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Some Remarks on Geodesic and Curvature Preserving Mappings

M. Belger and K.-U. Beyer

In memoriam Professor Paul Günther

Abstract. We ask for the converse of Gauss' theorema egregium. Because in general isocurved manifolds are not isometric we ask stronger for isocurved, geodesic equivalent manifolds. For these we give a local criterion from which there follows that two-dimensional manifolds $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ of that type essentially are isometric, or both are Euclidean with an affine mapping in the ordinary sense.

Keywords: *Curvature preserving mappings, geodesic mappings*

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0. Introduction

In [5: p. 159] M. P. do Carmo considers a diffeomorphism $f : \mathcal{M} \to \overline{\mathcal{M}}$ between Riemannian manifolds $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, \bar{g})$ which preserves the corresponding (0,4)- Riemannnian curvature tensors R and $\bar{R}$. Referring to R. S. Kulkarni [8] and S. T. Yau [16] he poses the problem of deciding whether f is an isometry. In two dimensions this problem can be viewed as the question for the convertibility of Gauss' theorema egregium. This converse is false: The surfaces

$$\begin{aligned} \mathcal{M}^2 : &\quad \mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, \log u^1) \\ \overline{\mathcal{M}}^2 : &\quad \mathbf{x}(u^1, u^2) = (u^1 \cos u^2, u^1 \sin u^2, u^2) \end{aligned}$$

in the Euclidean space $\mathbb{R}^3$ have the same Gauß curvature, but the mapping $f = \bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry (see [6: p. 180]). For the compact case the so-called "dumbell spaces" give essential knowledge about this question (see [5: p. 159] or [8: p. 327]).

It is well-known that Riemannian manifolds of the same constant curvature are locally isometric; two diffeomorphic manifolds of that kind are isocurved but need not be globally isometric. For sectional curvatures $K \neq \text{const}$ and $n = \dim \mathcal{M} \geq 4$ Kulkarni [8] proved that all curvature preserving diffeomorphisms are isometries. For $n = 3$ Yau constructed examples of Riemannian manifolds which permit non-isometric curvature preserving diffeomorphisms [16]. Furthermore, he proved that if $\mathcal{M}$, $\overline{\mathcal{M}}$ are nowhere

M. Belger: Univ. Leipzig, Math. Inst., Augustuspl. 10, D - 04109 Leipzig
K.-U. Beyer: Univ. Leipzig, Math. Inst., Augustuspl. 10, D - 04109 Leipzig

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constantly curved compact three manifolds, then any curvature preserving diffeomorphism is an isometry. B. Ruh [12] showed that Yau's examples are the only ones of this kind. He determines all Riemannian and pseudo-Riemannian manifolds, which permit non-trivial curvature preserving diffeomorphisms. The different behaviour of curvature preserving diffeomorphisms $f : \mathcal{M} \to \overline{\mathcal{M}}$ obviously depends on the dimension n of $\mathcal{M}$, namely $n = 2$, $n = 3$ or $n \geq 4$. To the reason for this appearance it should be mentioned that the higher the dimension the more conditions the diffeomorphism has to satisfy.

In this paper we investigate curvature preserving geodesic diffeomorphisms and give a local criterion about such mappings with the intention to find an isometry. Using the fact that (for $n = 2$) $\mathcal{M} \subset \mathbb{R}^3$ is a Liouville surface, if there exist a geodesic diffeomorphism $f : \mathcal{M} \to \overline{\mathcal{M}}$ between the surfaces $\mathcal{M}$ and $\overline{\mathcal{M}}$ (see [3: pp. 168 and 213]), we find that a curvature preserving geodesic diffeomorphism essentially is isometric or $\mathcal{M}$ and $\overline{\mathcal{M}}$ are Euclidean and we have an affine mapping in the ordinary sense.

Observe that the notion "curvature preserving" is used not uniformly in different publications. Here we use the (1,3)-curvature operators R and $\bar{R}$ of the Riemannian manifolds $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, \bar{g})$, because earlier on in the investigations we started firstly with linear connected isocurved manifolds, i.e. we used $f_* R = \bar{R} f_*$ or (mappings by means of the same coordinates x assumed) we defined equation (5) as curvature preserving condition. That is just what we need advantageously in formula (3) when we study curvature preserving geodesic mappings. We have to be carefully because, in general, from $R^h_{jkl}(x) = \bar{R}^h_{jkl}(x)$ it does not follow that $R_{ijkl}(x) = \bar{R}_{ijkl}(x)$ as used in [5] for a curvature preserving mapping. Kulkarni and Yau make use of the invariance of the sectional curvature.

1. Curvature preserving and geodesic mappings

Let $(\mathcal{M}, g)$ and $(\overline{\mathcal{M}}, \bar{g})$ be n-dimensional Riemannian manifolds and $f : \mathcal{M} \to \overline{\mathcal{M}}$ a local diffeomorphism – throughout the whole paper. We consider two arbitrary cards (φ, U) in $\mathcal{M}$ and $(\bar{\varphi}, \bar{U})$ in $\overline{\mathcal{M}}$ with $\bar{U} = f(U)$ and $\bar{\varphi} := \varphi \circ f^{-1}_{|\bar{U}}$ because geodesic and curvature preserving mappings will be formulated in the following as mappings by means of the same coordinates

$$x = (x^1, \ldots, x^n) \quad \text{where} \quad x = \varphi(p) = \bar{\varphi}(\bar{p}) \quad (\bar{p} = f(p) \in \bar{U},\ p \in U).$$

f is a geodesic mapping (i.e. $\mathcal{M}$ and $\overline{\mathcal{M}}$ are *geodesic equivalent*) if and only if the corresponding Christoffel symbols transform according to

$$\bar{\Gamma}^k_{ij}(x) = \Gamma^k_{ij}(x) - \psi_j(x)\delta^k_i + \psi_i(x)\delta^k_j \tag{1}$$

where the gradient (ψ_i) is given by

$$\psi_i(x) = \frac{1}{2(n+1)} \frac{\partial}{\partial x^i} \ln \frac{\bar{g}(x)}{g(x)} \qquad \begin{pmatrix} \bar{g} = \det \bar{g}_{ij} \\ g = \det g_{ij} \end{pmatrix}. \tag{2}$$

Furthermore, it is well-known that the coordinates of the (1,3)-type curvature tensor at the same time transform according to

$$\bar{R}^h_{ijk}(x) = R^h_{ijk}(x) - \psi_{ik}(x)\delta^h_j + \psi_{ij}(x)\delta^h_k \tag{3}$$

where

$$\psi_{ij}(x) = \psi_{i,j}(x) - \psi_i(x)\psi_j(x) \qquad (, = \nabla) \tag{4}$$

(see [13]).

The diffeomorphism f above is said to be a *curvature preserving* mapping if

$$\bar{R}^h_{ijk}(x) = R^h_{ijk}(x) \tag{5}$$

and as far as f additionally is geodesic, $(\psi_1, \ldots, \psi_n)$ have to solve the partial differential equation system $\psi_{ij} = 0$.

Remark 1. For a geodesic line $\gamma : I \to \mathcal{M}$ it is pointlessly to use an affine parametrization here because only the affine among the geodesic mappings preserve the affine parameter. So we define γ by means of a certain function $\varrho \in C^\infty(I)$ with the property

$$\nabla_{\dot\gamma}\dot\gamma = \varrho\dot\gamma. \tag{6}$$

If f is a geodesic mapping, then the curve $\bar\gamma := f \circ \gamma \subset \overline{\mathcal{M}}$ is the image geodesic line.

2. Criterion for curvature preserving geodesic mappings

Now we deal with the problem of deciding whether $\mathcal{M}$ and $\overline{\mathcal{M}}$ are locally isocurved and geodesic equivalent Riemannian manifolds. So, in regard to (2) - (5), for curvature preserving geodesic mappings we have to investigate the differential equation system

$$\psi_{ij}(x) = \partial_j\psi_i(x) - \Gamma^k_{ij}\psi_k(x) - \psi_i(x)\psi_j(x) = 0 \qquad (i, j = 1, \ldots, n) \tag{7}$$

in the unknown functions $\psi_1, \ldots, \psi_n$.

Let $N \subset M$ be a normal neighbourhood, $U \in N$ a coordinate neighbourhood of $p_0 \in M$ and, in U,

$$\gamma_{X_0} : x^i = x^i(s)$$

the arc length parametrization of the geodesic line $\gamma_{X_0} \subset N$, which passes through

$$p_0 = \varphi^{-1}(x_0) \quad \text{where} \quad x_0 = (x^1(0), \ldots, x^n(0))$$

in direction of the tangential unit vector

$$X_0 = x^{i\prime}(0)\,\frac{\partial}{\partial x^i}_{|p_0} \in M_{p_0}. \tag{8}$$

We consider equation (7) on γ_{X_0} and contract (7) by $x^{i\prime}(s)x^{j\prime}(s)$. Using $x^{j\prime}\frac{\partial}{\partial x^j} = \frac{d}{ds}$ we obtain

$$x^{i\prime}(s)\frac{d}{ds}\psi_i(x(s)) - x^{i\prime}(s)x^{j\prime}(s)\Gamma^k_{ij}(x(s))\psi_k(x(s)) - \left(x^{i\prime}(s)\psi_i(x(s))\right)^2 = 0. \tag{9}$$

With respect to the differential equation system

$$x^{k''}(s) + \Gamma^k_{ij}(x(s))x^{i'}(s)x^{j'}(s) = 0$$

of γ_{X_0} (where $\varrho = 0$ for an affine parameter s in (6)), and considering the relation

$$\frac{d}{ds}(x^{i'}\psi_i) = x^{i''}\psi_i + x^{i'}\frac{d}{ds}\psi_i$$

along γ_{X_0}, relation (9) leads to

$$\frac{d}{ds}\left(x^{i'}(s)\psi_i(x(s))\right) - \left(x^{i'}(s)\psi_i(x(s))\right)^2 = 0. \tag{10}$$

Naturally, now we have to introduce the new function

$$F_{X_0}(s) = x^{i'}(s)\psi_i(x(s)) \tag{11}$$

which is a γ_{X_0}-corresponding one. Equation (10) can be written as ordinary differential equation

$$F'_{X_0}(s) - F^2_{X_0}(s) = 0. \tag{12}$$

We investigate the initial value problem to this equation with initial condition

$$F'_{X_0}(0) = x^{i'}(0)\psi_i(x_0). \tag{13}$$

The form of the solution of (12) - (13) depends on the direction of X_0 relative to the direction of

$$Y_0 := \operatorname{grad}\psi_{|p_0} \qquad \text{with } \psi = \frac{1}{2(n+1)}\ln\frac{\overline{g}}{g}.$$

We will consider the following two cases.

(I) Trivial case: $Y_0 = 0$. Then we have $F_{X_0}(0) = 0$ for all X_0 with $|X_0| = 1$ (because of (8)) and $F_{X_0}(s) = 0$ is the solution of (12) - (13) for all these X_0. This leads by (11) to the integration of $\frac{d}{ds}\ln\frac{\overline{g}(x_0)}{g(x_0)}g(x)$ along the geodesic line throughout p_0 in an arbitrary direction X_0. We obtain

$$\overline{g}(x) = \frac{\overline{g}(x_0)}{g(x_0)}g(x) \tag{14}$$

in a neighbourhood of $x_0 = \varphi(p_0)$. It follows from (14) that $\psi_i(x) = 0$. Therefore the mapping f is trivial (as a geodesic mapping) and because of $\overline{\Gamma}^k_{ij}(x) = \Gamma^k_{ij}(x)$ (see (1)), f is an affine mapping. Conversely, any affine mapping is always geodesic and also curvature preserving.

(II) Non-trivial case: $Y_0 \neq 0$.

a) Firstly consider all directions in (8) with $X_0 \perp Y_0$. That means we have to consider the subset

$$V = \left\{X_0 \in \mathcal{M}_{p_0} \,\middle|\, F_{X_0}(0) = 0 \text{ and } |X_0| = 1\right\}$$

in $\mathcal{M}_{p_0}$. For $X_0 \in V$ the initial value problem gives once more $F_{X_0}(s) = 0$ as trivial solution of (12). With regard to case (I) the relation (14) is also valid here – but only for all $x = \varphi(\gamma_{X_0}(s))$ with $X_0 \in V$ and s so that $\gamma_{X_0}(s) \in U$.

b) Now consider the remaining part of directions X_0, non-orthogonal to Y_0. That means

$$W := \left\{ X_0 \in \mathcal{M}_{p_0} \middle| \, F_{X_0}(0) \neq 0 \text{ and } |X_0| = 1 \right\} \tag{15}$$

is to be investigated. For $X_0 \in W$ it follows from (12) - (13) that

$$F_{X_0}(s) = -\frac{1}{s + c(X_0)} \qquad (s \neq -c(X_0)) \tag{16}$$

where the integration constant c depends on X_0 and

$$c(X_0) = -\frac{1}{F_{X_0}(0)} \neq 0. \tag{17}$$

Now we have to integrate (16) with respect to s:

$$\overline{g}(x(s)) = K(s+c)^{-2(n+1)} \cdot g(x(s)) \qquad (0 < K = \text{const}).$$

For $s = 0$ we have

$$K = (c(X_0))^{2(n+1)} \cdot \frac{\overline{g}(x_0)}{g(x_0)}.$$

Because p_0 and $p \in U \subset N$ are connected by a unique geodesic line γ_{X_0}, we have, using still (17) for $X_0 \in W$, the relation

$$\overline{g}(x) = \frac{\overline{g}(x_0)}{g(x_0)} \left(\frac{1}{1 - sF_{X_0}(0)} \right)^{2(n+1)} \cdot g(x) \tag{18}$$

for all $x = \varphi(\gamma_{X_0}(s))$ and s not so great.

c) To include now the solution case (II)/a) in that of b) (i.e. to understand (14) for all $x = \varphi(\gamma_{X_0}(s))$ with $X_0 \in V$ and $\gamma_{X_0}(s) \in U$ as a special case of (18)), we observe the fact that $F_{X_0}(0) \to 0$ if $W \ni X_0 \to V$ (or, by (17), $c(X_0) \to \pm\infty$). Doing this, we see that the relation (18) changes into (14) for (II) a), as was to be expected. Therefore (18) alone describes already the solution of (12) - (13) for all $|X_0| = 1$ in the non-trivial case (II). In (18) the term

$$\phi(x) := s \cdot F_{X_0}(0) \tag{19}$$

is a differentiable function as can be seen using Riemannian normal coordinates $x^i = s \cdot p^i$ with respect to the origin x_0 $(p^i = x^{i'}(0)$ in (13)). Then we obtain namely $\phi(x) = x^i \psi_i(x_0)$.

For curvature preserving geodesic mappings $f : \mathcal{M} \to \overline{\mathcal{M}}$ (because of (2), (18) and (19)) it has to be necessarily

$$\psi_i = \frac{\phi_i}{1 - \phi} \qquad \text{with } \phi_i = \partial_i \phi. \tag{20}$$

From here there follows – carry out simple differentiations – that ψ_i from (20) fulfils indeed the equation $\psi_{ij}(x) = 0$, as far as we assume still that ϕ_i is a parallel gradient field in a neighbourhood of x_0. So case (II) is finished.

Putting together cases (I) and (II), we get now the following

Theorem. *Let $f : \mathcal{M} \to \overline{\mathcal{M}}$ be a local geodesic diffeomorphism between C^∞-Riemannian manifolds $\mathcal{M}$ and $\overline{\mathcal{M}}$. Then around each point $p_0 \in \mathcal{M}$ we find a neighborhood $U \subset \mathcal{M}$ which is isocurved to $\overline{U} := f(U) \subset \overline{\mathcal{M}}$ if and only if there is a C^∞-function ϕ, for which $g = \det g_{ij}$ and $\overline{g} = \det \overline{g}_{ij}$ enter into the relation*

$$\overline{g}(x) = \frac{\overline{g}(x_0)}{g(x_0)}(1 - \phi(x))^{-2(n+1)} \cdot g(x) \tag{21}$$

where $\phi_i = \partial_i \phi$ has to be a parallel gradient field ($x \in \varphi(U), \phi(x_0) = 0$; mappings by means of the same coordinates assumed).

Remark 2. **(I)** For $\phi(x) = \text{const}$ (i.e. by (21) inevitably $\phi(x) = 0$), we are situated in the trivial case $Y_0 = 0$ and f is an affine mapping (see (20); from there $\psi_i(x) = 0$ follows). Conversely, affine mappings are always geodesic and also curvature preserving. **(II)** For $\phi \neq \text{const}$ we have (because of (20) and (21)) the non-trivial case $Y_0 \neq 0$ and f can not be an affine mapping.

3. Isocurved geodesic equivalent surfaces in $\mathbb{R}^3$

Let us consider the Theorem especially for 2-dimensional surfaces $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ in $\mathbb{R}^3$ instead of n-dimensional Riemannian manifolds $\mathcal{M}$ and $\overline{\mathcal{M}}$. Then, according to the proof given below, we have the following local

Corollary. *Let* **G**, **C** *and* **A** *be the classes of geodesic, curvature preserving and affine local diffeomorphisms $f : \mathcal{M}^2 \to \overline{\mathcal{M}}^2$, respectively. Then* $\mathbf{A} = \mathbf{G} \cap \mathbf{C}$.

This corollary delivers still more detailed informations about affine local diffeomorphisms $f : \mathcal{M}^2 \to \overline{\mathcal{M}}^2$ if we take into account the following well-known assertion (see [9: Satz 13.4]):

An affine mapping between 2-dimensional Riemannian manifolds $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ is either a homothety or the spaces are Euclidean, and then the affine mapping is affine in the ordinary elementary sense.

Really this assertion is valid only for $n = 2$, unless the holonomy group of the common parallel displacement in $\mathcal{M}$ and $\overline{\mathcal{M}}$ is sufficently large. Such a result also follows directly from the Theorem for $n = 2$ if, more general, f is used as a curvature preserving geodesic mapping. Finally we can formulate the wished

Proposition. *A local diffeomorphism $f : \mathcal{M}^2 \to \overline{\mathcal{M}}^2$ between the C^∞-surfaces $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ is a curvature preserving geodesic mapping if and only if either*

a) *f is essentially an isometry*

or

b) *the intrinsic geometry of $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ is Euclidean and then f is an affine mapping in the ordinary elementary sense.*

Proof. a) May be that f a priori is essentially an isometry (homothetic mapping), because such a mapping by itself is curvature preserving and geodesic.

b) Assume now f is a curvature preserving geodesic, but non-homothetic mapping: for each U and constant k there are i,j with $\bar{g}_{ij} \neq k g_{ij}$ on U. Then $\bar{g}_{ij}$ and g_{ij} are constants.

In order to prove that, we use the Liouville parametrizations (22) and (23) for $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$. Dini [4] determined all pairs of geodesic equivalent surfaces $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$, insofar as $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ are non-homothetic even. They all are pairs of Liouville surfaces, i.e. with respect to a common orthogonal parameter system (u^1, u^2) the coordinates of the metric fundamental tensors of $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ are

$$g_{ij}(u^1, u^2) = (U_1 - U_2)\delta_{ij} \tag{22}$$

$$\bar{g}_{ij}(u^1, u^2) = (U_1 U_2 U_i)^{-1} \cdot g_{ij}(u^1, u^2). \tag{23}$$

Here the function $U_i = U_i(u^i)$ depends only on u^i $(i = 1,2)$, and because of $g_{ij}(u^1, u^2)$, $\bar{g}_{ij}(u^1, u^2) > 0$ it has to be

$$U_1(u^1) > U_2(u^2) > 0.$$

Observe that in the homothetic case a) ($\bar{g}_{ij} = k\, g_{ij}$) equations (22) and (23) would lead to $\bar{g}_{ij} = g_{ij} = 0$. For this reason a) is considered separately.

Before the proof of the Proposition will be continued, we give the following two remarks.

Remark 3. As a Liouville surface it will be considered sometimes stronger only the type of surfaces $\mathcal{M}^2$, which allows a parametrization with property (22) (s. [3]). Then in different place, for $\overline{\mathcal{M}}^2$ of type (23) we can read also "the other Liouville surface" [10]. The Liouville surface (22) is a generalization of the rotation surfaces; these and for example also the helicoids are Liouville surfaces of type (22).

Remark 4. It is easy to see that the metric fundamental tensors g_{ij} and $\bar{g}_{ij}$ from (22) and (23) satisfy criterion (1) for geodesic mappings; a straightforward calculation, starting at $\overline{\Gamma}^k_{ij}$ by means of (26) - (28), leads to the right-hand side of (1).

Continuing the proof we look at the assumption b). There is a function ϕ with property (21). In order to determine the function $\phi(u^1, u^2)$ from (21) and to use that $\phi_i = \partial_i \phi(u^1, u^2)$ shall be a parallel field, we need for these next steps on the whole the quantities g_{ij}, g^{hl}, $\bar{g}_{ij}$, $\bar{g}^{hl}$, g, $\bar{g}$, Γ^k_{ij}, $\overline{\Gamma}^k_{ij}$ and, to serve Remark 4, also ψ_i. From (22) and (23) there follows

$$g^{ij} = \frac{1}{U_1 - U_2}\, \delta^{ij} \qquad \text{and} \qquad \bar{g}^{ij} = U_1 U_2 U_i \cdot g^{ij} \tag{24}$$

(no summation over i)

$$g = \det(g_{ij}) = (U_1 - U_2)^2 \qquad \text{and} \qquad \bar{g} = \det(\bar{g}_{ij}) = \frac{g}{(U_1 U_2)^3} \tag{25}$$

and, if we set $U_i' = \frac{dU_i}{du^i}$, furthermore

$$\left.\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = -\Gamma^1_{22} &= \frac{U_1'}{2(U_1 - U_2)} \\ \Gamma^2_{22} = \Gamma_{21}1 = -\Gamma^2_{11} &= \frac{U_2'}{2(U_2 - U_1)} \end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{array}{l} \Gamma^1_{11} = \dfrac{2U_2 - U_1}{U_2} \, \Gamma^2_{12} = -\dfrac{2U_2 - U_1}{U_1} \, \Gamma^1_{22} = \dfrac{2U_2 - U_1}{2U_1(U_1 - U_2)} U_1' \\ \Gamma^2_{22} = \dfrac{2U_1 - U_2}{U_1} \, \Gamma^1_{21} = -\dfrac{2U_1 - U_2}{U_2} \, \Gamma^2_{11} = \dfrac{2U_1 - U_2}{2U_2(U_2 - U_1)} U_2'. \end{array}\right\} \tag{27}$$

Let us observe that one gets the second lines of (26) and (27), respectively each of them, by exchanging the indices 1,2 in the first lines. For $\psi_i = \frac{1}{2(n+1)} \partial_i \ln \frac{\bar{g}}{g}$ from (25) we obtain

$$\psi_i = -\frac{1}{2} \frac{U_i'}{U_i}. \tag{28}$$

Now, by means of (25) and for $n = 2$ and $x_0 = (\mathring{u}^1, \mathring{u}^2)$ condition (21) delivers

$$\phi = \kappa \cdot \sqrt{U_1 U_2} + 1 \qquad \text{with} \qquad \kappa = -\left(\frac{\bar{g}(\mathring{u}^1, \mathring{u}^2)}{g(\mathring{u}^1, \mathring{u}^2)} \right)^{\frac{1}{6}} \neq 0.$$

The corresponding gradient has the components $\phi_i = \frac{\kappa}{2} \frac{U_i'}{\sqrt{U_1 U_2}}$ and because it has to be a parallel field, we can evaluate $\nabla_j \phi_i = 0$. Using (27), only two conditions for the determination of U_1 and U_2 will be essentially:

$$\nabla_1 \phi_1 = 0 \qquad \text{and} \qquad \nabla_2 \phi_2 = 0 \tag{29}$$

($\nabla_1 \phi_2$ and $\nabla_2 \phi_1$ are identically zero). Because of the fact that U_i depends on u^i only, this differential equation system can be transformed into a separably written system for $U_1(u^1) > U_2(u^2) > 0$. In connection with (29) this leads to $U_1' = U_2' = 0$ and by (22) and (23) to

$$g_{ij}(u^1, u^2) = \text{const} \qquad \text{and} \qquad \bar{g}_{ij}(u^1, u^2) = \text{const}.$$

That means, the coordinates of the Christoffel symbols and then also those of the curvature tensor vanish, just as the Gauß curvatures of $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$. The intrinsic geometry of $\mathcal{M}^2$ and $\overline{\mathcal{M}}^2$ is Euclidean, f is a non-homothetic affine mapping ∎

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