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On a Class of Multilinear Operator Equations

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Abstract. By means of contraction principle in a Banach space E with a scale of norms $\|\cdot\|_\sigma$ $(\sigma \geq 0)$ existence, uniqueness and stability of solutions are proved for a general class of operator equations $u + G_0 u + G_1 u = g$ including multilinear ones where $G_0, G_1 \in (E \to E)$ are some operators. The theorems are applicable to equations with operators of generalized convolution type.

Keywords: *Nonlinear operator equations, scale of norms, existence, stability*

AMS subject classification: 47 H 15, 45 G 10, 45 D 05

0. Introduction

Recently, by Bukhgeim [2] and the authors [6], global existence and stability theorems for some types of abstract one-dimensional nonlinear convolution equations are proved using norms with exponential weights. This method has been applied also to multidimensional nonlinear Volterra equations of convolution type in Lebesgue spaces L_p with mixed norm [5].

In the present paper this approach is extended to a general class of nonlinear operator equations in a Banach space with a scale of norms. In particular, this class of operator equations includes some types of equations with multilinear operators. As a special case of such multilinear operators a class of operators of generalized convolution type in classical spaces C and L_∞ is dealt with.

1. Main theorem

We study the operator equation

$$u + G_0 u + G_1 u = g \tag{1.1}$$

in a Banach space E, which is endowed with a scale of norms $\|\cdot\|_\sigma$ $(\sigma \geq 0)$ satisfying the condition

$$\psi(\sigma) \cdot \|u\|_0 \leq \|u\|_\sigma \leq \|u\|_0 \qquad (u \in E,\ \sigma \geq 0) \tag{1.2}$$

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with a continuous positive function ψ. For the operators $G_0, G_1 \in (E \to E)$ there hold the following assumptions:

(A1) $\|G_0u_1 - G_0u_2\|_\sigma \le M_0(\|u_1\|_\sigma, \|u_2\|_\sigma, \sigma) \cdot \|u_1 - u_2\|_\sigma \quad (u_1, u_2 \in E,\ \sigma \ge 0)$.

(A2) $\|G_1u\|_\sigma \to 0$ as $\sigma \to \infty \quad (u \in E)$.

(A3) $\|G_1u_1 - G_1u_2\|_\sigma \le M_1(f, \|u_1 - f\|_\sigma, \|u_2 - f\|_\sigma, \sigma) \cdot \|u_1 - u_2\|_\sigma$ for every $u_1, u_2, f \in E$ and $\sigma \ge 0$ with $\|u_i - f\|_\sigma \le \rho_0$ $(i = 1,2)$ where ρ_0 is some given positive number and the coefficients M_0 and M_1 satisfy the following conditions:

(A4) $M_0 \in C(\mathbb{R}_+^3 \to \mathbb{R}_+)$
$M_0(\rho_1, \rho_2, \sigma)$ is increasing in ρ_1, ρ_2 and decreasing in σ
$\lim_{\sigma\to\infty} M_0(\rho_1, \rho_2, \sigma) = 0$ for any positive ρ_1, ρ_2.

(A5)$'$ $M_1 \in C(E \times [0, \rho_0]^2 \times \mathbb{R}_+ \to \mathbb{R}_+)$
$M_1(f, \rho_1, \rho_2, \sigma)$ is increasing in ρ_1, ρ_2 and decreasing in σ
$m_1(f, \rho_1, \rho_2) := \lim_{\sigma\to\infty} M_1(f, \rho_1, \rho_2, \sigma) \in C(E \times [0, \rho_0]^2 \to \mathbb{R}_+)$
$Q(f) := m_1(f, 0, 0) < 1$.

Here, as usual, $\mathbb{R}_+$ denotes the positive real semi-axis.

Let us first draw some simple conclusions from the conditions (A2), (A4) and (A5). Due to the monotonicity of M_0 in σ and the last condition of (A4), for every pair $\rho > 0$ and $\varepsilon > 0$ we can define

$$\sigma_0(\rho, \varepsilon) = \inf\Big\{\sigma_* \in [0, \infty) : M_0(\rho, \rho, \sigma) \le \varepsilon \text{ if } \sigma \ge \sigma_*\Big\}.$$

Moreover, it follows from the condition (A4) that, for any $\varepsilon > 0$,

$$M_0(\rho_1, \rho_2, \sigma) \le \varepsilon \quad \text{if } 0 \le \rho_i \le \rho\ (i = 1,2) \text{ and } \sigma \ge \sigma_0(\rho, \varepsilon) \tag{1.3}$$

$$\sigma_0 \in C(\mathbb{R}_+^2 \to \mathbb{R}_+). \tag{1.4}$$

Let us further denote

$$q(f) = \frac{2 + Q(f)}{3}. \tag{1.5}$$

Evidently, by (1.5) and the last condition of (A5) we have the inequalities

$$Q(f) < \frac{1 + Q(f)}{2} < q(f) < 1.$$

Thus, due to the last condition of (A5) and the monotonicity of M_1 in ρ_1, ρ_2 and σ, respectively, we can define

$$r_1(f) = \sup\Big\{r \in (0, \rho_0] : m_1(f, \rho_1, \rho_2) \le \frac{1 + Q(f)}{2} \text{ if } \rho_1, \rho_2 \le r\Big\}$$

and

$$\sigma_1(f) = \inf\Big\{\sigma_* \in [0, \infty) : M_1(f, r_1(f), r_1(f), \sigma) \le q(f) \text{ if } \sigma \ge \sigma_*\Big\}.$$

Then the first two conditions of (A5) imply

$$M(f,\rho_1,\rho_2,\sigma) \le q(f) < 1 \quad \text{if } 0 \le \rho_1,\rho_2 \le r_1(f) \text{ and } \sigma \ge \sigma_1(f) \tag{1.6}$$

$$q, r_1, \sigma_1 \in C(E \to \mathbb{R}_+). \tag{1.7}$$

Finally, in view of (A2), for every $\rho > 0$ and $f \in E$ we can define

$$\sigma_2(f,\rho) = \inf\left\{\sigma_* \in [0,\infty) : \|G_1 f\|_\sigma \le \frac{1-q(f)}{2}\rho \quad if\ \sigma \ge \sigma_*\right\}.$$

Then there holds

$$\|G_1 f\|_\sigma \le \frac{1-q(f)}{2}\rho \tag{1.8}$$

if $\sigma \ge \sigma_2(f,\rho)$ and due to the continuity of G_1 (see (A3)) and q we have

$$\sigma_2 \in C(E \times \mathbb{R}_+ \to \mathbb{R}_+). \tag{1.9}$$

Now we can formulate our main result.

Theorem 1. *Let* (1.2) *and the assumptions* (A1) - (A5) *be satisfied. Then for every $g \in E$ equation* (1.1) *has a unique solution $u \in E$. For the solutions u_1 and u_2 corresponding to data g_1 and g_2, respectively, the estimates*

(S1) $\|u_1 - u_2\|_\sigma \le \dfrac{2}{1-q(f_1)}\|g_1 - g_2\|_\sigma$ *if* $\sigma \ge \tilde\sigma(f_1, p(g_1,g_2))$ *and* $\|g_1 - g_2\|_0 \le \tilde\delta(f_1, p(g_1,g_2))$

(S2) $\|u_1 - u_2\|_0 \le \dfrac{2}{(1-q(f_1))\cdot\psi(\tilde\sigma(f_1,p(g_1,g_2)))}\|g_1 - g_2\|_0$ *if* $\|g_1 - g_2\|_0 \le \tilde\delta(f_1, p(g_1,g_2))$

hold. Here $p(g_1,g_2) = \max_{i=1,2}\{2\|G_0 g_i\|_0 + \|g_i\|_0\}$, *$f_1$ is the solution of the equation*

$$f + G_0 f = g \tag{1.10}$$

with $g = g_1$, q is defined by (1.5), *$\tilde\sigma(f_1,\cdot) \ge 0$ and $\tilde\delta(f_1,\cdot) > 0$ are certain continuous functions.*

Proof. Let us define the balls

$$B_{\rho,\sigma}(v) = \{u \in E : \|u - v\|_\sigma \le \rho\} \qquad (\rho > 0,\ \sigma \ge 0,\ v \in E)$$

in E.

Step 1. At first we show that the auxiliary equation (1.10) has a solution in the ball $B_{R,\sigma}(g)$, where $R = 2\|G_0 g\|_0$ and σ is chosen large enough. By assumption (A1) and (1.2) for the operator $A_0 f = g - G_0 f$ we derive the estimates

$$\begin{aligned}\|A_0 f - g\|_\sigma = \|G_0 f\|_\sigma &\le \|G_0 f - G_0 g\|_\sigma + \|G_0 g\|_\sigma \\ &\le M_0(\|f\|_\sigma, \|g\|_\sigma, \sigma)\cdot\|f - g\|_\sigma + \frac{R}{2}\end{aligned}$$

and

$$\|A_0 f_1 - A_0 f_2\|_\sigma = \|G_0 f_1 - G_0 f_2\|_\sigma \le M_0(\|f_1\|_\sigma, \|f_2\|_\sigma, \sigma) \cdot \|f_1 - f_2\|_\sigma.$$

If $f_1, f_2, f \in B_{R,\sigma}(g)$, then

$$\|f\|_\sigma \le R + \|g\|_0 \qquad \text{and} \qquad \|f_i\|_\sigma \le R + \|g\|_0 \ \ (i = 1,2).$$

Now by (1.3) we have

$$\|A_0 f - g\|_\sigma \le \frac{1}{2}R + \frac{R}{2} = R \qquad \text{and} \qquad \|A_0 f_1 - A_0 f_2\|_\sigma \le \frac{1}{2}\|f_1 - f_2\|_\sigma$$

if $f, f_i \in B_{R,\sigma}(g)$ with $\sigma \ge \sigma_0(R + \|g\|_0, \frac{1}{2})$. Thus, for such σ the operator A_0 maps $B_{R,\sigma}(g)$ into itself and is a contraction in $B_{R,\sigma}(g)$. This implies the existence of a solution to equation (1.10) in $B_{R,\sigma}(g)$, where $R = 2\|G_0 g\|_0$ and $\sigma \ge \sigma_0(R + \|g\|_0, \frac{1}{2})$.

Step 2. Next we are going to show that a unique solution of equation (1.1) exists in the ball $B_{\rho,\sigma}(f)$, where ρ is small enough, σ is large enough and f is a solution to equation (1.10). Let us denote

$$Au = g - G_0 u - G_1 u.$$

By virtue of (1.10) and the assumptions (A1) and (A3) we derive the estimates

$$\begin{aligned} &\|Au - f\|_\sigma \\ &\quad = \|G_0 f - G_0 u - G_1 u + G_1 f - G_1 f\|_\sigma \\ &\quad \le \|G_0 f - G_0 u\|_\sigma + \|G_1 f - G_1 u\|_\sigma + \|G_1 f\|_\sigma \\ &\quad \le \Big[M_0(\|f\|_\sigma, \|u\|_\sigma, \sigma) + M_1(f, 0, \|u - f\|_\sigma, \sigma)\Big] \cdot \|u - f\|_\sigma + \|G_1 f\|_\sigma \end{aligned} \tag{1.11}$$

if $\|u - f\|_\sigma \le \rho_0$ and

$$\begin{aligned} &\|Au_1 - Au_2\|_\sigma \\ &\quad \le \|G_0 u_1 - G_0 u_2\|_\sigma + \|G_1 u_1 - G_1 u_2\|_\sigma \\ &\quad \le \Big[M_0(\|u_1\|_\sigma, \|u_2\|_\sigma, \sigma) + M_1(f, \|u_1 - f\|_\sigma, \|u_2 - f\|_\sigma, \sigma)\Big] \cdot \|u_1 - u_2\|_\sigma \end{aligned} \tag{1.12}$$

if $\|u_i - f\|_\sigma \le \rho_0$ $(i = 1,2)$. We further estimate the coefficients M_0 and M_1 in (1.11) and (1.12). Suppose that $u, u_1, u_2 \in B_{\rho,\sigma}(f)$, where $\rho \le r_1(f)$. Then we have

$$\|u\|_\sigma \le r_1(f) + \|f\|_0 \qquad \text{and} \qquad \|u_i\|_\sigma \le r_1(f) + \|f\|_0 \ \ (i = 1,2) \tag{1.13}$$

and (1.3), (1.6) imply

$$M_0(\|f\|_\sigma, \|u\|_\sigma, \sigma) \le \frac{1 - q(f)}{2} \quad \text{and} \quad M_0(\|u_1\|_\sigma, \|u_2\|_\sigma, \sigma) \le \frac{1 - q(f)}{2} \tag{1.14}$$

if $\sigma \geq \sigma_0\left(r_1(f) + \|f\|_0, \frac{1-q(f)}{2}\right)$ and

$$M_1(f, 0, \|u - f\|_\sigma, \sigma) \leq q(f) \quad \text{and} \quad M_1(f, \|u_1 - f\|_\sigma, \|u_2 - f\|_\sigma, \sigma) \leq q(f) \tag{1.15}$$

if $\sigma \geq \sigma_1(f)$. Combining (1.11), (1.12) with (1.14), (1.15) and also taking (1.8) into account, we obtain

$$\|Au - f\|_\sigma \leq \left[\frac{1 - q(f)}{2} + q(f)\right] \rho + \frac{1 - q(f)}{2}\, \rho = \rho$$

and

$$\|Au_1 - Au_2\|_\sigma \leq \left[\frac{1 - q(f)}{2} + q(f)\right] \cdot \|u_1 - u_2\|_\sigma = \frac{1 + q(f)}{2} \cdot \|u_1 - u_2\|_\sigma$$

if $u, u_1, u_2 \in B_{\rho,\sigma}(f)$, $\rho \leq r_1(f)$ and $\sigma \geq \sigma_3(f, \rho)$ where

$$\sigma_3(f, \rho) = \max\left\{\sigma_1(f),\ \sigma_0(r_1(f) + \|f\|_0,\ \frac{1 - q(f)}{2},\ \sigma_2(f, \rho)\right\}. \tag{1.16}$$

Since $\frac{1+q(f)}{2} < 1$, we have that A maps $B_{\rho,\sigma}(f)$ into itself and is a contraction in $B_{\rho,\sigma}(f)$ if $\rho \leq r_1(f)$ and $\sigma \geq \sigma_3(f, \rho)$. Thus, equation (1.1) has a unique solution u in every ball $B_{\rho,\sigma}(f)$, where $\rho \leq r_1(f)$ and $\sigma \geq \sigma_3(f, \rho)$. Particularly, this proves the existence result of Theorem 1.

Step 3. Let us prove the uniqueness of the solution of equation (1.1) in E. Suppose that $u_1 \in E$ and $u_2 \in E$ are two arbitrary solutions of equation (1.1). Then

$$\begin{aligned} \|u_i - f\|_\sigma &= \|G_0 f - G_0 u_i - G_1 u_i\|_\sigma \\ &\leq \|G_0 f - G_0 u_i\|_\sigma + \|G_1 u_i\|_\sigma \\ &\leq M_0(\|f\|_\sigma, \|u_i\|_\sigma, \sigma) \cdot \|u_i - f\|_\sigma + \|G_1 u_i\|_\sigma \quad (i = 1, 2). \end{aligned}$$

Now it follows from (1.2) and the assumptions (A4), (A2) that $\|u_i - f\|_\sigma \leq r_1(f)$ if σ is greater than some number σ_4 which depends on u_1, u_2 and f. Thus, $u_i \in B_{r_1(f),\sigma}(f)$ if $\sigma \geq \sigma_4$. Taking $\sigma \geq \max\{\sigma_4, \sigma_3(f, r_1(f))\}$, the solutions u_1 and u_2 belong to a ball where the uniqueness of the solution has already been shown. Thus, $u_1 = u_2$.

Step 4. Now we derive a stability estimate for the solution of the auxiliary equation (1.10), which is uniquely determined as we have just shown. Suppose that f_1 and f_2 are the solutions of (1.10) with g replaced by g_1 and g_2, respectively. Then by assumption (A1)

$$\begin{aligned} \|f_1 - f_2\|_\sigma &\leq \|g_1 - g_2\|_\sigma + \|G_0 f_1 - G_0 f_2\|_\sigma \\ &\leq \|g_1 - g_2\|_\sigma + M_0(\|f_1\|_\sigma, \|f_2\|_\sigma, \sigma) \cdot \|f_1 - f_2\|_\sigma. \end{aligned} \tag{1.17}$$

From Step 1 it follows that $\|f_i - g_i\|_\sigma \leq R_i$ if $\sigma \geq \sigma_0(R_i + \|g_i\|_0, \frac{1}{2})$, where $R_i = 2\|G_0 g_i\|_0$. Thus, observing (1.2) we have

$$\|f_i\|_\sigma \leq 2\|G_0 g_i\|_0 + \|g_i\|_0 \leq \max_{j=1,2}\{2\|G_0 g_j\|_0 + \|g_j\|_0\} = p(g_1, g_2) \tag{1.18}$$

if $\sigma \geq \sigma_0(p(g_1,g_2),\frac{1}{2})$. In view of (1.18) condition (1.3) implies

$$M_0(\|f_1\|_\sigma,\|f_2\|_\sigma,\sigma) \leq \frac{1}{2} \qquad \text{if } \sigma \geq \sigma_0(p(g_1,g_2),\tfrac{1}{2}).$$

This together with (1.17) yields

$$\|f_1 - f_2\|_\sigma \leq 2\,\|g_1 - g_2\|_\sigma \qquad \text{if } \sigma \geq \sigma_0(p(g_1,g_2),\tfrac{1}{2}). \tag{1.19}$$

Step 5. Finally, let us derive the estimates (S1) and (S2). Suppose that u_1 and u_2 are the solutions of equation (1.1) with g replaced by g_1 and g_2, respectively. Then, by the assumptions (A1) and (A3),

$$\begin{aligned}\|u_1 - u_2\|_\sigma &\leq \|G_0u_1 - G_0u_2\|_\sigma + \|G_1u_1 - G_1u_2\|_\sigma + \|g_1 - g_2\|_\sigma \\ &\leq \Big[M_0(\|u_1\|_\sigma,\|u_2\|_\sigma,\sigma) + M_1(f_1,\|u_1 - f_1\|_\sigma,\|u_2 - f_1\|_\sigma,\sigma)\Big] \\ &\quad\times \|u_1 - u_2\|_\sigma + \|g_1 - g_2\|_\sigma\end{aligned} \tag{1.20}$$

if $\|u_i - f_1\|_\sigma \leq \rho_0$ $(i = 1,2)$ where as above f_1 is the solution of equation (1.10) for $g = g_1$. We estimate the quantities $\|u_1\|_\sigma, \|u_2\|_\sigma$ and $\|u_1 - f_1\|_\sigma, \|u_2 - f_1\|_\sigma$ in (1.20). It follows from Step 2 that

$$\|u_1 - f_1\|_\sigma \leq r_1(f_1) \qquad \text{if } \sigma \geq \sigma_3(f_1, r_1(f_1)) \tag{1.21}$$

and

$$\|u_2 - f_2\|_\sigma \leq \frac{1}{2}r_1(f_2) \qquad \text{if } \sigma \geq \sigma_3(f_2,\tfrac{1}{2}r_1(f_2)). \tag{1.22}$$

By virtue of the continuity properties (1.7), (1.4) and (1.9) the functional σ_3 defined by (1.16) is also continuous in its arguments. Thus, there exists $\delta(f_1) \in (0,\frac{1}{2}r_1(f_1))$ such that

$$\sigma_3(f_2,\tfrac{1}{2}r_1(f_2)) \leq 2\sigma_3(f_1,\tfrac{1}{2}r_1(f_1)) \qquad \text{and} \qquad \tfrac{1}{2}r_2(f_2) \leq r_1(f_1) - 2\delta(f_1)$$

if $\|f_1 - f_2\|_0 \leq 2\delta(f_1)$. From (1.22) and (1.2) we now obtain

$$\|u_2 - f_1\|_\sigma \leq \|u_2 - f_2\|_\sigma + \|f_2 - f_1\|_0 \leq r_1(f_1) \tag{1.23}$$

if $\sigma \geq 2\sigma_3(f_1,\frac{1}{2}r_1(f_1))$ and $\|f_1 - f_2\|_0 \leq 2\delta(f_1)$. Denote

$$\sigma_5(f_1) = \max\Big\{\sigma_3(f_1,r_1(f_1)), 2\sigma_3(f_1,\tfrac{1}{2}r_1(f_1))\Big\}.$$

The estimates (1.21) and (1.23) imply

$$\|u_i\|_\sigma \leq \|f_1\|_0 + r_1(f_1) \qquad (i = 1,2) \tag{1.24}$$

if $\sigma \geq \sigma_5(f_1)$ and $\|f_1 - f_2\|_0 \leq 2\delta(f_1)$. With the help of the bounds (1.21) and (1.23) for $\|u_i - f_1\|_\sigma$ and (1.24) for $\|u_i\|_\sigma$ and the conditions (1.3) and (1.6) we continue the estimation of $\|u_1 - u_2\|_\sigma$ in (1.20) obtaining

$$\|u_1 - u_2\|_\sigma \leq \frac{1 + q(f_1)}{2} \cdot \|u_1 - u_2\|_\sigma + \|g_1 - g_2\|_\sigma$$

if

$$\sigma \geq \sigma_5(f_1),\ \sigma \geq \sigma_0\left(\|f_1\|_0 + r_1(f_1), \frac{1-q(f_1)}{2}\right),\ \sigma \geq \sigma_1(f_1),\ \|f_1 - f_2\|_0 \leq 2\delta(f_1).$$

Since

$$\sigma_5(f_1) \geq \sigma_0\left(\|f\|_0 + r_1(f_1), \frac{1-q(f_1)}{2}\right) \qquad \text{and} \qquad \sigma_5(f_1) \geq \sigma_1(f_1)$$

we obtain

$$\|u_1 - u_2\|_\sigma \leq \frac{2}{1-q(f_1)} \cdot \|g_1 - g_2\|_\sigma \tag{1.25}$$

if $\sigma \geq \sigma_5(f_1)$ and $\|f_1 - f_2\|_0 \leq 2\delta(f_1)$. Taking (1.19) and (1.2) into account we see that (1.25) holds if

$$\sigma \geq \tilde{\sigma}\big(f_1, p(g_1, g_2)\big) = \max\left\{\sigma_5(f_1); \sigma_0\big(p(g_1, g_2), \tfrac{1}{2}\big)\right\}$$

and

$$\|g_1 - g_2\|_0 \leq \tilde{\delta}\big(f_1, p(g_1, g_2)\big) = \delta(f_1) \cdot \psi\big(\sigma_0(p(q_1, g_2), \tfrac{1}{2})\big).$$

Thus, we have proved the estimate (S1). But the estimate (S1) together with (1.2) implies the estimate (S2). Finally, since σ_0 is continuous, $\psi > 0$ and $\delta > 0$, the functions $\tilde{\sigma}(f_1, \cdot)$ and $\tilde{\delta}(f_1, \cdot)$ in the estimates (S1) and (S2) are also continuous and $\tilde{\delta} > 0$. The proof is complete ∎

2. Equation with multilinear operator

As a particular case of equation (1.1) we consider the operator equation

$$u + G_0 u + \sum_{k=2}^{N} \sum_{j=1}^{n_k} K_{k,j}\left[G^j_{k,1} u,\ \ldots, G^j_{k,k} u\right] = g \tag{2.1}$$

where $N \geq 2$ and $n_k \geq 1$, $G^j_{k,i} \in (E \to E^j_{k,i})$, $E^j_{k,i}$ $(1 \leq i \leq k)$ are Banach spaces and $K_{k,j}$ are multilinear operators from $E^j_{k,1} \times \ldots \times E^j_{k,k}$ into E. We suppose that the spaces $E^j_{k,i}$ are endowed with scales of norms $\|\cdot\|_{k,i,j,\sigma}$ $(\sigma \geq 0)$ which satisfy the condition

$$\|u\|_{k,i,j,\sigma} \leq \|u\|_{k,i,j,0} \qquad (u \in E^j_{k,i},\ \sigma \geq 0) \tag{2.2}$$

and for the operators $K_{k,j}$ and $G^j_{k,i}$ there hold the following assumptions:

(B1) $\|K_{k,j}[f_1, \ldots, f_k]\|_\sigma \leq c_{k,j} \prod_{i=1}^{k} \|f_i\|_{k,i,j,\sigma}$ for $f_i \in E^j_{k,i}$ $(1 \leq i \leq k,\ \sigma \geq 0,\ c_{k,j} \geq 0)$.

(B2) $\|K_{k,j}[f_1, \ldots, f_k]\|_\sigma \leq \lambda_{k,j}(\sigma) \cdot \prod_{l \neq i} \|f_l\|_{k,l,j,\sigma} \cdot \|f_i\|_{k,i,j,0}$ for $f_l \in E^j_{k,l}$ $(1 \leq l \leq k,\ \sigma \geq 0,\ i = 1, \ldots, k)$.

(B3) $\|G^j_{k,i}u_1 - G^j_{k,i}u_2\|_{k,i,j,\sigma} \le M^j_{k,i}(\|u_1\|_\sigma, \|u_2\|_\sigma) \cdot \|u_1 - u_2\|_\sigma$ for $u_1, u_2 \in E$ $(\sigma \ge 0)$ where the coefficients $\lambda_{k,j}$ and $M^j_{k,i}$ satisfy the following conditions:

(B4) $\lambda_{k,j} \in C(\mathbb{R}_+ \to \mathbb{R}_+), \lambda^i_{k,j}$ is decreasing, $\lim_{\sigma\to\infty} \lambda^i_{k,j}(\sigma) = 0$.

(B5) $M^j_{k,i} \in C(\mathbb{R}^2_+ \to \mathbb{R}_+), M^j_{k,i}(\rho_1, \rho_2)$ is increasing in ρ_1 and ρ_2.

Concerning the operator G_0 we assume (A1) and (A4).

Theorem 2. *Let* (1.2) *and* (2.2), *the assumptions* (B1) - (B5) *as well as the assumptions* (A1) *and* (A4) *be satisfied. Then equation* (2.1) *has for every $g \in E$ a unique solution $u \in E$. For the solutions u_1 and u_2 corresponding to data g_1 and g_2, respectively, the estimates*

(C1) $\|u_1 - u_2\|_\sigma \le 6\,\|g_1 - g_2\|_\sigma$ *if* $\sigma \ge \tilde{\sigma}(f_1, p(g_1, g_2))$ *and* $\|g_1 - g_2\|_0 \le \tilde{\delta}(f_1, p(g_1, g_2))$

(C2) $\|u_1 - u_2\|_0 \le \dfrac{6}{\psi(\tilde{\sigma}(f_1, p(g_1, g_2)))}\|g_1 - g_2\|_0$ *if* $\|g_1 - g_2\|_0 \le \tilde{\delta}(f_1, p(g_1, g_2))$

hold. Here $p(g_1, g_2)$ is defined as in Theorem 1, $\tilde{\sigma}(f_1, \cdot) \ge 0$ *and* $\tilde{\delta}(f_1, \cdot) > 0$ *are certain continuous functions.*

Proof. Theorem 2 reduces to Theorem 1 if the operator

$$G_1 u = \sum_{k=2}^{N}\sum_{j=1}^{n_k} K_{k,j}[G^j_{k,1}u, ..., G^j_{k,k}u] \tag{2.3}$$

satisfies the conditions (A2), (A3) and (A5) with $Q(f) = 0$.

The assumptions (B2) and (2.2) imply

$$\|K_{k,j}[f_1, ..., f_k]\|_\sigma \le \lambda_{k,j}(\sigma) \cdot \prod_{l\neq i} \|f_l\|_{k,l,j,p_l} \cdot \|f_i\|_{k,i,j,0} \tag{2.4}$$

for $f_l \in E^j_{k,l}$ and $p_l \in \{0, \sigma\}$ $(1 \le l \le k,\ \sigma \ge 0,\ i = 1, ..., k)$. Condition (A2) is a simple consequence of assumption (B4) and (2.4) with $(p_1, ..., p_k) = (0, ..., 0)$.

Let us show condition (A3). Due to the multilinearity of $K_{k,j}$ we can write

$$\begin{aligned} &K_{k,j}\left[G^j_{k,1}u_1, ..., G^j_{k,k}u_1\right] - K_{k,j}\left[G^j_{k,1}u_2, ..., G^j_{k,k}u_2\right] \\ &\qquad = \sum_{p=1}^{k}\sum_{l'_p} K_{k,j}\left[\chi^l_1, ..., \chi^l_{p-1}, G^j_{k,p}u_1 - G^j_{k,p}u_2, \chi^l_{p+1}, ..., \chi^l_k\right] \end{aligned}$$

where $l'_p = (l_1, ..., l_{p-1}, l_{p+1}, ..., l_k)$ with all $l_s \in \{0; 1\}$,

$$\chi^l_s = \begin{cases} G^j_{k,s}u_2 - G^j_{k,s}f & \text{if } l_s = 1 \text{ and } s \le p-1 \\ G^j_{k,s}u_1 - G^j_{k,s}f & \text{if } l_s = 1 \text{ and } s \ge p+1 \\ G^j_{k,s}f & \text{if } l_s = 0 \end{cases}$$

and u_1, u_2 and f are arbitrary elements in E. Taking into account the assumptions (B1) - (B3), we can estimate as follows:

$$\begin{aligned}
&\Big\| K_{k,j}\left[G^j_{k,1}u_1, \; \dots, G^j_{k,k}u_1\right] - K_{k,j}\left[G^j_{k,1}u_2, \; \dots, G^j_{k,k}u_2\right] \|_\sigma \\
&\quad \le \sum_{p=1}^{k} \Big\{ c_{k,j} \cdot M^j_{k,1}(\|u_2\|_\sigma, \|f\|_\sigma) \cdots M^j_{k,p-1}(\|u_2\|_\sigma, \|f\|_\sigma) \\
&\qquad \times \|u_2 - f\|_\sigma^{p-1} \cdot M^j_{k,p}(\|u_1\|_\sigma, \|u_2\|_\sigma) \cdot \|u_1 - u_2\|_\sigma \\
&\qquad \times M^j_{k,p+1}(\|u_1\|_\sigma, \|f\|_\sigma) \cdots M^j_{k,k}(\|u_1\|_\sigma, \|f\|_\sigma) \cdot \|u_1 - f\|_\sigma^{k-p} \\
&\qquad + \sum_{l'_p \neq (1,\dots,1)} \lambda_{k,j}(\sigma) \cdot \mu^l_1 \cdots \mu^l_{p-1} \\
&\qquad \times M^j_{k,p}(\|u_1\|_\sigma, \|u_2\|_\sigma) \cdot \|u_1 - u_2\|_\sigma \cdot \mu^l_{p+1} \cdots \mu^l_k \Big\}
\end{aligned}$$

for $u_i, f \in E$ $(i = 1, 2)$ and $\sigma \ge 0$, where $l'_p = (l_1, \dots, l_{p-1}, l_{p+1}, \dots, l_k)$ with all $l_s \in \{0; 1\}$ and

$$\mu^l_s = \begin{cases} M^j_{k,s}(\|u_2\|_\sigma, \|f\|_\sigma) \cdot \|u_2 - f\|_\sigma & \text{if } l_s = 1 \text{ and } s \le p - 1 \\ M^j_{k,s}(\|u_1\|_\sigma, \|f\|_\sigma) \cdot \|u_1 - f\|_\sigma & \text{if } l_s = 1 \text{ and } s \ge p + 1 \\ \|G^j_{k,s} f\|_{k,s,j,0} & \text{if } l_s = 0. \end{cases}$$

Estimating further, we have

$$\begin{aligned}
&\Big\| K_{k,j}\left[G^j_{k,1}u_1, \dots, G^j_{k,k}u_1\right] - K_{k,j}\left[G^j_{k,1}u_2, \dots, G^j_{k,k}u_2\right] \Big\|_\sigma \\
&\quad \le \overline{M}_{k,j}\Big(\|u_1\|_\sigma, \|u_2\|_\sigma, \|f\|_\sigma, \|G^j_{k,1}f\|_{k,1,j,0}, \; \dots \; , \|G^j_{k,k}f\|_{k,k,j,0}\Big) \\
&\quad \times \Bigg[\sum_{\substack{s_1, s_2 \ge 0 \\ s_1 + s_2 = k-1}} \|u_1 - f\|_\sigma^{s_1} \|u_2 - f\|_\sigma^{s_2} + \lambda_{k,j}(\sigma) \cdot \sum_{\substack{s_1, s_2 \ge 0 \\ s_1 + s_2 < k-1}} \|u_1 - f\|_\sigma^{s_1} \|u_2 - f\|_\sigma^{s_2} \Bigg] \\
&\quad \times \|u_1 - u_2\|_\sigma
\end{aligned}$$

for $u_i, f \in E$ and $\sigma \ge 0$ where due to assumption (B5) the function $\overline{M}_{k,j}$ is continuous and increasing in each of its arguments. Let us replace the arguments $\|u_1\|_\sigma, \|u_2\|_\sigma$ and $\|f\|_\sigma$ of $\overline{M}_{k,j}$ by their majorants $\|u_1 - f\|_\sigma + \|f\|_0, \|u_2 - f\|_\sigma + \|f\|_0$ and $\|f\|_0$, respectively, and take a sum over k, j to get an estimate for the operator G_1 defined by (2.3). We obtain

$$\begin{aligned}
\|G_1 u_1 - G_1 u_2\|_\sigma \le \overline{M}_1(f, \|u_1 - f\|_\sigma, \|u_2 - f\|_\sigma) \\
\times \Bigg[\sum_{\substack{s_1, s_2 \ge 0 \\ 1 \le s_1 + s_2 \le N-1}} \|u_1 - f\|_\sigma^{s_1} \cdot \|u_2 - f\|_\sigma^{s_2} \\
+ \lambda(\sigma) \sum_{\substack{s_1, s_2 \ge 0 \\ s_1 + s_2 < N-1}} \|u_1 - f\|_\sigma^{s_1} \cdot \|u_2 - f\|_\sigma^{s_2} \Bigg] \cdot \|u_1 - u_2\|_\sigma
\end{aligned}$$

for $u_i, f \in E$ and $\sigma \geq 0$ where due to the mentioned properties of $\overline{M}_{k,j}$, the continuity of $G^j_{k,i}$ and assumption (B4) the coefficients $\overline{M}_1$ and λ satisfy the conditions

$$\overline{M}_1 \in C(E \times \mathbb{R}^2_+ \to \mathbb{R}_+) \quad \text{and} \quad \overline{M}_1(f, \rho_1, \rho_2) \text{ is increasing in } \rho_1, \rho_2$$

and

$$\lambda \in C(\mathbb{R}_+ \to \mathbb{R}_+), \quad \lambda \text{ is decreasing,} \quad \lim_{\sigma \to \infty} \lambda(\sigma) = 0.$$

Hence there follow the assumptions (A3) and (A5) with $Q(f) = 0$ and an arbitrary ρ_0. The Theorem is proved ∎

3. Equations with generalized convolution operators

As an example for a multilinear operator we deal with the following integral operator of generalized convolution type:

$$K[f_1, ..., f_k](x) = \int_0^{x_n} \cdots \int_0^{x_1} m(x,y) \prod_{i=1}^{k} f_i(\alpha_i x - \beta_i y)\, dy_1 \cdots dy_n \tag{3.1}$$

where

$$x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in D = \prod_{j=1}^{n} (0, X_j) \qquad (0 < X_j < \infty).$$

We consider the operator K in the spaces $E = C(\overline{D})$ and $E = L_\infty(D)$ (for more general spaces L_p cp. [1]). The function m should have the form

$$m(x,y) = m_0(x,y) \prod_{i=1}^{k} m_i(\alpha_i x - \beta_i y) \tag{3.2}$$

with $m_0 \in C(\overline{D} \times \overline{D})$ or $m_0 \in L_\infty(D \times D)$, respectively. The parameters

$$\begin{aligned} \alpha_i &= (\alpha_i^1, ..., \alpha_i^n) \quad &&\text{and} \quad \beta_i = (\beta_i^1, ..., \beta_i^n) \\ \alpha_i x &= (\alpha_i^1 x_1, ..., \alpha_i^n x_n) \quad &&\text{and} \quad \beta_i y = (\beta_i^1 y_1, ..., \beta_i^n y_n) \end{aligned}$$

are in $\mathbb{R}^n$ where we suppose the componentwise inequalities

$$0 < \beta_i \leq \alpha_i \quad \text{or} \quad \alpha_i - 1 \leq \beta_i < 0 \leq \alpha_i \qquad (1 \leq i \leq k) \tag{3.3}$$

and

$$\sum_{i=1}^{k} \alpha_i \leq 1 \quad \text{and} \quad \sum_{i=1}^{k} \beta_i \geq 0 \tag{3.4}$$

so that $0 \leq \alpha_i x - \beta_2 y \leq X$ if $0 \leq y \leq x \leq X, X = (X_1, ..., X_n)$.

The operator K is defined on $F = \prod_{i=1}^{k} E_i, E_i = L_{p_i}(D)$ $(1 \le p_i \le \infty)$, and we assume that $m_i \in L_{q_i}(D)$ $(1 \le q_i < \infty)$ where

$$\sum_{i=1}^{k} \frac{1}{r_i} = 1 \qquad \text{and} \qquad \frac{1}{r_i} = \frac{1}{p_i} + \frac{1}{q_i} \tag{3.5}$$

with $r_i < \infty$. Due to these assumptions we have $K \in (F \to E)$ in both cases $E = C(\overline{D})$ and $E = L_\infty(D)$; for the proof in case $E = C(\overline{D})$ compare [1: Section 10/Theorem 1].

We have to show that the operator K fulfils the assumptions (B1) and (B2) with (B4) in suitably chosen scales of norms in E and F. For this purpose we use the well-known norms with exponential weights

$$\|u\|_\sigma = \|e^{-\sigma|x|}u\|_E \quad \text{and} \quad \|f_i\|_{i,\sigma} = \|e^{-\sigma|x|}f_i\|_{E_i} \qquad (\sigma \ge 0) \tag{3.6}$$

where $|x| = \sum_{j=1}^{n} x_j$. These norms fulfil condition (1.2) with $\psi(\sigma) = \exp(-\sigma|X|)$ and condition (2.2), respectively.

There holds $|m_0| \le M_0$ with a positive constant M_0 and by (3.4) we have

$$\begin{aligned} e^{-\sigma|x|} &= \prod_{i=1}^{k} e^{-\sigma|\alpha_i x - \beta_i y|} \exp\left(-\sigma\left|\left(1 - \sum_{i=1}^{k}\alpha_i\right)x\right|\right) \exp\left(-\sigma\left|\sum_{i=1}^{k}\beta_i y\right|\right) \\ &\le \prod_{i=1}^{k} e^{-\sigma|\alpha_i x - \beta_i y|}. \end{aligned}$$

Hence using the Hölder inequality in view of (3.5), we obtain

$$\begin{aligned} &\|K[f_1, \dots, f_k]\|_\sigma \\ &\quad \le M_0 \operatorname*{ess\,sup}_{x \in D} \int_0^{x_n} \dots \int_0^{x_1} \prod_{i=1}^{k} \Big|m_i(\alpha_i x - \beta_i y) f_i(\alpha_i x - \beta_i y)\Big| e^{-\sigma|\alpha_i x - \beta_i y|} \, dy_1 \cdots dy_n \\ &\quad \le C_0 \prod_{i=1}^{k} \left(\int_D |m_i(z) f_i(z)|^{r_i} e^{-\sigma|z| r_i} \, dz_1 \cdots dz_n\right)^{\frac{1}{r_i}} \end{aligned}$$

with

$$C_0 = M_0 \prod_{i=1}^{k} \left(\prod_{j=1}^{n} |\beta_i^j|^{-1}\right)^{\frac{1}{r_i}}.$$

Again by the Hölder inequality there hold the estimations

$$\int_D |m_i(z) f_i(z)|^{r_i} e^{-\sigma|z| r_i} \, dz_1 \cdots dz_n \le M_i^{r_i} \left(\int_D |f_i(z)|^{p_i} e^{-\sigma|z| p_i} \, dz_1 \cdots dz_n\right)^{\frac{r_i}{p_i}}$$

with $M_i = \left(\int_D |m_i(z)|^{q_i} dz_1 \cdots dz_n\right)^{\frac{1}{q_i}}$ and

$$\int_D |m_i(z)f_i(z)|^{r_i} e^{-\sigma|z|r_i} dz_1 \cdots dz_n \le N_i^{r_i}(\sigma) \left(\int_D |f_i(z)|^{p_i} dz_1 \cdots dz_n\right)^{\frac{r_i}{p_i}}$$

with $N_i(\sigma) = \left(\int_D |m_i(z)|^{q_i} e^{-\sigma|z|q_i} dz_1 \cdots dz_n\right)^{\frac{1}{q_i}}$ if $p_i < \infty$ (besides $q_i, r_i < \infty$). Therefore we have the desired inequalities

$$\|K[f_1, ..., f_k]\|_\sigma \le C \prod_{i=1}^k \|f_i\|_{i,\sigma} \tag{3.7}$$

with $C = C_0 \prod_{i=1}^k M_i$ and

$$\|K[f_1, ..., f_k]\|_\sigma \le \lambda(\sigma) \prod_{l \ne i} \|f_l\|_{l,\sigma} \cdot \|f_i\|_{i,0} \qquad (1 \le i \le k) \tag{3.8}$$

with $\lambda(\sigma) = C_0 \cdot \max_{i \in \{1,...,k\}} \left(\prod_{l \ne i} M_l \cdot N_i(\sigma)\right)$ where by the Lebesgue dominant convergence theorem $N_i(\sigma) \to 0$ as $\sigma \to \infty$, hence also $\lambda(\sigma) \to 0$ as $\sigma \to \infty$. Corresponding inequalities hold in case $p_i = \infty$ for some $i \in \{1, ..., k\}$.

We point out the particular case $k = p + 1$ ($p \ge 1$) with

$$\alpha_i = 0 \quad \text{and} \quad \beta_i = -\frac{1}{p} \quad (1 \le i \le p) \qquad \text{and} \qquad \alpha_k = \beta_k = 1$$

of (3.3) and (3.4), which leads for $f_i = u$ ($1 \le i \le k$) to the power operator of convolution type

$$K_p[u] = \int_0^{x_n} \ldots \int_0^{x_1} m(x, y) u^p(\tfrac{y}{p}) u(x - y)\, dy_1 \cdots dy_n. \tag{3.9}$$

Examples of operators $G_{k,i}^j$ fulfilling the Lipschitz conditions (B3) with (B5) in the weighted norms (3.6) are also given by powers of functions with deviating argument, for instance. So let us consider in the space $E = C(\overline{D})$ or $E = L_\infty(D)$ the operator

$$(Gu)(x) = u^p(h(x)) \qquad (p \ge 1, \text{entire}) \tag{3.10}$$

where $h \in C_n(\overline{D})$ is a continuous n-dimensional vector function satisfying $0 \le h(x) \le \frac{x}{p}$. We have

$$\begin{aligned}
&\left|e^{-\sigma|x|}[(Gu_1)(x) - (Gu_2)(x)]\right| \\
&\quad = \left|e^{-\sigma|x|}[u_1(h(x)) - u_2(h(x))] \sum_{j=0}^{p-1} u_1^j(h(x)) u_2^{p-1-j}(h(x))\right| \\
&\quad \le e^{-\sigma|x - ph(x)|} e^{-\sigma|h(x)|} |u_1(h(x)) - u_2(h(x))| \\
&\qquad \times \sum_{j=0}^{p-1} e^{-\sigma j|h(x)|} |u_1^j(h(x))| \cdot e^{-\sigma(p-1-j)|h(x)|} |u_2^{p-1-j}(h(x))| \\
&\quad \le \|u_1 - u_2\|_\sigma \sum_{j=0}^{p-1} \|u_1\|_\sigma^j \|u_2\|_\sigma^{p-1-j},
\end{aligned}$$

i.e. condition (B3) is fulfilled with $M(\rho_1, \rho_2) = \sum_{j=0}^{p-1} \rho_1^j \rho_2^{p-1-j}$.

As an application of the foregoing considerations we show that the initial-value problem for the one-dimensional integro-functional-differential equation

$$\left.\begin{aligned} & v'(t) + \Phi\left(t, v(t), v^2(\tfrac{t}{2}), \ldots, v^{n_0}(\tfrac{t}{n_0})\right) \\ & \quad + \int_0^t a_0(s,t) v(t-s)\, ds + \int_0^t b_0(s,t) v'(t-s)\, ds \\ & \quad + \int_0^t v(t-s) \sum_{k=1}^{n_1} a_k(s,t) v^k(\tfrac{s}{k})\, ds \\ & \quad + \int_0^t v'(t-s) \sum_{k=1}^{n_2} b_k(s,t) v^k(\tfrac{s}{k})\, ds \\ & \quad + \int_0^t v'(t-s) \sum_{k=1}^{n_3} c_k(s,t) [v'(\tfrac{s}{k})]^k ds = g(t), \quad v(0) = c_0 \end{aligned}\right\} \tag{3.11}$$

with $c_0 \in \mathbb{R}$ and entire $n_0, n_1, n_2, n_3 \geq 1$ has a unique solution $u \in C^1[0,T]$ for $g \in C[0,T]$ in any finite interval $[0,T]$ $(T > 0)$, if the functions a_k, b_k and c_k are continuous and the function $\Phi(t, v_1, v_2, \ldots, v_{n_0})$ is continuous and fulfills a uniform Lipschitz condition in the variables $(v_1, v_2, \ldots, v_{n_0})$.

The statement immediately follows from the above results by taking $u = v' \in C[0,T]$ with

$$v(t) = \int_0^t u(s)\, ds + c_0$$

as unknown function and observing that by the Young inequality

$$\|v_1 - v_2\|_\sigma \leq \min(T, \tfrac{1}{\sigma})\, \|u_1 - u_2\|_\sigma$$

for the norm (3.6) in $C[0,T]$ (cp. [6: Example 5]). So the term with the function Φ generates an operator of the form G_0, the integral terms are operators of the form (3.1). Of course, the functions $v^p(\frac{t}{p})$ in (3.11) can be replaced by other operators of the form (3.10). For functional-differential equations of the form (3.11) without integrals cp. [3, 4], for instance.

We finally remark that further examples related to the examples in [6] are possible, also for systems of differential and integral equations and for operators $G_{k,i}^j$ with functional dependence on u.

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