of connected components and smooth boundary. If the initial value $u_0 \not\equiv 0$ on each connected component of Ω_1, then either finite time blow-up occurs or there exists a $\bar{t} \ge 0$ and a stationary subsolution $\underline{u}_*$ of problem* (1.1) *such that*

$$\underline{u}_* > 0 \ \text{ in } \Omega_1 \qquad \text{and} \qquad u(\bar{t}, u_0) \ge \underline{u}_*. \tag{4.7}$$

Proof. If finite time blow-up occurs the lemma is proved. Hence we assume that $u(t; u_0)$ is global. By Lemma 4.2 (with Ω_+ replaced by Ω_1), there exists a stationary subsolution $\underline{u}_0$ of problem (1.1) such that $\underline{u}_0 \leq u_0$ and $\underline{u}_0 \neq 0$ on each connected component of Ω_1. Consider the problem

$$\left.\begin{aligned} &\partial_t v = \Delta v^m + h v^m && \text{in } (0,+\infty)\times\Omega_1 \\ &v = 0 && \text{in } (0,+\infty)\times\partial\Omega_1 \\ &v = \underline{u}_0 && \text{in } \{0\}\times\Omega_1. \end{aligned}\right\} \tag{4.8}$$

Then $u(\cdot\,; \underline{u}_0)$ is a supersolution of problem (4.8); by the comparison results in Proposition 2.1 we have

$$v(t; \underline{u}_0) \leq u(t; \underline{u}_0) \tag{4.9}$$

for any $t > 0$. By [10] the existence of a $\bar{t} > 0$ with $v(\bar{t}; \underline{u}_0) > 0$ in Ω_1 follows, which in turn implies $u(\bar{t}; \underline{u}_0) > 0$ in Ω_1. On the other hand, $u(t;\, \underline{u}_0)$ is a stationary subsolution of problem (1.1) for any $t > 0$. Thus (4.7) follows setting $\underline{u}_* = u(\bar{t}; \underline{u}_0)$. Indeed, by the comparison results in Proposition 2.1 we have $u(t; \underline{u}_0) \leq u(t; u_0)$ for all $t \geq 0$ ∎

Proof of Proposition 3.3. The idea underlying this proof is similar to that of Proposition 3.2/(ii) for $A > 0$. However, in this case the integral which bounds T is not finite if the initial value u_0 vanishes somewhere, hence the proof becomes more technical.

Since $h = 0$, we have $\varphi_0 = 1$. Hence $A = \int_\Omega a(x)\, dx$. Take an open subset $\Omega_1 \subseteq \Omega^+$, with a finite number of connected components and smooth boundary, such that

$$\int_{\Omega\setminus\Omega_1} a^+(x)\, dx < \frac{A}{8} \tag{4.10}$$

(here and in the following we set $r^\pm = \frac{|r|\pm r}{2}$ for $r \in \mathbb{R}$). Since $u_0 \neq 0$ on each connected component of Ω_+, we may choose Ω_1 such that $u_0 \neq 0$ on each connected component of Ω_1 as well. By Lemma 4.3 and the comparison results in Proposition 2.1, we only have to prove finite time blow-up taking the stationary subsolution $\underline{u}_*$ of problem (1.1) (see Lemma 4.3) as initial value. Thus without loss of generality we assume that $u_0 = \underline{u}_*$ is a stationary subsolution of problem (1.1) and $u_0 > 0$ on Ω_1.

Let $\delta > 0$ such that for any measurable set $E \subseteq \Omega$

$$|E| < \delta \Longrightarrow \int_E |a(x)|\, dx < \frac{A}{8}$$

and define

$$E_n = \left\{x \in \Omega_1 \,\middle|\, 0 < u_0^p(x) \leq \frac{1}{n}\right\} \qquad (n \in \mathbb{N}).$$

It is easily seen that $|E_n| \to 0$ as $n \to \infty$. Let $\bar{n} \in \mathbb{N}$ such that $|E_n| < \delta$ for any $n \geq \bar{n}$, define

$$A^+ = \int_\Omega a^+(x)\, dx \qquad \text{and} \qquad A^- = \int_\Omega a^-(x)\, dx, \tag{4.11}$$

fix ε such that

$$\varepsilon \in \left(0, \tfrac{A}{[2\overline{n}(A^+ + A^-)]}\right) \tag{4.12}$$

and define

$$u_k(t) = u\left(t;\, u_0 + \tfrac{1}{k}\right) \qquad (k \in \mathbb{N}).$$

Then u_k is strictly positive in $\overline{\Omega}$ for any t in the maximal existence interval of $[0, T_k)$; moreover,

$$u_k(t) \geq u(t) \quad \text{for any } t \in [0, T_k)$$

by comparison results. Since u_k is a classical solution, from the first equation in (1.1) we obtain easily for any $\tau \in [0, T_k)$

$$\int_\Omega dx \int_{u_0(x)+\frac{1}{k}}^{u_k(\tau,x)} \frac{ds}{s^p + \varepsilon} = \iint_{Q_\tau} \frac{\Delta u_k^m}{u_k^p + \varepsilon}\, dt dx + \iint_{Q_\tau} a(x) \frac{u_k^p}{u_k^p + \varepsilon}\, dt dx. \tag{4.13}$$

By definition u_k solves the initial-boundary value problem (1.1) (with initial value $u_k(0) = u_0 + \frac{1}{k}$). Hence $\frac{\partial u_k^m}{\partial n} = 0$ in $(0, \tau) \times \partial\Omega$. We obtain

$$\iint_{Q_\tau} \frac{\Delta u_k^m}{u_k^p + \varepsilon}\, dt dx = mp \int_0^\tau dt \int_\Omega \frac{u_k^{m+p-2}}{(u_k^p + \varepsilon)^2} |\nabla u_k|^2 \geq 0 \tag{4.14}$$

for all $\tau \in [0, T_k)$.

As for the second term in the right-hand side of (4.13), observe that for any $t \in [0, \tau]$

$$\begin{aligned} \int_\Omega a(x) \frac{u_k^p(t,x)}{u_k^p(t,x) + \varepsilon}\, dx &= \int_\Omega a^+(x) \frac{u_k^p(t,x)}{u_k^p(t,x) + \varepsilon}\, dx - \int_\Omega a^-(x) \frac{u_k^p(t,x)}{u_k^p(t,x) + \varepsilon}\, dx \\ &\geq \int_{\Omega_1 \setminus E_{\overline{n}}} a^+(x) \frac{u_k^p(t,x)}{u_k^p(t,x) + \varepsilon}\, dx - A^-. \end{aligned} \tag{4.15}$$

Since u_0 is a stationary subsolution of problem (1.1), by the comparison results in Proposition 2.1 we have $u_k \geq u \geq u_0$ in Q_τ for any $k \in \mathbb{N}$. Since by definition $u_0^p > \frac{1}{\overline{n}}$ in $\Omega_1 \setminus E_{\overline{n}}$, from inequality (4.15), from (4.10), (4.12) and the choice of $\overline{n}$ we obtain easily

$$\begin{aligned} \int_\Omega a(x) \frac{u_k^p(t,x)}{u_k^p(t,x) + \varepsilon}\, dx &\geq \frac{1}{1 + \varepsilon\overline{n}} \int_{\Omega_1 \setminus E_{\overline{n}}} a^+(x)\, dx - A^- \qquad (0 \leq t \leq \tau < T_k). \\ &> \frac{2(A^+ + A^-)}{3A^+ + A^-}\left(A^+ - \frac{A}{4}\right) - A^- = \frac{A}{2} \end{aligned} \tag{4.16}$$

The above inequality gives immediately

$$\iint_{Q_\tau} a(x)\frac{u_k^p}{u_k^p+\varepsilon}\,dtdx \geq \frac{A}{2}\tau \qquad (\tau \in [0,T_k)). \tag{4.17}$$

On the other hand, it is immediately seen that

$$\int_\Omega dx \int\limits_{u_o(x)+\frac{1}{k}}^{u_k(t,x)} \frac{ds}{s^p+\varepsilon} \leq |\Omega| \int\limits_0^\infty \frac{ds}{s^p+\varepsilon} < +\infty. \tag{4.18}$$

From equality (4.13) and inequalities (4.14), (4.17) and (4.18) we obtain

$$\tau \leq \frac{2|\Omega|}{A}\int\limits_0^\infty \frac{ds}{s^p+\varepsilon} \qquad (\tau \in [0,\, T_k)),$$

hence

$$T_k \leq \frac{2|\Omega|}{A}\int\limits_0^\infty \frac{ds}{s^p+\varepsilon} = T \qquad (k \in \mathbb{N}). \tag{4.19}$$

Let $T > 0$ as defined in the right-hand side of (4.19). Assume by contradiction that the solution u of problem (1.1) is global. Then, for any $T' > T$, $\|u\|_{L^\infty(Q_{T'})} < +\infty$. We prove below the continuous dependence of the solution on the initial data; then there exists $k_{T'} \in \mathbb{N}$ such that, for any integer $k \geq k_{T'}$, the solution u_k exists in $[0,T']$. This is in contradiction with inequality (4.19), hence the conclusion follows. Indeed, let $[0,\tau]$ be the interval of local existence mentioned in Lemma 2.1 for any initial value $u_0 \in L^\infty(\Omega)$ such that

$$\|u_0\|_\infty \leq M = \|u\|_{L^\infty(Q_{T'})} + 1.$$

By the results in [12] the sequence $\{u_k\}_{k\geq 1}$ is equicontinuous and uniformly bounded in $[0,\tau]$. Moreover,

$$u_{k+1} \leq u_k \quad \text{in } [0,\tau]\times\overline{\Omega} \qquad (k \in \mathbb{N})$$

by comparison results. Hence $u_k \to u$ as $k \to \infty$, uniformly in $[0,\tau]$. Then there exists a $k_1 \in \mathbb{N}$ such that $\|u_k(t) - u(t)\|_\infty \leq 1$ for any $t \in [0,\tau]$ and any integer $k \geq k_1$. If $\tau \leq T'$, this entails $\|u_k(t)\|_\infty \leq M$ for any $t \in [0,\tau]$ and any $k \geq k_1$. Since problem (1.1) is time independent, taking $u_k(\tau)$ as initial datum, we have that the solution u_k exists in $[0,2\tau]$ for any integer $k \geq k_1$.

Iterating the above argument proves that the solution u_k of problem (1.1) exists in $[0,m\tau]$, where $m = \left[\frac{T'}{\tau}\right]+1$, for any $k \geq k_{m-1}$. Since $m\tau > T'$ the conclusion follows ∎

The general blow-up result in Proposition 3.5 is a consequence of the following lemma, which can be proved using some ideas in [14, 18].

Lemma 4.4. *Assume $m < p$ and $\Omega_+ \neq \emptyset$. Let B be a connected open set such that $\overline{B} \subseteq \Omega_+$ and ∂B is regular. Further, let $\lambda_0 > 0$ and $\eta_0 \in C(\overline{B}) \cap C^2(B)$ be such that*

$$\left.\begin{aligned} -\Delta\eta_0 &= \lambda_0\eta_0 && \text{in } B \\ \eta_0 &= 0 && \text{on } \partial B \\ \eta_0 &> 0 && \text{in } B \\ \textstyle\int_B \eta_0\, dx &= 1. \end{aligned}\right\}$$

If $\gamma = \min_{\overline{B}}\{-\lambda_0 + h(x)\} \geq 0$, then the solution $u(t;u_0)$ of problem (1.1) blows up in finite time for any initial value $u_0 \neq 0$. If $\gamma < 0$, then finite time blow-up occurs if $u_0 > 0$ in $\overline{\Omega}$ and

$$\int_B \eta_0 u_0\, dx > \left(\frac{|\gamma|}{\min_{\overline{B}} a}\right)^{\frac{1}{p-m}}. \tag{4.20}$$

Proof. By Lemma 2.2, without loss of generality we can assume that $u_0 > 0$ in $\overline{\Omega}$. Hence problem (1.1) is satisfied in classical sense for any $t \in [0,T)$, where $T \leq +\infty$ is the maximal existence time. Let us multiply the first equation in (1.1) by η_0 and integrate on B. Integrating by parts and setting $\alpha = \min_{\overline{B}} a(x)$, we get

$$\begin{aligned} \int_B \eta_0 u_t\, dx &= -\int_{\partial B} u^m \frac{\partial \eta_0}{\partial n}\, ds + \int_B \Big(u^m \Delta\eta_0 + hu^m\eta_0 + a(x)u^p\eta_0\Big) dx \\ &\geq \int_B u^m\Big(-\lambda_0 + h + \alpha u^{p-m}\Big)\eta_0\, dx. \end{aligned} \tag{4.21}$$

Let $\gamma = \min_{\overline{B}}\{-\lambda_0 + h(x)\}$. If $\gamma \geq 0$, define $\psi(s) = \alpha s^p + \gamma s^m$ for any $s \geq 0$; if $\gamma < 0$, define

$$\psi(s) = \begin{cases} -(p-m)\left(\frac{|\gamma|}{p}\right)^{\frac{p}{p-m}}\left(\frac{m}{\alpha}\right)^{\frac{m}{p-m}} & \text{if } s \in \left[0, \left(\frac{|\gamma| m}{p\alpha}\right)^{\frac{1}{p-m}}\right] \\ \alpha s^p - |\gamma| s^m & \text{if } s > \left(\frac{|\gamma| m}{p\alpha}\right)^{\frac{1}{p-m}}. \end{cases}$$

In both cases

$$\psi(s) \leq s^m\big(-\lambda_0 + h(x) + \alpha s^{p-m}\big) \qquad \text{in } B$$

for any $s \geq 0$ and ψ is convex in $[0,+\infty)$. Hence, by (4.21), applying Gårding's inequality (since $\int_B \eta_0 = 1$), we get

$$\left(\int_B \eta_0 u\, dx\right)_t = \int_B \eta_0 u_t\, dx \geq \int_B \psi(u)\eta_0\, dx \geq \psi\left(\int_B \eta_0 u\, dx\right).$$

The solution of the scalar problem

$$\left.\begin{aligned} z'(t) &= \psi(z(t)) \quad (t > 0) \\ z(0) &= z_0 \geq 0 \end{aligned}\right\}$$

blows up in finite time for any initial value $z_0 > 0$ if $\gamma \geq 0$, and for $z_0 > \left(\frac{|\gamma|}{\alpha}\right)^{\frac{1}{p-m}}$ if $\gamma < 0$. The blow-up of z implies the blow-up of $\int_B \eta_0 u(t)\, dx$, hence of $u(t,\cdot)$. Thus if $\gamma \geq 0$, we get finite time blow-up for any initial value u_0. If $\gamma < 0$, we get finite time blow-up for initial values satisfying (4.20). This completes the proof ∎

Proof of Proposition 3.5. The first part follows by Lemma 4.4. Assertions (i) and (ii) for $A < 0$: By Proposition 3.4/(i) and 3.4/(ii), respectively, there exists a non-trivial stationary solution u^* of problem (1.1). Hence for any initial value u_0 such that $0 \le u_0 \le u^*$ in $\overline{\Omega}$ the solution of problem (1.1) is global.

Assertions (ii) and (iii) for $A > 0$: By Lemma 2.2 we can assume $u_0 > 0$ in $\overline{\Omega}$ without loss of generality. Then the proof follows as in the proof of Proposition 3.2/(ii) and 3.2/(iii) for $A > 0$.

Assertion (iii): Let $A = 0$. Since $a \neq 0$, we have $\Omega_+ \neq \emptyset$. Using the notation in Lemma 4.4, if B exists such that $\gamma \geq 0$, the solution blows up for any initial value $u_0 \neq 0$. Thus we only need to prove the result if for any B we have $\gamma < 0$.

Let us remark that for any $k \in (0, k_0]$, $k_0 > 0$ suitably chosen, $\underline{u} = k\varphi_0^{\frac{1}{m}}$ is a stationary subsolution of problem (1.1). Indeed,

$$(\Delta + h)\underline{u}^m + a\underline{u}^p \geq k^m \varphi_0\big(|\mu_0| - \|a\|_\infty \|\varphi_0\|_\infty^{\frac{p-m}{m}} k^{p-m}\big) \geq 0$$

provided that $k \leq k_0 = \big(\frac{|\mu_0|}{\|a\|_\infty}\big)^{\frac{1}{p-m}} \|\varphi_0\|_\infty^{-\frac{1}{m}}$. Hence, by Proposition 2.1/(ii), $u(t; \underline{u})$ is monotonically non-decreasing in time.

Since the constant k can be chosen arbitrarily small, for any initial value $u_0 > 0$ in $\overline{\Omega}$ there exists a k such that $\underline{u} \leq u_0$. Thus we only need to prove that $u(t; \underline{u})$ blows up in finite time. We apply Lemma 4.1 to $u(t; \underline{u})$ and get

$$\begin{aligned} \frac{1}{p-1}\int_\Omega \varphi_0^{\frac{p}{m}+1}\underline{u}^{-(p-1)} &\geq |\mu_0| \iint_{Q_\tau} \varphi_0^{\frac{p}{m}+1} u^{m-p} \\ &\geq |\mu_0| (\min_{\overline{\Omega}}\varphi_0)^{\frac{p}{m}+1} \iint_{Q_\tau} u^{m-p} \end{aligned}$$

for any $\tau \in (0, T)$, where $T \leq +\infty$ is the maximal existence time of $u(t; \underline{u})$. Thus there exists a constant $C > 0$ such that

$$\iint_{Q_\tau} u^{m-p} \leq C \qquad \text{for all } \tau \in (0, T). \tag{4.22}$$

Assume by contradiction that $T = +\infty$. Since $m < p$ and $u(t; \underline{u})$ is non-decreasing, (4.22) implies that u is unbounded as $t \to +\infty$. Namely, we first prove that (4.22) implies the unboundedness of $\int_\Omega u(\tau)$ as $\tau \to +\infty$ and that this in turn implies that the hypotheses of Lemma 4.4 are satisfied for $u_0 = u(\tau; \underline{u})$, for sufficiently large τ. Then finite time blow-up will follow, which contradicts the hypothesis $T = +\infty$ and the proof will be complete.

Take B and γ as in Lemma 4.4. As remarked above we only need to prove the case $\gamma < 0$. Let B^* be an open set such that $\overline{B}^* \subseteq B$. Then by the Hölder inequality and

(4.22) we get

$$
\begin{aligned}
|B_*|\tau &= \iint_{B_*\times[0,\tau]} 1 \\
&= \iint_{B_*\times[0,\tau]} u^{\frac{p-m}{p-m+1}} \cdot u^{-\frac{p-m}{p-m+1}} \\
&\le \left(\iint_{B_*\times[0,\tau]} u\right)^{\frac{p-m}{p-m+1}} \left(\iint_{B_*\times[0,\tau]} u^{-(p-m)}\right)^{\frac{1}{p-m+1}} \\
&\le C^{\frac{1}{p-m+1}}\, \tau^{\frac{p-m}{p-m+1}} \left(\int_{B_*} u(\tau)\right)^{\frac{p-m}{p-m+1}}
\end{aligned}
$$

for all $\tau \in (0, +\infty)$. Hence, for a constant $C_1 > 0$,

$$\int_{B^*} u(\tau) \ge C_1 \tau^{\frac{1}{p-m}}$$

for all $\tau \in (0, T)$. Moreover, for some constant $C_2 > 0$,

$$\int_B \eta_0 u(\tau) \ge \min_{B_*} \eta_0 \int_{B_*} u(\tau) \ge C_2 \tau^{\frac{1}{p-m}} > \left(\frac{|\gamma|}{\min_{\overline{B}} a}\right)^{\frac{1}{p-m}}$$

where the last inequality holds for sufficiently large τ. Then for sufficiently large τ condition (4.20) is satisfied and finite time blow-up follows by Lemma 4.4.

Let $a < 0$ in $\overline{\Omega}$. Then, by Proposition 3.4/(iii), there exists a non-trivial stationary solution u^* of problem (1.1). Hence, for any initial value u_0 such that $0 \le u_0 \le u^*$ in $\overline{\Omega}$, the solution of problem (1.1) is global. This completes the proof ∎

References

[1] Alama, S. and G. Tarantello: *On semilinear elliptic equations with indefinite nonlinearities.* Calc. Var. and Part. Diff. Equ. 1 (1993), 439 - 475.

[2] Aronson, D. G., Crandall, M. G. and L. A. Peletier: *Stabilization of solutions of a degenerate nonlinear diffusion problem.* Nonlin. Anal. 6 (1982), 1001 - 1022.

[3] Aronson, D. G. and L. A. Peletier: *Large time behaviour of solutions of the porous medium equation in bounded domains.* J. Diff. Equ. 39 (1981), 378 - 412.

[4] Bandle, C. and M. A. Pozio: *Nonlinear parabolic equations with sinks and sources.* In: Nonlinear Diffusion Equations and Their Equilibrium States, Vol. I (Math. Sci. Res. Inst. Publ. (Berkeley/USA): Vol. 12 - 13). Proc. of a Microprogram held August 25 - September 12, 1986 (ed.: W.-M. Ni). New York et al.: Springer-Verlag 1988, pp. 207 - 216.

[5] Bandle, C. and M. A. Pozio: *On a class of nonlinear Neumann problems.* Ann. Mat. Pura Appl. 157 (1990), 161 - 182.

[6] Bandle, C., Pozio, M. A. and A. Tesei: *The asymptotic behaviour of the solutions of degenerate parabolic equations.* Trans. Amer. Math. Soc. 303 (1987), 487 - 501.

[7] Bandle, C., Pozio, M. A. and A. Tesei: *Existence and uniqueness of solutions of nonlinear Neumann problems.* Math. Z. 199 (1988), 257 - 278.

[8] Berestycki, H., Capuzzo-Dolcetta, I. and L. Nirenberg: *Solutions positives de problèmes elliptiques indéfinis et théorèmes de type Liouville non linéaires.* C.R. Acad. Sci. Paris 317 (1993), 945 - 950.

[9] Berestycki, H., Capuzzo-Dolcetta, I. and L. Nirenberg: *Variational methods for indefinite superlinear homogeneous elliptic problems.* Nonlin. Diff. Equ. Appl. 2 (1995), 553 - 572.

[10] Bertsch, M. and L. A. Peletier: *A positivity property of solutions of nonlinear diffusion equations.* J. Diff. Equ. 53 (1984), 30 - 47.

[11] de Mottoni, P., Schiaffino, A. and A. Tesei: *Attractivity properties of nonnegative solutions for a class of nonlinear degenerate parabolic problems.* Ann. Mat. Pura Appl. 136 (1984), 35 - 48.

[12] Di Benedetto, E.: *Continuity of weak solutions to a general porous medium equation.* Indiana Univ. Math. J. 32 (1983), 83 - 118.

[13] Galaktionov, V. A.: *On a blow-up set for the quasilinear heat equation* $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$. J. Diff. Equ. 101 (1993), 66 - 79.

[14] Kaplan, S.: *On the growth of solutions of quasilinear parabolic equations.* Comm. Pure Appl. Math. 16 (1963), 305 - 330.

[15] Kersner, R.: *Nonlinear heat conduction with absorption: space localization and extinction in finite time.* SIAM J. Appl. Math. 43 (1983), 1274 - 1285.

[16] Ladyzenskaja, O. A., Solonnikov, V. A. and N. N. Ural'tceva: *Linear and Quasilinear Equations of Parabolic Type* (Transl. Math. Monographs: Vol. 23). Providence (R.I.): Amer. Math. Soc. 1968.

[17] Namba, T.: *Density-dependent dispersal and spatial distribution of a population.* J. Theor. Biol. 86 (1980), 351 - 363.

[18] Ni, W.-M., Sacks, P. E. and J. Tavantzis: *On the asymptotic behavior of solutions of certain quasilinear parabolic equations.* J. Diff. Equ. 54 (1984), 97 - 120.

[19] Okubo, A.: *Diffusion and Ecological Problems: Mathematical Models* (Biomathematics: Vol. 10). Berlin et al.: Springer-Verlag 1980.

[20] Peletier, L. A. and A. Tesei: *Global bifurcation and attractivity of stationary solutions of a degenerate diffusion equation.* Adv. Appl. Math. 7 (1986), 435 - 454.

[21] Pozio, M. A. and A. Tesei: *Support properties of solutions for a class of degenerate parabolic problems.* Comm. Part. Diff. Equ. 12 (1987), 47 - 75.

Received 12.06.1996