A Survey on Huygens' Principle

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Dedicated to the memory of Paul Günther (1926-1996)

The authors remember with much gratitude their academic teacher.
They were deeply influenced by his work and personality
and felt a great loss with his passing.

Abstract. There are two classes of physical wave phenomena: with (e.g. water waves) or without (e.g. flash or bang in space) after effects. The second class is said to obey Huygens' principle. The mathematical formulation concerns Cauchy's initial value problem to a given linear hyperbolic differential equation, and is generalized to arbitrary dimensions as well as to curved spacetimes, i.e. Lorentzian manifolds. The original conjecture that every Huygens-type equation is transformable to the wave equation in Minkowski spacetime was refuted by counter-examples found by K. L. Stellmacher and by P. Günther. Since then, many results accumulated, but a general characterization of the equations which satisfy Huygens' principle is not yet known. Some classes of examples show interesting relations to other branches of physics or mathematics: the new higher spinor field equations of Buchdahl and Wünsch solve the long-standing inconsistency problem, Huygensian wave equations on symmetric spaces are treated by means of Lie-theoretical methods, far-reaching generalizations of the Stellmacher-Lagnese examples are related to Coxeter groups and to integrable dynamical systems. The present paper surveys the research on Huygens' principle – from Hadamard up to recent results.

Keywords: *Hyperbolic equations, Huygens' principle, Hadamard's coefficients, Hadamard's fundamental solution, tail term, characteristic conoid, conformal invariants, conformal derivatives, moments, higher spin field equations, Riesz kernels, mean value operators*

AMS subject classification: 35 L 10, 58 G 16, 35 Q 75

0. Introduction

The principle which is treated in this survey has a physical meaning and a mathematical one, whereby the latter evolves by abstraction and generalization from the former. The *physical Huygens' principle* is a special feature of some wave phenomena. Namely waves can show fundamentally different appearances. If a stone is thrown into water, new waves come from the centre of disturbance even when the stone has vanished. Behind a sharp forward wave front the whole interior domain is excited by waves, which spread out in form of concentric circles; there is no backward wave front. In contrast, a flash of

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light or a bang is momentary, i.e. we observe a sharp forward wave front and a backward wave front and smoothness between these.

The stone, the flash and the bang have in common that they are a limited disturbance. If the waves produced by it show a sharp backward front, then Huygens' principle is said to be valid, if not, then the Huygens' principle is said to be violated.

The transition from physics to mathematics involves the following.

Model building: Description of a wave phenomenon by a partial differential equation of *hyperbolic* type.

Generalization: Every hyperbolic equation might be taken into consideration, let it be realized in nature or not. Up to now, an exact theory of Huygens' principle has been developed only for *linear* hyperbolic equations.

Geometrization: Modern differential geometry with its key concepts *manifold, Riemannian metric, fibre bundle,* ... becomes the appropriate language.

The *mathematical Huygens' principle* is expressed in terms of Cauchy's initial value problem to a given hyperbolic differential equation. Huygens principle is satisfied if the solution of such a problem taken in a point x depends only on the Cauchy data (and their derivatives) on the intersection of the initial hypersurface with the past characteristic conoid issuing from x.

Essentially, two classes of linear hyperbolic equations have been investigated: second-order equations with a Laplace-like principal part, and first-order equation systems which can be reduced to Laplace-like equations with special initial values. The D' Alembert equation and the Klein-Gordon equation for scalar fields, differential forms or other fields belong to the second-order class. The Weyl and Dirac equations, the Maxwell and Proca equations and some more belong to the first-order class. The spinor formalism has proved to be extremely useful in order to deal with such first order equations. Thereby the long-standing problem to describe higher spin fields on a curved spacetime has found a satisfactory solution.

Three books [21, 35, 40] are landmarks in the history of the concept which we consider here. J. Hadamard [40] analysed three variants A, B, C of wave-like behaviour, and he called B *Huygens' minor premise*. R. Courant and D. Hilbert [21] reduced the name to *Huygens' principle*, though the latter existed already with another meaning. The problem to characterize all Huygens-type linear hyperbolic equations turned out to be more and more complex the longer it was investigated. Nowadays, there is a bulk of results and the problem is solved for some subclasses of equations, on the one hand. But Hadamard's problem remains unsolved in general and even in the physically important particular case of the wave equation on a four-dimensional spacetime on the other hand [4]. P. Günther's monograph [35] presented most of the results on Huygens' principle at its time and developed a systematic theory. The greater part of the results in review article [3] is due to the author of the book and his school.

There are, additionally to the books [21, 35, 40], papers which give an introduction to Huygens' principle. Each of them stresses a special aspect: R. G. McLenaghan's paper [61] is centered on plane-wave spacetimes and their generalizations, P. Günther's

article [36] is aimed at an intuitive and historical understanding of Huygens principle, J. J. Duistermaat [22] compares several different approaches to wave equations. Many informations on a few pages are contained in B. Ørsted's review [67] of P. Günther's book [35].

1. The D' Alembert equation on $\mathbb{R}^n$

Let us consider a simple example of a linear hyperbolic equation – the wave equation or D' Alembert equation

$$\Box\, u \equiv \left(\frac{\partial^2}{\partial t^2} - \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} \right) u = 0$$

for a scalar field u on the flat n-dimensional spacetime $\mathbb{R}^n$ – as a first step towards a mathematical formulation of Huygens' principle. Note that $\Box\, u = 0$ is, for instance, a realistic linear model for sound if $n \leq 4$. The spacetime $\mathbb{R}^n$ is equipped with the Minkowski metric

$$g = dt^2 - \sum_{k=1}^{n-1} dx_k^2$$

where $t = x_0$ is the time, and $x_k = x_1, \ldots, x_{n-1}$ are Cartesian coordinates in space. The same information as in g is contained in the so-called world function

$$\sigma = \sigma(x,y) = \frac{1}{2} \left[(x_0 - y_0)^2 - \sum_{k=1}^{n-1} (x_k - y_k)^2 \right] \ .$$

The invariant distance $r = r(x,y)$ between two points $x, y \in \mathbb{R}^n$ is given by

$$r(x,y) = |2\sigma(x,y)|^{\frac{1}{2}} \ .$$

Some elements of Minkowski geometry are defined in terms of σ:

$$D(x) = \left\{ y \in \mathbb{R}^n | \, \sigma(x,y) \geq 0 \right\}$$

is the *solid light cone* with vertex x, its boundary

$$C(x) := \partial D(x) = \left\{ y \in \mathbb{R}^n | \, \sigma(x,y) = 0 \right\}$$

is the *light cone surface*. Note that both $D(x)$ and $C(x)$ decompose into *future parts* $D^+(x), C^+(x)$ and *past parts* $D^-(x), C^-(x)$ the points or events of which are later or earlier, respectively, than x. Let us further introduce the zero time hyperplane

$$H = \{0\} \times \mathbb{R}^{n-1} \ ,$$

the $(n-1)$-dimensional ball in H

$$B(x) = D(x) \cap H \ ,$$

and the $(n-2)$-dimensional sphere in H

$$S(x) = C(x) \cap H \ .$$

Now we are able to pose Cauchy's problem: the wave equation $\Box\, u = 0$ is completed by initial conditions

$$\left.\begin{array}{l} u_{|H} = u_0 \\[2mm] \dfrac{\partial u}{\partial t}\Big|_H = u_1 \ . \end{array}\right\}$$

It is well-known that for sufficiently smooth initial data u_0 and u_1 there is a unique solution $u = u(x) = u[u_0, u_1]$ and that

$$u[u_0, u_1] = u[0, u_1] + \frac{\partial}{\partial t} u[0, u_0] \ .$$

Thus, the general Cauchy problem is reduced to the special one

$$\left.\begin{array}{l} u_{|H} = 0 \\[2mm] \dfrac{\partial u}{\partial t}\Big|_H = u_1 \ . \end{array}\right\}$$

Let us analyse the well-known solution formulas to the latter, namely

$$2\,u(t, x_1) = \int\limits_{x_1 - t}^{x_1 + t} u_1(y)\, dy$$

$$2\pi\, u(t, x_1, x_2) = \iint\limits_{B(t, x_1, x_2)} \frac{u_1(y_1, y_2)\, dy_1\, dy_2}{\sqrt{t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}}$$

$$4\pi t\, u(t, x_1, x_2, x_3) = \iint\limits_{S(t, x_1, x_2, x_3)} u_1(y_1, y_2, y_3)\, d_y S$$

for $n = 2, 3, 4$ named after D'Alembert, Volterra and Poisson, respectively [21]. For $n = 2$ and $n = 3$ the value $u(x)$ of the solution u depends on the data on the whole of the ball $B(x)$, while for $n = 4$ it depends only on the data on $S(x)$. If we reverse the direction of time and consider $u(x)$ as a disturbance concentrated at the point x, then we arrive at the situation of the introduction. In this definition, by analogy, Huygens principle is satisfied here for $n = 4$ and is violated for $n = 2$ and $n = 3$. The analysis can be extended to arbitrary dimensions and shows that the wave equation $\Box\, u = 0$ on Minkowski spacetime $\mathbb{R}^n$ is Huygens' for even $n \geq 4$ and is non-Huygens' for odd $n \geq 3$ and also for $n = 2$.

Some visualization may help to understand the step from spatial to spacetime geometry. Imagine a wave process and a series of photographs of it. If the snapshots are piled up in their temporal order and if they are, in thought, interpolated to a continuum, then we obtain the picture of the wave process in Minkowski spacetime. For instance, the future light cone $C^+(x)$ is the picture of a point-like disturbance at x, which just appeared at the start of our physical discussion in the introduction.

2. Huygens' principle for second-order equations

Let us introduce, taking advantage of differential-geometric concepts, a far-reaching generalization of the precceding example.

- The Minkowski spacetime $\mathbb{R}^n$ is replaced by an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ of Lorentzian signature $(+ - \cdots -)$.

- The independent variables are now local coordinates $x^a = x^1, x^2, \ldots, x^n$ of a point $x \in \mathcal{M}$.

- The dependent variable or unknown function generalized to a section $u = u(x)$ of a given vector bundle E over $\mathcal{M}$.

- The wave equation on $\mathbb{R}^n$ is replaced by a linear partial differential equation of the form

$$L[u] \equiv g^{ab} D_a D_b u + Wu = f ,$$

where u and f are sections of E, $(g^{ab}) = (g_{ab})^{-1}$ belongs to the metric $g = g_{ab} \, dx^a dx^b$, D is a covariant derivation, and W is a section of the endomorphism bundle $\operatorname{End} E$ to E.

- The world function of $(\mathcal{M}, g)$ can be introduced as the solution $\sigma = \sigma(x, y)$ of the differential equation

$$g^{ab}(\nabla_a \sigma)(\nabla_b \sigma) = 2\sigma$$

together with the initial conditions

$$\left. \begin{array}{r} (\nabla_a \sigma)(x, x) = 0 \\ (\nabla_a \nabla_b \sigma)(x, x) = g_{ab}(x) \end{array} \right\}$$

where the ∇_a are the Levi-Civita derivatives with respect to the first argument x.

- The light cone is replaced by the characteristic conoid. More precisely, the solid conoid $D(x)$ is defined by $\sigma(x, y) \geq 0$ and the conoid surface $C(x)$ is defined by $\sigma(x, y) = 0$.

- The time coordinate is replaced by a time function, that is a function $t = t(x)$ such that

$$g^{ab}(\nabla_a t)(\nabla_b t) > 0 .$$

The zero time hyperplane is replaced by the spacelike hypersurface

$$H = \{x \in \mathcal{M} \mid t(x) = 0\} .$$

We set then

$$B(x) = D(x) \cap H \qquad \text{and} \qquad S(x) = C(x) \cap H .$$

- Finally, the equation $L[u] = f$ is completed by initial conditions

$$\left. \begin{array}{r} u_{|H} = u_0 \\ g^{ab}(\nabla_a t)(D_b u)_{|H} = u_1 \end{array} \right\}$$

to a Cauchy problem.

Some comments are in order.

1. We apply the usual notations of tensor calculus, in particular the Einstein summation convention with respect to repeated indices in products.

2. All objects are assumed to be smooth, i.e. of differentiability class C^∞.

3. The world function $\sigma = \sigma(x, y)$ exists, in general, only locally, that means for x and y sufficiently near to each other. Therefore, $D(x)$ and $S(x)$ are restricted to some neighbourhood of the point $x \in \mathcal{M}$, and all constructions based on these elements are local ones.

4. The Levi-Civita covariant derivatives ∇_a are given in terms of the Christoffel symbols Γ^c_{ab} to the metric g. In particular, we have for a function or scalar field $s = s(x)$

$$\nabla_a s = \partial_a s \qquad \text{and} \qquad \nabla_a \nabla_b s = \partial_a \partial_b s - \Gamma^c_{ab} \partial_c s$$

where $\partial_a = \frac{\partial}{\partial x^a}$ are the partial derivatives with respect to the local coordinates x^a.

5. Covariant derivatives D_a of sections u of E are characterized by the Leibniz rule

$$D_a(su) = (\nabla_a s)u + s D_a u$$

for scalar fields s. The second derivatives $D_a D_b$ are defined by another Leibniz rule which combines D and ∇.

6. The differential operator L introduced above has the same principal symbol as the Laplacian or D' Alembertian

$$\Delta = g^{ab} \nabla_a \nabla_b$$

acting on scalar or tensor fields. Therefore, we call L *Laplace-like* or of *Laplace type*. Notice that Δ and L are hyperbolic if the metric g has signature $(+ - \cdots -)$.

Huygens' principle. Now we can give a mathematical formulation of Huygens' principle. A Laplace-like hyperbolic operator L is *Huygens'* if for every H, x, f, u_0, u_1 the solution $u = u(x)$ of Cauchy's problem taken at x depends only on the data u_0, u_1 and its dervatives taken on $S(x)$, but not on the values of u_0, u_1 in the interior of $B(x)$. That means, if the data differ only in $B(x) \setminus S(x)$, then the solution $u(x)$ at x will be the same. This can be made more precise, using the linearity of L. *Huygens' principle* becomes the following peculiar property of a differential operator L: If the initial data u_0, u_1 have support in the interior of $B(x)$, i.e.

$$\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B(x) \setminus S(x) ,$$

then the solution u of the Cauchy problem (with $f = 0$)

$$\left. \begin{array}{r} L[u] = 0 \\ u_{|H} = u_0 \\ g^{ab}(\nabla_a t)(D_b u)_{|H} = u_1 \end{array} \right\}$$

vanishes at the point x.

There are several theories which give existence, uniqueness, and a construction of a solution of the above Cauchy problem. Each of them leads to its own criterion for Huygens principle. Let us mention the following:

• J. Hadamard [40, 41] constructed a solution by means of "finite parts of divergent integrals". He found that Huygens' principle holds for the differential operator L if and only if n is even, $n \geq 4$, and the formal adjoint L^* to L admits a logarithm-free elementary solution.

• M. Riesz [68] introduced a semigroup of integral operators with kernels $V(x, y, \lambda)$, λ being the semigroup parameter. He solved the Cauchy problem by means of analytical continuation to $\lambda = 2$. Huygens' principle holds, for even $n \geq 4$, if and only if $V(x, y, 2) = 0$.

• S. L. Sobolew [79, 80] and L. Asgeirsson [6, 7] studied, for even $n \geq 4$, $L[u] = f$ together with timelike derivatives of order $1, 2, \ldots, \frac{n-4}{2}$ of this differential equation and derived from the resulting system an integral equation for the solution of Cauchy's problem. Huygens' principle holds if and only if a certain' integral kernel, called "diffusion kernel" vanishes. In this case, the integral equation becomes a solution formula.

• F. G. Friedlander [23] and P. Günther [35] reformulated the Cauchy problem in terms of distributions and constructed distributional solutions. Huygens' principle holds, for even $n \geq 4$, if and only if the fundamental solution of L^* has its support on the characteristic conoid surface (and not in the interior of the solid conoid). This is equivalent to the condition that the "tail term" to L^* vanishes.

It is not suprising that all the necessary and sufficient criteria for Huygens' principle from different authors turn out to be equivalent. They can all be reduced to one more explicit condition, which is accessible to evaluations and calculations. We have, in order to present this condition, to introduce the so-called Hadamard coefficients $H_k = H_k(x, y)$ $(k = 0, 1, 2, \ldots)$ to L. These two-point quantities are recursively defined by

$$g^{ab}(\nabla_a \sigma) D_b H_0 + \mu H_0 = 0 , \quad H_0(x, x) = I$$
$$g^{ab}(\nabla_a \sigma) D_b H_k + (\mu + k) H_k = L[H_{k-1}] \quad \text{for } k \geq 1$$

where

$$\mu = \frac{1}{2}(\Delta \sigma - n)$$

and where all differentiations refer to the first argument x. Each $H_k = H_k(x, y)$ behaves like a section of E with respect to $x \in \mathcal{M}$ and like a section of the dual bundle E^* with respect to $y \in \mathcal{M}$. Thus, a diagonal value $H_k(x, x)$ can be interpreted as a section of $\text{End} E$. In particular, $H_0(x, x) = I$ is required to be the unit matrix. The differential-recursion system for the Hadamard coefficients has a remarkable property: it can be shown that there is a neighbourhood of the diagonal of $\mathcal{M} \times \mathcal{M}$ where solutions $H_k = H_k(x, y)$ $(k = 0, 1, 2, \ldots)$ exist and are unique.

Theorem 1. *Huygens' principle never holds for $n = 2$ or for odd $n \geq 3$. So, let $n = 2m + 2 \geq 4$ be even. Huygens' principle holds for the formal adjoint L^* of*

the Laplace-like hyperbolic operator L if and only if the m-th Hadamard coefficient to L contains the world function $\sigma = \sigma(x, y)$ as a factor, that means there is a regular two-point function $R = R(x, y)$ such that

$$H_m(x, y) = \sigma(x, y)\, R(x, y)\ . \tag{1}$$

An explicit or implicit proof of this criterion can be found in each of the papers [6, 23, 35, 40, 68, 79]. Notice that L and its formal adjoint L^* can easily interchange their roles since $L^{**} = L$. The two-point condition (1) implies a sequence of one-point conditions, namely conditions for the Taylor coefficients of H_m with respect to the running point x and the origin y. We need, in order to present these, elements of a calculus of symmetric differential forms.

A symmetric p-form

$$u = u_p = u_{a_1 a_2 \dots a_p} dx^{a_1} dx^{a_2} \dots dx^{a_p}$$

is a special notation for a totally symmetric covariant tensor field of valence p. For instance, the Riemannian metric $g = g_{ab} dx^a dx^b$ is a symmetric 2-form. The multiplication of symmetric forms is the tensor multiplication followed by symmetrization. A metric g defines a trace operator tr by

$$\left.\begin{array}{l} \operatorname{tr} u_0 = 0 \\[4pt] \operatorname{tr} u_1 = 0 \\[4pt] \operatorname{tr} u_2 = g^{ab} u_{ab} \\[4pt] \operatorname{tr} u_p = g^{ab} u_{ab a_3 \dots a_p} dx^{a_3} \dots dx^{a_p} \quad \text{for } p \geq 3\ . \end{array}\right\}$$

Every p-form u_p admits a unique decomposition into a part proportional to g and a trace-free part $TS\, u_p$:

$$u_p = g \cdot u_{p-2} + TS\, u_p\ , \qquad \operatorname{tr}(TS\, u_p) = 0\ ,$$

where " $\cdot$ " indicates symmetric multiplication. All these facts are naturally generalized to $(\operatorname{End} E)$-valued symmetric forms.

Theorem 2. *Let $n = 2m + 2 \geq 4$ be even. If Huygens' principle holds for L^*, then*

$$TS\,(D_{a_1} D_{a_2} \dots D_{a_p} H_m)(x, x) dx^{a_1} dx^{a_2} \dots dx^{a_p} = 0 \tag{2}$$

for $p = 0, 1, 2, \dots$. If, in particular, the objects $\mathcal{M}, g$ and L are analytic, then the conditions (2) are not only necessary but also sufficient for Huygens' principle.

The proof of Theorem 2 evaluates (1) and is based on

$$(\nabla_a \nabla_b\, \sigma)(x, x) = g_{ab}(x)$$
$$(\nabla_{a_1} \nabla_{a_2} \cdots \nabla_{a_p}\, \sigma)(x, x) dx^{a_1} dx^{a_2} \cdots dx^{a_p} = 0 \quad \text{for } p \geq 3\ .$$

Let us explain the left-hand side of (2): Covariant derivatives $D_{a_1}, D_{a_2}, \dots, D_{a_p}$ with respect to x are applied to the m-th Hadamard cofficient $H_m(x, y)$, the result is restricted

to the diagonal $x = y$, finally the symmetric and trace-free part of the resulting p-tensor is taken.

Standard arguments of the theory of invariants show that the (End E)-valued tensor components

$$(D_{a_1} D_{a_2} \ldots D_{a_p} H_m)(x, x)$$

are polynomials in the variables

$$g_{ab}, \quad g^{ab}, \quad R_{abcd}, \quad \nabla_{c_1} R_{abcd}, \quad \nabla_{c_1} \nabla_{c_2} R_{abcd}, \ldots$$
$$F_{ab}, \quad D_{c_1} F_{ab}, \quad D_{c_1} D_{c_2} F_{ab}, \ldots$$

where the components of the Riemannian curvature R_{abcd} and the components of the gauge field curvature F_{ab} are defined through the Ricci identities

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v_c = R_{abcd}\, v^d$$
$$(D_a D_b - D_b D_a) u = F_{ab} u$$

for a 1-form $v = v_a\, dx^a$, $v^a := g^{ab} v_b$ and for a section u of E, respectively.

For low values of n and p the conditions (2) have been made explicit.

Theorem 3. *Let $n = 4$. If a differential operator*

$$L = g^{ab} D_a D_b + W$$

satisfies Huygens' principle, then

(i) $W - \frac{R}{6} I = 0$

(ii) $D^b F_{ab} = 0$

(iii) $-\frac{2}{5} B_{ab} I = F_{abc} F_b{}^c + F_{bc} F_a{}^c - \frac{1}{2} g_{ab} F_{cd} F^{cd}$.

Here and throughout the paper we use the following notations:

$$R_{ab} = g^{cd} R_{acdb}$$

are the components of the Ricci tensor,

$$R = g^{ab} R_{ab}$$

is the scalar curvature,

$$W_{abcd}$$

are the components of the conformal curvature tensor (the definition of which we omit here) and

$$B_{ab} = \nabla^c \nabla^d W_{acdb} - \frac{1}{2} R^{cd} W_{acdb}$$

are the components of the Bach tensor.

A proof of Theorem 3 was given for the scalar case in [27] and for the vector-bundle case in [74].

The above conditions (i) - (iii) in Theorem 3 admit the following interpretations: (i) means that Cotton's invariant $C = W - \frac{R}{6} I$ vanishes; (ii) is the Yang-Mills equation for the gauge potentials A_a in $D_a u = \partial_a u + A_a u$, and the endomorphism trace of (iii) is a gravitational field equation with a Yang-Mills source.

Theorem 4. *Let $n = 6$. If a differential operator*

$$L = g^{ab}D_aD_b + W$$

satisfies Huygens' principle, then

$$30g^{ab}D_aD_bC + 6RC + W_{abcd}W^{abcd}I + 15F_{ab}F^{ab} + 90C^2 = 0 \ ,$$

where $C = W - \frac{R}{5}I$ is the so-called Cotton endomorphism to L.

A proof of Theorem 4 was given for the scalar case in [29] and for the general case in [73]. The result also appeared earlier in the context of spectral geometry [25, 26].

Note that the formula in Theorem 4 also admits an interpretation, namely as a nonlinear Higgs equation for $C = C(x)$ with some source terms.

4. Trivial and non-trivial Huygens' equations

The class of Laplace-like differential operators

$$L = g^{ab}D_aD_b + W$$

is form-invariant under diffeomorphisms, when the objects g, D and W are carried along with the diffeomorphism. Clearly, it is also form-invariant under coordinate transformations. Further, we have form-invariance under gauge transformations

$$\overline{L} = \Lambda^{-1}L\Lambda$$

where $\Lambda = \Lambda(x)$ is a section of the automorphism bundle $\operatorname{Aut} E$ to E, and under conformal transformations

$$\overline{L} = e^{-(m+2)\varphi}Le^{m\varphi}$$

where $\varphi = \varphi(x)$ is a smooth function, $m = \frac{n-2}{2}$, and functions are notationally identified with multiplication operators. A gauge transformation with Λ induces

$$\overline{g} = g \ , \qquad \overline{D}_a = \Lambda^{-1}D_a\Lambda \ , \qquad \overline{W} = \Lambda^{-1}W\Lambda$$

while a conformal transformation with φ induces [35, 66, 74]

$$\overline{g} = e^{2\varphi}g \qquad \text{and} \qquad \overline{W} = e^{-2\varphi}(W + Ie^{-m\varphi}\Delta e^{m\varphi}) \ .$$

One can show that multiple covariant derivatives D of sections u of E behave like Levi-Civita derivatives of scalar fields under a conformal transformation [74]. In particular,

$$\overline{D}_au = D_au \ .$$

Two Laplace-like operators L and $\overline{L}$ are called *equivalent* if there is a combination of coordinate transformations, gauge transformations, and conformal transformations

which changes L into $\overline{L}$. A Laplace-like operator L is called *trivial* if it is locally equivalent to an operator

$$\overline{L} = g^{ab}\partial_a\partial_b$$

where $g^{ab} = $ const and $\partial_a = \frac{\partial}{\partial x^a}$. Obviously, all elements in the definition of Huygens' principle are invariant under diffeomorphisms, gauge transformations, and conformal transformations. (The characteristic conoid is, in particular, conformally invariant.) Hence:

> *Huygens' principle is a covariant, gauge invariant, and conformally invariant property!*

A hyperbolic operator L is trivial if and only if it is locally equivalent to the wave equation on $\mathbb{R}^n$. The latter is Huygens', if $n \geq 4$ is even. So we come to the conclusion:

> *If $n \geq 4$ is even, then a trivial hyperbolic operator L satisfies Huygens' principle.*

Naturally, the question arises whether there are non-trivial Huygens' equations? M. Mathisson [58] in 1939 gave a negative answer for a scalar-type operator L on the four-dimensional Minkowski spacetime

$$L = g^{ab}(\partial_a + A_a)(\partial_b + A_b) + W \qquad (a, b = 1, 2, 3, 4)$$

$$g^{ab} = \text{const}, \quad A_a = A_a(x^b), \quad W = W(x^b) .$$

If such a hyperbolic operator is Huygens', then it is trivial. The proof of this fact in [58] is an evaluation of the conditions

$$H_1(x, x) = 0, \qquad (D_a H_1)(x, x) = 0, \qquad (D_a D_b H_1)(x, x) \approx g_{ab} .$$

Later, K. L. Stellmacher [81] in 1953 found a positive answer to the above question: he constructed a non-trivial Huygens' equation in $n = 6$ dimensions. His example is scalar-type and the underlying manifold is flat, i.e. $g^{ab} = $ const. It reads, in a notation which differs from Stellmacher's,

$$L = g^{ab}\partial_a\partial_b - 2\langle e, x\rangle^{-2}$$

where

$$\langle x, y\rangle = g_{ab}x^a y^b$$

is the scalar product of $x, y \in \mathbb{R}^n$ with respect to the flat metric and $e \in \mathbb{R}^n$ is an arbitrary unit vector, i.e. $\langle e, e\rangle = 1$. In a second paper [82] he generalized the example to any even dimension $n \geq 6$ by introducing a parameter l:

$$L = g^{ab}\partial_a\partial_b - l(l + 1)\langle e, x\rangle^{-2} .$$

The Hadamard coefficients H_k to this L can be explicitly calculated [77]:

$$H_k = (-1)^k k! \binom{l}{k}\binom{l+k}{k} \langle e, x\rangle^{-k}\langle e, y\rangle^{-k} .$$

If $l = 0, 1, \ldots, m-1$, then obviously $H_m = 0$, hence Huygens' principle holds. Moreover, if $l \neq 0$, then the operator is non-trivial. Stellmacher's example does not cover $n = 4$. This gap was filled by P. Günther [32] in 1965. He showed the following:

For every even $n \geq 4$ the scalar wave equation $\Delta u = 0$ to a plane wave metric

$$g = 2dx^0 dx^1 - a_{ij}(x^0)dx^i dx^j \qquad (i, j = 2, 3, \ldots, n-1),$$

where the matrix $a(x^0) = (a_{ij}(x^0))$ is positive definite, satisfies Huygens' principle.

The proof relies on an explicit formula for the world function:

$$\sigma = (x^0 - y^0)(x^1 - y^1) - \frac{1}{2}u_{ij}(x^0, y^0)(x^i - y^i)(x^j - y^j)$$

where $u_{ij}(x^0, y^0)$ are the elements of the matrix

$$u(x^0, y^0) = \left(\int_{y^0}^{x^0} a(t)^{-1}dt\right)^{-1}.$$

The plane wave metrics admit a physical interpretation as plane gravitational waves. They are characterized by the existence of a vector field $l = l^a \partial_a$ such that

$$\nabla_a l_b = 0, \qquad R_{ab[cd}l_{e]} = 0, \qquad l_{[f}\nabla_{e]}R_{abcd} = 0$$

where the brackets [ ] indicate antisymmetrization. The group of motions of a general plane wave spacetime is isomorphic to the $(2n - 3)$-dimensional Heisenberg group; it is generated by translations and null rotations [72].

After P. Günther's paper [32], non-scalar operators L too have been studied with a plane wave background. The following results have been found in [71] for a), b) and in [88] for c) (cf. also [47, 64]).

Theorem 5. *Let $(\mathcal{M}, g)$ be a plane wave manifold of even dimension n. Then the following equations satisfy Huygen's principle in (M, g):*

a) *The Hodge-de Rham wave equation*

$$(d\delta + \delta d)u = 0$$

for alternating p-forms u $(0 \leq p \leq n)$, if $n \geq 6$ (see also Section 7 and there the result a) with footnote [1]).

b) *The Maxwell equations*

$$du = 0 \qquad and \qquad \delta u = 0$$

for alternating p-forms u $(1 \leq p \leq n - 1)$, if $n \geq 4$ (see also Section 7 and there the results (M), (V) and behind that the Corollary).

c) *The D' Alembert equation*

$$\Delta u = 0$$

on 1-spinor fields u, if n = 4.

There are more theorems of this type where plane wave metrics are sufficient for Huygens' principle. There are also many theorems where plane wave metrics appear as necessary conditions for Huygens' principle [56, 58]. Notice that we do not present here a theory of Huygens' principle for hyperbolic first-order systems. Let us only mention that the latter are equivalent to hyperbolic Laplace-like systems with special initial values in Cauchy's problem.

An important subclass of the linear hyperbolic equations of second order are those with constant coefficients. The above question can be modified: Are there non-trivial Huygens' equations with constant coefficients? M. Mathisson's result gives, again, a negative answer for $n = 4$ and one-component (i.e. number-valued) fields u. A positive answer can be given for N-component fields, $N \geq 2$. The following systems with constant coefficients non-trivially satisfy Huygens' principle [75]:

For $N = 2$, $n \geq 4$ even:

$$\Box u_1 + (\partial_1 - \partial_2)u_1 + \partial_3 u_2 = 0$$
$$\Box u_2(\partial_1 - \partial_2)u_2 = 0.$$

For $N = 3$, $n \geq 4$ even:

$$\Box u_1 + \partial_2 u_2 = 0$$
$$\Box u_2 + \partial_3 u_3 = 0$$
$$\Box u_3 = 0.$$

For $N = n \geq 4$ even:

$$\Box u_1 = 0$$
$$\Box u_{i+1} + \partial_i u_i = 0 \quad (i = 1, 2, \ldots, n - 1).$$

Here $\Box = \partial_0^2 - \partial_1^2 - \ldots - \partial_{n-1}^2$ is the wave operator of the flat spacetime $\mathbb{R}^n$.

5. Linear differential equations of mathematical physics

Let us consider in this section a (possibly curved) spacetime of dimension $n = 4$. The principle of first quantization associates kinds of elementary particles to field equations. Thus, the linear field equations of mathematical physics can be classified by the rest mass m and by the spin quantum number s of the associated particles. If s is integer, then the particle is called a *boson* and the field is a tensor of valence s. If s is half-integer, then the particle is called a *fermion* and the field is a spinor of valence $2s$.

For $m = 0$ and absent sources there are the following field equations:

$s = 0$: D'Alembert equation $\Delta u = 0$.

$s = \frac{1}{2}$:   Weyl's equation $\nabla^{\dot{X}}_A \varphi_{\dot{X}} = 0$.

$s = 1$:   Maxwell's equation $\nabla^b F_{ab} = 0$ for $F_{ab} := \nabla_a A_b - \nabla_b A_a$.

$s > 1$:   Several equation systems have been proposed. Among these,

$$\nabla^{\dot{X}}_{(A}\varphi_{A_1,\ldots,A_p)\dot{X}} = 0 \qquad (p = 2s - 1)$$

is distinguished by particularly good properties [48, 50].

For $m > 0$, $\mu := \frac{mc}{\hbar}$, where c is the velocity of light and $\hbar$ is Planck's constant, and absent sources there are the following field equations:

$s = 0$:   Klein-Gordon equation $\Delta u + \mu^2 u = 0$.

$s = \frac{1}{2}$:   Dirac equation $\gamma^a \nabla_a u + \mu u = 0$.

$s = 1$:   Proca equation $\nabla^b F_{ab} - \mu^2 A_a = 0$ for $F_{ab} := \nabla_a A_b - \nabla_b A_a$.

$s > 1$:   Several equation systems have been proposed.

Most of these equations exhibit disadvantages if they are, by means of minimal gravitational coupling, extended to a curved spacetime: either they require additional structures or they imply inacceptibly strong integrability conditions. The following system discovered by H. A. Buchdahl and brought into a convenient form by V. Wünsch is free of such inconsistencies [49 - 51, 87, 88, 90, 91, 93]:

$$\nabla^B_{\dot{X}} \varphi_{B A_1,\ldots,A_p} + \mu \chi_{A_1,\ldots,A_p \dot{X}} = 0$$
$$\nabla^{\dot{X}}_{(A} \varphi_{A_1,\ldots,A_p)\dot{X}} - \mu \chi_{A A_1,\ldots,A_p} = 0 \quad (p = 2s - 1).$$

Notice that in the above scheme the field equations for $s = \frac{1}{2}$ and $s = 1$ can be subsumed to the equations given for $s > 1$. R. Illge [50, 51] constructed a Lagrangian and an energy-momentum tensor to these higher spin field equations.

Observed elementary particles can be associated to most but not all of the above linear field equations. For $m = 0$ one has only

$s = \frac{1}{2}$:   Neutrinos ν

$s = 1$:   Photon γ.

For $m > 0$ one has

$s = 0$:   Scalar mesons $\pi, K, D, \eta, \ldots$

$s = \frac{1}{2}$:   Leptons e, μ, τ, baryons $p, n, \Lambda, \Sigma, \Xi, \ldots$

$s = 1$:   Vector mesons $\varrho, \psi, \omega, \Phi, \ldots$

$s > 1$:   Resonances.

Notice that all the above fundamental equations in spacetimes are of hyperbolic type, i.e. they have wave-like character. Elliptic equations emerge from the hyperbolic ones by the assumption of stationarity; parabolic equations emerge as macroscopical models by means of thermodynamics.

It makes sense to ask about Huygens' principle for each of the fundamental equations. The following has been proven in [87, 88, 90, 91, 93].

Theorem 6. *The spinor field equations of Buchdahl and Wünsch*

$$\nabla^B_{\dot X}\varphi_{BA_1,\dots,A_p} + \mu\chi_{A_1,\dots,A_p\dot X} = 0$$

$$\nabla^{\dot X}_{(A}\varphi_{A_1,\dots,A_p)\dot X} - \mu\chi_{AA_1,\dots,A_p} = 0$$

for $p = 2s - 1$, $s \geq \frac{1}{2}$, $\mu > 0$ satisfy Huygens' principle if and only if the underlying spacetime has constant curvature and the value of the scalar curvature equals $R = 12\,s\,\mu^2$.

Spinors are needed for the description of fermionic fields, but they are also used as technical tools: the tensorial conditions for Huygens' principle automatically decompose into their irreducible parts if they are translated into spinorial form [88, 89, 92, 94, 98, 99].

5. Relations to conformal differential geometry

Let T be a geometric object which is generally defined on any Riemannian manifold $(\mathcal{M}, g)$. It is called a *concomitant* of the metric g if its components are coordinate-independent functions of

$$g_{ab}, \qquad \partial_{a_1}g_{ab}, \qquad \partial_{a_1}\partial_{a_2}g_{ab}, \dots .$$

A concomitant $T = T[g]$ of g is called *polynomial* if its components are coordinate-independent polynomials in

$$g^{ab}, \qquad g_{ab}, \qquad \partial_{a_1}g_{ab}, \qquad \partial_{a_1}\partial_{a_2}g_{ab}, \dots .$$

An important theorem states that the components of a polynomial concomitant $T = T[g]$ are coordinate-independent polynomials in

$$g^{ab}, \qquad g_{ab}, \qquad R_{abcd}, \qquad \nabla_{a_1}R_{abcd}, \qquad \nabla_{a_1}\nabla_{a_2}R_{abcd}, \dots .$$

It is desirable to describe the conformal behaviour of $T[g]$. P. Günther and V. Wünsch [38, 39, 85, 94, 99] solved this task; let us shortly review their theory.

A polynomial concomitant T is said to have the *conformal weight* $\omega = $ const if for every $\varphi = $ const

$$T[e^{2\varphi}g] = e^{2\omega\varphi}T[g] .$$

It is said to have the *conformal order* k if $T[e^{2\varphi}g]$ depends for every function $\varphi \in C^\infty(\mathcal{M})$ only on φ and its derivatives up to the k-th order, but not on derivatives of a higher order. Further, T is a *(relative) conformal invariant of weight* $\omega = $ const, if for every function $\varphi \in C^\infty(\mathcal{M})$

$$T[e^{2\varphi}g] = e^{2\omega\varphi}T[g] .$$

(Clearly, such a T has conformal order 0.) For example, the Weyl conformal tensor for g in $n \geq 4$ and the Bach tensor (for $n = 4$ only) are conformal invariants of weight -1. The Schouten tensor $L = L_{ab}dx^a dx^b$ to g, which is defined by

$$(n - 2)L_{ab} = R_{ab} - \frac{R}{2(n - 1)}g_{ab}$$

has conformal weight 0 and conformal order 2, since $\bar{g} = e^{2\varphi}g$ implies

$$\overline{L}_{ab} = L_{ab} + \nabla_a \nabla_b \varphi - (\nabla_a \varphi)(\nabla_b \varphi) + \frac{1}{2}g_{ab}(\nabla_c \varphi)(\nabla^c \varphi) \ .$$

Let T be a polynomial concomitant of conformal weight ω. Then the infinitesimal conformal transform of T is defined as the limit

$$X_\varphi T = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(T[e^{2\epsilon\varphi}g] - e^{2\omega\epsilon\varphi}T[g]\right).$$

P. Günther and V. Wünsch [38, 39] proved that T is a relative conformal invariant if and only if

$$X_\varphi T = 0$$

for every $\varphi \in C^\infty(\mathcal{M})$. If, in particular, T is a polynomial concomitant of conformal order 1, then $\overset{c}{\nabla} T \equiv (\overset{c}{\nabla}_a T) \otimes dx^a$ given by

$$\overset{c}{\nabla}_a T = \nabla_a T - L_{ab}X^b T$$

is again polynomial and of conformal order 1. Here the so-called infinitesimal generators X^a are defined by

$$X_\varphi T =: (X^a T)\nabla_a \varphi \ ,$$

where the $X^a T$ do not depend on φ. The operators $\overset{c}{\nabla}_a$ are called the *conformal derivatives*.

P. Günther and V. Wünsch were able to construct sequences of conformal invariants by means of these tools, thus contributing to conformal differential geometry. Strictly speaking, the objects of conformal differential geometry are not the Riemannian metrics g but their conformal classes

$$\{e^{2\varphi}g \mid \varphi \in C^\infty(\mathcal{M})\} \ .$$

The theory of [38, 39] applies to Huygens' principle, as follows. Let $n = 2m + 2 \geq 4$ be even, $H_m = H_m(x,y)$ be the m-th Hadamard coefficient to $L = g^{ab}D_a D_b + W$, and abbreviate

$$c_{kl} = (-1)^k \binom{-m}{2}\binom{k}{l}\binom{k+m-1}{l} \bigg/ \binom{-m}{l}\binom{2k+2m-2}{k}$$

$$I_{kl}(x) = TS\left(\overset{c}{\underset{x}{\nabla}}_{a_1} \cdots \overset{c}{\underset{x}{\nabla}}_{a_l} \overset{c}{\underset{y}{\nabla}}_{a_{l+1}} \cdots \overset{c}{\underset{y}{\nabla}}_{a_k} H_m\right)(x,x) \, dx^{a_1} dx^{a_2} \cdots dx^{a_k}$$

$$I_k(x) = \sum_{l=0}^{k} c_{kl} I_{kl}(x).$$

Here $\overset{c}{\underset{x}{\nabla}}_a$ are the conformal derivatives with respect to the first argument x, and $\overset{c}{\underset{y}{\nabla}}_a$ are the conformal derivatives with respect to the second argument y; a symbol TS means the trace-free part of a symmetric form.

P. Günther [35] called the trace-free symmetric forms $I_k = I_k(x)$ $(k = 0,1,2,\ldots)$ the *moments* of the differential operator L and proved the following

Theorem 7. *Let $n = 2m + 2 \geq 4$ be even. Each moment I_k $(k = 0, 1, 2, \ldots)$ is a conformal invariant of weight $-m$. Moreover, if L satisfies Huygens' principle, then $I_k = 0$ for $k = 0, 1, 2, \ldots$.*

V. Wünsch [85 - 94] extended the "method of moments" for $n = 4$ to the Maxwell equations in the version

$$du = 0, \quad \delta u = 0$$

for an alternating 2-form u and to Weyl's equation

$$\nabla_A^{\dot{X}} \varphi_{\dot{X}} = 0 .$$

He constructed in both cases a sequence $\{I_k\}_{k \geq 0}$ of trace-free symmetric k-forms with the following two properties:

- Each I_k is a conformal invariant of weight $-m$.

- If the field equation under consideration satisfies Huygens' principle, then $I_k = 0$ for $k \geq 0$.

We omit the concrete expressions for the I_k because of their complexity (cf. [60, 100]). The moment equations $I_k = 0$ are determined explicitly at present for $0 \leq k \leq 4$. Using some results on the theory of conformally invariant tensors one obtains information about the algebraic structure of the moments for $0 \leq k \leq 6$ (s. [99]).

A step towards the determination of all Huygens' metrics for the conformally invariant field equations

(E_1) Scalar wave equation $g^{ab} \nabla_a \nabla_b u - \frac{1}{6} R = 0$

(E_2) Maxwell's equation $\nabla^b F_{ab} = 0$ for $F_{ab} = \nabla_a, A_b - \nabla_b A_a$

(E_3) Weyl's equation $\nabla_A^{\dot{X}} \varphi_{\dot{X}} = 0$

in an arbitrary four-dimensional spacetime is a program outlined by J. Carminati and R. G. McLenaghan, based on the conformally invariant Petrov classification of the Weyl tensor [19, 96, 99]. One obtains, in particular, the following:

Theorem 8. *Huygens' principle for the conformally invariant equations $(E_1) - (E_3)$ is valid only for conformally flat and for plane wave metrics within the classes of centrally symmetric, Petrov type N, D-spacetimes, or spacetimes with $\nabla_a R_{bc} = 0$.*

6. Relations to the Korteweg-de Vries equation

J. E. Lagnese [56] considerably generalized Stellmacher's examples of non-trivial Huygens-type equations. He introduced, in order to formulate his result, a sequence of polynomials $P_k = P_k(t)$ $(k \geq 1)$ of one real variable t through the recursive differential equation

$$\frac{d}{dt}\left(\frac{P_{k+1}}{P_{k-1}}\right) = (2k-1)\left(\frac{P_k}{P_{k-1}}\right)^2$$

together with the initial condition
$$P_1 = 1.$$

It is a highly non-trivial fact that this recursion is solved by polynomials. From

$$\frac{P_{k+1}}{P_{k-1}} = (2k-1)\int \left(\frac{P_k}{P_{k-1}}\right)^2 dt$$

it is clear that P_k depends on $k-1$ integration constants $a_1, a_2, \ldots, a_{k-1}$:

$$P_k = P_k(t) = P_k(t; a_1, a_2, \ldots, a_{k-1}) \ .$$

One constant, say a_1, is merely a translation of t; that means t and a_1 appear in the combination $t + a_1$. We set $a_1 = 0$ and present the first polynomials:

$$\left.\begin{aligned}
P_2 &= t \\
P_3 &= t^3 + a_2 \\
P_4 &= t^6 + 5a_2 t^3 + a_3 t - 5a_2^2 \ .
\end{aligned}\right\}$$

Generally, P_k is a monic polynomial of degree $\binom{k}{2}$, that means

$$P_k = t^{\binom{k}{2}} + \text{lower terms}.$$

The sequence of the P_k has been discovered three times, at least. First, by J. L. Burchnall and T. W. Chaundy [17] in 1929, second, by Lagnese [56] in 1969, and third by M. Adler and J. Moser [1] in 1978. The last mentioned paper unveiled a relation to the Korteweg-de Vries equation: the functions

$$u(x) = 2\frac{d^2}{dx^2} \log P_k(x)$$

are the rational solutions of the Kordeweg-de Vries equation which vanish at infinity. Moreover, these $u = u(x)$ are finite-gap potentials in the one-dimensional time-free Schrödinger equation

$$Ly \equiv y'' + u(x)y = \lambda y \ .$$

The name of these special potentials refers to the fact that the components of the spectrum of L are separated by finitely many intervals, called gaps. J. E. Lagnese [56] proved the following.

Theorem 9. *Let a flat spacetime $\mathbb{R}^n$ of even dimension $n = 2m+2 \geq 4$ be equipped with a Lorentzian scalar product*

$$\langle x, y \rangle = g_{ab} x^a x^b, \quad g_{ab} = \text{const}$$

and let $e \in \mathbb{R}^n$ be a unit vector, i.e $\langle e, e \rangle = 1$. A hyperbolic operator of the form

$$L = g^{ab} \partial_a \partial_b + w(\langle e, e \rangle)$$

satisfies Huygens' principle if and only if there is a $k \leq m$ and a polynomial P_k, as described above, such that

$$w(t) = 2 \frac{d^2}{dt^2} \log P_k(t) .$$

Yu. Yu. Berest and A. P. Veselov [10 - 13] had the idea to generalize the Stellmacher class of Huygens-type equation from one-variable potentials to many-variable potentials. They discovered an unexpected relation to Coxeter groups, i.e. to finite reflection groups of an Euclidean space.

Theorem 10. *Let G be a Coxeter group on $\mathbb{R}^{n-1}$, Δ_+ a system of positive normals to reflection hyperplanes of G, m_α a positive integer attached to each $\alpha \in \Delta_+$. The operator*

$$L = \partial_0^2 - \partial_1^2 - \ldots - \partial_{n-1}^2 - w(x) ,$$

$$w(x) = w(x_1, \ldots, x_{n-1}) = \sum_{\alpha \in \Delta_+} \frac{m_\alpha(m_\alpha + 1)\langle \alpha, \alpha \rangle}{\langle \alpha, x \rangle^2}$$

satisfies Huygens' principle if and only if $n = 2m + 2 \geq 4$ is even and $\sum_{\alpha \in \Delta_+} m_\alpha \leq m$.

The functions $w = w(x)$ given in Theorem 10 are known as the Calogero-Moser potentials [2, 65]. They again represent solutions of the Kordeweg-de Vries equation, and the motion of their poles defines some integrable dynamical system, called Calogero-Moser system. In this way, distant areas of mathematics are unexpectedly connected to each other.

7. Huygens' principle on symmetric spaces, especially of constant curvature

The class of Riemannian manifolds which we will consider now admits a geometrical definition as well as a Lie-theoretical one.

Let $(\mathcal{M}, g)$ be a Riemannian manifold, and $x = x(t)$ denote a local geodesic starting at $x(0) = y$. The map s_y which sends $x(t)$ to $x(-t)$ is defined in a neighbourhood of y; it is called the *geodesic reflection* at y. If, in particular, every s_y is a global isometry of $(\mathcal{M}, g)$, then $(\mathcal{M}, g)$ is called a *symmetric space*. It turns out that such a manifold can be represented as a homogeneous space, that means $M = G/H$ is the factor manifold of a Lie group G with respect to a Lie subgroup $H \subseteq G$. More precisely, here G is semisimple, H is the identity component of the subgroup of elements which are

invariant under some involutive automorphism, and $\mathcal{M}$ is equipped with a biinvariant metric g which is induced by the Killing form of G.

The analysis on symmetric spaces can use both, analytical and Lie-theoretical methods. This advantage led to new examples of the validity of Huygens' principle. Namely, consider the modified wave equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \frac{R}{6} u$$

for a scalar field $u = u(x,t)$ $(x \in \mathcal{M}, t \in \mathbb{R})$ on a symmetric space $\mathcal{M} = G/H$, where $\Delta = g^{ab}\nabla_a\nabla_b$ is the Laplacian to the positve definite biinvariant metric g of $\mathcal{M}$ and R denotes the scalar curvature. (Note that the Riemannian curvature tensor of a symmetric space is covariantly constant, hence $R = \text{const.}$) S. Helgason [42 - 45], T. Branson and G. Olafsson [14], and P. Günther [35] proved the following:

The modified wave equation above satisfies Huygens' principle in the following cases of odd-dimensional symmetric spaces.

(i) $\mathcal{M} = G$ *is a simple compact Lie group.*

(ii) $\mathcal{M} = G/H$, *where G is a connected semisimple Lie group and $H \subset G$ a maximal compact subgroup, is a symmetric space of non-compact type and the Lie algebra $\mathbf{g}$ to G admits a complex structure.*

(iii) $\mathcal{M} = G/H$ *is a symmetric space of non-compact type and all Cartan subgroups of G are conjugate to each other.*

The authors of [14, 35, 42 - 45] solved the Cauchy problem in the cases (i) - (iii) by means of Lie-theoretical methods and then read Huygens' principle from the solution formulas.

Riemannian manifolds of constant curvature are globally or locally symmetric spaces and are covered by the aforesaid cases. Thus we get the following example. Let $(\mathcal{M}, g)$ be a manifold of constant curvature K and odd dimension $n \geq 3$. Then the modified wave equation or Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \left(\frac{n-1}{2}\right)^2 K u$$

for a scalar field $u = u(x,t)$ $(x \in \mathcal{M}, t \in \mathbb{R})$ satisfies Huygens' principle. This fact has been repeatedly discovered or rediscovered and the Cauchy problem to this Klein-Gordon equation has been solved by means of several methods [16, 53, 57, 77].

A manifold $(\mathcal{M}, g)$ of constant curvature K can have any signature of g. Let us assume Lorentzian signature $(+ - \cdots -)$ and consider the Klein-Gordon equation

$$\Delta u + \lambda u = 0, \qquad \lambda = \text{const}$$

for a scalar field $u = u(x)$ $(x \in \mathcal{M})$. R. Schimming and H. Schlichtkrull proved [78] that

$$\Delta u + \left[l(l+1) - m(m+1)\right] K u = 0 \qquad (m = \tfrac{n-2}{2},\ n \geq 4 \text{ even})$$

satisfies Huygens' principle for $l = 0, 1, \ldots, m - 1$.

The following "negative results" are known for Riemannian manifolds $(\mathcal{M}, g)$ of constant curvature K and of dimension n (where these results are obtained by means of purely analytical methods [8, 9, 34]):

a) The Hodge-de Rahm wave equation

$$(d\delta + \delta d)u = 0$$

for p-forms u $(1 \le p \le n - 1, n \ge 4$ even) satisfies Huygens' principle only if $K = 0$.[1]

b) The modified wave equation

$$\frac{\partial^2 u}{\partial t^2} + (d\delta + \delta d)\, u = c\, u$$

for p-forms u $(0 \le p \le n, n$ odd) satisfies Huygens' principle only if $p = 0$ and $c = \left(\frac{n-1}{2}\right)^2 K$.

c) The Maxwell equations on the Lorentzian product $(\mathbb{R} \times \mathcal{M}, dt^2 - g)$

$$\frac{\partial u}{\partial t} = \delta v\,, \quad \delta u = 0$$

$$\frac{\partial v}{\partial t} = -d u\,, \quad d v = 0$$

for a $(p-1)$-form u and a p-form v satisfy Huygens' principle if and only if $p = \frac{n+1}{2}$.

The positive part of the last result is explained as follows. If $(\mathcal{M}, g)$ has constant curvature, then $(\mathbb{R} \times \mathcal{M}, dt^2 - g)$ is conformally flat, on the one hand. The Maxwell equations are conformally invariant if p equals one half of the dimension of the spacetime, which means here $p = \frac{n+1}{2}$, on the other hand. So the problem can be conformally transformed to Minkowski spacetime.

In order to strive for above results a) - c) (and much more besides), P. Günther introduced special tools in spaces of constant curvature [33, 34], namely

• geodesic p-forms $\gamma_p(x, y)$ which are turned out in [8, 9] as pseudo-orthogonal invariants among the p-stepped double differential forms (two-point-forms),

• spherical mean values $M_t^\tau[u]$ and $M_t^\sigma[u]$ of ordinary p-stepped differential forms $u(x)$ in which two special geodesic p-forms $\tau_p(x, y)$ and $\sigma_p(x, y)$ appear as the kernels (t: radius of the sphere).

These render (in Huygens' case explicit) solution formulas for several wave equations, for the D'Alembert equation, Euler-Poisson-Darboux equation and in close connection with the latter the modified wave equation, furthermore Maxwell's equations. So from here the above Huygens' principle assertions a) - c) follow.

General criterions for the Huygens' principle behaviour of Maxwell's equations in pseudo-Riemannian manifolds $(\mathcal{M}, g)$ of signature $(+ - \cdots -)$ and dimension n are found in [31]; weightily (e.g. for corollaries in spaces of constant curvature) there is the following assertion:

---

[1] In an arbitrary curved space $(\mathcal{M}, g)$ with $n = 4$ all Huygens' equations among these equations are explicitely determined (see Wünsch [90, 93]).

In $(\mathcal{M}, g)$ the Maxwell equations for differential forms $u(x)$ of the degree p

(**M**) $du = 0$, $\delta u = 0$ $(1 \le p \le n - 1)$

satisfy Huygens' principle if and only if

(**V**) $d\delta\, V_p(x, y; 2) = 0$ $(x, y \in \mathcal{M})$

(where Riesz's kernel form V_p essentially is the factor of the logarithmic term in Hadamard's elementary solution to the Hodge-deRahm wave equation, see the "negative result" a) above). In spaces of constant curvature P. Günther founds the four linear independent geodesic solutions $\gamma_p^{(i)}(x, y)$ of $(d\delta + \delta d)\, u = 0$ and because one of them (in [8, 34] just $\gamma_p^{(4)}$) is Hadamard's elementary solution, this leads in [9] to the Huygens' principle assertion a). Another solution, $\gamma_p^{(1)}$, at the same time solves (M) and because of $V_p = c \cdot \gamma_p^{(1)}$ with c constant P. Günther's criterion (V) means that

Corollary. *Maxwell's equations* (M) *in spaces of constant curvature satisfy Huygens' principle.*

On the strength of their origin as geometric invariants or as solution type for certain differential equations (of mathematical physics) the geodesic forms or also the spherical mean values disclosed new insights into many other mathematical facts and therefore their study is interesting in its own right.

8. A review of other problems

This survey is not intended to be complete. Let us mention here subjects which we have not treated, namely certain extensions of Huygens' principle to other classes of hyperbolic equations, some conceptual generalizations of Huygens' principle, and some open problems.

We have considered two classes of differential operators only: Laplace-like second-order operators and first-order operators the iteration of which is Laplace-like. The very definition of Huygens' principle can also be applied to the following classes.

- Systems of mixed first and second orders; for instance

$$(d\delta + \delta d)\, u = 0, \quad \delta u = 0$$

where u is an alternating p-form.

- Singular Cauchy problems, like the Euler-Poisson-Darboux equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\alpha}{t}\frac{\partial u}{\partial t} - \Delta u = 0, \quad \alpha = \text{const}$$

with special initial values

$$\left.\begin{aligned} u(0, x) &= u_0(x) \\ \frac{\partial u}{\partial t}(0, x) &= 0 \ . \end{aligned}\right\}$$

• Linear gauge field equations of hyperbolic type; for instance Maxwell's equations in the form
$$du = 0, \quad \delta u = j \ ,$$
the linearized Yang-Mills equation, and the linearized Einstein equations [95, 97].

• Semilinear hyperbolic equations; the simplest type reads
$$\Box u + F(u) = 0 \ .$$
There are well-developed theories for some mixed systems, for the Euler-Poisson-Darboux equation, and for the first-order Maxwell system. There are further preliminary results for the linearizations of the Yang-Mills and Einstein equations [95]. As a curiosity, we mention speculations on Huygens' principle for the fully nonlinear Einstein equations [70, 83].

Huygens' principle admits conceptual generalizations in different directions; there exists results on each of the following items.

• Replacement of the exact validity of Huygens' principle by an approximate validity. The standard perturbation technique defines orders of approximation. The first and second orders are the most interesting ones [30, 35, 84].

• Replacement of the local Huygens' principle (treated here) by a global – in space and in time – Huygens' principle. This global problem is akin to scattering theory.

• Replacement of the Cauchy problem by a characteristic initial value problem, where the data are prescribed on a future characteristic conoid (*conoid problem*) or on two interesting null hypersurfaces (*Goursat problem*).

• Study of higher-order hyperbolic equations the principal part of which is not defined by a Lorentzian metric. The characteristic conoid is then replaced by a system of cones which are defined by means of polynomial equations. Domains between such cones which do not contribute to the solutions of Cauchy problems are called *lacunas*. This concept naturally generalizes Huygens' principle.

• A natural generalization of Huygens' principle to Laplace-like equations of any mathematical type – elliptic, hyperbolic, ultrahyperbolic, or complex-holomorphic – is the property that, for even dimension, there exists a logarithm-free elementary solution in Hadamard's sense. Note that for the elliptic type the elementary solution generalizes the Newtonian potential (which belongs to the scalar Laplacian).

Let us finally try to formulate open problems of topical interest:

• Proof of the conjecture that for the dimension $n = 4$ every Huygens-type linear hyperbolic equation belongs either to a conformally flat metric or to a conformal image of a plane-wave metric (cf. [18 - 20, 59 - 63, 96 - 99]).

• Construction of Huygens-type modified wave equations on symmetric spaces for non-scalar fields, e.g. alternating differential forms.

• Estimation of effects due to the violation of Huygens' principle in our actual universe; derivation of approximative formulas which can be given to the hands of astrophysicsts.

• Development of a theory of Huygens' principle for semilinear hyperbolic equations.

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