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An Explicit Determination of the Non-self-adjoint Wave Equations that Satisfy Huygens' Principle on Petrov Type III Background Space-times

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Abstract. It is shown that the validity of Huygens' principle for the non-self-adjoint wave equation on a Petrov type III space-time implies that the space-time is conformally related to one in which every repeated null vector field of the Weyl tensor is recurrent. It is further shown that, given a certain mild assumption imposed on the covariant derivative of the Weyl curvature spinor, there are no Petrov type III space-times on which the non-self-adjoint scalar wave equation satisfies Huygens' Principle.

Keywords: *Huygens' principle, scalar wave equation, Petrov type III space-times*

AMS subject classification: Primary 35 B 30, secondary 35 L 10, 83 C 60

It is with great respect and admiration that we dedicate this article to the memory of Professor Paul Günther. Professor Günther made seminal contributions to the problem of the validity of Huygens' principle for the scalar wave equation on curved space-time and other relativistic wave equations that have had a profound influence on our work. The present paper examines some consequence of the set of necessary conditions for the validity of Huygens' principle for the non-self-adjoint scalar wave equation that Professor Günther's derived in his 1952 paper entitled "Zur Gültigkeit des Huygensschen Princips bei partiellen Differentialgleichungen von normalen hyperbolischen Typus" in the case of Petrov type III background space-times. We shall employ some additional necessary condition that we have recently derived to obtain a partial characterization of such equations. It will be seen that Professor Günther's conditions play an essential role in our proof.

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1. Introduction

This paper is devoted to the solution of Hadamard's problem for the general non-self-adjoint scalar wave equation in four dimensions which may be written in coordinate invariant form as

$$F[u] := g^{ab}\nabla_a\nabla_b u + A^a\partial_a u + Cu = 0 \tag{1.1}$$

where g^{ab} denotes the contravariant components of a pseudo-Riemmanian metric with signature $(+,-,-,-)$ on the space-time manifold V_4 and ∇_a denotes the covariant derivative with respect to the Levi-Civita connection. We assume that g^{ab}, A^a and C are C^∞-functions and restrict our consideration to a geodesically convex domain.

Hadamard [14] defined *Huygens' principle* to hold for equation (1.1) if for every Cauchy initial value problem, and for each point $x_0 \in V_4$, the solution of equation (1.1) depends only on the Cauchy data in an arbitrarily small neighbourhood of $S \cap C^-(x_0)$, where S denotes the Cauchy surface and $C^-(x_0)$ the past null conoid of x_0. Such an equation is called a *Huygens' equation.*

Hadamard's problem is that of determining, up to equivalence, all Huygens' equations. Two equations of form (1.1) are *equivalent* if one may be transformed into the other by any combination of the following *trivial transformations* which preserve the Huygens' nature of the equation:

(a) Coordinate transformations.

(b) Multiplication of both sides of equation (1.1) by a conformal factor $e^{-2\phi(x)}$, which is equivalent to a conformal transformation of the metric $\tilde{g}_{ab} = e^{2\phi(x)}g_{ab}$.

(c) Replacement of the dependent variable u by $\lambda(x)u$ where $\lambda(x)$ is nowhere vanishing.

The solution to Hadamard's problem has been found for the case when V_4 is locally conformally flat. In this case it has been shown [3, 15, 17] that every Huygens' equation is equivalent to the ordinary wave equation

$$\Box u := g^{ab}\nabla_a\nabla_b u = 0 \tag{1.2}$$

on flat space-time. A detailed review of this work can be found in Professor Günther's treatise "Huygens' Principle and Hyperbolic Equations" [13]. The problem has also been solved for (1.1) for space-times of Petrov type N [6, 12, 22, 31] and for the self-adjoint equation (1.1) ($A^a \equiv 0$) for Petrov type D [7, 23, 33]. Results have also been obtained in this case for Petrov types III [8] and II [4], although for type III it was necessary to place a mild restriction on the Weyl tensor in order to solve the problem (see Theorem 2 below), while for type II only a partial result is available. McLenaghan and Walton [22, 31] show that any non-self-adjoint equation (1.1) on any Petrov type N background space-time satisfies Huygens' principle if and only if it is equivalent to the pure wave equation (1.2) on an exact plane wave space-time with metric

$$ds^2 = 2dv\Big\{du + [D(v)z^2 + \bar{D}(v)\bar{z}^2 + e(v)z\bar{z}]\Big\}dv - 2dzd\bar{z} \tag{1.3}$$

in a special coordinate system, where D and e denote arbitrary C^{∞}-functions. Their theorem depends on a key result proven by Günther [12], namely, that *Huygens' principle is satisfied by equation* (1.2) *on every exact plane wave space-time.*

The main purpose of the present paper is to extend the analysis to Petrov type III background space-times. Recall that a Petrov type III space-time is defined by the existence of a null vector field l_a called a *repeated principal null vector field* which satisfies

$$C_{bcd[e}l_{f]}l^d = 0 \tag{1.4}$$

where C_{abcd} is the Weyl tensor defined below. For such a space-time Carminati and McLenaghan have proven the following two theorems [6].

Theorem 1 (see [6]). *The validity of Huygens' principle for the conformally invariant scalar wave equation on any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated principal null vector field l_a of the Weyl tensor is recurrent, that is*

$$l_{[a}l_{b;c]} = 0. \tag{1.5}$$

Theorem 2 (see [6]). *If any one of the following three conditions*

$$\Psi_{ABCD;EE'}\iota^A\iota^B\iota^C o^D \iota^E \bar{o}^{E'} = 0 \qquad (1.6)_a$$

$$\Psi_{ABCD;EE'}\iota^A\iota^B\iota^C o^D o^E \bar{\iota}^{E'} = 0 \qquad (1.6)_b$$

$$\Psi_{ABCD;EE'}\iota^A\iota^B\iota^C \iota^D o^E \bar{o}^{E'} = 0 \qquad (1.6)_c$$

is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation

$$\Box u + \frac{1}{6}Ru = 0 \tag{1.7}$$

satisfies Huygens' principle.

In Theorem 2, Ψ_{ABCD} is the Weyl spinor, which we will introduce explicitly in the next section. We note that Carminati and McLenaghan also prove this result for the Weyl equation and Maxwell's equations [6]. It should also be mentioned that the restrictions (1.5) of Theorem 2 in the case of these equations has been recently removed by McLenaghan and Sasse [24].

The proofs of the Theorems 1 and 2 and the other results for scalar wave equations of the form (1.1) are based on the following necessary conditions for (1.1) to be a Huygens' equation:

(I) $C = \frac{1}{2}A^a{}_{;a} + \frac{1}{4}A_aA^a + \frac{1}{6}R$

(II) $H^k{}_{a;k} = 0$

(III) $S_{abk;}{}^k - \frac{1}{2}C^k{}_{ab}{}^l L_{kl} = -5\left(H_{ak}H_b{}^k - \frac{1}{4}g_{ab}H_{kl}H^{kl}\right)$

(IV) $TS\left[3S_{abk}H^k{}_c + C^k{}_{ab}{}^l H_{ck;l}\right] = 0$

(V) $TS\Big[3C_{kcdl;m}C^k{}_{ef}{}^l{}_;{}^m + 8C^k{}_{cd}{}^l{}_{;e}S_{klf} + 40S_{cd}{}^kS_{efk} - 8C^k{}_{cd}{}^lS_{kle;f}$

$$-24C^k{}_{cd}{}^l S_{efk;l} + 4C^k{}_{cd}{}^l C_l{}^m{}_{ek} L_{fm} + 2C^k{}_{cd}{}^l D^m{}_{efl} L_{km}$$

$$+12H_{kc;de}H^k{}_f - 16H_{kc;d}H^k{}_{e;f} - 84H^k{}_c C_{cdel} H^l{}_f - 18H_{kc}H^k{}_d L_{ef}\Big] = 0$$

(VI) $$TS\Big[36C^k{}_{ab}{}^l C_{lcdm;k} H^m{}_e - 6C^k{}_{ab}{}^l{}_{;c} C_{lde}{}^m H_{km} - 138 S_{ab}{}^k C_{kcdl} H^l{}_e$$
$$+6S_{abk}H^k{}_{c;de} + 6C^k{}_{ab}{}^l{}_{;c} H_{kd;le} - 24 S_{abk;c} H^k{}_{d;e}$$
$$+12C^k{}_{ab}{}^l L_{kc} H_{ld;e} - 9C^k{}_{ab}{}^l{}_{;c} L_{kd} H_{le} - 9 S_{abk} L_{cd} H^k{}_e\Big] = 0.$$

where

$$\left.\begin{aligned} H_{ab} &= A_{[a,b]} \\ C_{abcd} &= R_{abcd} - 2g_{[a[d}L_{b]c]} \\ S_{abc} &= L_{a[b;c]} \\ L_{ab} &= -R_{ab} + \tfrac{1}{6} R g_{ab} \end{aligned}\right\} \tag{1.8}$$

and R_{abcd} denotes the Riemann curvature tensor, R_{ab} the Ricci tensor and $TS[\ ]$ the operator which takes the trace free symmetric part of the enclosed tensor.

The history of these conditions is as follows:

Conditions (I) - (IV) were proved by Günther [11]. Condition (V) was derived by Wünsch [32] for the self-adjoint case and McLenaghan [19] for the non-self-adjoint case. Condition (VI) is due to Anderson and McLenaghan [2]. The solution of Hadamard's problem for Petrov type N also required the use of a seventh necessary condition (VII) which has been derived by Rinke and Wünsch [30]. In the present paper these results are extended to the case of equation (1.1) on Petrov type III background space-times. The main results will be the proof of the following extensions of Theorems 1 and 2:

Theorem 3. *The validity of Huygens' principle for any non-self-adjoint scalar wave equation* (1.1) *in any Petrov type III background space-time implies that the space-time is conformally related to one in which every repeated null vector field of the Weyl tensor, l_a, is recurrent.*

Theorem 4. *There exist no non-self-adjoint equations* (1.1) *which satisfy Huygens' principle on any Petrov type III background space-time for which equations* (1.6) *hold.*

The plan of this paper is as follows:

Section 2 consists of a description of the formalisms used. Section 3 prepares for the proof of Theorems 3 and 4 by applying the formalisms of Section 2 to the necessary conditions. Sections 4 and 5 contain the proofs of Theorems 3 and 4, respectively. Finally, Section 6 contains our concluding remarks.

2. Formalism

We employ the two-component spinor formalism of Penrose [26, 29] and the spin coefficient formalism of Newman and Penrose [25], whose conventions we follow. Spinors are transformed to and from tensors via the Infeld-Van der Waerden symbols $\sigma_a{}^{AA'}$ ($a \in \{0, \ldots, 3\}$) which are Hermitian in the spinor indices $A, A' \in \{0, 1\}$. The role of a metric on the spinor space is played by the skew symmetric spinors ϵ_{AB} and $\epsilon_{A'B'}$ where we will define $\epsilon_{01} = \epsilon_{0'1'} = 1$. Spinor indices are raised and lowered according to the convention

$$\zeta^A = \epsilon^{AB}\zeta_B, \quad \zeta_A = \zeta^B\epsilon_{BA} \tag{2.1}$$

where ζ_A is an arbitrary spinor.

We will use extensively the spinorial equivalents of the tensors defined in (1.7). They are

$$\left.\begin{aligned} H_{ab} &\leftrightarrow \phi_{AB}\epsilon_{A'B'} + \overline{\phi}_{A'B'}\epsilon_{AB} \\ L_{ab} &\leftrightarrow 2(\Phi_{ABA'B'} - \Lambda\epsilon_{AB}\epsilon_{A'B'}) \\ C_{abcd} &\leftrightarrow \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ S_{abc} &\leftrightarrow \Psi^D{}_{ABC;DA'}\epsilon_{C'B'} + \overline{\Psi}^{D'}{}_{A'B'C';D'A}\epsilon_{CB} \end{aligned}\right\} \tag{2.2}$$

where the spinor $\phi_{AB} = \phi_{(AB)}$ is called the *Maxwell spinor*, the spinor $\Phi_{ABA'B'} = \Phi_{(AB)(A'B')}$ is the *Ricci spinor*, and corresponds to the trace-free part of the Ricci tensor, $\Lambda = \frac{1}{24}R$, and $\Psi_{ABCD} = \Psi_{(ABCD)}$ is called the *Weyl spinor*.

We define the connection of the spin space to satisfy $\sigma_a{}^{AA'}{}_{;BB'} = \epsilon_{AB;CC'} = 0$. The basis of the spin space is denoted by the spin dyad $\{o_A, \iota_A\}$ which satisfies the completeness relation $o_A\iota^A = 1$. We define the null tetrad $\{l, n, m, \overline{m}\}$, which is a basis of M_4, according to

$$l^a = \sigma^a{}_{AA'}o^A\bar{o}^{A'}, \qquad n^a = \sigma^a{}_{AA'}\iota^A\bar{\iota}^{A'}, \qquad m^a = \sigma^a{}_{AA'}o^A\bar{\iota}^{A'}. \tag{2.3}$$

Let us look now at how all the spinorial quantities introduced above are denoted in the Newman-Penrose formalism. We begin with the Newman-Penrose spin connection which one denotes according to [7]

$$\left.\begin{aligned} \nabla_{BB'}o_A &= I_{BB'}o_A + II_{BB'}\iota_A \\ \nabla_{BB'}\iota_A &= III_{BB'}o_A - I_{BB'}\iota_A \end{aligned}\right\} \tag{2.4}$$

where

$$\left.\begin{aligned} I_{BB'} &= \gamma o_B\bar{o}_{B'} - \alpha o_B\bar{\iota}_{B'} - \beta\iota_B\bar{o}_{B'} + \epsilon\iota_B\bar{\iota}_{B'} \\ II_{BB'} &= -\tau o_B\bar{o}_{B'} + \rho o_B\bar{\iota}_{B'} + \sigma\iota_B\bar{o}_{B'} - \kappa\iota_B\bar{\iota}_{B'} \\ III_{BB'} &= \nu o_B\bar{o}_{B'} - \lambda o_b\bar{\iota}_{B'} - \mu\iota_B\bar{o}_{B'} + \pi\iota_B\bar{\iota}_{B'}. \end{aligned}\right\} \tag{2.5}$$

Along with the spin connection, the spinor covariant derivative also contains four first-order differential operators, denoted in Newman-Penrose notation by

$$\nabla_{AA'}S = \delta S o_A\bar{o}_{A'} - \bar{\delta}S o_A\bar{\iota}_{A'} - \delta S\iota_A\bar{o}_{A'} + DS\iota_A\bar{\iota}_{A'} \tag{2.6}$$

where S is an arbitrary scalar field. The curvature spinors and Maxwell spinor are also decomposed according to the spinor dyad in the Newman-Penrose formalism. They take the form

$$\begin{aligned}\Psi_{ABCD} = {} & \Psi_0\, \iota_{ABCD} - 4\,\Psi_1\, o_{(A}\iota_{BCD)} \\ & + 6\,\Psi_2\, o_{(AB}\iota_{CD)} - 4\,\Psi_3\, o_{(ABC}\iota_{D)} + \Psi_4\, o_{ABCD}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\Phi_{ABA'B'} = {} & \frac{1}{2}\Phi_{22}\, o_{AB}\bar{o}_{A'B'} - 2\,\Phi_{12}\, o_{(A}\iota_{B)}\bar{\iota}_{A'B'} + \Phi_{02}\, \iota_{AB}\bar{o}_{A'B'} \\ & + 2\,\Phi_{11}\, o_{(A}\iota_{B)}\bar{o}_{(A'}\iota_{B')} - 2\,\Phi_{01}\, \iota_{AB}\bar{o}_{(A'}\bar{\iota}_{B')} + \frac{1}{2}\Phi_{00}\iota_{AB}\bar{\iota}_{A'B'} \\ & + \text{c.c.}\end{aligned} \tag{2.8}$$

$$\phi_{AB} = \phi_0\, \iota_{AB} - 2\,\phi_1\, o_{(A}\iota_{B)} + \phi_2\, o_{AB} \tag{2.9}$$

where the convenient notation $\iota_{A_1 \ldots A_N} = \iota_{A_1} \cdots \iota_{A_N}$ has been introduced and "c.c." denotes the complex conjugate of the proceeding terms. The Newman-Penrose field equations are the equations relating the curvature components to the spin coefficients. Throughout this paper we will refer to the Newman-Penrose equations according to their enumeration in [25] and the Bianchi identities according to the order in which they occur at the end of Pirani [29: Chapter 4].

There are two types of transformations corresponding to the trivial transformations of Section 1 that will be employed in the proof of Theorems 3 and 5. The first is the dyad transformation

$$\left.\begin{aligned} o' &= e^{\frac{w}{2}} o \\ \iota' &= e^{-\frac{w}{2}}(\iota + q o) \end{aligned}\right\} \tag{2.10}$$

where w is also complex. The second is the conformal transformation. There is considerable freedom in choosing conformal weights for spinorial quantities. We will follow Penrose [27] and choose

$$\tilde{\sigma}_a{}^{AA'} = e^{\phi}\sigma_a{}^{AA'} \qquad \text{and} \qquad \tilde{\sigma}^a{}_{AA'} = e^{-\phi}\sigma^a{}_{AA'}. \tag{2.11}$$

We must also assign conformal weights to the spin basis. Following McLenaghan and Walton [22] we will choose

$$\tilde{o}_A = e^{\frac{r-1}{2}\phi} o_A \qquad \text{and} \qquad \tilde{\iota}_A = e^{\frac{1-r}{2}\phi}\iota_A, \tag{2.12}$$

where r is a real parameter. These two transformations lead to transformations of the Newman-Penrose spin coefficients, curvature quantities, and differential operators. The transformations of some of these quantities are listed in [8] and [22]. In adition to the transformations listed in [8] and [22] we require the following transformations of the components of the Maxwell spinor induced by (2.12):

$$\tilde{\phi}_0 = e^{(r-3)\phi}\phi_0, \qquad \tilde{\phi}_1 = e^{-2\phi}\phi_1, \qquad \tilde{\phi}_2 = e^{-(r+1)\phi}\phi_2. \tag{2.13}$$

In Section 3,, we will outline the conversion of the necessary conditions (I) - (VI) into Newman-Penrose form in preparation for the proof of Theorems 3 and 5. This will give us, when combined with the Newman-Penrose field equations and the Neman-Penrose form of the Bianchi identities, a large set of algebraic equations in the Newman-Penrose quantities. In Sections 4 and 5, we will prove Theorems 3 and 4 by showing that this set of algebraic equations is incompatible unless Theorems 3 and 4 hold.

3. Necessary conditions: Newman-Penrose form

As outlined at the end of Section 2, the goal here will be to prove Theorems 3 and 4 by showing that the Newman-Penrose field equations, Maxwell's equations, the Bianchi identities and the six necessary conditions are incompatible for a Petrov type III space-time unless Theorems 3 and 4 hold. In order to do this, we will need to express the Petrov type III condition (1.4) and the six necessary conditions (I) - (VI) in Newman-Penrose form. As an intermediate step, we will express them in spinorial form. The method is straightforward, and we will present only examples here.

We begin with the Petrov Type III condition. It is well known [29] that (1.4) implies

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\delta_{D)}.$$

We can easily convert this to Newman-Penrose form by choosing our spin basis so that α_A is proportional to o_A and δ_D to ι_D. We may further use a dyad transformation to set $\Psi_3 = -1$, so that

$$\Psi_{ABCD} = 4o_{(A}o_B o_C\iota_{D)}.$$

This form of the Weyl spinor will be used to simplify the expansion of the necessary conditions (I) - (VI).

We now proceed with the conversion of the necessary conditions. We will use the sixth necessary condition (VI) as an example of how the expansion is carried out. It is convenient to convert condition (VI) to spinorial form term by term (note that the TS operation is distributive over addition). Let us begin with the term $6S_{abk}H^k{}_{c;de}$. Then, according to $(2.2)_1$ and $(2.2)_4$ we have

$$6\,TS(S_{abk}H^k{}_{c;de}) \leftrightarrow 6\,S\Big[\Big(\Psi^K{}_{ABC;KA'}\epsilon_{B'C'} + \overline{\Psi}^{K'}{}_{A'B'C';AK'}\epsilon_{BC}\Big) \Big(\phi^K{}_C\epsilon^{K'}{}_{C'} + \overline{\phi}^{K'}{}_{C'}\epsilon^K{}_C\Big)_{;DD'EE'}\Big]$$

where we introduce the notation

$$\epsilon^A{}_B := \epsilon^{AC}\epsilon_{CB} = -\delta^A_B.$$

Recalling that $\nabla\epsilon_{AB} = 0$ and noting that the symmetric part of ϵ_{AB} vanishes since ϵ_{AB} is skew-symmetric, we have

$$6\,TS(S_{abk}H^k{}_{c;de}) \leftrightarrow 6\,S\Big[-\overline{\Psi}^{K'}{}_{A'B'C';AK'}\phi_{BC;DD'EE'} - \Psi^K{}_{ABC;KA'}\phi_{B'C';DD'EE'}\Big]$$

where the S operator on a spinor takes the symmetric part.

The remaining terms in condition (VI) are converted to spinorial form in a similar manner, whence we obtain

$$\textbf{(VI)}_S\;\; 0 = S\Big[-6\Psi^K{}_{ABC;KA'}\bar{\phi}_{B'C';DD'EE'} + 6\Psi_{ABC}{}^K{}_{;DD'}\bar{\phi}_{A'B';KC'EE'}$$

$$
\begin{aligned}
&+24\Psi^{K}{}_{ABC;KA'DD'}\bar{\phi}_{B'C';EE'} + 24\Psi^{K}{}_{ABC}\Phi_{KDA'D'}\bar{\phi}_{B'C';EE'} \\
&-18\Psi^{K}{}_{ABC;EE'}\Phi_{KDA'D'}\bar{\phi}_{B'C'} + 18\Psi^{K}{}_{ABC;KA'}\Phi_{DED'E'}\bar{\phi}_{B'C'} \\
&-36\Psi^{K}{}_{ABC}\bar{\Psi}_{B'C'D'E';KA'}\phi_{DE} - 138\Psi^{K}{}_{ABC;KA'}\bar{\Psi}_{B'C'D'E'}\phi_{DE} \\
&+6\Psi^{K}{}_{ABC;DD'}\bar{\Psi}_{B'C'E'A'}\phi_{KE} + 6\Psi_{ABCD;EE'}\bar{\Psi}^{K}{}_{A'B'C'}\phi_{KD} \\
&+\text{c.c.}\Big].
\end{aligned}
$$

One may convert the first five necessary conditions (I) - (V) through the process illustrated above. The results of this conversion may be summarized in the following proposition:

Proposition 1. *If Huygens' principle holds for the equation (1.1), then the following conditions are satisfied:*

$$(\mathbf{I})_S \; 0 = C$$

$$(\mathbf{II})_S \; 0 = \phi_{AK;}{}^{K}{}_{A'}$$

$$(\mathbf{III})_S \; 0 = \Psi_{ABKL;}{}^{K}{}_{A'}{}^{L}{}_{B'} + Psi_{AB}{}^{KL}\Phi_{KLA'B'} + 5\phi_{AB}\bar{\phi}_{A'B'}$$

$$
\begin{aligned}
(\mathbf{IV})_S \; 0 = {}& 3\Psi_{ABCK;}{}^{K}{}_{(A'}\phi_{B'C')} + 3\bar{\Psi}_{A'B'C'K';}{}^{K'}{}_{(A}\phi_{BC)} \\
&-\Psi_{ABC}{}^{K}\bar{\phi}_{(A'B';C')K} - \bar{\Psi}_{A'B'C'}{}^{K'}\phi_{(AB;C)K}
\end{aligned}
$$

$$
\begin{aligned}
(\mathbf{V})_S \; 0 = {}& 3\Psi_{ABCD;KK'}\bar{\Psi}_{A'B'C'D';}{}^{KK'} + 4\Psi^{K}{}_{(ABC;D)(A'}\bar{\Psi}_{B'C'D')L';K}{}^{L'} \\
&+4\bar{\Psi}^{K'}{}_{(A'B'C';D')(A}\Psi_{BCD)L;}{}^{L}{}_{K'} - 40\Psi_{(ABC|K|;}{}^{K}{}_{(A'}\bar{\Psi}_{B'C'D')K';}{}^{K'}{}_{D)} \\
&-4\Psi^{K}{}_{(ABC}\bar{\Psi}_{(A'B'C'|K';K|}{}^{K'}{}_{D)D')} - 4\bar{\Psi}^{K'}{}_{(A'B'C'}\Psi_{(ABC|K;}{}^{K}{}_{K'|D)D')} \\
&+12\Psi^{K}{}_{(ABC}\bar{\Psi}_{(A'B'C'|K'|;D)}{}^{K'}{}_{KD')} - 12\bar{\Psi}^{K'}{}_{(A'B'C'}\Psi_{(ABC|K|;}{}^{K}{}_{D')D)K'} \\
&-16\Psi^{K}{}_{(ABC}\Phi_{D)KK'(A'}\bar{\Psi}_{B'C'D')}{}^{K'} - 32\Lambda\Psi_{ABCD}\bar{\Psi}_{A'B'C'D'} \\
&-6\phi_{(AB;CD)(C'D'}\bar{\phi}_{A'B')} - 6\bar{\phi}_{(A'B';(CDC'D')}\phi_{AB)} \\
&+16\phi_{(AB;C(C'}\bar{\phi}_{A'B';D')D)} - 42\phi_{(AB}\phi_{CD)}\bar{\Psi}_{A'B'C'D'} \\
&-42\bar{\phi}_{(A'B'}\bar{\phi}_{C'D')}\Psi_{ABCD} - 36\phi_{(AB}\Phi_{CD)(C'D'}\bar{\phi}_{A'B')}.
\end{aligned}
$$

We remark that [23: equation (2.7)] has been used to obtain a stronger form of condition $(III)_S$.

We now have obtained the necessary conditions in a form suitable for conversion to the Newman-Penrose formalism. To perform this conversion, two methods were employed. The first, which was used for the expansion of the conditions (I) - (IV), is the `NPspinor` package for the MAPLE symbolic algebra system. For cases where larger expressions are involved, the `NPspinor` package may not be suitable for conversion from spinor to Newman-Penrose form. In the case of the necessary conditions (V) and (VI), we modified MAPLE code written specifically for the conversion of these conditions to Newman-Penrose form by G. C. Williams. A listing of the MAPLE code for the modified Williams routines can be found in [1].

Because the tensorial form of the necessary conditions always involves the trace free symmetric part of a tensorial expression, the spinorial form of all the necessary conditions will involve spinors of the form $S_{(A_1 \ldots A_n)(A'_1 \ldots A'_n)}$. Thus, when we label the dyad components of such equations, it is sufficient to specify the number of unprimed indices with the value one and the number of primed indices with the value one. For instance, we can identify the dyad component $S_{(1110\ldots0)(1'1'0'\ldots0')}$ as the $(3,2)$ component of the spinor S, and likewise for the conditions themselves. Thus, the component of condition (IV) with three 1 and two 1′ indices would be denoted by (IV)/(3,2). This component labeling scheme is the one used throughout the rest of this paper.

4. Proof of Theorem 3

As in Section 3 we choose our dyad so that $\Psi_{ABCD} = 4o_{(ABC}\iota_{D)}$. This form of Ψ_{ABCD} is invariant under conformal transformations if we choose the conformal parameter $r = 1$ from the transformation, which we shall. Since we wish to deal only with the non-self-adjoint wave equation, we shall restrict our considerations to the case

$$\phi_{AB} \neq 0. \tag{4.1}$$

We start by proving the following lemma.

Lemma 1. *The validity of Huygens' principle for the scalar wave equation* (1.1) *on a Petrov type III background implies that, with respect to a spinor dyad* $\{o, \iota\}$, *where* o *is the repeated principal spinor of the type III Weyl spinor, there exists a conformal gauge for which*

$$\kappa = \sigma = \phi_0 = \phi_1 = \rho = \epsilon = \tau = \Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0.$$

Proof. Consider first condition (IV)/(2,3), which for the selected dyad yields $\kappa\bar{\phi}_0 = 0$. Assume

(i) $\kappa \neq 0$

which implies

$$\phi_0 = 0. \tag{4.2}$$

Thus, under assumption (i), condition (IV)/(1,3) yields $\bar{\kappa}\bar{\phi}_1 = 0$ which, by that same assumption (i), yields

$$\phi_1 = 0. \tag{4.3}$$

Using (4.2) and (4.3) we see that condition (IV)/(1,2) yields $\bar{\kappa}(3\phi_2 + \bar{\phi}_2) = 0$ which, again due to assumption (i), implies

$$3\phi_2 + \bar{\phi}_2 = 0 \quad \Longrightarrow \quad \phi_2 = 0. \tag{4.4}$$

However, (4.2) - (4.4) contradict our initial criteria (4.1), and so we have by contradiction that assumption (i) must be false, or

$$\kappa = 0. \tag{4.5}$$

It follows immediately from condition (III)/(2,2) that

$$\phi_0 = 0. \tag{4.6}$$

Inspection of [8: equation (2.23)] and (2.12) reveals that (4.5) and (4.6) are invariant under conformal transformations.

The results above and condition (IV)/(0,3) imply $\bar{\sigma}\bar{\phi}_1 = 0$. If we now make the assumption

(ii) $\sigma \neq 0$

then it follows that

$$\phi_1 = 0. \tag{4.7}$$

Substituting (4.5) - (4.7) into condition (III)/(0,2) we have $\sigma(9\phi_2 - \bar{\phi}_2) = 0$ which, on account of assumption (ii), implies $\phi_2 = 0$, which, with (4.6) and (4.7), again contradicts (4.1) and thus proves by contradiction that assumption (ii) is false, or that

$$\sigma = 0. \tag{4.8}$$

Further, by [8: equation (2.23)] we note that (4.8) is invariant under conformal transformations.

Using the results obtained so far, we have from conditions (II)/(1,1) and (IV)/(1,2), respectively,

$$\left.\begin{aligned} D\phi_1 &= 2\rho\phi_1 \\ 6(\epsilon - \rho)\bar{\phi}_1 + D\bar{\phi}_1 + \bar{\rho}\bar{\phi}_1 &= 0 \end{aligned}\right\} \tag{4.9}$$

Substituting the conjugate of $(4.9)_1$ into $(4.9)_2$ one obtains

$$\bar{\phi}_1(2\epsilon - 2\rho + \bar{\rho}) = 0. \tag{4.10}$$

Assume

(iii) $\phi_1 \neq 0$.

Then (4.10) implies that

$$\epsilon = \rho - \frac{1}{2}\bar{\rho}. \tag{4.11}$$

Exploiting the conformal invariance of the results in the proof thus far, we now use the conformal transformation law for ρ (i.e. [8: equation (2.23)]) to set

$$\bar{\rho} = -\rho \tag{4.12}$$

which gives (4.11) the form

$$\epsilon = \frac{3}{2}\rho. \tag{4.13}$$

Equations (4.12) and (4.13) imply that [25: equation (4.2a)] may be written as $D\rho = -\rho^2 + \Phi_{00}$ which, when added to its complex conjugate, gives

$$\Phi_{00} = -\rho^2. \tag{4.14}$$

Furthermore, from condition (II)/(1,1) we have

$$D\phi_1 = 2\phi_1\rho. \tag{4.15}$$

Substituting (4.14) and (4.15) and its conjugate into condition (V)/(1,1) gives $\phi_1\bar{\phi}_1\rho^2 = 0$. Since $\phi_1 = 0$ contradicts our assumption, we conclude

$$\rho = 0, \tag{4.16}$$

which in turn yields

$$\epsilon = \Phi_{00} = 0. \tag{4.17}$$

The results of assumption (iii) thus far imply, in conjunction with [25: equations (4.2d), (4.2e) and (4.2k)] that

$$D\tau = D\beta = \Phi_{01} = 0. \tag{4.18}$$

Substituting (4.5), (4.6), (4.8), (4.16) - (4.18) into condition (III)/(1,2) yields $\phi_1 = 0$ which contradicts assumption (iii). We conclude that

$$\phi_1 = 0, \tag{4.19}$$

which, by (2.13), is also conformally invariant.

Next, consider the Pfaffian derivatives of the remaining Maxwell component yielded, by conditions (II)/(0,0) and (II)/(0,1),

$$\left.\begin{aligned} \delta\phi_2 &= -2\phi_2\beta + \phi_2\tau \\ D\phi_2 &= -2\phi_2\epsilon + \phi_2\rho. \end{aligned}\right\} \tag{4.20}$$

Substituting these into condition (IV)/(0,1) one gets

$$\bar{\delta}\phi_2 = (-6\bar{\beta} + 6\bar{\tau} - 2\alpha)\phi_2 + (\bar{\tau} + 2\alpha + 4\pi)\bar{\phi}_2.$$

¿From this point on we will assume that these Pfaffians of the Maxwell spinor have already been substituted for according to these equations.

Now, by adding [25: equation (4.2a)] to its complex conjugate we obtain $\Phi_{00} = -\rho^2$. Thus, [25: equation (4.2a)], conditions (III)/(1,2), (IV)/(1,1) and (VI)/(2,2) become

$$\left.\begin{aligned} D\rho &= (\epsilon + \bar{\epsilon})\rho \\ D\epsilon &= -\frac{3}{2}\rho^2 + 3\epsilon\rho - \rho\bar{\epsilon} - \epsilon^2 + \epsilon\bar{\epsilon} + D\rho \\ 0 &= 3\phi_2\rho + 2\epsilon\bar{\phi}_2 - 3\rho\bar{\phi}_2 + 2\bar{\epsilon}\phi_2 \\ 0 &= -13\rho^3\phi_2 + 13\rho^3\bar{\phi}_2 + 3\bar{\phi}_2\rho^2\bar{\epsilon} + 3\bar{\phi}_2\rho^2\epsilon + 2\bar{\phi}_2\rho\epsilon\bar{\epsilon} \\ &\quad + 3\phi_2\rho^2\bar{\epsilon} - 2\phi_2\rho\epsilon\bar{\epsilon} + 3\phi_2\rho^2\epsilon + 2\phi_2\rho\bar{\epsilon}^2 + 2\phi_2\rho D\bar{\epsilon} \\ &\quad - 3\phi_2\rho D\rho - 3\bar{\phi}_2\rho D\rho - 2\bar{\phi}_2\rho\epsilon^2 - 2\bar{\phi}_2\rho D\epsilon, \end{aligned}\right\} \tag{4.21}$$

respectively. By eliminating $D\epsilon$ and $D\rho$ from $(4.21)_4$ using $(4.21)_1$ and $(4.21)_2$, multiplying the resulting equation by two and then adding to it $4\rho^2$ times $(4.21)_3$ we obtain $4\rho^3(-\bar{\phi}_2 + \phi_2) = 0$ which implies that either $\rho = 0$ or $\bar{\phi}_2 = \phi_2$. Let us assume

(iv) $\rho \neq 0$.

Then $\bar{\phi}_2 = \phi_2$. It follows immediately from $(4.21)_3$ that, if $\phi_2 \neq 0$, $\bar{\epsilon} = -\epsilon$ which in turn implies from $(4.21)_1$ that $D\rho = 0$ and from $(4.21)_2$, when it is subtracted from its conjugate, that $D\epsilon = 0$. Thus, $(4.20)_2$ when respectively subtracted from and added to its conjugate yields

$$\left.\begin{aligned} D\phi_2 &= 0, \\ \epsilon &= \frac{1}{2}\rho. \end{aligned}\right\}$$

Now consider condition (V)/(2,2) which, in light of the results just obtained, takes the form

$$\rho^2(\phi_2 - 2)(\phi_2 + 2) = 0. \tag{4.22}$$

Considering assumption (iv), (4.22) implies $\phi_2 = \pm 2$. We proceed with the case $\phi_2 = 2$, which immediately implies, by $(4.20)_1$, that $\tau = 2\beta$ which in turn implies, by way of condition (IV)/(0,1), that $\bar{\pi} = -2\beta$.

Thus [25: equations (4.2c), (4.2d), and (4.2e)] and condition (VI)/(2,3) may be written as

$$\left.\begin{aligned} D\beta &= \beta\rho + \frac{1}{2}\Phi_{01} \\ D\alpha &= \frac{1}{2}(\bar{\delta}\rho - \alpha\rho - 7\bar{\beta}\rho) + \Phi_{10} \\ \delta\rho &= 2D\beta + 3\beta\rho + \rho\bar{\alpha} \\ 0 &= \rho(\beta\rho - \rho\bar{\alpha} + 2D\bar{\alpha} - \delta\rho - 2\Phi_{01} - 4D\beta) \end{aligned}\right\} \tag{4.23}$$

respectively. Eliminating all the derivative operators from these equations, one is left with $\rho(3\beta\rho + \rho\bar{\alpha} + 2\Phi_{01}) = 0$ which, since $\rho \neq 0$, implies that

$$\Phi_{01} = -\frac{1}{2}\rho(\bar{\alpha} + 3\beta).$$

Next, consider the first Bianchi identity of [29], which we may now write as

$$\rho(11\beta\rho + 5\rho\bar{\alpha} + 3D(\beta) + D(\bar{\alpha}) - 4\delta(\rho)) = 0. \tag{4.24}$$

Combining $(4.23)_1$ - $(4.23)_3$ and (4.24) we have $0 = \rho^2(\beta - \bar{\alpha})$ which, under assumption (iv), implies $\beta = \bar{\alpha}$. Substituting this into $(4.23)_2$ we also obtain $D\alpha = 0$ while substituting it into $(4.23)_3$ yields $\delta\rho = 4\rho\bar{\alpha}$.

Let us now consider [25: equation (4.2p)] and the second Bianchi identity of [29] which at this point have the forms

$$\left.\begin{aligned} \delta\bar{\alpha} &= \frac{1}{2}\bar{\lambda}\rho + 2\bar{\alpha}^2 + \frac{1}{2}\Phi_{02} \\ D(\Phi_{02}) &= \rho(\bar{\lambda}\rho - 2\delta(\bar{\alpha}) + 4\bar{\alpha}^2 + \Phi_{02}). \end{aligned}\right\} \tag{4.25}$$

These equations imply $D\Phi_{02} = 0$. On the other hand, [25: equation (4.2g)] now takes the form

$$D\lambda = -2\bar{\delta}\alpha + \rho\lambda - 4\alpha^2 + \Phi_{20}$$

which with $(4.25)_1$ implies $D\lambda = 0$.

Next, consider condition (VI)/(2,4), which can now be written as

$$0 = \rho(\rho\bar{\lambda} - 2\delta\bar{\alpha} - 4\bar{\alpha}^2 - \Phi_{02}). \tag{4.26}$$

By combining $(4.25)_1$ with (4.26) we obtain $0 = \rho(-8\bar{\alpha}^2 - 2\Phi_{02})$ which implies that $\Phi_{02} = -4\bar{\alpha}^2$ which in turn implies by $(4.25)_1$ that $\bar{\delta}\alpha = -\frac{1}{2}\rho\lambda$.

Now, let us consider the [25: equations (4.2f) and (4.2h)] and condition (III)/(0,1), which now may be written as

$$\left.\begin{aligned} 2D\gamma - \delta\rho &= -(\gamma + \bar{\gamma})\epsilon - 8\alpha\bar{\alpha} - 2\Lambda + 2\Phi_{11} \\ D\mu + 2\delta\alpha &= -\rho\mu + 4\alpha\bar{\alpha} + 2\Lambda \\ 6\delta\alpha + 2D\gamma - \delta\rho + 4D\mu - 2\bar{\delta}\bar{\alpha} &= \alpha\bar{\alpha} + 14\epsilon\mu + 6\epsilon\gamma - 2\epsilon\bar{\mu} - 4\Phi_{11} - \epsilon\bar{\gamma} \end{aligned}\right\} \tag{4.27}$$

respectively. These equations imply that

$$14\delta\alpha - 2\bar{\delta}\bar{\alpha} = 6\Phi_{11} - 4\rho\gamma + 6\Lambda - 11\rho\mu + \rho\bar{\mu}. \tag{4.28}$$

On the other hand, from [25: equation (4.21)] we have

$$\delta\alpha - \bar{\delta}\bar{\alpha} = \mu\rho + 2\gamma\rho + \frac{1}{2}(\mu - \bar{\mu})\rho + \Lambda + \Phi_{11}. \tag{4.29}$$

Solving (4.28) and (4.29) for $\delta\alpha$ and $\bar{\delta}\bar{\alpha}$ we get

$$\left.\begin{aligned} \delta\alpha &= \frac{1}{2}(\Phi_{11} - \mu\rho + \Lambda) \\ \bar{\delta}\bar{\alpha} &= \frac{1}{2}(-\Phi_{11} - 4\mu\rho - \Lambda - 4\gamma\rho + \rho\bar{\mu}). \end{aligned}\right\} \tag{4.30}$$

Subtracting $(4.30)_1$ from the complex conjugate of $(4.30)_2$ we get

$$-\Phi_{11} - 2\mu\rho - \Lambda - 2\gamma\rho = 0 \tag{4.31}$$

while adding (4.29) to its conjugate gives

$$\mu\rho + \gamma\rho - \rho\bar{\mu} + \Lambda + \Phi_{11} - \rho\bar{\gamma} = 0. \tag{4.32}$$

Adding (4.31) and (4.32) and recalling assumption (iv) we obtain

$$\mu + \gamma + \bar{\mu} + \bar{\gamma} = 0. \tag{4.33}$$

However, condition (IV)/(0,0) now has the form

$$2\gamma + 3\mu + 2\bar{\gamma} + 3\bar{\mu} = 0. \tag{4.34}$$

Solving (4.33) and (4.34) we get $\bar{\gamma} = -\gamma$ and $\bar{\mu} = -\mu$.

Next, upon substituting $(4.30)_1$ into $(4.27)_2$ and adding and subtracting the resulting equation from its conjugate we get

$$\left.\begin{aligned} D\mu &= 0 \\ 4\alpha\bar{\alpha} + \Lambda - \Phi_{11} &= 0. \end{aligned}\right\} \tag{4.35}$$

We further note that (V)/(3,3) now reads

$$\Phi_{11} - \Lambda + 5\mu\rho - 4\bar{\alpha}\alpha - 2\gamma\rho = 0. \tag{4.36}$$

Solving (4.36), $(4.35)_2$ and (4.31) for μ, Φ_{11} and Λ we obtain

$$\left.\begin{aligned} \mu &= \frac{2}{5}\gamma \\ \Lambda &= -\frac{7}{5}\gamma\rho - 2\alpha\bar{\alpha} \\ \Phi_{11} &= -\frac{7}{5}\gamma\rho + 2\alpha\bar{\alpha}. \end{aligned}\right\} \tag{4.37}$$

Thus, by $(4.35)_1$, $(4.37)_1$ and $(4.27)_1$ we have $D\gamma = 0$ and $\delta\rho = 0$.

Now, consider [25: equations (4.2o) and (4.2r)], the fourth and seventh Bianchi identities of [29] and [25: equations (4.2i), and (4.2m)], which can be written respectively as

$$\left.\begin{aligned} 0 &= \delta\gamma - \delta\bar{\alpha} + \frac{4}{5}\gamma\bar{\alpha} + \frac{1}{2}\rho\bar{\nu} - \alpha\bar{\lambda} - \Phi_{12} \\ 0 &= \delta\alpha - \bar{\delta}\gamma - \frac{3}{2}\rho\nu + 3\bar{\alpha}\lambda + \frac{8}{5}\alpha\gamma + 1 \\ 0 &= \bar{\delta}\Phi_{21} + \frac{14}{5}\lambda\gamma\rho + \alpha(8\delta\alpha + 6 + 4\nu\rho - 4\bar{\alpha}\lambda + \frac{27}{5}\alpha\gamma - 2\Phi_{21}) \\ 0 &= D\nu + 2\delta\alpha + 4\gamma\alpha + \rho\nu + 1 - \Phi_{21} \\ 0 &= 2D\Phi_{12} + 2\bar{\lambda}\rho\alpha + \frac{88}{5}\gamma\rho\bar{\alpha} + \frac{14}{5}\rho\delta\gamma + 2\rho\delta\bar{\alpha} - \bar{\nu}\rho^2 + 2\rho\Phi_{12} \\ 0 &= \delta\lambda - \frac{2}{5}\bar{\delta}\gamma - 2\rho\nu + \frac{4}{5}\gamma\alpha + 2\bar{\alpha}\lambda - 1 - \Phi_{21}, \end{aligned}\right\}$$

We may solve these equations to obtain expressions for each of the Pfaffian operators. In this manner we obtain

$$\left.\begin{aligned} \delta\lambda &= -\frac{4}{25}\rho(4\gamma\alpha - 10\rho\bar{\nu} + 10\bar{\alpha}\lambda - 5\Phi_{21} - 5) \\ \bar{\delta}\gamma &= -\frac{1}{10}(4\gamma\alpha - 10\rho\bar{\nu} + 10\bar{\alpha}\lambda - 5\Phi_{21} + 5) \\ \delta\alpha &= -\frac{1}{10}(12\gamma\alpha - 5\rho\bar{\nu} + 20\bar{\alpha}\lambda + 5\Phi_{21} - 5) \\ D\nu &= -\frac{2}{5}(4\gamma\alpha - 5\rho\bar{\nu} - 10\bar{\alpha}\lambda - 5\Phi_{21} + 5) \\ D\Phi_{21} &= -\frac{6}{25}\rho(44\gamma\alpha - 10\rho\bar{\nu} + 10\bar{\alpha}\lambda - 5\Phi_{21} - 5) \\ \bar{\delta}\Phi_{21} &= -\frac{2}{5}(12\gamma\alpha^2 - 20\alpha\rho\bar{\nu} - 50\alpha\bar{\alpha}\lambda - 15\alpha\Phi_{21} + 25\alpha + 7\lambda\gamma\rho). \end{aligned}\right\}$$

Substituting these derivatives into the commutator $[\bar{\delta},\delta]\alpha$, the fifth Bianchi condition in [29], conditions (III)/(0,0), (VI)/(3,4) and (VI)/(3,5) one obtains

$$\left.\begin{aligned} \rho(-85\rho\nu + 244\gamma\alpha + 110\bar{\alpha}\lambda - 5\Phi_{21} - 55) &= 0 \\ \rho(5\rho\nu - 152\gamma\alpha + 20\bar{\alpha}\lambda - 40\Phi_{21} - 10) &= 0 \\ \rho(5\rho\nu - 12\gamma\alpha + 20\bar{\alpha}\lambda + 15\Phi_{21} + 65) &= 0 \\ \rho(-230\rho\nu - 448\gamma\alpha + 280\bar{\alpha}\lambda + 260\Phi_{21} - 565) &= 0 \\ \alpha(-60\rho\nu + 184\gamma\alpha + 60\bar{\alpha}\lambda + 70\Phi_{21} - 155) &= 0. \end{aligned}\right\} \tag{4.38}$$

Solving the first four equations of (4.38) for the four unknowns $\rho\nu$, $\gamma\alpha$, $\bar{\alpha}\lambda$ and Φ_{21} we get

$$\left.\begin{aligned} \rho\nu &= -\frac{53}{12} \\ \gamma\alpha &= -\frac{95}{96} \\ \bar{\alpha}\lambda &= -\frac{5}{6} \\ \Phi_{12} &= -\frac{61}{24}. \end{aligned}\right\} \tag{4.39}$$

Substituting the above values into $(4.38)_5$ we get immediately $\alpha = 0$. However, this clearly contradicts $(4.39)_2$, and thus we conclude that assumption (iv) fails in the case $\phi_2 = 2$. Exactly the same steps lead to a contradiction for the case $\phi_2 = -2$. Therefore, assumption (iv) is false, and we conclude that

$$\rho = 0. \tag{4.40}$$

This equation has immediate consequences. First, from [25: equations (4.2k) and (4.2a)] we have $\Phi_{01} = 0$ and $\Phi_{00} = 0$ which, in conjunction with the third and ninth Bianchi identities of [29] implies $D\Phi_{11} = 0$ and $D(\Lambda) = 0$.

Next, from $(4.21)_2$ and condition (V)/(2,2) we have

$$\left.\begin{aligned} \bar{\epsilon}^2 + 10\epsilon\bar{\epsilon} + \epsilon^2 + D\bar{\epsilon} + D\epsilon &= 0 \\ \epsilon^2 - \epsilon\bar{\epsilon} + D\epsilon &= 0 \end{aligned}\right\}$$

from which we conclude that

$$\epsilon = 0. \tag{4.41}$$

We then have immediately from [25: equations (4.2c), (4.2d) and (4.2e)] and the second Bianchi identity of [29]

$$\left.\begin{aligned} D\tau &= 0 \\ D\alpha &= 0 \\ D\beta &= 0 \\ D\Phi_{02} &= 0 \end{aligned}\right\} \tag{4.42}$$

while $(4.42)_2$ and condition (III)/(0,2) yield

$$D\pi = 0. \tag{4.43}$$

We now consider the conformal invariance of our results under the assumption $D\phi = 0$. Recalling [8: equation (2.23)] and (2.2) we observe that equations (4.5), (4.6), (4.8), (4.19), (4.40) and (4.41) are invariant under a conformal transformation. Thus far, we have only exploited the conformal invariance by specifying $D\phi$ to obtain condition (4.12). We now wish to make the further specification

$$\left.\begin{aligned} \delta\phi &= \tau \\ \bar{\delta}\phi &= \bar{\tau} \end{aligned}\right\}$$

while preserving $D\phi = 0$. This will imply $\tau = 0$ in the new tetrad according to [8: equation (2.23)]. In order to check whether or not this specification of $\delta\phi$ is compatible with (4.12) we must verify the integrability conditions generated by considering $[D,\delta]\,\phi$, $[D,\bar{\delta}]\,\phi$ and $[\delta,\bar{\delta}]\,\phi$. The expressions obtained for these commutators are

$$\left.\begin{aligned} D\tau &= 0 \\ D\bar{\tau} &= 0 \\ -\bar{\delta}\tau + \delta\bar{\tau} &= -(\alpha - \bar{\beta})\tau - (-\bar{\alpha} + \beta)\bar{\tau}. \end{aligned}\right\} \tag{4.44}$$

That $(4.44)_1$ and $(4.44)_2$ are satisfied is clear upon inspection of [25: equations (4.2c) and (4.2k)] while $(4.4)_3$ can be seen to hold due to [25: equations (4.2p) and (4.2q)]. Thus, we can make the transformation (4.43) and conclude as a result that $\tau = 0$ [8]. We can now recover results which were not conformally invariant, since we have from [25: equation (4.2k)] that $\Phi_{01} = 0$. Thus from the second, third and ninth Bianchi identities of [29], NP (4.2d), and condition (III)/(0,2) we obtain

$$\left.\begin{aligned} D\Phi_{11} &= 0 \\ D\Lambda &= 0 \\ D\alpha &= 0 \\ D\Phi_{02} &= 0 \\ D\pi &= 0 \end{aligned}\right\}$$

Further [25: equation (4.2e)] implies that $D\beta = 0$. It follows immediately, via [25: equaiton (4.2p) and (4.2q)], that $\Phi_{02} = 0$ and $\Lambda = 0$ which concludes the proof of Lemma 1 ∎

In order to establish Theorem 3 we need the following second lemma from [8].

Lemma 2. *If, for any space-time, there exists a spinor dyad $\{o_a, \iota_a\}$ and a conformal transformation ϕ such that*

$$\left.\begin{aligned} \kappa = \sigma = \rho = \tau = \epsilon = 0 \\ \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \Psi_3 = -1 \\ \Phi_{00} = \Phi_{01} = \Phi_{02} = \Lambda = 0 \end{aligned}\right\}$$

then every repeated principal null vector field of the Weyl tensor is recurrent.

Proof of Theorem 3. The proof follows trivially from Lemma 1 and Lemma 2 ∎

5. Proof of Theorem 4

The idea of the proof is to show

$$H_{ab} = 0. \tag{5.1}$$

If this equation holds, it is a well known result (see for example [22: p. 273]) that the equation (1.1) is equivalent to the conformally invariant equation (1.7). Equations $(2.2)_1$ and (2.9) imply that $H_{ab} = 0$ if and only if $\phi_0 = \phi_1 = \phi_2 = 0$. Since we have already shown in Section 4 that equation (1.1) can be a Huygens' equation only if $\phi_0 = \phi_1 = 0$ in the specified gauge, we need only prove that $\phi_2 \neq 0$ is incompatible with the necessary conditions when the equations (1.6) hold. In our special conform gauge these equations reduce to

$$\alpha = \beta = \pi = 0. \tag{5.2}$$

With the results of Lemma 1 and this assumption, we see that conditions (V)/(2,2), (II)/(0,1) and (III)/(1,1) give $\epsilon = 0$. Also, from NP (5.21) (???) we see immediately that $\Phi_{11} = 0$. Finally, conditions (II)/(0,0), (II)/(0,1) and (IV)/(0,2) now require three of the Pfaffians of ϕ_2 to vanish, namely

$$D\phi_2 = \delta\phi_2 = \bar{\delta}\phi_2 = 0. \tag{5.3}$$

Many of the Newman-Penrose equations now have very simple forms. The following relationships are found amongst them:

$$\left.\begin{aligned} \delta\gamma &= \Phi_{12} \\ \bar{\delta}\gamma &= -1 \\ D\gamma &= 0 \\ D\lambda &= 0 \\ D\mu &= 0 \\ D\nu &= \Phi_{21} - 1 \\ D\Phi_{12} &= 0 \\ \delta\Phi_{12} &= 0. \end{aligned}\right\} \tag{5.4}$$

¿From the commutator $[\bar{\delta}, \delta]\gamma$ we further obtain $\delta\Phi_{21} = 0$ which with final Bianchi identity of [29] gives us $D\Phi_{22} = 0$.

¿From the commutators $[\Delta, D]\phi_2$, $[\Delta, \delta]\phi_2$ and $[\Delta, \bar{\delta}]\phi_2$ we have the mixed Pfaffians

$$\left.\begin{aligned} D\Delta\phi_2 &= 0 \\ \delta\Delta\phi_2 &= 0 \\ \bar{\delta}\Delta\phi_2 &= 0. \end{aligned}\right\} \tag{5.5}$$

Now, by applying the δ operator to condition (IV)/(0,0) and substituting for $\bar{\delta}\mu$ from condition (V)/(4,3) we obtain

$$(16\phi_2 + 60\bar{\phi}_2 + 45{\phi_2}^2\bar{\phi}_2)\Phi_{12} + (20 + 39{\phi_2}^2 + 276\phi_2\bar{\phi}_2)\bar{\phi}_2 = 0. \tag{5.6}$$

We may solve this equation for Φ_{12} provided $T := 16\phi_2 + 60\bar{\phi}_2 + 45\phi_2{}^2\bar{\phi}_2 \neq 0$. We observe that $T = 0$ implies that $\phi_2 = \pm 2i\sqrt{55}/15$. Further, removal of $\bar{\delta}\mu$ from δ of conditions (IV)/(0,0) and (V)/(4,3) with the above values of ϕ_2 yields a contradiction, and we therefore conclude that $T \neq 0$. Thus we can solve equation (5.1) for Φ_{12} to obtain

$$\Phi_{12} = -\frac{(20 + 39\phi_2{}^2 + 276\phi_2\bar{\phi}_2)}{16\phi_2 + 60\bar{\phi}_2 + 45\phi_2{}^2\bar{\phi}_2}\,\bar{\phi}_2. \tag{5.7}$$

Now, take δ of condition (VI)/(4,5), [25: equation (5.2m)] and condition (V)/(4,3) and eliminate the Pfaffians. The resulting equation is

$$\begin{aligned} &-1020\phi_2 + 1700\bar{\phi}_2\Phi_{21} - 93\phi_2{}^2\bar{\phi}_2 + 199\bar{\phi}_2{}^2\Phi_{21}\phi_2 \\ &+4913\bar{\phi}_2{}^2\phi_2 + 660\bar{\phi}_2 + 461\phi_2{}^2\Phi_{12}\bar{\phi}_2 + 996\Phi_{12}\bar{\phi}_2\Phi_{21} \\ &+45\Phi_{12}\bar{\phi}_2{}^2\phi_2\Phi_{21} + 39\phi_2\bar{\phi}_2{}^2\Phi_{12} + 180\Phi_{12}\bar{\phi}_2 + 228\phi_2\Phi_{12} \\ &+244\phi_2\Phi_{12}\Phi_{21} - 45\Phi_{21}\phi_2{}^2\bar{\phi}_2\Phi_{12} - 39\Phi_{21}\phi_2{}^2\bar{\phi}_2 + 180\Phi_{21}\phi_2 = 0. \end{aligned} \tag{5.8}$$

By substituting for Φ_{12} from (5.2) herein we obtain for ϕ_2 the equation

$$\begin{aligned} &18240\phi_2\bar{\phi}_2{}^2 + 211776\phi_2{}^2\bar{\phi}_2 + 445792\phi_2{}^3\bar{\phi}_2{}^2 \\ &+206544\phi_2{}^4\bar{\phi}_2 + 305824\phi_2{}^2\bar{\phi}_2{}^3 - 502608\phi_2{}^4\bar{\phi}_2{}^3 \\ &-116400\phi_2\bar{\phi}_2{}^4 - 3802704\phi_2{}^3\bar{\phi}_2{}^4 + 51840\phi_2{}^3 \\ &-28800\bar{\phi}_2{}^3 - 340875\phi_2{}^4\bar{\phi}_2{}^5 + 38340\phi_2{}^5\bar{\phi}_2{}^2 \\ &+320625\phi_2{}^5\bar{\phi}_2{}^4 - 1055628\bar{\phi}_2{}^5\phi_2{}^2 = 0. \end{aligned} \tag{5.9}$$

Let us define $\phi_2 = x + iy$. Then the real and imaginary parts of (5.9) are

$$\begin{aligned} 0 = y\Big(&92400\,x^4 + 548500\,x^6 + 55125\,y^8 + 330750\,y^4x^4 + 9408\,y^2 \\ &+ 77152\,y^2x^2 + 220500\,y^2x^6 + 916188\,y^4x^2 + 220500\,y^6x^2 - 15248\,y^4 \\ &+ 183844\,y^6 + 1280844\,y^2x^4 + 55125\,x^8 + 36288\,x^2\Big) \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} 0 = x\Big(&60750\,y^4x^4 + 10125\,y^8 + 5949324\,y^2x^4 + 40500\,y^6x^2 + 40500\,y^2x^6 \\ &- 240592\,y^4 + 3914748\,y^4x^2 - 661472\,y^2x^2 - 80448\,y^2 + 626724\,y^6 \\ &+ 10125\,x^8 - 420880\,x^4 + 2661300\,x^6 - 126528\,x^2\Big), \end{aligned} \tag{5.11}$$

respectively. Clearly,

$$x = y = 0 \tag{5.12}$$

is a solution of equations (5.10) and (5.11). If it is the only real solution, then we have proved that ϕ_2 must vanish for a Huygens' equation on our background, and hence have proved the theorem. Now, suppose $x = 0$ but $y \neq 0$. Then (5.10) implies

$$55125\,y^6 + 183844\,y^4 - 15248\,y^2 + 9408 = 0$$

which has no real solutions for y. Thus there are no real solutions of equations (5.10) and (5.11) with $x = 0$ but (5.12). Next, suppose $y = 0$ but $x \neq 0$. Then equation (5.11) implies

$$10125\,x^6 + 2661300\,x^4 - 420880\,x^2 - 126528 = 0.$$

This equation has two real roots $x^2 = {x_s}^2$, so it would seem that there are real values of $\phi_2 \neq 0$ for which equation (5.10) has a solution. However, suppose $\phi_2 = \pm x_s$. The proof that this case is impossible and is now described without giving the explicit forms of the equations obtained, since they are very long. We begin by noting that $\Delta\phi_2 = 0$. Now apply the δ operator to condition (V)/(4,4) and eliminate the mixed Pfaffians using the $[\delta,\Delta]\gamma$, $[\delta,\Delta]\bar{\gamma}$, $[\delta,\Delta]\mu$, and $[\delta,\Delta]\bar{\mu}$ commutators. Eliminate $\delta\bar{\lambda}$ and $\bar{\delta}\Phi_{22}$ from the resulting equation using $\bar{\delta}$ of condition (VI)/(5,4) and the final Bianchi identity of [29], respectively. Apply the δ operator to the complex conjugate of the resulting equation and eliminate the remaining $\delta\bar{\lambda}$'s using $\bar{\delta}$ of condition (VI)/(5,4). Finally, substitute for Φ_{12} in the equation thus obtained from δ of condition (IV)/(0,0) to obtain

$$\begin{aligned}260287171875\,{x_s}^{12} - 87747543000\,{x_s}^{10} + 2309587376400\,{x_s}^8 \\ -1845238913280\,{x_s}^6 + 383433397504\,{x_s}^4 - 27548129280\,{x_s}^2 \\ +11925549056 = 0\end{aligned}$$

which implies $0 = 1$ and is therefore a contradiction. Thus, we have proven that the only solution of equations (5.10) and (5.11) for which $x = 0$ or $y = 0$ is (5.12).

It remains to prove that there are no solutions of equations (5.10) and (5.11) for which neither x nor y vanish. Under this assumption these equations imply

$$\left.\begin{aligned}N_1 :=&92400\,x^4 + 548500\,x^6 + 55125\,y^8 + 330750\,y^4x^4 + 9408\,y^2 \\ &+ 77152\,y^2x^2 + 220500\,y^2x^6 + 916188\,y^4x^2 + 220500\,y^6x^2 - 15248\,y^4 \\ &+ 183844\,y^6 + 1280844\,y^2x^4 + 55125\,x^8 + 36288\,x^2 = 0 \\ N_2 :=&60750\,y^4x^4 + 10125\,y^8 + 5949324\,y^2x^4 + 40500\,y^6x^2 + 40500\,y^2x^6 \\ &- 240592\,y^4 + 3914748\,y^4x^2 - 661472\,y^2x^2 - 80448\,y^2 + 626724\,y^6 \\ &+ 10125\,x^8 - 420880\,x^4 + 2661300\,x^6 - 126528\,x^2 = 0\end{aligned}\right\} \quad (5.13)$$

At this point we use the package **grobner** in the computer algebra system Maple for the bivariate polynomial equations defined by the polynomials N_1 and N_2. In particular we employ the function `gsolve`. It computes a collection of reduced (lexicographic) Gröbner bases corresponding to a set of polynomials. The system corresponding to the set is first subdivided by factorization. Then a variant of Buchberger's algorithm which factors all intermediate results is applied to each subsystem. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables have been successively eliminated and separated as far as possible. In the present case we obtain the two systems given by

$$G_1 = [61893\,y^4 + 18704\,y^2 + 5120, 3474\,y^2 + 6165\,x^2 + 332] \quad (5.14)$$

and

$$
\left.\begin{aligned}
G_2 = [&219244532352355678500\, x^6 - 1292859421254386511155\, x^4 \\
&+ 1492411985766321405 2\, x^2 - 6570939064338291552, \\
&366408006623973544854674148 71\, y^2 \\
&+ 7560445247769811407838973100 0\, x^4 \\
&+ 1172657269486572194615334388 85\, x^2 \\
&- 8963925418010020951740016645.]
\end{aligned}\right\} \quad (5.15)
$$

By inspection the system (5.14) clearly has no solution. On the other hand, equation $(5.15)_1$ has two real solutions $x = \pm x_0$. However, upon substituting $x^2 = {x_0}^2$ into equation $(5.15)_2$ we find that y must be imaginary. Thus, the only real solutions of the system (5.13) are $x = y = 0$. We have therefore shown that the only solutions in the specified gauge with $\alpha = \beta = \gamma = 0$ are those for which $H_{ab} \equiv 0$. We conclude that equation (1.1) is equivalent to the equation (1.7). We complete the proof of the theorem by noting that the conditions of Theorem 2 are now satisfied since the equations (5.2) imply the equations (1.6) in the conformal gauge we are using. ∎

Conclusion. We have demonstrated in this paper that Theorems 1 and 2 of [8] for the self-adjoint scalar wave equation (1.1) (with $A_a \equiv 0$) have counterparts, Theorems 3 and 4, for the case of the non-self-adjoint scalar wave equation. In particular, Theorems 3 and 4 now supercede Theorems 1 and 2. This extends the original work by two of us in [22] to extend the program begun by Carminati and McLenaghan [5, 6] to the case of non-selfadjoint wave equations.

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