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Some Discrete Inequalities

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Abstract. A number of inequalities with finite differences which are connected with weighted, quasiarithmetic and logarithmic means and some well-known general inequalities are considered.

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1. Introduction

In [3] one can find the following generalization of the so-called Pólya inequality (see [3] and [5: Vol. I/p. 57 and Vol. II/p. 114]) and of a result of Balzer (see [1]).

Theorem A. *Let $x_i : [a,b] \to \mathbb{R}$ $(1 \le i \le n)$ be non-negative increasing functions with continuous first derivative, p_i $(1 \le i \le n)$ real positive numbers with $\sum_{i=1}^n p_i = 1$ and $f : [a,b] \to \mathbb{R}$ a non-negative function.*

a) *If f is non-decreasing, then*

$$\int_a^b \left(\prod_{i=1}^n (x_i(t))^{p_i} \right)' f(t)\,dt \ge \prod_{i=1}^n \left(\int_a^b x_i'(t) f(t)\,dt \right)^{p_i} \tag{1}$$

is valid.

b) *If f is non-increasing and $x_i(a) = 0$ $(1 \le i \le n)$, then in* (1) *the reverse inequality is valid.*

In this paper we give a discrete analogue of inequality (1) and some other discrete results.

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2. Main results

2.1 Results in connection with general inequalities. In the following theorems we state some inequalities for finite differences. It is interesting that every of them have its reverse version. It could be done because in the proof we combine general inequalities such as those of Hölder or Minkowski with their reverses such as those of Popoviciu and Bellman. Generally, the proof is given with all details only for one case and the other ones can be proved analogously using related inequalities.

Theorem 1. *Let $w = (w_i)_{1\le i\le n}$ and $a_j = (a_{ji})_{1\le i\le n}$ $(1 \le j \le m)$ be non-negative n-tuples and the sums $\sum_{i=1}^{n-1} w_i \Delta a_{ji}$ $(1 \le j \le m)$ be non-negative. Further, let p_j $(1 \le j \le m)$ be real numbers such that $\sum_{j=1}^{m} p_j = 1$.*

a) *Let $p_j \ge 0$ $(1 \le j \le m)$. If w is non-decreasing, then*

$$\sum_{i=1}^{n-1} w_i\, \Delta(a_{1i}^{p_1} \cdots a_{mi}^{p_m}) \ge \prod_{j=1}^{m} \left(\sum_{i=1}^{n-1} w_i\, \Delta a_{ji} \right)^{p_j} \tag{2}$$

is valid where $\Delta a_{ji} = a_{j(i+1)} - a_{ji}$ and $\Delta(a_{1i}^{p_1} \cdots a_{mi}^{p_m}) = a_{1(i+1)}^{p_1} \cdots a_{m(i+1)}^{p_m} - a_{1i}^{p_1} \cdots a_{mi}^{p_m}$. If w is non-increasing and $a_{j1} = 0$ $(1 \le j \le m)$, then in (2) the reverse inequality is valid.

b) *Let $p_1 > 0$ and $p_j < 0$ $(2 \le j \le m)$. If w is non-increasing and $a_{j1} = 0$ $(1 \le j \le m)$, then (2) is valid.*

Proof. First, let us recall Popoviciu's inequality (see [2: p. 118]): Let $w = (w_i)_{1\le i\le n}$ and $a_j = (a_{ji})_{1\le i\le n}$ $(1 \le j \le m)$ be non-negative n-tuples such that

$$w_1 a_{j1} - w_2 a_{j2} - \ldots - w_n a_{jn} \ge 0 \qquad (1 \le j \le m)$$

and let p_j $(1 \le j \le m)$ be real numbers such that $\sum_{j=1}^{m} p_j = 1$.

(i) If $p_j > 0$ $(1 \le j \le m)$, then

$$\begin{aligned} &\prod_{j=1}^{m} (w_1 a_{j1} - w_2 a_{j2} - \ldots - w_n a_{jn})^{p_j} \\ &\quad \le w_1 a_{11}^{p_1} a_{21}^{p_2} \ldots a_{m1}^{p_m} - w_2 a_{12}^{p_1} a_{22}^{p_2} \ldots a_{m2}^{p_m} - \ldots - w_n a_{1n}^{p_1} a_{2n}^{p_n} \ldots a_{mn}^{p_m} \end{aligned} \tag{3}$$

is valid.

(ii) If $p_1 > 0$ and $p_j < 0$ $(2 \le j \le m)$, then in (3) the reverse inequality is valid.

Now, for proving assertion a) define $\Delta w_{i-1} = w_i - w_{i-1}$. If w is non-decreasing, then

$\Delta w_{i-1} \geq 0$ and we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} w_i\, \Delta(a_{1i}^{p_1} \cdots a_{mi}^{p_m}) \\
&\quad = w_n\, a_{1n}^{p_1} a_{2n}^{p_n} \cdots a_{mn}^{p_m} - w_1 a_{11}^{p_1} a_{21}^{p_2} \cdots a_{m1}^{p_m} - \sum_{i=2}^{n} a_{i1}^{p_1} a_{i1}^{p_2} \cdots a_{i1}^{p_m} \Delta w_{i-1} \\
&\quad \geq \prod_{j=1}^{m} \left(w_n a_{jn} - w_1 a_{j1} - \sum_{i=2}^{n} a_{ji}\, \Delta w_{i-1} \right)^{p_j} \\
&\quad = \prod_{j=1}^{m} \left(\sum_{i=1}^{n-1} w_i\, \Delta a_{ji} \right)^{p_j}
\end{aligned}$$

where inequality (3) is used. If w is non-increasing, then the Hölder inequality is used instead that of Popoviciu. The proof of assertion b) is similar to the previous one ∎

Theorem 2. *Let $w = (w_i)_{1\leq i\leq n}$ and $a_j = (a_{ji})_{1\leq i\leq n}$ $(1 \leq j \leq m)$ be non-negative n-tuples such that for some $p \in \mathbb{R}$ the sums $\sum_{i=1}^{n-1} w_i\, \Delta a_{ji}^p$ $(1 \leq j \leq m)$ are non-negative.*

a) *Let w be non-decreasing. If $p > 1$ or $p < 0$, then*

$$\sum_{i=1}^{n-1} w_i\, \Delta(a_{1i} + \ldots + a_{mi})^p \geq \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{n-1} w_i\, \Delta a_{ji}^p \right)^{1/p} \right)^p \tag{4}$$

is valid. If $0 < p < 1$, then in (4) the reverse inequality is valid.

b) *Let w be non-increasing and $a_{j1} = 0$ $(1 \leq j \leq m)$. If $0 < p < 1$, then (4) is valid. If $p > 1$, then in (4) the reverse inequality is valid.*

Proof. For proving assertion a) we will use the same idee as in the previous theorem and the Bellman inequality (see [2: p. 118]): Let $a = (a_i)_{1\leq i\leq n}$ and $b = (b_i)_{1\leq i\leq n}$ be two non-negative n-tuples such that

$$a_1^p - a_2^p - \ldots - a_n^p \geq 0 \qquad \text{and} \qquad b_1^p - b_2^p - \ldots - b_n^p \geq 0$$

where $p > 1$ or $p < 0$. Then

$$\begin{aligned}
&\left((a_1^p - a_2^p - \ldots - a_n^p)^{1/p} + (b_1^p - b_2^p - \ldots - b_n^p)^{1/p} \right)^p \\
&\qquad \leq (a_1 + b_1)^p - (a_2 + b_2)^p - \ldots - (a_n + b_n)^p
\end{aligned} \tag{5}$$

is valid. If $0 < p < 1$, then in (5) the reverse inequality is valid. An analogous formula states for m-tuples a_j $(1 \leq j \leq m)$. Further, analogously assertion b) can be proved using the Minkowski inequality ∎

Remark 2. An integral version of the previous theorem is given in [6].

Theorem 3. *Let $g = (g_i)_{1\le i\le n}$ and $h = (h_i)_{1\le i\le n}$ be non-negative and non-decreasing n-tuples such that $g_1 = h_1 = 0$. If $f = (f_i)_{1\le i\le n}$ is a non-negative and non-increasing n-tuple with $f_1 \neq 0$, then*

$$f_1 \sum_{i=1}^{n-1} f_i\,\Delta(g_i h_i) \ge \left(\sum_{i=1}^{n-1} f_i\,\Delta g_i\right)\left(\sum_{i=1}^{n-1} f_i\,\Delta h_i\right) \tag{5}$$

is valid.

Proof. Using the Čebyšev inequality we obtain

$$\begin{aligned}\sum_{i=1}^{n-1} f_i\,\Delta(g_i h_i) &= f_n g_n h_n - \sum_{i=2}^{n} g_i h_i\,\Delta f_{i-1}\\ &= f_n g_n h_n + \sum_{i=2}^{n} g_i h_i\,\Delta \overline{f_{i-1}}\\ &\ge \frac{1}{f_n + \sum_{i=2}^{n}\Delta\overline{f_{i-1}}}\\ &\quad\times\left(f_n g_n + \sum_{i=2}^{n} g_i\Delta\overline{f_{i-1}}\right)\left(f_n h_n + \sum_{i=2}^{n} h_i\Delta\overline{f_{i-1}}\right)\\ &= \frac{1}{f_1}\left(\sum_{i=1}^{n-1} f_i\,\Delta g_i\right)\left(\sum_{i=1}^{n-1} f_i\,\Delta h_i\right)\end{aligned}$$

where $\overline{f_i} = -f_i$ ∎

2.2 Results in connection with weighted, quasiarithmetic and logarithmic means. The previous results are connected with general inequalities as of Hölder, Minkowski and Čebyšev and their reverse versions. In the following theorem we deal with weighted mean. So, let us recall the definition of that mean.

Definition 1. Let $a = (a_i)_{1\le i\le n}$ and $p = (p_i)_{1\le i\le n}$ be positive n-tuples, $\sum_{1=i}^{n} p_i = 1$ and $r \in \mathbb{R}$. Then $M_p^{[r]}(a)$ defined by

$$M_p^{[r]}(a) = \begin{cases} \left(\sum_{i=1}^{n} p_i\, a_i^r\right)^{1/r} & \text{for } r \neq 0\\ \prod_{i=1}^{n} a_i^{p_i} & \text{for } r = 0\end{cases}$$

is the *weighted mean* of order r of a with weight p.

Theorem 4. *Let $a = (a_i)_{1\le i\le n}$ and $b = (b_i)_{1\le i\le n}$ be non-negative and non-decreasing n-tuples such that $a_1 = b_1$ and $a_n = b_n$, let p_1 and p_2 be positive real numbers such that $p_1 + p_2 = 1$, and let r and s be arbitrary real numbers. Further, let $f = (f_i)_{1\le i\le n}$ be a non-negative n-tuple.*

a) *Let f be non-decreasing. If $r, s < 1$, then*

$$\sum_{i=1}^{n-1} \Delta M_p^{[r]}(a_i, b_i) f_i \ge M_p^{[s]}\left(\sum_{i=1}^{n-1} f_i\,\Delta a_i, \sum_{i=1}^{n-1} f_i\,\Delta b_i\right) \tag{7}$$

is valid. If $r, s > 1$, then in (7) the reverse inequality is valid.

b) *Let f be non-increasing. If $r < 1 < s$, then (7) is valid. If $r > 1 > s$, then in (7) the reverse inequality is valid.*

Proof. For proving assertion a) let us suppose that $r, s < 1$. Using the inequality between means we obtain

$$
\begin{aligned}
M_p^{[s]}&\left(\sum_{i=1}^{n-1} f_i\,\Delta a_i, \sum_{i=1}^{n-1} f_i\,\Delta b_i\right) \\
&\le M_p^{[1]}\left(\sum_{i=1}^{n-1} f_i\,\Delta a_i, \sum_{i=1}^{n-1} f_i\,\Delta b_i\right) \\
&= \sum_{i=1}^{n-1}(p_1\,\Delta a_i + p_2\,\Delta b_i) f_i \\
&= f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) - \sum_{i=2}^{n} M_p^{[1]}(a_i, b_i)\Delta f_i \\
&\le f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) - \sum_{i=2}^{n} M_p^{[r]}(a_i, b_i)\Delta f_i \\
&= f_n M_p^{[1]}(a_n, b_n) - f_1 M_p^{[1]}(a_1, b_1) \\
&\quad - \left(f_n M_p^{[r]}(a_n, b_n) - f_1 M_p^{[r]}(a_1, b_1) - \sum_{i=1}^{n-1} \Delta M_p^{[r]}(a_i, b_i) f_i\right) \\
&= \sum_{i=1}^{n-1} \Delta M_p^{[r]}(a_i, b_i) f_i
\end{aligned}
$$

which is the first assertion. The other cases can be proved analogously using the inequality between means ∎

Definition 2. Let $f : [a, b] \to \mathbb{R}$ be a monotone function with inverse f^{-1}, and let $p = (p_i)_{1 \le i \le n}$ and $a = (a_i)_{1 \le i \le n}$ be real n-tuples. Then $M_f(a;p)$ defined by

$$M_f(a;p) = f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i f(a_i)\right)$$

with $P_n = \sum_{i=1}^{n} p_i$ is the *quasiarithmetic f-mean* of a with weight p.

If p is non-negative, $P_n = 1$ and $f(x) = x^r$ $(r \ne 0)$ or $f(x) = \ln x$, then the quasiarithmetic mean $M_f(a;p)$ is the weighted mean r $M_p^{[r]}(a)$ of order r.

Theorem 5. *Let $p = (p_i)_{1 \le i \le n}$ be a positive n-tuple, $x_i = (x_{ij})_{1 \le j \le m}$ $(1 \le i \le n)$ non-negative m-tuples with $x_{i'1} = x_{i''1}$ and $x_{i'm} = x_{i''m}$ for $1 \le i', i'' \le n$, and $w = (w_j)_{1 \le j \le m}$ a non-negative m-tuple. Further, let f and g be real functions and suppose that all quasiarithmetic means below are well defined.*

a) *Let w be non-decreasing. If f and g are convex increasing or concave decreasing, then*

$$M_f\left(\left(\sum_{k=1}^{m-1} w_k\,\Delta x_{ik}\right)_i ; p\right) \geq \sum_{k=1}^{m-1} w_k\,\Delta M_g((x_{ik})_i; p) \tag{8}$$

is valid. If f and g are concave increasing or convex decreasing, then in (8) *the reverse inequality is valid.*

b) *Let w be non-increasing. If f is convex increasing or concave decreasing and g concave increasing or convex decreasing, then* (8) *is valid. If f is concave increasing or convex decreasing and g convex increasing or concave decreasing, then in* (8) *the reverse inequality is valid.*

Proof. Let us suppose that f and g are convex increasing. We will use the well-known Jensen inequality, namely, if $0 < p_i \in \mathbb{R}$ and $x_i \in [a,b]$ $(1 \leq i \leq n)$ are such that $\frac{1}{P_n}\sum_{i=1}^n p_i x_i \in [a,b]$, then for every convex function $f : [a,b] \to \mathbb{R}$ we have

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)$$

where $P_n = \sum_{i=1}^n p_i$. So, we obtain

$$\begin{aligned}
&M_f\left(\left(\sum_{k=1}^{m-1} w_k\,\Delta x_{ik}\right)_i ; p\right)\\
&\quad = f^{-1}\left(\frac{1}{P_n}\sum_{i=1}^n p_i\, f\left(\sum_{k=1}^{m-1} w_k\,\Delta x_{ik}\right)\right)\\
&\quad \geq \frac{1}{P_n}\sum_{i=1}^n p_i \sum_{k=1}^{m-1} w_k\,\Delta x_{ik}\\
&\quad = \sum_{k=1}^{m-1} \frac{1}{P_n}\left(\sum_{i=1}^n p_i\,\Delta x_{ik}\right) w_k\\
&\quad = \frac{1}{P_n}\left(\sum_{i=1}^n p_i\, x_{im}\right) w_m - \frac{1}{P_n}\left(\sum_{i=1}^n p_i\, x_{i1}\right) w_1\\
&\qquad - \sum_{k=2}^{m} \frac{1}{P_n}\left(\sum_{i=1}^n p_i\, x_{ik}\right) \Delta w_k\\
&\quad \geq \frac{1}{P_n}\left(\sum_{i=1}^n p_i\, x_{im}\right) w_m - \frac{1}{P_n}\left(\sum_{i=1}^n p_i\, x_{i1}\right) w_1\\
&\qquad - \sum_{k=2}^{m} g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^n p_i\, g(x_{ik})\right)\right) \Delta w_k
\end{aligned}$$

$$= \sum_{k=1}^{m-1} \Delta g^{-1}\left(\frac{1}{P_n}\left(\sum_{i=1}^{n} p_i\, g(x_{ik})\right)\right) w_k$$

$$= \sum_{k=1}^{m-1} w_k\, \Delta M_g((x_{ik})_i; p)$$

which is the first assertion. The other cases can be proved analogously ∎

Remark 3. If $p_1 > 0$ and $p_i < 0$ $(2 \leq i \leq n)$, then using the reverse version of the Jensen inequality we can state similar results as in the previous theorem. For another weaker condition on p see [2: p. 6].

Definition 3. Let us define the *logarithmic mean* $L_r(x,y)$ by

$$L_r(x,y) = \begin{cases} \left(\dfrac{1}{y-x}\dfrac{y^{r+1}-x^{r+1}}{r+1}\right)^{1/r} & \text{for } r \neq -1, 0 \\ \dfrac{1}{e}\left(\dfrac{y^y}{x^x}\right)^{1/(y-x)} & \text{for } r = 0 \\ \dfrac{y-x}{\ln y - \ln x} & \text{for } r = -1 \end{cases}$$

if $x > 0$ and $y > 0$ are such that $x \neq y$, and by $L_r(x,x) = x$ (see [2: p. 41]).

Theorem 6. *Let $a = (a_i)_{1 \leq i \leq n}$ and $b = (b_i)_{1 \leq i \leq n}$ be non-negative and non-decreasing n-tuples such that $a_1 = b_1$ and $a_n = b_n$, and $w = (w_i)_{1 \leq i \leq n}$ a non-negative n-tuple. Further, let r and s be real numbers.*

a) *Let w be non-decreasing. If $r, s \leq 1$, then*

$$L_r\left(\sum_{j=1}^{n-1} w_j\, \Delta a_j,\ \sum_{j=1}^{n-1} w_j\, \Delta b_j\right) \leq \sum_{j=1}^{n-1} w_j\, \Delta L_s(a_j, b_j) \tag{9}$$

is valid. If $r, s \geq 1$, then in (9) the reverse inequality is valid.

b) *Let w be non-increasing. If $r < 1 < s$, then (9) is valid. If $r > 1 > s$, then in (9) the reverse inequality is valid.*

Theorem 6 can be proved using the inequality for logarithmic mean, i.e. $L_r(x,y) \leq L_s(x,y)$ for $r \leq s$.

Remark 4. An integral version of Theorems 4 - 6 is given in [4].

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