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About Systems of Differential Equations Related to Geodesic Double Differential Forms Mean Values and Harmonic Spaces Part I

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Dedicated to Prof. Dr. P. Günther†

Abstract. In spaces of constant curvature P. Günther has introduced geodesic double differential forms and mean value operators for differential forms. As solutions of differential equations for forms (for instance, Weyl-De Rham equations) the form equations so as the mean values suffice certain ordinary systems of differential equations. We generalize such systems and study properties which are determined by differential geometric structures (parallel translation of double differential forms with respect to geodesic lines, the construction of closed, coclosed and harmonic components of differential forms and double forms and the telescopage theorem of McKean and Singer). As special cases we get the systems known for geodesic double differential forms or mean value operators in real and complex hyperbolic spaces by the application of structural information without any special information about these spaces.

Keywords: *Geodesic double differential forms, harmonic spaces, parallel translations, Laplace operator, telescopage theorem, mean value operators*

AMS subject classification: 34 A 30, 58 A 10, 53 B 20, 11 F 72, 35 Q 05

1. Introduction

In order to illustrate the geometric background of certain systems of differential equations and transformations we want to remind their meaning in spaces of constant curvature in which they were indroduced and studied by Günther [5, 7, 8]. Beginning with Section 2 we will study, for which classes of functions and related spaces the underlying formalisms determined by differential geometric principles will work.

Using the geodesic distance $r(x,y)$ of the points x and y in spaces of constant curvature k Günther [5] has introduced the geodesic double differential forms

$$\begin{aligned} \sigma_0(x,y) &= 1 & \tau_0 &= 0 \\ \sigma_1(x,y) &= \tfrac{\sin kr(x,y)}{k}\, d\hat{d}r(x,y) & \tau_1(x,y) &= dr(x,y)\hat{d}r(x,y) \\ \sigma_p &= \tfrac{1}{p}\sigma_{p-1} \wedge \hat{\wedge}\sigma_1 & \tau_p &= \sigma_{p-1} \wedge \hat{\wedge}\tau_1. \end{aligned} \tag{1}$$

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We suppose that y lies in a sufficiently small neighbourhood of x. In hyperbolic spaces (using $k = -1$), we have $\sigma_1(x,y) = \sinh r(x,y)\, d\hat{d}r(x,y)$. The symbols d, $\wedge$ are related to the variable x and $\hat{d}$, $\hat{\wedge}$ shall be related to the variable y. Belger [1] has considered the real pseudo Riemannian and harmonic case.

The so-called transport form $T_p(x,y)$ (cf. [11]) for the parallel displacement of differential forms $\alpha_p(x)$ along the geodesic γ from x to y is given by

$$T_p(x,y) = (-1)^p\,(\sigma_p + \tau_p)(x,y).$$

The double differential form $\sigma_p + \tau_p$ is the sum of their component σ_p orthogonal to the geodesic line from x to y (for the definition cf. [5 - 8, 11]) and the component τ_p which is taken with respect to the direction of this geodesic line. The covariant derivatives of these components in direction to the geodesic considered are disappearing (cf. [11]). Therefore we can state

$$\Delta\big(h(r)\Phi_p(x,y)\big) = \Delta h(r)\Phi_p(x,y) + h(r)\Delta\Phi_p(x,y) \tag{2}$$

with the Laplace operator

$$\Delta = d\delta + \delta d, \tag{3}$$

a sufficiently smooth function $h = h(r)$ of the geodesic distance $r = r(x,y)$ and $\Phi_p = \sigma_p$ or $\Phi_p = \tau_p$. The Laplace operator $\Delta\Phi$ shall be taken with respect to the point x. d and δ denote the differential and codifferential operator for p-forms (cf. [4, 11, 17, 18]).

The application of the Laplace operator with respect to differential forms will play an essential role in our formulas. Harmonic spaces are connected in a natural way with invariance properties of the Laplace operator (Günther [7 - 10], Helgason [14], Selberg [29], Vanhecke [31]). That is the reason to study the relevant differential equations in connection with harmonic spaces. We remark that the trace in the 1-transport-form $-(\sigma_1 + \tau_1)(x,x)$ is leading to the metric (cf. Günther and Schimming [11]).

In harmonic spaces one has (remind the sign of Δ resulting from (3))

$$\Delta h(r) = -h''(r) - F^*(r)h'(r)\,. \tag{4}$$

The function F^* is related to the polar density function f by $F^* = \frac{f'}{f}$ (cf. [4]). For symmetric spaces of rank one (with adequate normalization) we get (cf. [9, 10, 14])

$$F^*(r) = (n-1)\frac{\cosh r}{\sinh r} + (d-1)\frac{\sinh r}{\cosh r} \tag{5}$$

with $d = 1,2,4,8$ for the hyperbolic spaces over the real and complex numbers, the quaternions and the Caylay numbers, respectively. For real hyperbolic spaces with dimension n and curvature -1 we have

$$F^*(r) = (n-1)\coth r\,. \tag{6}$$

According to [11, 25] one has

$$\begin{aligned}\Delta(\sigma_p + \tau_p) &= \Delta(\sigma_1 + \tau_1) \wedge \hat{\Lambda}(\sigma_{p-1} + \tau_{p-1}) \\ &\quad - L(\sigma_1 + \tau_1, \sigma_1 + \tau_1) \wedge \hat{\Lambda}(\sigma_{p-2} + \tau_{p-2}) \end{aligned} \tag{7}$$

$$\begin{aligned}\Delta(\sigma_p - \tau_p) &= \Delta(\sigma_1 - \tau_1) \wedge \hat{\Lambda}(\sigma_{p-1} - \tau_{p-1}) \\ &\quad - L(\sigma_1 - \tau_1, \sigma_1 - \tau_1) \wedge \hat{\Lambda}(\sigma_{p-2} - \tau_{p-2}) \end{aligned} \tag{8}$$

for all harmonic spaces. In Section 4 we study linear equations, which must be satisfied for classes of solutions of the system (21) of differential equations based on (7) and (8). The same results can be reached by quadratic transformations which we will study in Section 3. This interference of linear and quadratic structures determine conditions for relevant classes of solutions and related spaces.

Using special properties of spaces of constant curvature, Günther [5] has proved that σ_p, τ_p satisfy the following system of differential equations. We can refer to several versions to prove this. The original concept uses specially adapted coordinates in combination with the Hodge dualization. Another strategy is based on the parallel translation and its components with respect to a geodesic line. Belger [2] has used this for complex hyperbolic spaces, but it works in the same way for real hyperbolic spaces. We recall

$$\begin{aligned}\Delta\sigma_p &= \left(-p(n-p-1) + 2p\,\tfrac{1}{\sinh^2 r}\right)\sigma_p - 2(n-p)\,\tfrac{\cosh r}{\sinh^2 r}\,\tau_p \\ \Delta\tau_p &= \left(-(p-1)(n-p) + 2(n-p)\,\tfrac{1}{\sinh^2 r}\right)\tau_p - 2p\,\tfrac{\cosh r}{\sinh^2 r}\,\sigma_p .\end{aligned} \tag{9}$$

If we are looking for harmonic double differential forms $u(r)\sigma_p + v(r)\tau_p$, the following system has to be satisfied:

$$\Delta\left(u(r)\sigma_p + v(r)\tau_p\right) = 0 . \tag{10}$$

The relations (2), (4), (5) - (9) are leading to (cf. [7, 8])

$$\begin{aligned} u_p''(r) &+ (n-1)\,u_p'(r)\coth r + p(n-p-1)\,u_p(r) \\ &- 2p\,\tfrac{1}{\sinh^2 r}u_p(r) + 2(n-p)\tfrac{\cosh r}{\sinh^2 r}v_p(r) = 0 \\ v_p''(r) &+ (n-1)\,v_p'(r)\coth r + (p-1)(n-p)\,v_p(r) \\ &- 2(n-p)\,\tfrac{1}{\sinh^2 r}v_p(r) + 2p\tfrac{\cosh r}{\sinh^2 r}u_p(r) = 0 .\end{aligned} \tag{11}$$

In order to solve these equations, Günther [5, 7] has introduced the functions

$$\begin{aligned}\sinh r\, l_p(r) &= u_p'(r) + p\,\tfrac{\cosh r}{\sinh r}\,u_p(r) - (n-p)\,\tfrac{1}{\sinh r}\,v_p(r) \\ \sinh r\, m_p(r) &= v_p'(r) + (n-p)\,\tfrac{\cosh r}{\sinh r}\,v_p(r) - p\,\tfrac{1}{\sinh r}\,u_p(r) .\end{aligned} \tag{12}$$

These definitions are related to the construction of closed and coclosed forms (cf. [7, 8, 21, 25]). Using definition (12), we can write (11) in the form

$$\begin{aligned} u_p''(r) + (n+1)\coth r\, u_p'(r) + p(n-p+1)\,u_p(r) &= 2\cosh r\, l_p(r) \\ v_p''(r) + (n+1)\coth r\, v_p'(r) + (p+1)(n-p)\,v_p(r) &= 2\cosh r\, m_p(r) .\end{aligned} \tag{13}$$

We get

$$\begin{array}{lllllllll} l''_p(r) & + & (n+1)\coth r & l'_p(r) & + & (p+1)(n-p) & l_p(r) & = & 0 \\ m''_p(r) & + & (n+1)\coth r & m'_p(r) & + & p(n+1-p) & m_p(r) & = & 0. \end{array} \tag{14}$$

Mean value operators can be used to define or characterize subspaces of harmonic spaces (cf. [3, 12, 31]). Mean value operators and their kernels have been used in [21] to derive the Selberg trace formula for p-spectra and to use the Selberg zeta-function in order to get spectral estimates and to solve lattice problems (cf. [15, 23, 24, 26, 27, 29]).

Let $S(x,r)$ denote the geodesic sphere around the point x of a real hyperbolic space with radius r. Günther [5 - 8] has introduced the mean value operators M_p^σ and M_p^τ for p-forms α_p (dy shall denote the survey element):

$$\begin{aligned} M_p^\sigma[\alpha](x,r) &= \frac{(-1)^p\Gamma(n/2)}{2\pi(n/2)} \int_{S(x,r)} \sigma_p(x,y)\cdot\alpha_p(y)\,dy \\ M_p^\tau[\alpha](x,r) &= \frac{(-1)^p\Gamma(n/2)}{2\pi(n/2)} \int_{S(x,r)} \tau_p(x,y)\cdot\alpha_p(y)\,dy\,. \end{aligned} \tag{15}$$

We remark that we could integrate over the geodesic ball instead of the geodesic sphere and adapted kernel functions in order to get natural generalizations of related differential equations (cf. Schuster [22 - 24]) using an additional parameter λ. If we study eigenform expansions in connection with properly discontinous groups of isometries of a real hyperbolic space, the generalization mentioned is leading to better convergence properties. The related recursion formulas with respect to the generalization mentioned, we will discuss in Part II. The introduced mean values are solutions of a system of differential equations (for p-forms) of almost the same structure as (11). We get the complete analogy, if we consider an eigenform problem instead of (10) or if we consider the following system for harmonic forms α:

$$\begin{aligned} &\frac{d^2}{dr^2}M_p^\sigma[\alpha](x,r) + (n-1)\coth r\,\frac{d}{dr}M_p^\sigma[\alpha](x,r) \\ &\quad +p(n-p-1)\,M_p^\sigma[\alpha](x,r) + \Delta_x M_p^\sigma[\alpha](x,r) \\ &-2p\,\frac{1}{\sinh^2 r}M_p^\sigma[\alpha](x,r) + 2(n-p)\frac{\cosh r}{\sinh^2 r}M_p^\tau[\alpha](x,r) = 0 \\ &\frac{d^2}{dr^2}M_p^\tau[\alpha](x,r) + (n-1)\coth r\,\frac{d}{dr}M_p^\tau[\alpha](x,r) \\ &\quad +(p-1)(n-p)\,M_p^\tau[\alpha](x,r) + \Delta_x M_p^\tau[\alpha](x,r) \\ &-2(n-p)\,\frac{1}{\sinh^2 r}M_p^\tau[\alpha](x,r) + 2p\frac{\cosh r}{\sinh^2 r}M_p^\sigma[\alpha](x,r) = 0\,. \end{aligned} \tag{16}$$

Günther [7, 8] has introduced mean value operators

$$\begin{aligned}\sinh r A_p[\alpha](x,r) &= \frac{d}{dr}M_p^\sigma[\alpha](x,r) \\ &\quad + p\coth r M_p^\sigma[\alpha](x,r) - (n-p)\frac{1}{\sinh r}M_p^\tau[\alpha](x,r) \\ \sinh r B_p[\alpha](x,r) &= \frac{d}{dr}M_p^\tau[\alpha](x,r) \\ &\quad + (n-p)\coth r M_p^\tau[\alpha](x,r) - p\frac{1}{\sinh r}M_p^\sigma[\alpha](x,r).\end{aligned} \tag{17}$$

One fundamental geometric property which plays (implicitely) a central role in the formulas of Section 2 is the (locally defined) decomposition of p-forms into harmonic, closed and coclosed components (cf. [4, 11, 16, 18, 19]). The connection between the decomposition of p-forms into harmonic, closed and coclosed components and the parallel translation of p-forms and their components with respect to geodesic lines is described by the formulas

$$\begin{aligned}\delta A_p[\alpha](x,r) &= 0 \\ dA_p[\alpha](x,r) &= B_p[d\alpha](x,r) \\ dB_p[\alpha](x,r) &= 0 \\ \delta B_p[\alpha](x,r) &= A_p[\delta\alpha](x,r) \\ A_p[\alpha](x,r) &= -\frac{1}{p}M_p^\tau[\delta d\alpha](x,r) \\ B_p[\alpha](x,r) &= -\frac{1}{n-p}M_p^\sigma[d\delta\alpha](x,r).\end{aligned} \tag{18}$$

It is an essential aspect of the generalization mentioned to guarantee that these equations remain true. In analogy to (14) one has

$$\begin{aligned}&\frac{d^2}{dr^2}A_p[\alpha](x,r) + (n+1)\frac{\cosh r}{\sinh r}\frac{d}{dr}A_p[\alpha](x,r) \\ &\qquad + \big((p+1)(n-p)+\Delta\big)A_p[\alpha](x,r) = 0 \\ &\frac{d^2}{dr^2}B_p[\alpha](x,r) + (n+1)\frac{\cosh r}{\sinh r}\frac{d}{dr}B_p[\alpha](x,r) \\ &\qquad + \big(p(n+1-p)+\Delta\big)B_p[\alpha](x,r) = 0.\end{aligned} \tag{19}$$

Using (17), we can write (16) in the form (cf. [7])

$$\begin{aligned}&\frac{d^2}{dr^2}M_p^\sigma[\alpha](x,r) + (n+1)\coth r\frac{d}{dr}M_p^\sigma[\alpha](x,r) \\ &\qquad + \big(p(n-p+1)+\Delta\big)M_p^\sigma[\alpha](x,r) = 2\cosh r A_p[\alpha](x,r) \\ &\frac{d^2}{dr^2}M_p^\tau[\alpha](x,r) + (n+1)\coth r\frac{d}{dr}M_p^\tau[\alpha](x,r) \\ &\qquad + \big((p+1)(n-p)+\Delta\big)M_p^\tau[\alpha](x,r) = 2\cosh r B_p[\alpha](x,r).\end{aligned} \tag{20}$$

We remark that the operators on the left-hand sides of $(19)_1$ and $(19)_2$ coinside with $(20)_2$ and $(20)_1$, respectively. The right-hand sides of (20) are zero for harmonic forms α. In contrast to this, the right-hand sides of (13) are not necessarily zero (looking for harmonic geodesic double forms $u(r)\sigma_p + v(r)\tau_p$). We will study properties of the solutions of (10) in Section 6 and in Part II. Thereby the fundamental solution of (10) will be of special interest. The real hyperbolic case $n = 4$ with signature $(+ - - -)$ is of physical importance, we will give details in Part II. A generalization of the system (13), (14) with the transformation (12) (or in the case of mean values a generalization of (19), (20) with (17)) will lead to the system (21), (25) with (22). We will discuss a related eigenvalue problem in Part II. For further results related to Riemannian, harmonic and hyperbolic spaces using differential geometric methods we refer to Belger [3], Helgason [13, 14], de Rham [18], Ruce, Walker and Willmore [20], Selberg [29], Simon and Wissner [30] and Yano [32]. For further aspects with respect to differential and double differential forms we refer to Belger [3], Hodge [16] and de Rham [19].

2. Basic differential equations and structural considerations

We look for functions u_p, v_p which satisfy a generalization of (11). Using the fact that (11) is a consequence of (12) and (13), we want to start with generalizations of these systems (cf. (21), (22) below). p shall get the interpretation of the degree of related double differential forms. In another interpretation (cf. [1 - 3, 7, 8, 23, 24]) u_p and v_p may be double differential forms. The geometric background for the mentioned transformations and equations is the construction of closed, coclosed and harmonic parts of double differential forms or mean value operators. We look for harmonic spaces (cf. [3, 19, 20, 31, 32]), in which the formalism described in this section works in combination with the algebraic considerations of Section 4. All functions shall depend on r (with the interpretation of geodesic distance). The derivatives are taken with respect to r. As a generalization of (12), (13) we consider the system

$$\begin{aligned} u_p'' + F u_p' + g_p \, u_p &= 2\alpha' \, l_p \\ v_p'' + F v_p' + g_{p+1} \, v_p &= 2\alpha' \, m_p \end{aligned} \tag{21}$$

with

$$\begin{aligned} \alpha \, l_p &= u_p' - S_p \, u_p + C_p^* v_p \\ \alpha \, m_p &= v_p' - R_p \, v_p + F_p^* u_p \, . \end{aligned} \tag{22}$$

Thereby F, g_p, g_{p+1}, α, S_p, R_p, C_p^* and F_p^* are functions depending on r. The function F will be related to the Laplace operator for radial functions by

$$\Delta f(r) = -f''(r) - F^*(r) f'(r) \qquad \text{with} \quad F^* = F - 2\frac{\alpha'}{\alpha} \, . \tag{23}$$

For the discussion of the radial part of the Laplace operator in symmetric spaces we refer to Helgason [14]. We study (21), (22) independent of this interpretation. The solution of (21) determines structural properties of the harmonic space used. The telescopage structure (cf. [4]) is coming into the considerations by the functions g_p and g_{p+1}. As initial functions we use

$$\begin{aligned} g_0(r) &= 0 \\ F_0^*(r) &= 0 \\ S_0(r) &= 0\,. \end{aligned} \tag{24}$$

Later on we will get $\alpha(r) = \sinh r$ for hyperbolic spaces ($\alpha(r) = \sin r$ is related to the spherical case). One central demand in order to determine the structure of the solutions of (21), (22) and related spaces is given by the following generalization of (14):

$$\begin{aligned} l_p'' + F\,l_p' + g_{p+1}\,l_p &= 0 \\ m_p'' + F\,m_p' + g_p\,m_p &= 0\,. \end{aligned} \tag{25}$$

We will discuss another point of view in Section 4. We also demand that (21), (22) will lead to a first order system of differential equations for l_p and m_p which is independent of u_p and v_p.

Definition 2.1. We call (21), (22), (24), (25) a *mean-hyperbolic system of type I*.

We use this terminology because the well-known examples are describing mean values in hyperbolic spaces.

Lemma 2.2. *Using* $C_p^* = \frac{C_p}{\alpha}$, $F_p^* = \frac{F_p}{\alpha}$ *(cf. Proposition* 2.5) (21) *and* (22) *are equivalent to the system*

$$\begin{aligned} u_p'' + \left(F - 2\tfrac{\alpha'}{\alpha}\right) u_p' + \left(g_p + 2\tfrac{\alpha'}{\alpha} S_p\right) u_p - 2C_p \tfrac{\alpha'}{\alpha^2} v_p &= 0 \\ v_p'' + \left(F - 2\tfrac{\alpha'}{\alpha}\right) v_p' + \left(g_{p+1} + 2\tfrac{\alpha'}{\alpha} R_p\right) v_p - 2F_p \tfrac{\alpha'}{\alpha^2} u_p &= 0\,. \end{aligned} \tag{26}$$

Proof. Straight-forward calculations ∎

Remark 2.3. We also could start with a system

$$\begin{aligned} u_p'' + \bar{F}u_p' - \bar{S}_p u_p - \bar{C}_p v_p &= 0 \\ v_p'' + \bar{F}v_p' - \bar{F}_p v_p - \bar{F}_p u_p &= 0 \end{aligned} \tag{27}$$

as a modification of (26) which looks more general at first sight. But, the analogy between (21) and (25) is getting lost. Using (27), it is much more complicated to describe the telescopage mechanism. Starting with (27), one can show that respective assumptions about the structure of the system after the transformation (22) will lead to (26).

If we differentiate (22) and substitute the second derivatives u_p'' and v_p'' by (21), we get

$$\begin{aligned} &\big[-\alpha' + (F + S_p)\alpha\big] l_p + \alpha l_p' - \alpha C_p^* m_p \\ &\qquad = \big\{-(F + S_p)S_p - C_p^* F_p^* - (g_p + S_p')\big\} u_p \\ &\qquad\quad + \big\{(F + S_p)C_p^* + C_p^* R_p + C_p^{*\prime}\big\} v_p \\ &\big[-\alpha' + (F + R_p)\alpha\big] m_p + \alpha m_p' - \alpha F_p^* l_p \\ &\qquad = \big\{-(F + R_p)R_p - C_p^* F_p^* - (g_{p+1} + R_p')\big\} v_p \\ &\qquad\quad + \big\{(F + R_p)F_p^* + F_p^* S_p + F_p^{*\prime}\big\} u_p\,. \end{aligned} \tag{28}$$

The demand noted above, that the first order system for l_p and m_p is independent of u_p and v_p shall be understood in the sense that the coefficients of u_p and v_p in (28) are zero.

Definition 2.4. If the coefficients of u_p and v_p in (28) are zero, we say that the systems (21), (22) or (26) have *transformation property I.*

Proposition 2.5. *The system* (21), (22) *has the transformation property I if and only if the following equations are true:*

$$\begin{aligned} C_p^{*'}/C_p^* = F_p^{*'}/F_p^* &=: \Omega \\ F + S_p + R_p + \Omega &= 0 \\ (F + S_p)S_p + C_p^* F_p^* + g_p + S_p' &= 0 \\ (F + R_p)R_p + C_p^* F_p^* + g_{p+1} + R_p' &= 0 . \end{aligned} \tag{29}$$

We can use

$$\begin{aligned} C_p &= \alpha\, C_p^* \\ F_p &= \alpha\, F_p^* \\ \Omega &= -\frac{\alpha'}{\alpha} . \end{aligned} \tag{30}$$

Proof. If the coefficients of u_p and v_p in (28) are zero, we get

$$\begin{aligned} F + S_p + R_p + C_p^{*'}/C_p^* = 0 \\ F + S_p + R_p + F_p^{*'}/F_p^* = 0 . \end{aligned}$$

It follows

$$\frac{C_p^{*'}}{C_p^*} = \frac{F_p^{*'}}{F_p^*} . \tag{31}$$

We define Ω by this value. From (31) it follows $C_p^* = F_p^* k$ with a constant k. We can use $C_p^* = \frac{C_p}{\alpha}$ and $F_p^* = \frac{F_p}{\alpha}$ with constant numbers C_p and F_p. The choise of α in (21), (22) is determined by convenience. It is obvious that the conditions (29) are independent of α. As a consequence of $(29)_1$, $(30)_1$ and $(30)_2$ we get $(30)_3$ ∎

Definition 2.6. A system (21), (22) satisfying (29) and (30) we will call a *mean-hyperbolic system of type II.*

We remark that we did not suppose (25) at this point.

Proposition 2.7. *If there are given the functions R_p, S_p, F and α of a mean-hyperbolic system of type II, the functions g_p are inductively defined. On the other hand, g_p (for all degrees p), F, α and initial values $R_p(1)$ and $S_p(1)$ determine the functions R_p and S_p. Both is reached by*

$$\begin{aligned} (R_p - S_p)' - (R_p - S_p)\Omega &= g_p - g_{p+1} \\ R_p + S_p &= -F - \Omega . \end{aligned} \tag{32}$$

Proof. Straight-forward calculations ∎

Using elementary calculations, we get the next two propositions.

Proposition 2.8. *The functions l_p and m_p of a mean-hyperbolic system of type II are solutions of the first order system*

$$\begin{aligned} l_p' &= R_p\, l_p + \tfrac{C_p}{\alpha}\, m_p \\ m_p' &= S_p\, m_p + \tfrac{F_p}{\alpha}\, l_p\,. \end{aligned} \tag{33}$$

If we consider an eigenvalue problem instead of (10), the resulting system (33) will contain terms depending on u_p and v_p.

Proposition 2.9. *Functions, which are satisfying a mean-hyperbolic system of type II are also satisfying a mean-hyperbolic system of type I:*

$$\begin{aligned} l_p'' + F\, l_p' + g_{p+1}\, l_p &= 0 \\ m_p'' + F\, m_p' + g_p\, m_p &= 0\,. \end{aligned} \tag{34}$$

We remark that the calculations to prove (33) are using transformation property I. We also have to use transformation property I as well as (21) and (22) in order to prove (34) supposing (33).

We want to point out that the possibility to calculate g_p inductively with increasing degree p with the initial function (24) (as it was stated in Proposition 2.7), is a direct consequence of the telescopage structure (cf. [4, 11]) using g_p, g_{p+1} in the definition of a mean-hyperbolic system of type I. We could have done all calculations with coefficient functions h_p instead of g_{p+1} (without the possibility to calculate g_p inductively). In contrast to this alternative, our definition explicitly includes a telescopage structure. This telescopage structure will be essential for the following considerations.

We have to remind that R_p, S_p, F and α are functions, which we want to determine under certain conditions. Only if we suppose that these functions are known, we can directly determine g_p as it was stated in Proposition 2.7. Summarizing the considered first order differential equations, we get

Proposition 2.10. *An mean-hyperbolic system of type II satisfies the following differential equations:*

$$\begin{aligned} u_p' &= S_p\, u_p - (C_p/\alpha)\, v_p + \alpha\, l_p \\ v_p' &= R_p\, v_p - (F_p/\alpha)\, u_p + \alpha\, m_p \\ l_p' &= R_p\, l_p + (C_p/\alpha)\, m_p \\ m_p' &= S_p\, m_p + (F_p/\alpha)\, l_p \\ S_p' &= -(F+S_p)S_p - C_p\, F_p/\alpha^2 - g_p \\ R_p' &= -(F+R_p)R_p - C_p\, F_p/\alpha^2 - g_{p+1} \\ \alpha' &= \alpha(F+S_p+R_p)\,. \end{aligned} \tag{35}$$

3. The "Verhulst method" to solve dynamical systems in harmonic spaces

Verhulst equations are often used in Medicine and Biology to describe growth processes with finite carrying capacity as a first step in direction to more precise models (cf. [28]). We consider simple additional invariance properties of the related logistic functions, which are important to find solutions of the dynamical system (35).

It is interesting to look for solutions of a fixed dimension (as Banach spaces over $\mathbb{R}$), which are independent of the degree p. In the applications which we will consider in Section 5, the mentioned dimension of the space of solutions shall be independent of the dimension n of the related harmonic space, which is determined by F.

By a simple translation one can transform the standard form $\gamma' = \gamma(1-\gamma)$ of a Verhulst equation into $\gamma' = 1-\gamma^2$. Thereby we use $\gamma = \gamma(r)$ and the derivatives are taken with respect to r:

$$' = \frac{d}{dr}.$$

Lemma 3.1. *The Verhulst equation*

$$\gamma' = 1 - \gamma^2 \tag{36}$$

is invariant with respect to inversion:

$$\left(\frac{1}{\gamma}\right)' = 1 - \left(\frac{1}{\gamma}\right)^2. \tag{37}$$

Lemma 3.2. *If we demand that the solution of the autonomous differential equation*

$$\gamma' = h(\gamma) \tag{38}$$

with an analytic function h with $h(0) = 1$ is invariant under inversion in the sense that one has

$$\left(\frac{1}{\gamma}\right)' = h\left(\frac{1}{\gamma}\right), \tag{39}$$

we get the uniquely determined function

$$h(\gamma) = 1 - \gamma^2. \tag{40}$$

The Verhulst equation is uniquely determined by the invariance property given.

Proof. Straight-forward calculations using power expansions ∎

The solution of $\gamma' = 1 - \gamma^2$ with the initial value $\gamma(0) = 1$ is given by

$$\gamma = \tanh r. \tag{41}$$

In spite of the structure of $(30)_3$ and (35), it is useful to write the solution of the Verhulst equation as a logarithmic derivative:

$$\frac{1}{\gamma} = \frac{\alpha'}{\alpha} = (\ln \alpha)'. \tag{42}$$

We define

$$\beta = \alpha' . \tag{43}$$

One directly could use

$$\begin{aligned} \alpha &= \sinh r \\ \beta &= \cosh r \end{aligned} \tag{44}$$

which follows from (42) with a suitable normation (see below) in order to derive the following formulas. But, with respect to a possible generalization of (38), (39) to more complex symmetric structures, we only want to use (36) and the definition (42).

As a first example, we want to remark that one could exchange (39) by the anti-symmetric invariance

$$\begin{aligned} \gamma' &= \hat{h}(\gamma) \\ \left(\frac{1}{\gamma}\right)' &= -\hat{h}\left(\frac{1}{\gamma}\right) . \end{aligned} \tag{45}$$

This leads to

$$\hat{h}(\gamma) = 1 + \gamma^2 \tag{46}$$

instead of (40) and

$$\gamma = \tan r \tag{47}$$

instead of (41). The equation (40) will lead us to hyperbolic spaces and (46) will give the respective results for spherical spaces. We will derive the formulas only for the hyperbolic case, but the changes for the spherical case are obvious.

Using (42), we get

$$\begin{aligned} \alpha' &= \frac{\alpha}{\gamma} \\ \left(\frac{1}{\alpha}\right)' &= -\frac{1}{\gamma}\frac{1}{\alpha} . \end{aligned} \tag{48}$$

Using (42) and (36), we get

$$\beta' = \left(\frac{\alpha}{\gamma}\right)' = \frac{\alpha'}{\gamma} - \frac{\alpha}{\gamma^2}\gamma' = \frac{\beta}{\gamma} - \frac{\beta}{\gamma}\gamma' = \frac{\beta}{\gamma} - \frac{\beta}{\gamma}(1-\gamma^2) = \beta\gamma$$

and thereby

$$\begin{aligned} \beta' &= \beta\gamma \\ \beta' &= \alpha \\ \left(\tfrac{1}{\beta}\right)' &= -\gamma\tfrac{1}{\beta} . \end{aligned} \tag{49}$$

The equations (43) and $(49)_2$ imply $(\beta^2 - \alpha^2)' = 0$. It follows that $\beta^2 - \alpha^2$ has a constant value. Multiplying α and β by a constant and eventually exchanging α and β we can use without loss of generality the normalization

$$\beta^2 - \alpha^2 = 1 . \tag{50}$$

It follows

$$\begin{aligned} \frac{1}{\alpha^2} &= -1 + \frac{1}{\gamma^2} \\ \frac{1}{\alpha\beta} &= \frac{1}{\gamma} - \gamma \\ \frac{1}{\beta^2} &= 1 - \gamma^2 . \end{aligned} \tag{51}$$

We try to find a solution of the dynamical system (35) by the "Verhulst method". That shall mean that we look for solutions with

$$\begin{aligned} \gamma &= \frac{\alpha}{\alpha'} \\ R_p &= A_p\gamma + B_p\frac{1}{\gamma} \\ S_p &= D_p\gamma + E_p\frac{1}{\gamma} \\ F &= a_0\frac{1}{\gamma} + a_1\gamma \end{aligned} \tag{52}$$

with constant numbers A_p, B_p, D_p, E_p, a_0 and a_1 with the background of Lemma 3.2. We have recalled $(52)_1$ (cf. (42)) in order to point out that the function γ used is defined by the function α from (35). We suppose that (36) and (48) - (51) are valid. We want to point out that it is not useful to look for functions u_p, v_p, l_p and m_p in the form of a linear combination of γ and $\frac{1}{\gamma}$. There is a main difference between the different functions in the dynamical system (35). The functions S_p and R_p reflect transformation properties, and u_p, v_p, l_p and m_p reflect properties of the solutions of equations like (10). May be that a certain generalization of the "Verhulst method" will lead to a unique point of view.

Using $(30)_3$ we get

$$\Omega = -\frac{1}{\gamma} \tag{53}$$

and by $(29)_2$ it follows

$$\begin{aligned} a_0 &= 1 - B_p - E_p \\ a_1 &= -A_p - D_p . \end{aligned} \tag{54}$$

From $(35)_5$ we get

$$\begin{aligned} -g_p = {} & \frac{1}{\gamma^2}(-B_p\,E_p + C_p\,F_p) \\ & + (2D_p + E_p - B_p\,D_p - A_p\,E_p - C_p\,F_p) \\ & - \gamma^2 D_p(A_p + 1) . \end{aligned} \tag{55}$$

From $(35)_4$ it follows

$$\begin{aligned} -g_{p+1} = {} & \frac{1}{\gamma^2}(-B_p\,E_p + C_p\,F_p) \\ & + (2A_p + B_p - E_p\,A_p - D_p\,B_p - C_p\,F_p) \\ & - \gamma^2 A_p(D_p + 1) . \end{aligned} \tag{56}$$

As a consequence, we have

$$-B_p\,E_p + C_p\,F_p = -B_{p+1}\,E_{p+1} + C_{p+1}\,F_{p+1}$$

and by (24) it follows

$$B_p\,E_p = C_p\,F_p\,. \tag{57}$$

We get

$$\begin{aligned} 2D_{p+1} + E_{p+1} - B_{p+1}D_{p+1} - A_{p+1}E_{p+1} - B_{p+1}E_{p+1} & \\ = 2A_p + B_p - E_pA_p - D_pB_p - B_pE_p & \\ D_{p+1}(1 + A_{p+1}) = A_p(1 + D_p)\,. & \end{aligned} \tag{58}$$

By $(54)_2$ the solution of the quadratic equation $(58)_2$ is given by

$$D_{p+1} = \frac{1 - a_1}{2} \pm \left(\frac{a_1 + 1}{2} + D_p\right)\,. \tag{59}$$

Case 1: The plus sign in (59) gives $D_{p+1} = D_p$. From (24) and $(52)_3$ we get $D_0 = 0$. It follows

$$\begin{aligned} D_p &= p \\ A_p &= -a_1 - p\,. \end{aligned} \tag{60}$$

Case 2: The minus sign in (59) gives $D_{p+1} = -a_1 - D_p$. From $(54)_2$ we get

$$\begin{aligned} D_{p+1} &= A_p \\ A_{p+1} &= D_p\,. \end{aligned} \tag{61}$$

Using $D_0 = 0$, it follows

$$\begin{aligned} D_{2k} &= 0 \\ A_{2k} &= -a_1 \end{aligned} \tag{62}$$

and

$$\begin{aligned} D_{2k+1} &= -a_1 \\ A_{2k+1} &= 0\,. \end{aligned} \tag{63}$$

We use $(54)_1$, (59), (62) and (63) in order to solve the quadratic equation $(58)_1$.

Case 1: We suppose (60). It follows

$$E_{p+1} = -\left(1 + p + \frac{a_0 + a_1}{2}\right) \pm \left(E_p - 1 + p + \frac{a_0 + a_1}{2}\right)\,. \tag{64}$$

Subcase 1.1: The plus sign in (64) gives

$$E_{p+1} = E_p - 2\,. \tag{65}$$

It follows

$$D_{p+1} = E_p - 2 \tag{66}$$

and thereby

$$\begin{aligned} E_p &= -2p \\ B_p &= 1 - a_0 + 2p\,. \end{aligned} \tag{67}$$

Subcase 1.2: The minus sign in (64) gives

$$E_{p+1} = -2p - a_0 - a_1 - E_p \tag{68}$$

and, as a consequence,

$$E_{p+2} = E_p - 2\,, \tag{69}$$

which has the same structure as (65). The equations (24) and $(52)_3$ imply $E_0 = 0$. It follows

$$\begin{aligned} E_{2k} &= -2k \\ B_{2k} &= 1 + 2k - a_0 \end{aligned} \tag{70}$$

and

$$\begin{aligned} E_{2k+1} &= -2k - 2 - a_0 - a_1 \\ B_{2k+1} &= 2k + 3 + a_1\,. \end{aligned} \tag{71}$$

Case 2: We suppose (62). It follows

$$E_{2k+1} = \frac{a_1 - a_0}{2} \pm \left(E_{2k} - 1 + \frac{a_0 + a_1}{2}\right). \tag{72}$$

Subcase 2.1: The plus sign in (72) gives

$$E_{2k+1} = a_1 - 1 + E_{2k}\,. \tag{73}$$

It follows

$$B_{2k+1} = 1 + a_1 - 2a_0 - B_{2k}\,. \tag{74}$$

Subcase 2.2: The minus sign in (72) gives

$$E_{2k+1} = -a_0 + 1 - E_{2k}\,. \tag{75}$$

It follows

$$B_{2k+1} = 1 - a_0 - B_{2k}\,. \tag{76}$$

Case 3: We suppose (63). It follows

$$E_{2k+2} = -\frac{a_1 + a_0}{2} \pm \left(E_{2k+1} - 1 + \frac{a_0 - a_1}{2}\right). \tag{77}$$

Subcase 3.1: The plus sign in (77) gives

$$E_{2k+2} = -a_1 - 1 - E_{2k+1}\,. \tag{78}$$

It follows

$$B_{2k+2} = 3 - 2a_0 + a_1 - B_{2k+1}\,. \tag{79}$$

Subcase 3.2: The minus sign in (77) gives

$$E_{2k+2} = -a_0 + 1 - E_{2k+1} \, . \tag{80}$$

It follows

$$B_{2k+2} = 1 - a_0 - B_{2k+1} \, . \tag{81}$$

For $a_1 = 0$ (this equation is true for real hyperbolic spaces) the Cases 2 and 3 give the same formulas. In Section 5 we will see, in which way real and complex hyperbolic spaces are special examples of the equations of this section. We remark that we have taken the same sign to solve the quadratic equations (58) for all degrees p. We also could discuss, which other possibilities are compatible with the formulas of the next section.

In order to determine C_p and F_p (not only their product by (57)), we additionally assume (in accordance with the real hyperbolic case mentioned in Section 1)

$$\begin{aligned} F_p &= B_p \\ C_p &= E_p \, . \end{aligned} \tag{82}$$

4. Linear equations for coefficients of double differential forms

Let σ_p and τ_p be the components of the transport form $T_p = (-1)^p(\sigma_p + \tau_p)$ with respect to a geodesic line in a harmonic space.

Definition 4.1. We say that a harmonic space has *transformation property II*, if the components of the transport form are satisfying the following system:

$$\begin{aligned} \Delta\sigma_p &= K_p\sigma_p + L_p\tau_p \\ \Delta\tau_p &= M_p\tau_p + N_p\sigma_p \, . \end{aligned} \tag{83}$$

Thereby K_p, L_p, M_p and N_p shall be functions of the geodesic distance $r(x, y)$ of the points x and y which are connected by the geodesic considered.

In Section 1 we have mentioned that in real hyperbolic spaces one has (cf. (9))

$$\begin{aligned} K_p(r) &= -\; p(n-p-1) + 2p\frac{1}{\sinh^2 r} \\ L_p(r) &= -\; 2(n-p)\frac{\cosh r}{\sinh^2 r} \\ M_p(r) &= -\; (p-1)(n-p) + 2(n-p)\frac{1}{\sinh^2 r} \\ N_p(r) &= -\; 2p\frac{\cosh r}{\sinh^2 r} \, . \end{aligned} \tag{84}$$

We consider the complex hyperbolic case in Section 5. We use (2) - (5) and in comparison with (26) we state

$$\begin{aligned} K_p &= g_p + 2\tfrac{\alpha'}{\alpha} S_p \\ L_p &= -C_p \tfrac{\alpha'}{\alpha^2} \\ M_p &= g_{p+1} + 2\tfrac{\alpha'}{\alpha} R_p \\ N_p &= -2F_p \tfrac{\alpha'}{\alpha^2}\,. \end{aligned} \tag{85}$$

In Section 3 we have determined S_p and R_p by the help of quadratic equations. Under the considered additional assumption (82) g_p and g_{p+1} are inductively determined. In contrast to this, we will state linear equations to reach the same goal in this section using (85), but one already has to know the structure for the degrees $p = 1$ and $p = 2$.

Proposition 4.2. *The functions K_p, L_p, M_p and N_p depending on r are satisfying the recursion equations*

$$\begin{aligned} K_p &= (2-p)pK_1 + \tfrac{(p-p1)}{2}K_2 \\ L_p &= (2-p)L_1 + (2-p)(p-1)N_1 + (p-1)L_2 + \tfrac{(p-1)(p-2)}{2}N_2 \\ M_p &= (2-p)(p-1)K_1 + (2-p)M_1 + \tfrac{(p-1)(p-2)}{2}K_2 + (p-1)M_2 \\ N_p &= (2-p)pN_1 + \tfrac{(p-1)p}{2}N_2\,. \end{aligned} \tag{86}$$

We use (cf. (1))

$$\begin{aligned} \sigma_p &= \tfrac{1}{p}\sigma_{p-1} \wedge \hat{\Lambda}\sigma_1 \\ \tau_p &= \sigma_{p-1} \wedge \hat{\Lambda}\tau_1 \\ 0 &= \tau_1 \wedge \hat{\Lambda}\tau_1\,. \end{aligned} \tag{87}$$

By straight-forward calculations we get

Lemma 4.3. *The double forms $\sigma_p + \tau_p$ and $\sigma_p - \tau_p$ are satisfying the following recursion formulas:*

$$\begin{aligned} \Delta(\sigma_p + \tau_p) &= \tfrac{1}{p}(\sigma_{p-1} + \tau_{p-1}) \wedge \hat{\Lambda}(\sigma_1 + \tau_1) \\ \Delta(\sigma_p - \tau_p) &= \tfrac{1}{p}(\sigma_{p-1} - \tau_{p-1}) \wedge \hat{\Lambda}(\sigma_1 - \tau_1)\,. \end{aligned} \tag{88}$$

The system (88) is symmetric with respect to $\sigma_p + \tau_p$ and $\sigma_p - \tau_p$ in contrast to the formulas (87) which are not symmetric with respect to σ_p and τ_p. The calculations will be shorter, if we use this symmetry. We can put (83) into the form

$$\begin{aligned} \Delta(\sigma_p + \tau_p) &= \widetilde{K_p}(\sigma_p + \tau_p) + \widetilde{L_p}(\sigma_p - \tau_p) \\ \Delta(\sigma_p - \tau_p) &= \widetilde{M_p}(\sigma_p - \tau_p) + \widetilde{N_p}(\sigma_p + \tau_p)\,. \end{aligned} \tag{89}$$

We define

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}. \tag{90}$$

It follows $\mathbf{A}^2 = 4\,\mathbf{I}$ with the identity matrix $\mathbf{I}$ and we get

$$\begin{aligned} 2\,(K_p, L_p, M_p, N_p)^t &= \mathbf{A}\,(\widetilde{K_p}, \widetilde{L_p}, \widetilde{M_p}, \widetilde{N_p})^t \\ 2\,(\widetilde{K_p}, \widetilde{L_p}, \widetilde{M_p}, \widetilde{N_p})^t &= \mathbf{A}\,(K_p, L_p, M_p, N_p)^t \end{aligned} \tag{91}$$

where $(\cdot)^t$ denotes the transposition. We remind (7) and (8):

$$\begin{aligned} \Delta(\sigma_p + \tau_p) = \Delta(\sigma_1 + \tau_1) \wedge \hat{\Lambda}(\sigma_{p-1} + \tau_{p-1}) \\ - L(\sigma_1 + \tau_1, \sigma_1 + \tau_1) \wedge \hat{\Lambda}(\sigma_{p-2} + \tau_{p-2}) \\ \Delta(\sigma_p - \tau_p) = \Delta(\sigma_1 - \tau_1) \wedge \hat{\Lambda}(\sigma_{p-1} - \tau_{p-1}) \\ - L(\sigma_1 - \tau_1, \sigma_1 - \tau_1) \wedge \hat{\Lambda}(\sigma_{p-2} - \tau_{p-2}) . \end{aligned} \tag{92}$$

If we apply (89) for $p = 1$ and $p = 2$, it follows

$$\begin{aligned} L(\sigma_1 + \tau_1, \sigma_1 + \tau_1) &= (-\widetilde{K_2} + 2\widetilde{K_1} + \widetilde{L_1})(\sigma_2 + \tau_2) + (\widetilde{L_1} - \widetilde{L_2})(\sigma_2 - \tau_2) \\ L(\sigma_1 - \tau_1, \sigma_1 - \tau_1) &= (-\widetilde{M_2} + 2\widetilde{M_1} + \widetilde{N_1})(\sigma_2 - \tau_2) + (\widetilde{N_1} - \widetilde{N_2})(\sigma_2 + \tau_2) . \end{aligned} \tag{93}$$

Lemma 4.4. *We state*

$$\begin{aligned} (\sigma_1 - \tau_1) \wedge \hat{\Lambda}(\sigma_{p-1} + \tau_{p-1}) &= (p-1)(\sigma_p + \tau_p) + (\sigma_p - \tau_p) \\ (\sigma_2 - \tau_2) \wedge \hat{\Lambda}(\sigma_{p-2} + \tau_{p-2}) &= \tfrac{(p-1)(p-2)}{2}(\sigma_p + \tau_p) + (p-1)(\sigma_p - \tau_p) \\ (\sigma_2 + \tau_2) \wedge \hat{\Lambda}(\sigma_{p-2} - \tau_{p-2}) &= (p-1)(\sigma_p + \tau_p) + \tfrac{(p-1)(p-2)}{2}(\sigma_p - \tau_p) . \end{aligned} \tag{94}$$

Proof of Proposition 4.2. Summarizing the formulas above, we get

$$\begin{aligned} \begin{pmatrix} \widetilde{K_p} \\ \widetilde{L_p} \\ \widetilde{M_p} \\ \widetilde{N_p} \end{pmatrix} &= \begin{pmatrix} p(2-p) & (2-p)(p-1) & 0 & 0 \\ 0 & 2-p & 0 & 0 \\ 0 & 0 & p(2-p) & (2-p)(p-1) \\ 0 & 0 & 0 & 2-p \end{pmatrix} \begin{pmatrix} \widetilde{K_1} \\ \widetilde{L_1} \\ \widetilde{M_1} \\ \widetilde{N_1} \end{pmatrix} \\ &+ \begin{pmatrix} \frac{(p-1)p}{2} & \frac{(p-1)(p-2)}{2} & 0 & 0 \\ 0 & p-1 & 0 & 0 \\ 0 & 0 & \frac{(p-1)p}{2} & \frac{(p-1)(p-2)}{2} \\ 0 & 0 & 0 & p-1 \end{pmatrix} \begin{pmatrix} \widetilde{K_2} \\ \widetilde{L_2} \\ \widetilde{M_2} \\ \widetilde{N_2} \end{pmatrix} . \end{aligned} \tag{95}$$

If we multiply (95) by $\mathbf{A}$ from the left, we get the assertion ∎

5. Real and complex hyperbolic spaces

The real hyperbolic case. We will get the results for the real hyperbolic spaces, if we suppose

$$\begin{aligned} a_0 &= n+1 \\ a_1 &= 0 \end{aligned} \tag{96}$$

and take the appropriate cases in the discussion of Section 3.

Because of the fact that these results coincide with the known systems mentioned in Section 1, we will not show that they indeed satisfy the original system (26). Of course, one can verify this directly.

In order to determine A_p and D_p, we use (62) and (63) (Case 2 of the discussion in Section 3). We get

$$\begin{aligned} A_p &= 0 \\ D_p &= 0\,. \end{aligned} \tag{97}$$

For $a_1 = 0$ the formulas (73) and (78) imply $E_{p+1} = -1 + E_p$. Using $(24)_3$ and $(52)_3$ and thereby $E_0 = 0$, we get

$$E_p = -p\,. \tag{98}$$

By (74) and (79) we get

$$B_p = -n + p\,. \tag{99}$$

From (57) we get

$$C_p F_p = p(n-p)\,. \tag{100}$$

The additional assumption (82) gives

$$\begin{aligned} F_p &= -n+p \\ C_p &= -p\,. \end{aligned} \tag{101}$$

The equations (52) imply

$$\begin{aligned} \gamma &= \tanh r \\ R_p &= (p-n)\coth r \\ S_p &= -p\coth r \\ F &= (n+1)\coth r\,. \end{aligned} \tag{102}$$

From (55), we get

$$g_p = p(n-p+1)\,. \tag{103}$$

It follows

$$\begin{aligned} F - 2\tfrac{\alpha'}{\alpha} &= (n-1)\coth r \\ -\left(g_p + 2\tfrac{\alpha'}{\alpha}S_p\right) &= p(n-p-1) - 2p\frac{1}{\sinh^2 r} \\ -2C_p\tfrac{\alpha'}{\alpha^2} &= 2(n-p)\frac{\cosh r}{\sinh^2 r} \\ -\left(g_{p+1} + 2\tfrac{\alpha'}{\alpha}R_p\right) &= (p-1)(n-p) - 2(n-p)\frac{1}{\sinh^2} \\ -2F_p\tfrac{\alpha'}{\alpha^2} &= 2p\frac{\cosh r}{\sinh^2 r}\,. \end{aligned} \tag{104}$$

Thereby (26) agrees with (11). Further on, straight-forward calculations show that the functions K_p, L_p, M_p and N_p defined by (84) are satisfying the equations (85).

The complex hyperbolic case. We will get the results for the complex hyperbolic spaces, if we suppose

$$\begin{aligned} a_0 &= n+1 \\ a_1 &= 1 \end{aligned} \tag{105}$$

and take the appropriate cases in the discussion of Section 3.

Belger [2] has considered the transport form in complex hyperbolic spaces (using self-conjugate complex coordinates in Fubini and Kähler spaces). He has proved that

$$\sigma_1^{(0,1)} = -d'r\hat{d}''r \tag{106}$$

$$\tau_1^{(0,1)} = \frac{2}{\cosh r} d'r\hat{d}''r \tag{107}$$

are the normal and tangential component of the (0,1) transport form. Thereby the decomposition $d = d' + d''$ with respect to complex and conjugate complex coordinates is used (cf. [32]). We can write the transport form as a sum of a (0,1) and a (1,0) transport form (cf. [2] for more details). With respect to the terminology used above, Belger has proved (cf. [2, 25])

$$\begin{aligned} \Delta\sigma_p = {} & \left(p(n+4-2p) - (n-p)p\cosh^2 r + p^2\frac{1}{\cosh^2 r}\right)\frac{1}{\sinh^2 r}\sigma_p \\ & + 2(2p-n)\frac{\cosh r}{\sinh^2 r}\tau_p \\ \Delta\tau_p = {} & -4p\frac{\cosh r}{\sinh^2 r}\sigma_p + \Bigg((p+1)(n-2p-2) \\ & + (-pn-2p+p^2+1+n)\cosh^2 r + \frac{(p+1)^2}{\cosh^2 r}\Bigg)\frac{1}{\sinh^2 r}\tau_p\,. \end{aligned} \tag{108}$$

We want to see that we get

$$\begin{aligned} K_p &= \left(p(n+4-2p) - (n-p)p\cosh^2 r + p^2\frac{1}{\cosh^2 r}\right)\frac{1}{\sinh^2 r} \\ L_p &= 2(2p-n)\frac{\cosh r}{\sinh^2 r} \\ M_p &= \left((p+1)(n-2p-2) + (-pn-2p+p^2+1+n)\cosh^2 r + \frac{(p+1)^2}{\cosh^2 r}\right) \\ N_p &= -4p\frac{\cosh r}{\sinh^2 r} \end{aligned} \tag{109}$$

as a special case of Section 3. In order to determine A_p and D_p, we use (60) (Case 1 of the discussion in Section 3). We get

$$\begin{aligned} A_p &= -(p+1) \\ D_p &= p\,. \end{aligned} \tag{110}$$

Using (67) (Subcase 1.1), it follows

$$\begin{aligned} E_p &= -2p \\ B_p &= -n + 2p\,. \end{aligned} \tag{111}$$

From (57) we get

$$C_p F_p = 2p(n - 2p)\,. \tag{112}$$

The additional assumption (82) gives

$$\begin{aligned} F_p &= -n + 2p \\ C_p &= -2p\,. \end{aligned} \tag{113}$$

The equations (52) imply

$$\begin{aligned} \gamma &= \tanh r \\ R_p &= -(p+1)\tanh r + (2p - n)\coth r \\ S_p &= p\tanh r - 2p\coth r \\ F &= (n+1)\coth r + \tanh r\,. \end{aligned} \tag{114}$$

From (55), we get

$$g_p = p(n+2) - p^2 \tanh^2 r\,. \tag{115}$$

It follows

$$\begin{aligned} F - 2\frac{\alpha'}{\alpha} &= (n-1)\coth r + \tanh r \\ -\left(g_p + 2\frac{\alpha'}{\alpha} S_p\right) &= \left(p(n + 4 - 2p) - (n-p)p\cosh^2 r + p^2 \frac{1}{\cosh^2 r}\right)\frac{1}{\sinh^2 r} \\ -2C_p \frac{\alpha'}{\alpha^2} &= 2(2p - n)\frac{\cosh r}{\sinh^2 r} \\ -\left(g_{p+1} + 2\frac{\alpha'}{\alpha} R_p\right) &= \Big((p+1)(n - 2p - 2) \\ &\quad + (-pn - 2p + p^2 + 1 + n)\cosh^2 r + \frac{(p+1)^2}{\cosh^2 r}\Big) \\ -2F_p \frac{\alpha'}{\alpha^2} &= -4p\frac{\cosh r}{\sinh^2 r}. \end{aligned} \tag{116}$$

Using (83), (109) gives (108). Further on, straight-forward calculations show that the functions K_p, L_p, M_p and N_p defined by (85) are satisfying the equations (85).

6. The structure of the solution of the considered harmonic systems

We want to determine the structure of the solutions of (10)

$$\Delta\big(u(r)\sigma_p + v(r)\tau_p\big) = 0$$

or, by another point of view (cf. (26)),

$$\begin{aligned} u_p'' &+ \left(F - 2\tfrac{\alpha'}{\alpha}\right) u_p' - \left(g_p + 2\tfrac{\alpha'}{\alpha} S_p\right) u_p - 2C_p \tfrac{\alpha'}{\alpha^2} v_p = 0 \\ v_p'' &+ \left(F - 2\tfrac{\alpha'}{\alpha}\right) v_p' - \left(g_{p+1} + 2\tfrac{\alpha'}{\alpha} R_p\right) v_p - 2F_p \tfrac{\alpha'}{\alpha^2} u_p = 0. \end{aligned}$$

The functions F, S_p, R_p, C_p and F_p are determined by the demand that the transformation properties I and II are satisfied. In Section 3 we have studied a special solution using the Verhulst method. In Part II we will discuss hypotheses about a related classification of harmonic spaces. For the cases determined by the Verhulst method we will discuss recursion formulas and explicit formulas for the cases which are of special interest for physics.

Applying the Verhulst method, as a consequence of (55) and (57) we have

$$-g_p = (2D_p + E_p - B_p D_p - A_p E_p - C_p F_p) - \gamma^2 D_p(A_p + 1). \tag{117}$$

Using $(52)_4$, we see that the equations (34) have the form

$$m'' + \left(\frac{a_0}{\gamma} + a_1\gamma\right) m' + \left(b_0 + b_1\gamma^2\right) m = 0 \tag{118}$$

with $m = m_p$ or $m = l_p$.

Proposition 6.1. *The solutions of* (118) *with* $\gamma = \frac{\alpha}{\beta}$, $\alpha' = \beta$, $\beta' = \alpha$ *are given by*

$$m(r) = (\alpha(r))^{1-a_0} (\beta(r))^{1-a_1}\, \widetilde{m}\left(\beta^2(r)\right) \tag{119}$$

where $m(R)$ *is a solution of a hypergeometric equation.*

Proof. We put $m = \alpha^c\beta^d\widetilde{m}(R)$ with $R = \beta^2$. The derivations shall be defined in the sense $m'(r) = \frac{dm}{dr}$ and $\widetilde{m}'(R) = \frac{d\widetilde{m}}{dR}$ where R is a function of r. Straight-forward calculations give

$$R(1-R)\widetilde{m}'' + \widetilde{m}'(R)\left(\frac{a_1 - a_0}{2}R + \frac{a_0 - 3}{2}\right) + \frac{-4 + 2a_0 + 2a_1 - b_0}{4}\widetilde{m}(R) = 0.$$

This proves the assertion ∎

There are solutions u_p, v_p of (21) with $l_p = 0$ and $m_p = 0$. In this cases the system (21) has the form (25) and the solutions are described by Proposition 6.1. The hypergeometric equations used have two linearly independent solutions. The related solutions of (25) we will denote by g_p^i, m_p^i $(i = 1, 2)$.

Proposition 6.2. *The system* (21) *has* 4 *linearly independent solutions. Two of them are given by* g_p^i, m_p^i $(i = 1, 2)$. *The other two are described by*

$$\begin{aligned} c_p^m\, u_p^1 &= m_p^1 \int 2\beta\, m_p^2\, f\, l_p^1\, dr - m_p^2 \int 2\beta\, m_p^1\, f l_p^1\, dr \\ c_p^l\, v_p^1 &= l_p^1 \int 2\beta\, l_p^2\, f\, m_p^1\, dr - l_p^2 \int 2\beta\, l_p^1\, f\, m_p^1 dr \end{aligned} \tag{120}$$

and

$$\begin{aligned} c_p^m\, u_p^2 &= m_p^1 \int 2\beta\, m_p^2\, f\, l_p^2\, dr - m_p^2 \int 2\beta\, m_p^1\, f l_p^2\, dr \\ c_p^l\, v_p^2 &= l_p^1 \int 2\beta\, l_p^2\, f\, m_p^2\, dr - l_p^2 \int 2\beta\, l_p^1\, f\, m_p^2 dr \end{aligned} \tag{121}$$

with

$$f = \ln(\alpha^{a_0}\, \beta^{a_1}) \tag{122}$$

and constants c_m, c_l *defined by the Wronskian determinant:*

$$\begin{aligned} c_p^m &= f\left((m_p^1)'\, m_p^2 - (m_p^2)'\, m_p^1\right) \\ c_p^l &= f\left((l_p^1)'\, l_p^2 - (l_p^2)'\, l_p^1\right). \end{aligned} \tag{123}$$

Proof. We can write $F = \frac{f'}{f} = (\ln f)'$ with the definition (122). From (25) we get

$$\begin{aligned} (m_p^1)''m_p^2 + \frac{f'}{f}(m_p^1)'m_p^2 + g_p m_p^1 m_p^2 &= 0 \\ (m_p^2)''m_p^1 + \frac{f'}{f}(m_p^2)'m_p^1 + g_p m_p^2 m_p^1 &= 0 \end{aligned}$$

and thereby

$$\left((m_p^1)'m_p^2 - (m_p^2)'m_p^1\right)' + \frac{f'}{f}\left((m_p^1)'m_p^2 - (m_p^2)'m_p^1\right) = 0\,. \tag{124}$$

It follows $c_p^m = f\left((m_p^1)'\, m_p^2 - (m_p^2)'\, m_p^1\right)$ with a constant c_p^m. This constant is different from zero as a consequence of the linear independence of m_p^1 and m_p^2. The assertion follows from straigt-forward calculations ∎

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