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εk^0-Subdifferentials of Convex Functions

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Abstract. The paper as a contribution to convex analysis in ordered linear topological spaces. For any convex function f from a Banach space X into a partially ordered one Y endowed with a convex cone K some properties of the εk^0-subdifferential $\partial^{\leq}_{\varepsilon k^0} f(x)$ of f are examined. The non-emptyness of $\partial^{\leq}_{\varepsilon k^0} f(x)$ is proved, whenever Y is a normal order complete vector lattice and f belongs to the class of functions which are continuous and convex with respect to the cone K. For the real-valued case Bronsted and Rockafellar have proved that the set of subgradients of a lower semicontinuous function f on a Banach space X is dense in the set of ε-subgradients [21]. We deduce a similar result for a class of εk^0-subdifferentials of functions which takes values in an ordered linear topological space Y.

Keywords: *Subdifferentials, ε-subdifferentials, order complete vector lattices, scalarization, properly efficient elements*

AMS subject classification: 90 C 25, 90 C 48

1. Introduction

The notion of ε-subdifferential of a real-valued convex function can be found, for instance, in the papers of Bronsted and Rockafellar [6] and Moreau [18]. During the last years many mathematicians have studied convex functions taking values in ordered linear topological spaces and their subdifferentials. Among them, we can quote [3 - 5, 7, 8, 10, 13, 18, 19, 27, 29, 30, 33]. Especially, Borwein [2, 4] has proved many conditions for subdifferentials to be non-empty. Borwein [2], Kutateladze [14] and Thera [26] have examined the notion of ε-subdifferentials for convex functions taking values in an ordered linear topological space and for ε in the positive cone of this space. Loridan [15] has considered this concept for finite-dimensional spaces. Also an operational calculus was established by these authors.

Now let us go into particulars. Let X and Y be linear topological spaces, Y ordered by a convex cone $K \subset Y$, and $C \subset X$ a convex set. For a convex function $f : C \to Y$ and $\varepsilon \in K$ the *ε-subdifferential* $\partial^{\varepsilon} f(x_0)$ at $x_0 \in C$ is the set of all continuous linear functions $T : X \to Y$ such that

$$T(x - x_0) \in f(x) - f(x_0) + \varepsilon - K$$

for all $x \in C$. Hiriart-Urruty [9] has proved that the ε-subdifferential multifunction of a lower semicontinuous real-valued convex function f is of locally Lipschitz type

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on its set of continuity points. The ε-subdifferential for the vector-valued case has some impressive properties, but the above locally Lipschitz-type behaviour is absent, if $\operatorname{int} K = \emptyset$. Therefore, Thibault [28] has introduced the notion of *V-subdifferentials* which enjoys the locally Lipschitz behaviour whenever V is a neighbourhood of zero.

Let $x, y \in Y$. The *order intervall* $[x, y]$ between x and y is defined by

$$[x, y] = \{(\{x\} + K) \cap (\{y\} - K)\}.$$

Further, taking $V = [-\varepsilon, \varepsilon]$, the notion of V-subdifferential includes that of ε-subdifferential, since $\partial_V f(x_0)$ is the set of all continuous linear maps $T : X \to Y$ such that

$$T(x - x_0) \in f(x) - f(x_0) + V - K$$

for all $x \in C$. Thibault [28] has proved the non-emptyness of V-subdifferentials of functions f which are at any point of their domain the limit of nested families of continuous affine minorants.

Thierfelder [29] has introduced different subdifferentials of functions which take their values in ordered linear topological spaces and has examined their applicability for the formulation of optimality conditions of vector optimization problems. Tammer [25] has generalized these definitions by introducing ε-subdifferentials of functions acting from a Banach space into an ordered one.

Our paper is concerned with the study of properties of ε-subdifferentials. In Section 2 we recall some preliminary definitions and results used later. In Section 3 we prove non-emptyness of the ε-subdifferential for the class of maps which are both continuous and convex with respect to a cone K and acting from a Banach space X into an order complete normal vector lattice Y. The main idea of the proof is a separation theorem of Elster and Nehse [8] for sets in product spaces. Section 3 contains also our main result (Theorem 2). We prove that for an order complete Banach lattice Y a subdifferential in the sense of Thierfelder [29] of functions which are continuous and convex with respect to a cone K is dense in the ε-subdifferential. The proof is based upon the variational principle of Ekeland and some results of Thierfelder [29] concerning some classes of subdifferentials. Additionally, our proof requieres a computation of the subdifferential of a special vector-valued function.

2. Preliminaries

Now we give necessary definitions and notational conventions. Let (Y, K) be an ordered linear topological space, $K \subset Y$ being a closed convex cone. Additionally, K is supposed to be pointed, i.e. $K \cap -K = \{0\}$. The order induced by K is denoted by $\leq_K$, i.e. we write $z \leq_K y$ for $z, y \in Y$ if $y - z \in K$.

A function $f : X \to Y$ is called *K-convex* on X if, for all $x_1, x_2 \in X$ and $t \in [0, 1]$,

$$f(tx_1 + (1-t)x_2) \leq_K tf(x_1) + (1-t)f(x_2)$$

holds. The (topological) *dual cone* K^* is defined by

$$K^* = \left\{y^* \in Y^* : y^*(y) \geq 0 \text{ for all } y \in K\right\}$$

where Y^* denotes the topological dual space of Y. The partial ordering induced by the dual cone K^* is called *dual partial ordering.* The set

$$(K^*)^0 = \left\{ y^* \in Y^* : y^*(y) > 0 \text{ for all } y \in K \setminus \{0\} \right\}$$

is called the *quasi-interior* of the dual cone K^*. A convex subset B of a convex cone $K \neq \{O_Y\}$ is called a *base* for K, if each $y \in K \setminus \{O_Y\}$ has a unique representation $y = \lambda b$ for some $\lambda > 0$ and $b \in B$.

Lemma 1 (see, e.c., Borwein [3: Proposition 2.7]). *Let K be a cone in a real locally convex space Y. If K has a base, then $(K^*)^0 \neq \emptyset$.*

Further, we need some definitions and results concerning vector lattices and refer to Jamesson [12], Meyer-Nieberg [17], Peressini [20], Schaefer [22] and Schwarz [24].

Let (Y, K) be an ordered vector space, $M \subseteq Y$ a subset and $y \in Y$. If $m \leq_K y$ for all $m \in M$, then M is said to be *bounded from above* (or *majorized*). Any element with this property is called an *upper bound* (or *majorant*) of M. Subsets bounded from below and lower bounds are defined analogously. If $y \in Y$ is an upper bound of the set $M \subseteq Y$, then y is called the *supremum* of M and denoted by $y = \sup M$ if $y \leq_K u$ for any other upper bound u of M. The infimum of a set $M \subseteq Y$ denoted by $\inf M$ is defined analogously. In particular, if $x, y \in (Y, \leq_K)$, then we write

$$x \vee y = \sup(x, y) \qquad \text{and} \qquad x \wedge y = \inf(x, y).$$

An ordered vector space Y is called a *vector lattice* or *Riesz space*, if for all $x, y \in Y$ the supremum $x \vee y$ exists. We use the following notations:

$$\begin{aligned} x^+ &= x \vee 0 && (\textit{positive part} \text{ of } x) \\ x^- &= (-x) \vee 0 && (\textit{negative part} \text{ of } x) \\ |x| &= x \vee (-x) && (\textit{absolute value} \text{ of } x). \end{aligned}$$

Remark that by definition $x^+, x^-, |x| \in K$. Further, x^+ and x^- are *disjoint*, i.e. $x^+ \wedge x^- = 0$. The cone $K \subset Y$ is called *reproducing*, if $K - K = Y$.

The following results are standart ones.

Lemma 2 (see, e.c., Meyer-Nieberg [17: Theorem 1.1.1] and Jamesson [12: Theorem 2.3.10]). *Let (Y, K) be a vector lattice and $x, y \in Y$. Then:*

(i) $|x| = x^+ + x^-$.

(ii) $x \leq_K y$ *if and only if* $x^+ \leq_K y^+$ *and* $y^- \leq_K x^-$.

(iii) $|x| \leq_K y$ *if and only if* $\pm x \leq_K y$, *i.e.*, $-y \leq_K x \leq_K y$.

A norm $\|\cdot\|$ on a vector lattice (Y, K) is called *lattice norm*, if $|x| \leq |y|$ $(x, y \in Y)$ implies $\|x\| \leq \|y\|$. Then $(Y, \|\cdot\|)$ is called *normed vector lattice* and, in case of norm completeness, *Banach lattice*.

Lemma 3 (see, e.c., Meyer-Nieberg [17: Proposition 1.1.6]). *In a normed vector lattice (Y, K) the cone K is closed.*

A partially ordered linear topological space (Y, K) is *normal*, if there exists a base of neighbourhoods V at zero with

$$V = (V - K) \cap (K - V). \tag{1}$$

Such neighbourhoods are said to be *full* or *saturated*. There are many equivalents for and consequences of equation (1) for which the reader is referred to the standart literature [12, 20, 22]. We will interchangeably refer to K and Y as being *normal*.

Lemma 4 (see, e.c., Borwein [3: Example 2.2]). *Any Banach lattice or locally convex lattice is normal.*

A vector lattice Y is called *order complete* (or *Dedekind complete*, or *conditionally complete*), if every non-empty subset B of Y which has an upper bound has a supremum. Order completeness plays in the theory of vector lattices a similar fundamental role as topological completeness in functional analysis. But order completeness and norm completeness are independent properties.

Let us consider some general relationsships about dual vector spaces. A vector lattice (Y, K) is called *Archimedean*, if $x \leq_K 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above. Let (Y, K) and (Z, P) be Archimedean vector lattices. An operator $T \in L(X, Y)$ is called

(i) *positive*, if $T(K) \subset P$;

(ii) *regular*, if it is the difference of two positive operators.

We denote by $L^r(Y, Z)$ the collection of all regular operators $T : X \to Y$ and by $Y^\sim = L^r(Y, R)$ the order dual of Y.

Lemma 5 (see, e.c., Meyer-Nieberg [17: Proposition 1.3.7]). *The dual Y^* of a normed vector lattice Y is a Banach lattice. If Y is a Banach lattice, then $Y^* = Y^\sim$.*

Lemma 6 (see, e.c., Borwein et al. [5]). *All dual Banach lattices are order complete.*

Example 1. Let us illustrate by some examples which classical Banach lattices are order complete ones and which corresponding positive cones are based. We refer the reader to Jahn [11] and Borwein et al. [5].

a) The classical Lebesgue spaces $L_p(\Omega)$ $(1 \leq p < \infty)$ with cone $K_{L_p} = \{f \in L_p(\Omega) : f(x) \geq 0 \text{ a.e. on } \Omega\}$ are order complete Banach lattices and K_{L_p} has a base.

b) The classical space $L_\infty(\Omega)$ with cone $K_{L_\infty} = \{f \in L_\infty(\Omega) : f(x) \geq 0 \text{ a.e. on } \Omega\}$ is an order complete Banach lattice and K_{L_∞} has a base.

c) The classical sequence spaces l_p $(1 \leq p < \infty)$ with cone $K_{l_p} = \{x \in l_p : \xi_i \geq 0 \text{ for all } i \in \mathbb{N}\}$ are order complete Banach lattices and K_{l_p} has a base.

d) The classical sequence space l_∞ with cone $K_{l_\infty} = \{x \in l_\infty : \xi_i \geq 0 \text{ for all } i \in \mathbb{N}\}$ is an order complete Banach lattice and K_{l_∞} has a base.

e) The classical space $C[0,1]$ of all continuous functions $f : [0,1] \to \mathbb{R}$ with cone of nowhere-negative functions is a non-complete lattice. Consequently, this space is not useful for our considerations.

Now, let us consider functions $f : X \to Y$ between linear spaces X and Y, X real and Y ordered by a convex cone K. The set

$$\operatorname{epi}(f) = \Big\{(x,y) \in X \times Y : y \geq_K \{f(x)\}\Big\}$$

is called the *epigraph* of f. It is well known that a convex function $f : X \to Y$ can be characterized by epi(f).

Lemma 7 (see, e.c., Jahn [11: Theorem 2.6]). *Let X and Y be linear spaces, X real and Y ordered by a convex cone K. Further, let $\emptyset \neq W \subset Y$ and $f : W \to Y$ a given function. Then f is K-convex if and only if* epi(f) *is convex.*

Now, we have to consider some known aspects about continuity of functions and refer to [16]. A real-valued function $f : S \subseteq X \to \mathbb{R} \cup \{\infty\}$ is *lower semicontinuous* at x_0, if for each $\varepsilon > 0$ there exists a neighbourhood $U(x_0)$ such that $f(x) \geq f(x_0) - \varepsilon$ for all $x \in U(x_0) \cap S$. In the case of vector-valued functions (maps) we get an analogous notion:

Let X and Y be linear topological spaces, Y ordered by a convex cone K, and $S \subset X$. A function $f : S \to Y$ is called *K-continuous* at $x_0 \in S$, if for any neighbourhood $V(f(x_0))$ there exists a neighbourhood $U(x_0)$ such that $f(x) \in V + K$ for all $x \in U(x_0) \cap S$. The function f is K-continuous on S, if it is K-continuous at any point of S. Whenever $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ (the cone of non-negative real numbers), K-continuity is the same as lower semicontinuity. It is easy to see that any continuous function $f : S \subseteq X \to Y$ is K-continuous (compare with [16]). Especially, any linear continuous functional $z^* \in (K^*)^0$ is lower semicontinuous.

Now we study an important property of the composition of cone-continuous functions [16]. Let X and Y be real linear topological spaces, ordered by convex cones H and K, respectively. Further, let $f : W \subseteq X \to Y$ be a given function. Then f is said to be *non-decreasing* (or *monotonic*) at $x_0 \in X$ with respect to (H, K), if

$$x \in W \cap (x_0 - H) \quad \Longrightarrow \quad f(x) \in f(x_0) - K.$$

For a continuous linear functional $z^* \in Y^*$ this definition is equal to the usual definition of monotonicity of functionals, i.e.

$$y \in Y \cap (y_0 - K) \quad \Longrightarrow \quad z^*(y) \leq z^*(y_0).$$

Especially, each functional $z^* \in (K^*)^0$ is non-decreasing.

Lemma 8 (see, e.c., Luc [16: Theorem 5.11]). *Let X, Y and Z be linear topological spaces, Y and Z ordered by cones K and L, respectively. Further, let $W \subset X$ have at least one accumulation point x_0, let $f : W \to Y$ and $g : Y \to Z$ be given functions, f being K-continuous at x_0. Then the composition $g \circ f : W \to Z$ is L-continuous at x_0 if and only if g is L-continuous and non-decreasing on Y.*

For the special case that $f : X \to Y$ is K-continuous and $z^* \in (K^*)^0$ there follows lower semicontinuity of $z^* \circ f$ for all $x \in X$.

Finally, we need two special classes of Banach spaces and refer to [21: p. 35; p. 95]. A Banach space X is called a *Gâteaux differentiability space* provided the set G of points of Gâteaux differentiability of a convex continuous function defined on a convex subset $D \subset X$ is necessarily dense in D. It is well known that the class of Gâteaux differentiability spaces contains the class of separable Banach spaces (compare with Phelps [21: p. 35, p. 95]).

A Banach space X is called *weakly compactly generated* provided there exists a weakly compact subset $W \subset X$ whose linear span is dense in X. It is well known that any separable or reflexive Banach space is weakly compactly generated. Further, any weakly compactly generated Banach space is a Gâteaux differentiability space.

3. Properties of εk^0-subdifferentials

3.1 The vector-valued subdifferential. Many authors were concerned with properties of subdifferentials of vector-valued functions (see, e.g., [3 - 5, 8, 10, 13, 18, 19, 27, 29 - 31, 33]). As in [3, 5] we adjoin an abstract maximal element infty (∞) to Y and K and denote the new objects by $\dot{Y}$ and $\dot{K}$. The element *infty* satisfies

$$a \cdot \infty = \infty, \qquad y + \infty = \infty, \qquad 0 \cdot \infty = 0, \qquad y \leq \infty$$

for any positive real α and any $y \in Y$. The reason for adjoning ∞ is to allow us, as in the real-valued case, to work with functions which are only defined on a subset of the space X. For any $z^* \in (K^*)^0$ we set $z^*(\infty) = \infty$ and so extend z^* to $\dot{Y}$. So it is useful to introduce the set

$$\operatorname{dom}(f) = \left\{ x \in X : z^*(f(x)) < \infty \text{ for some } z^* \in (K^*)^0 \right\}$$

which is called the *effective domain* of the function $f : X \to Y$.

Definition 1. Let X and Y be linear topological spaces, Y ordered by a cone K, $f : X \to Y$ a given function and $x_0 \in \operatorname{dom}(f)$ a given point. Then the set

$$\partial^{\leq} f(x_0) = \left\{ T \in L(X,Y) : T(h) \lesssim_K f(x_0 + h) - f(x_0) \text{ for all } h \in X \right\}$$

is called the *subdifferential* of f at x_0, and each $T \in \partial^{\leq} f(x_0)$ is called a *subgradient* of f at x_0. Also, by definition, if $x_0 \notin \operatorname{dom}(f)$, then $\partial^{\leq} f(x_0) = \emptyset$.

In Section 1 we mentioned that Thierfelder [29] defined different subdifferentials of functions which take their values in ordered linear topological spaces. Let us quote some properties of these subdifferentials that will be used later on.

Definition 2. Let X and Y be linear topological spaces, Y ordered by a cone K, and $f : X \to Y$ a given function. Further, let $k^0 \in K \setminus \{0\}$ and $\varepsilon > 0$. Then we set:

a) $$\partial^{(\leq)} f(x_0) = \left\{ T \in L(X,Y) \;\middle|\; \begin{array}{l} \text{for all } z^* \in K^* \text{ and all } h \in X : \\ (z^* \circ T)(h) \leq (z^* \circ f)(x_0 + h) - (z^* \circ f)(x_0) \end{array} \right\}$$

b) $\partial^{\not>} f x_0) = \left\{ T \in L(X,Y) \middle| T(h) \not>_K f(x_0+h) - f(x_0) \text{ for all } h \in X \right\}$
where $y_1 \not>_K y_2$ means $y_1 - y_2 \notin K \setminus \{0\}$.

c) $\partial^{(\not>)} f(x_0) = \left\{ T \in L(X,Y) \middle| \begin{array}{l} \text{there exists } z^* \in (K^*)^0 \text{ such that, for all } h \in X, \\ (z^* \circ T)(h) \leq (z^* \circ f)(x_0+h) - (z^* \circ f)(x_0) \end{array} \right\}.$

Thierfelder [29] has given characterizations of solutions of vector optimization problems by means of vector-valued subdifferentials. For the following definition we refer the reader to Schönfeld [23].

Definition 3. Let X and Y be linear topological spaces, Y ordered by a cone K, and $f : X \to Y$ a function. Then $x_0 \in X$ is called *properly efficient element* of f, if there exists a $z^* \in (K^*)^0$ such that $(z^* \circ f)(x_0) \leq (z^* \circ f)(x)$ for all $x \in X$.

Lemma 9 (Thierfelder [29]). *Let X and Y be linear topological spaces, Y ordered, and $f : X \to Y$ a function. Then $x_0 \in X$ is a properly efficient element of f if and only if $O \in \partial^{(\not>)} f(x_0)$.*

Lemma 10 (Thierfelder [29]). *Let X and Y be linear topological spaces, Y ordered by a cone K, and $f : X \to Y$ a function. Then for $x_0 \in X$ the equality $\partial^{\leq} f(x_0) = \partial^{(\leq)} f(x_0)$ holds, if K is closed.*

Further we need the following rule of sums for subdifferentials introduced by Thierfelder.

Definition 4. Let Y be a linear topological space ordered by a convex cone K. Then the order structure is called *conditionally complete* if any set bounded from below has an infimum.

Lemma 11 (Thierfelder [29]). *Let X and Y be linear topological spaces, Y ordered by a cone K with weakly compact base, and let $f_1, f_2 : X \to Y$ be convex functions, continuous at $x_0 \in X$. Additionally, let*

(i) *the order structure of Y be conditionally complete*

or

(ii) *X be a Gâteaux differentiability space.*

Then $\partial^{(\leq)} f_1(x_0) + \partial^{(\not>)} f_2(x_0) = \partial^{(\not>)}(f_1 + f_2)(x_0)$.

Remarks. Many versions of this lemma with different assumptions on X and Y are known. An essential part of their proof is the equation

$$z^* \circ \partial^{\leq} f(x_0) = \partial(z^* \circ f(x_0)) \tag{2}$$

for all $z^* \in (K^*)^0$. For example, Zowe [31, 32] has shown (2) for the special case where X is weakly compactly generated and Y is locally convex with K having a weakly compact base. Borwein [3] has proved (2) under the assumption that X is a Gâteaux differentiability space, Y has weakly compact intervalls, K is closed and normal and $(K^*)^0 \neq \emptyset$. Jahn [11] has examined (2) under the assumptions that X and Y are reflexive Banach spaces and K has a weakly compact base. Valadier [39] proved that the above result applies to any separable and reflexive Banach lattice Y and to Y equal the Lebesgue space L_1.

3.2 The Non-emptyness of the εk^0-subdifferential. The following definition is a direct transcription of the usual definition of ε-subdifferential of real-valued convex functions which goes back to Tammer [25].

Definition 5. Let X and Y be linear topological spaces, Y ordered by a cone K, and $f : X \to Y$ a function. Further, let $k^0 \in K \setminus \{0\}$, $x_0 \in \operatorname{dom}(f)$ and $\varepsilon > 0$. Then the set

$$\partial^{\leq}_{\varepsilon k^0} f(x_0) = \left\{ T \in L(X,Y) \,\middle|\, T(h) \leq_k f(x_0 + h) - f(x_0) + \varepsilon k_0 \text{ for every } h \in X \right\}$$

is called the εk^0-*subdifferential* of f at x_0 and each $T \in \partial^{\leq}_{\varepsilon k^0} f(x_0)$ is called εk^0-*subgradient of f at x_0. Also, by definition, in case $x_0 \notin \operatorname{dom}(f)$ we set $\partial^{\leq}_{\varepsilon k^0} f(x_0) = \emptyset$.*

Definition 6. The operator $P_X : X \times Y \to X$ defined by $P_X(x, y) = x$ for $(x, y) \in X \times Y$ is called *projector* of $X \times Y$ onto X. Further, for any non-void set $A \subseteq X \times Y$ the set $P_X(A) = \{x \in X \mid (x, y) \in A \text{ for some } y \in Y\}$ is called *projection* of A onto X.

The following separation theorem of Elster and Nehse [8] for convex sets in product spaces is a principal auxilliary result for our examinations.

Lemma 12 (Elster and Nehse [8]). *Let X be a linear topological space, (Y, K) a normal order complete vector lattice, and let $A, B \subset X \times Y$ be K-convex with $\operatorname{int} P_X(A) \cap P_X(B) \neq \emptyset$. Further, suppose the following:*

(i) $$\left.\begin{matrix} (x, y_1) \in A \\ (x, y_2) \in B \end{matrix}\right\} \implies y_1 \geq_K y_2.$$

(ii) *For any $x_0 \in \operatorname{int} P_X(A) \cap P_X(B)$ there exists a $\tilde{y} \in Y$ with $(x_0, \tilde{y}) \in A$, and for any neighbourhood $V(0) \subset Y$ there exists a neighbourhood $U(0) \subset X$ such that*

$$\left.\begin{matrix} x \in x_0 + U(0) \\ (x, y) \in A \end{matrix}\right\} \implies y \in \tilde{y} + V(0).$$

Then there exist $T \in L(X, Y)$ and $y_0 \in Y$ such that

$$T(x_1) - y_1 \leq_K y_0 \leq_K T(x_2) - y_2$$

for all $(x_1, y_1) \in A$ and all $(x_2, y_2) \in B$.

To ensure the applicability of Lemma 12 we have to prove that epi(f) has a non-empty interior.

Lemma 13. *Let X and Y be linear topological spaces, Y ordered by a cone K, and let $f : X \to Y$ be a function continuous at some $x_0 \in \operatorname{dom}(f)$ and K-continuous at dom(f). Then dom(f) and epi(f) have non-empty interior.*

Proof. Let $x_0 \in \operatorname{dom}(f)$. Then, by definition, there exists $z^* \in (K^*)^0$ such that $z^*(f(x_0)) < \infty$. The composition $z^* \circ f$ is continuous at x_0. Consequently, there exists for any $\varepsilon > 0$ a neighbourhood $U(x_0)$ such that $|z^*(f(x)) - z^*(f(x_0))| < \varepsilon$ for all $x \in U(x_0)$. The triangle inequality yields, for all $x \in U(x_0)$,

$$|z^*(f(x))| - |z^*(f(x_0))| \leq |z^*(f(x)) - z^*(f(x_0))| < \varepsilon,$$

i.e.

$$|z^*(f(x))| < \varepsilon + |z^*(f(x_0))|.$$

So we have

$$z^*(f(x)) \leq |z^*(f(x))| < \varepsilon + |z^*(f(x_0))| < \infty$$

for all $x \in U(x_0)$ by the assumption $x_0 \in \operatorname{dom}(f)$. Therefore $U(x_0) \subset \operatorname{dom}(f)$. Consequently, x_0 is not a boundary point of $\operatorname{dom}(f)$.

Further, because of the fact proved in the first part and the K-continuity of f at $\operatorname{dom}(f)$ we have for any neighbourhood $V(f(x_0)) \subset Y$ a neighbourhood $W(x_0) \subset \operatorname{dom}(f)$ such that $f(x) \in V(f(x_0)) + K$ for all $x \in W(x_0)$. So $W(x_0) \times V(f(x_0))$ is a neighbourhood of $(x_0, f(x_0)) \in \operatorname{epi}(f)$ which entirely lies in $\operatorname{epi}(f)$ ∎

Now we give an important condition about the non-emptyness of $\partial^{\leq}_{\varepsilon k^0} f(x)$.

Theorem 1. *Suppose that X is a Banach space and (Y, K) an oder complete normal vector lattice. Further, let $f : X \to Y$ be a function continuous at $x_0 \in \operatorname{dom}(f)$ as well as K-continuous and K-convex. Then $\partial^{\leq}_{\varepsilon k^0} f(x_0) \neq \emptyset$ for any $\varepsilon > 0$ and $k^0 \in K \setminus \{0\}$.*

Proof. We set $A = \operatorname{epi}(f)$ and $B = (x_0, f(x_0) - \varepsilon k^0)$. Because of the K-convexity of f the set A is convex, by Lemma 7. By definition the one-point set B is convex. From the proof of Lemma 13 it follows that $x_0 \in \operatorname{int} P_X(A) \cap P_X(B)$. The definition of the epigraph of f implies

$$\left.\begin{array}{r} (x_0, y) \in A \\ (x_0, f(x_0) - \varepsilon k^0) \in B \end{array}\right\} \quad \Longrightarrow \quad y \geq_K f(x_0) - \varepsilon k^0.$$

Together with the K-continuity of f all assumptions of Lemma 12 are fulfilled. Then there exists an operator $T \in L(X, Y)$ such that $T(x_1) - y_1 \leq_K T(x_0) - (f(x_0) - \varepsilon k^0)$ for all $(x_1, y_1) \in A$. In particular, if we set $x_1 = x$ and $y_1 = f(x)$, we get

$$T(x) - f(x) \leq_K T(x_0) - (f(x_0) - \varepsilon k^0)$$

for all $x \in \operatorname{dom}(f)$, i.e. $T(x - x_0) \leq_K f(x) - f(x_0) + \varepsilon k^0$ ∎

Remarks. The order complete Banach lattices L_p $(1 < p \leq \infty)$ and l_p $(1 \leq p \leq \infty)$ are normal and can be substituted for the space Y in Theorem 1. A result similar to Theorem 1 was proved by Thibault [28] showing that non-emptyness of the V-subdifferential for the class of functions which are at any point of their domain the limit of nested families of continuous affine minorants.

Thierfelder [29] introduced a series of vector-valued subdifferentials as mentioned in Subsection 3.1. Tammer [25] applied these definitions to εk^0-subdifferentials. Following the pattern of Tammer [25] we get some new definitions.

Definition 7. Let X and Y be linear topological spaces, Y ordered by a cone K, and $f : X \to Y$ a given function. Further, let $k^0 \in K \setminus \{0\}$ and $\varepsilon > 0$. Then we set:

a) $\partial^{(\leq)}_{\varepsilon k^0} f(x_0)$

$$= \left\{ T \in L(X,Y) \middle| \begin{array}{l} \text{for all } z^* \in K^* \text{ and } h \in X : \\ (z^* \circ T)(h) \leq (z^* \circ f)(x_0 + h) - (z^* \circ f)(x_0) + \varepsilon z^*(k^0) \end{array} \right\}.$$

b) $\partial^{\not>}_{\varepsilon k^0} f(x_0) = \left\{ T \in L(X,Y) \middle| T(h) \not>_K f(x_0 + h) - f(x_0) + \varepsilon k^0 \text{ for all } h \in X \right\}$

where $y_1 \not>_K y_2$ means $y_1 - y_2 \notin K \setminus \{0\}$.

c) $\partial^{(\not>)}_{\varepsilon k^0} f(x_0)$

$$= \left\{ T \in L(X,Y) \middle| \begin{array}{l} \text{there exists } z^* \in (K^*)^0 \text{ such that, for all } h \in X, \\ (z^* \circ T)(h) \leq (z^* \circ f)(x_0 + h) - (z^* \circ f)(x_0) + \varepsilon z^*(k^0) \end{array} \right\}.$$

Analogous to Thierfelder [29] the relation

$$\partial^{\leq}_{\varepsilon k^0} f(x_0) \subseteq \partial^{(\leq)}_{\varepsilon k^0} f(x_0) \subseteq \partial^{(\not>)}_{\varepsilon k^0} f(x_0) \subseteq \partial^{\not>}_{\varepsilon k^0} f(x_0)$$

holds because $(K^*)^0 \neq \emptyset$ and K is closed. Therefore, we get the following

Corollary 1. *Let X be a Banach space and (Y, K) an order complete normal vector lattice. Further, let there exist a point $x_0 \in \operatorname{dom}(f)$ such that the function $f : X \to Y$ is continuous and let f be as well as K-continuous and K-convex at $\operatorname{dom}(f)$. Then*

$$\partial^{(\leq)}_{\varepsilon k^0} f(x_0) \neq \emptyset, \qquad \partial^{(\not>)}_{\varepsilon k^0} f(x_0) \neq \emptyset, \qquad \partial^{\not>}_{\varepsilon k^0} f(x_0) \neq \emptyset$$

for any $\varepsilon > 0$ and $k^0 \in K \setminus \{0\}$.

3.3 A density result for εk^0-subdifferentials. Now we are able to formulate and to prove our main result. It is a generalization of a result of Bronsted and Rockafellar [21].

For real-valued functions the following definitions are special cases of Definitions 1 and 5, respectively.

Definition 8. Let f be a proper lower semicontinuous function on a Banach space X and $x_0 \in \operatorname{dom}(f)$. Then the set

$$\partial f(x_0) = \left\{ x^* \in X^* \middle| x^*(h) \leq f(x_0 + h) - f(x_0) \text{ for all } h \in X \right\}$$

is called the *subdifferential* of f at x_0, and each $x^* \in \partial f(x_0)$ is called a *subgradient* of f at x_0. Also, if $x_0 \notin \operatorname{dom}(f)$, then by defintion $\partial f(x_0) = \emptyset$.

Definition 9. Let f be a proper lower semicontinuous function on a Banach space X, $x_0 \in \operatorname{dom}(f)$ and $\varepsilon > 0$. Then the set

$$\partial_\varepsilon f(x_0) = \left\{ x^* \in X^* \middle| x^*(h) \leq f(x_0 + h) - f(x_0) + \varepsilon \text{ for all } h \in X \right\}$$

is called *ε-subdifferential* of f at x_0, and each $x^* \in \partial_\varepsilon f(x_0)$ is called a *ε-subgradient* of f at x_0. Also, if $x_0 \notin \operatorname{dom}(f)$, then by definition $\partial_\varepsilon f(x_0) = \emptyset$.

Lemma 14 (see, e. c., Phelps [21: Theorem 3.17]). *Let f be a convex proper lower semicontinuous function on a Banach space X. Then, given any point $x_0 \in \operatorname{dom}(f), \varepsilon > 0, \lambda > 0$ and $x_0^* \in \partial_\varepsilon f(x)$, there exists $x \in \operatorname{dom}(f)$ and $x^* \in X^*$ such that*

$$x^* \in \partial f(x), \qquad \|x - x_0\| \le \frac{\varepsilon}{\lambda}, \qquad \|x^* - x_0^*\| \le \lambda.$$

In particular, the domain of ∂f is dense in $\operatorname{dom}(f)$.

Further, for the proof of our main result we need the following real-valued variational principle of Ekeland.

Lemma 15 (see, e.c., Aubin and Ekeland [1: p. 261/262]). *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{\infty\}$ a proper, lower semicontinuous and bounded from below function. For $\varepsilon > 0$ choose $x_0 \in X$ such that*

$$f(x_0) \le \inf_{x \in X} f(x) + \varepsilon.$$

Then for each $\mu > 0$ there exists an element $x_\varepsilon \in X$ such that

1. $f(x_\varepsilon) \le \inf_{x \in X} f(x) + \varepsilon$
2. $\|x_\varepsilon - x_0\| \le \frac{1}{\mu}$
3. $f_{\varepsilon\mu}(x_\varepsilon) \le f(x) + \mu\varepsilon \|x - x_\varepsilon\|$ $(x \ne x_\varepsilon)$ *where* $f_{\mu\varepsilon}(x) = f(x) + \mu\varepsilon\|x - x_\varepsilon\|$.

Now we come to our density theorem.

Theorem 2. *Let X and Y be Banach spaces, Y lattice ordered by a cone K with weakly compact base, and let the function $f : X \to Y$ be K-convex and continuous. Further, let $k^0 \in K \setminus \{0\}$ with $\|k^0\|_Y = 1$ and suppose*

(i) *the order structure of Y is conditionally complete*

or

(ii) *X is a Gâteaux differentiability space.*

Then, for any $x_0 \in \operatorname{dom}(f), \varepsilon > 0, \mu > 0$ and $T_0 \in \partial^{\le}_{\varepsilon k^0} f(x_0)$, there exist $x \in \operatorname{dom}(f)$ and $T \in L(X,Y)$ such that

$$T \in \partial^{(\not>)} f(x), \qquad \|x - x_0\| \le \frac{1}{\mu}, \qquad \|T - T_0\| \le \mu\varepsilon.$$

In particular, the domain of $\partial^{(\not>)} f$ is dense in $\operatorname{dom}(f)$.

Proof. By the assumption $T_0 \in \partial^{\le}_{\varepsilon k^0} f(x_0)$ it follows

$$T_0(h) \le_K f(x_0 + h) - f(x_0) + \varepsilon k^0 \qquad \text{for all } h \in X. \tag{3}$$

Set

$$g(x) = f(x) - T_0(x) \qquad (x \in X). \tag{4}$$

We derive a scalar relation for (3) by means functionals $z^* \in (K^*)^0$. This is really possible, because the cone K has a base and therefore $(K^*)^0 \neq \emptyset$ by Lemma 1. The elements of $(K^*)^0$ are non-decreasing ones. Then we get

$$z^*[T_0(h)] \leq z^*[f(x_0 + h)] - z^*[f(x_0)] + \varepsilon z^*(k^0) \tag{5}$$

for all $z^* \in (K^*)^0$. The function g is the difference of two continuous ones. Because of the considerations after Lemma 8, the composition $z^* \circ g$ is lower semicontinuous. Now, we have to show that $z^* \circ g$ is bounded from below on X. By the assumption $T_0 \in \partial^{\leq}_{\varepsilon k^0} f(x_0)$ we have $T_0(x - x_0) \leq_K f(x) - f(x_0) + \varepsilon k^0$ for all $x \in X$, i.e.

$$f(x) \geq_K T_0(x - x_0) + f(x_0) - \varepsilon k^0.$$

Then equation (4) yields

$$g(x) = f(x) - T_0(x) \geq_K T_0(x) - T_0(x_0) + f(x_0) - \varepsilon k^0 - T_0(x)$$

for all $x \in X$, i.e.

$$g(x) \geq_K f(x_0) - T_0(x_0) - \varepsilon k^0.$$

This means g is bounded from below on X. The desired result follows by the monotonicity and the continuity of $z^* \in (K^*)^0$.

In a further step we show that

$$z^*[g(x_0)] \leq \inf_{x \in X} (z^* \circ g) + \varepsilon z^*(k^0)$$

for all $z^* \in (K^*)^0$. From inequality (5)

$$z^*[f(x_0)] \leq z^*[f(x_0 + h)] - z^*[T_0(h)] + \varepsilon z^*(k^0)$$

for all $z^* \in (K^*)^0$ follows. By adding here the term $-z^*[T_0(x_0)]$ we get

$$z^*[f(x_0)] - z^*[T_0(x_0)] \leq z^*[f(x_0 + h)] - z^*[T_0(h)] - z^*[T_0(x_0)] + \varepsilon z^*(k^0),$$

i.e.

$$z^*[f(x_0) - T_0(x_0)] \leq z^*[f(x_0 + h)] - z^*[T_0(x_0 + h)] + \varepsilon z^*(k^0).$$

So we have

$$z^*[g(x_0)] \leq z^*[g(x_0 + h)] + \varepsilon z^*(k^0)$$

for all $h \in X$ and $z^* \in (K^*)^0$. From Lemma 15 we get that for any $\mu > 0$ there exists an $x_\varepsilon \in \operatorname{dom}(f) = \operatorname{dom}(g)$ such that, for all $x \in X$,

(α) $z^*[g(x_\varepsilon)] \leq \inf_{x \in X} z^*[g(x)] + \varepsilon z^*(k^0)$

(β) $\|x_0 - x_\varepsilon\| \leq \frac{1}{\mu}$

(γ) $z^*[g_{\mu\varepsilon k^0}(x_\varepsilon)] \leq z^*[g(x)] + \mu\varepsilon \|x - x_\varepsilon\| z^*(k^0)$

where $g_{\mu\varepsilon k^0}(x) = g(x) + \mu\varepsilon\|x - x_\varepsilon\|k^0$. In general, the element x_ε depends on $z^* \in (K^*)^0$. Because of $g_{\mu\varepsilon k^0}(x_\varepsilon) = g(x_\varepsilon)$ it follows from (γ) that

$$z^*[g(x_\varepsilon)] \le z^*[g(x)] + \mu\varepsilon\|x - x_\varepsilon\|z^*(k^0). \tag{6}$$

Now we set $m(x) = \mu\varepsilon\|x - x_\varepsilon\|\,k^0$ for all $x \in X$. It is obvious that the function m depends on $z^* \in (K^*)^0$. From (6) it follows

$$0 \le z^*[g(x)] + \mu\varepsilon\|x - x_\varepsilon\|z^*(k^0) - z^*[g(x_\varepsilon)],$$

i. e. $0 \le z^*(g+m)(x) - z^*(g+m)(x_\varepsilon)$. This means x_ε is a properly efficient element of the function $g + m$ on X. Then $O \in \partial^{(\not>)}(g+m)(x_\varepsilon)$ by Lemma 9. By Lemma 11 it follows that there exists operators $\overline{S} \in \partial^{(\not>)}g(x_\varepsilon)$ and $-\overline{S} \in \partial^{(\le)}m(x_\varepsilon)$. From Lemma 3 we get that the cone of Y is closed. Therefore $\partial^{(\le)}m(x_\varepsilon) = \partial^{\le}m(x_\varepsilon)$ by Lemma 10.

There remains the task of computing the subdifferential $\partial^{\le}m(x_\varepsilon) = \{S \in L(X,Y) : \|S\| \le \mu\varepsilon\}$. By definition,

$$\partial^{\le}m(x_\varepsilon) = \Big\{S \in L(X,Y) :\ S(l) \le_K m(x_\varepsilon + l) - m(x_\varepsilon) \text{ for all } l \in X\Big\}. \tag{7}$$

Using $-l$ instead of l we get

$$\partial^{\le}m(x_\varepsilon) = \Big\{S \in L(X,Y) :\ S(-l) \le_K m(x_\varepsilon - l) - m(x_\varepsilon) \text{ for all } l \in X\Big\}. \tag{8}$$

Relation (7) implies $S(l) \le_K \mu\varepsilon\|x_\varepsilon + l - x_\varepsilon\|k^0$, i.e.

$$S\left(\frac{l}{\|l\|}\right) \le_K \mu\varepsilon k^0 \tag{9}$$

for all $l \in X$. On the other hand, (8) yields $S(-l) \le_K \mu\varepsilon\|x_\varepsilon - l - x_\varepsilon\|k^0 = \mu\varepsilon\|l\|k^0$, i.e.

$$-S\left(\frac{l}{\|l\|}\right) \le_K \mu\varepsilon k^0 \tag{10}$$

for all $l \in X$. Because of Lemma 2 relations (9) and (10) imply $|S(\frac{l}{\|l\|})| \le_K \mu\varepsilon k^0$ for all $l \in X$. The element $k^0 \in K$ fulfills $k^0 = |k^0|$ and the norm in Y is monotone. So $\|S(\frac{l}{\|l\|})\| \le \mu\varepsilon\|k^0\|$ for all $l \in X$. The vector k^0 has norm 1 which yields $\|S(\frac{l}{\|l\|})\| \le \mu\varepsilon$ for all $l \in X$, i. e.

$$\|S\| := \sup_l \|S(\frac{l}{\|l\|})\| \le \mu\varepsilon. \tag{11}$$

Let $T = S + T_0$. By setting $f_1 = f - g$ and $f_2 = g$ Lemmata 10 and 11 yield

$$\partial^{\le}f_1(x_\varepsilon) + \partial^{(\not>)}f_2(x) = \partial^{(\not>)}(f_1 + f_2)(x_\varepsilon). \tag{12}$$

By the definition of f_1 and f_2 we have $\partial^{(\not>)}(f_1 + f_2)(x_\varepsilon) = \partial^{(\not>)}((f(x_\varepsilon) - g(x_\varepsilon)) + g(x_\varepsilon))$, i.e.

$$\partial^{(\not>)}(f_1 + f_2)(x_\varepsilon) = \partial^{(\not>)}f(x_\varepsilon). \tag{13}$$

From (4) we get $f - g = T_0$, i.e. $f_1 = T_0$. Then by $\partial^{\le}(T_0(x)_{|x=x_\varepsilon}) = T_0$ it follows $\partial^{\le}(f_1(x)_{|x=x_\varepsilon}) = T_0$. Together with $\overline{S} \in \partial^{(\not>)}f_2(x_\varepsilon)$ we have by (12) and (13) the relation $T \in \partial^{(\not>)}f(x_\varepsilon)$. Further,

$$\|T - T_0\| \le \mu\varepsilon$$

by (11) and $\|x_\varepsilon - x_0\| \le \frac{1}{\mu}$ by (β) ∎

Remark. From the considerations after Lemma 11 it follows that the assumptions of Theorem 2 are fullfilled if Y is substituted by (L_p, K_{L_p}) $(1 \leq p < \infty)$ or (l_p, K_{l_p}) $(1 < p < \infty)$.

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