On Affine Connections
whose Holonomy is a Tensor Representation

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Dedicated to the memory of Professor Paul Günther

Abstract. In 1955, Berger [4] gave a list of irreducible reductive representations which can occur as the holonomy of a torsion-free affine connection. While this list was stated to be complete in the case of metric connections, the situation in the general case remained unclear. The (non-metric) representations which are missing from this list are called exotic. In recent years, it has been determined that exotic holonomies do exist. Thus, Berger's classification is yet to be completed in the non-metric case. In this paper, we investigate certain holonomy representations of reductive Lie groups whose semi-simple part is not simple.

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1. Introduction

An affine connection is one of the basic objects of interest in differential geometry. It provides a simple and invariant way of transferring information from one point of a connected manifold M to another and, not surprisingly, enjoys lots of applications in many branches of mathematics, physics and mechanics. Among the most informative characteristics of an affine connection is its (restricted) holonomy group which is defined, up to conjugacy, as the subgroup of $Gl(T_tM)$ consisting of all automorphisms of the tangent space T_tM at $t \in M$ induced by parallel translations along t-based loops in M.

Which reductive Lie groups G can be irreducibly acting holonomies of affine connections?

By a result of Hano and Ozeki [13], any (closed) Lie group representation $G \subseteq Gl(V)$ can be realized in this way. The same question, if posed in the subclass of *torsion-free* affine connections, has a very different answer. Long ago, Berger [4] presented a very restricted list of possible holonomies of a torsion-free affine connection which, as he suggested, is complete up to a finite number of missing terms. His list is separated into two parts. The first part corresponds to the holonomies of metric connections, the second part to the non-metric ones. While Berger gave detailed arguments for the proof of the metric part, the proof of the second part was omitted.

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The list of metric connections has been studied extensively in the intervening years. Most of the entries of this list are by now known to occur as holonomies of torsion-free connections. Especially for Riemannian manifolds, the possible holonomies and their relation to both the geometry and the topology of the underlying manifold have been the subject of tremendous research efforts during the past decades (see [1, 6, 7, 14, 19] as well as the surveys in [5, 18]).

On the other hand, the classification of non-metric holonomies is far from being complete. In fact, it turns out that Berger's classification in this case is incorrect. Examples of exotic holonomies, i.e. holonomies which are missing from Berger's list, were first found in [8]. Further exotic holonomies were discovered in [10, 11]; the former reference even establishes an infinite family of exotic holonomies.

These results necessitate a more thorough investigation of the possible candidates for exotic holonomies. In this article, we restrict our attention to representations of reductive Lie algebras whose semi-simple part is not simple. This subclass already embraces many interesting geometries. In the metric case, conformal 4-manifolds and quaternionic Kähler manifolds are included as well as the Grassmannians. In the non-metric case, this class contains the paraconformal geometries, for example connections with holonomy $Sp(1)Gl(n, \mathbb{H})$, $Sl(2, \mathbb{R})GL(n, \mathbb{R})$ or $Sl(2, \mathbb{C})GL(n, \mathbb{C})$, which have been studied by many authors (see the fundamental papers [3, 17], the books [5, 18], and the references cited therein). Finally, the exotic holonomies discovered in [10, 11] are both of this type.

All these representations have in common that they (or at least their complexifications) are tensor products of irreducible representations, in the sense that there are Lie algebras $\mathfrak{g}_i$ which act irreducibly on V_i $(i = 1, 2)$ such that the holonomy representation is equivalent to the induced representation of the Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on $V = V_1 \otimes V_2$ via the tensor representation. Moreover, in all of these representations, $\dim V_1 = 2$.

In this paper, we investigate the irreducible tensor representations which can occur as holonomies under the additional condition that $\dim V_i \geq 3$. It turns out that, in this case, there are no exotic examples; in fact, the only possible irreducible representations are the "generic" and the symmetric ones. More precisely, our main result is the following theorem (cf. Proposition 3.1, Theorem 3.3 and Corollary 3.4).

Main Theorem. *Let V_i $(i = 1, 2)$ be two finite-dimensional vector spaces over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, with $\dim V_i \geq 3$. Let $V = V_1 \otimes V_2$, and let $\mathfrak{g} \subseteq \mathbb{F} \oplus \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \subseteq \mathfrak{gl}(V)$ be an irreducibly acting Lie subalgebra. Suppose that $\mathfrak{g}$ occurs as the holonomy algebra of a torsion-free affine connection on a manifold M. Then one of the following must hold.*

1. $\mathfrak{g} = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$.

2. $\mathfrak{g} = \mathfrak{sl}(\mathbb{H}^n) \oplus \mathfrak{sl}(\mathbb{H}^m)$.

3. $\mathfrak{g} = \mathbb{F} \oplus \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$ *and the associated G-structure is locally flat.*

4. $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{sl}(\mathbb{H}^n) \oplus sl(\mathbb{H}^m)$ *and the associated G-structure is locally flat.*

5. *The connection is locally equivalent to the symmetric connection on the Grassmannian*

$$SO(p, q)/(SO(p_1, q_1)SO(p_2, q_2))$$

with $p = p_1 + p_2$ and $q = q_1 + q_2$.

6. *The connection is locally equivalent to the symmetric connection on the Grassmannian*

$$Sp(n + m)/\big(Sp(n)Sp(m)\big).$$

In Section 2, we introduce the notion of G-structures on an n-dimensional (real or complex) manifold, where $G \subseteq Gl(n, \mathbb{F})$ is a Lie subgroup with corresponding Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{F})$. We give a brief account of the Spencer complex of $\mathfrak{g}$ and use it to introduce the notions of intrinsic torsion, intrinsic curvature, and local flatness (for a more detailed exposition we refer the interested reader to [9, 12, 13]). In Section 3, we calculate the Spencer cohomologies for the tensor representation, and then prove the results that lead to the Main Theorem.

2. G-structures and intrinsic torsion

Let M^n be a (real or complex) manifold of dimension n. Let $\pi : \mathcal{F} \to M$ be the *coframe bundle* of M, i.e. each $u \in \mathcal{F}$ is a linear isomorphism $u : T_{\pi(u)}M \xrightarrow{\sim} V$, where V is a fixed n-dimensional (real or complex) vector space. Then $\mathcal{F}$ is naturally a principal right $Gl(V)$-bundle over M, where the right action $R_g : \mathcal{F} \to \mathcal{F}$ is defined by $R_g(u) = g^{-1} \circ u$. The *tautological 1-form* θ on $\mathcal{F}$ with values in V is defined by letting $\theta(\xi) = u(\pi_*(\xi))$ for $\xi \in T_u\mathcal{F}$. For θ, we have the $Gl(V)$-equivariance $R_g^*(\theta) = g^{-1}\theta$.

Let $G \subseteq Gl(V)$ be a closed Lie subgroup and let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be the Lie algebra of G. A *G-structure* on M is, by definition, a G-subbundle $F \subseteq \mathcal{F}$. For any G-structure, we will denote the restrictions of π and θ to F by the same letters. Given $A \in \mathfrak{g}$ we define the vector field A_* on F by

$$(A_*)_u = \frac{d}{dt}\big(R_{exp(tA)}(u)\big)\big|_{t=0},$$

which is called a *fundamental vertical vector field on F*. It is evident that $\pi_*(A_*) = 0$ and thus $\theta(A_*) = 0$ for all $A \in \mathfrak{g}$; in fact, $\{A_* | A \in \mathfrak{g}\} = \ker(\pi_*)$. Moreover, for $A, B \in \mathfrak{g}$ it is well-known that $[A_*, B_*] = [A, B]_*$.

Let $x \in M$ and $u \in \pi^{-1}(x)$. The Lie algebra $\mathfrak{g}_x = u^{-1}\mathfrak{g}_u \subseteq \mathfrak{gl}(T_xM)$ is independent of the choice of u, and the union $\mathfrak{g}_F = \bigcup_x \mathfrak{g}_x$ is a vector subbundle of $T^*M \otimes TM$.

Given a Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, we define the *$k$-th prolongation of $\mathfrak{g}$* by the formula

$$\mathfrak{g}^{(k)} = \big(\mathfrak{g} \otimes S^k(V^*)\big) \cap \big(V \otimes S^{k+1}(V^*)\big)$$

where both spaces are regarded as subspaces of $V \otimes V^* \otimes S^k(V^*)$. The *Spencer complex* $\mathfrak{g}(C^{*,*}(\mathfrak{g}), \delta)$ is then defined by setting

$$C^{p,q}(\mathfrak{g}) = \mathfrak{g}^{(p-1)} \otimes \Lambda^q V^* \subseteq V \otimes S^p(V^*) \otimes \Lambda^q(V^*)$$

and $\delta^{p,q} : C^{p,q} \to C^{p-1,q+1}$ is given by the restriction of the composition

$$V \otimes S^pV^* \otimes \Lambda^q V^* \xrightarrow{id \otimes d \otimes id} V \otimes S^{p-1}V^* \otimes V \otimes \Lambda^q V^*$$
$$\longrightarrow V \otimes S^{p-1}V^* \otimes \Lambda^{q+1}V^*.$$

Here $d : S^k(V^*) \rightarrow S^{k-1}V^* \otimes V^*$ refers to the exterior derivative of a polynomial map of degree k. Note that $d \circ \iota : S^k V^* \rightarrow S^k V^*$ with the symmetrization map $\iota : S^{k-1}V^* \otimes V^* \rightarrow S^k V^*$ is a multiple of the identity. We leave it to the reader to verify that $\delta^{p,q}(C^{p,q}) \subseteq C^{p-1,q+1}$ and that $\delta^{p-1,q+1} \circ \delta^{p,q} = 0$ for all p and q. As usual, we denote the cycle groups and the cohomology groups of the Spencer complex by $\mathcal{Z}^{p,q}(\mathfrak{g})$ and $H^{p,q}(\mathfrak{g})$, respectively.

If we consider the Spencer complex of $\mathfrak{g}_x \subseteq T_x^* M \otimes T_x M$, then we get a complex of vector bundles over M, and in particular, we obtain the vector bundles $\mathcal{Z}_F^{p,q} = \cup_{x \in M} \mathcal{Z}^{p,q}(\mathfrak{g}_x)$ and $H_F^{p,q} = \cup_{x \in M} H^{p,q}(\mathfrak{g}_x)$, respectively. We denote the natural projection map by $pr : \mathcal{Z}_F^{p,q} \rightarrow H_F^{p,q}$.

We shall continue to denote points in M by x and points in F by u. Moreover, ξ and ξ' denote tangent vectors on F and we let $\underline{\xi}_u = \pi_*(\xi_u)$, $\underline{\xi}'_u = \pi_*(\xi'_u)$ etc.

A *connection* on F is a $\mathfrak{g}$-valued 1-form ω on F satisfying the conditions

$$\omega(A_*) = A \qquad \text{for all} \quad A \in \mathfrak{g}$$
$$R_g^*(\omega) = g^{-1}\omega g \qquad \text{for all} \quad g \in G. \tag{1}$$

Given a connection ω, its *torsion* Θ is the V-valued 2-form given by

$$\Theta = d\theta + \omega \wedge \theta. \tag{2}$$

From (1) and (2) it follows that there is a section Tor of $\Lambda^2 T^* M \otimes TM$ satisfying

$$\Theta(\xi_u, \xi'_u) = u\big(Tor(\underline{\xi}_u, \underline{\xi}'_u)\big) \qquad \text{for all} \quad \xi_u, \xi'_u \in T_u F \text{ and } u \in F. \tag{3}$$

The connection ω is called *torsion free* if $\Theta = 0$. Note that Tor is a section of $\mathcal{Z}_F^{0,2} = \Lambda^2 T^* M \otimes TM$ and thus induces a section $\tau = pr(Tor)$ of $H_F^{0,2}$.

Now let ω' be another connection on F, and let Θ' and Tor' represent its torsion. From (1) it follows that there is a section α of the bundle $T^* M \otimes \mathfrak{g}_F$ such that

$$(\omega' - \omega)(\xi_u) = u \, \alpha(\underline{\xi}_u) u^{-1}. \tag{4}$$

From (2) - (4) we obtain for the torsion

$$(\Theta' - \Theta)(\xi_u, \xi'_u) = u \big(\alpha(\underline{\xi}_u) \cdot \underline{\xi}'_u - \alpha(\underline{\xi}'_u) \cdot \underline{\xi}_u \big)$$

and hence

$$(Tor' - Tor)(\underline{\xi}, \underline{\xi}') = \alpha(\underline{\xi}) \cdot \underline{\xi}' - \alpha(\underline{\xi}') \cdot \underline{\xi} = \delta^{1,1}(\alpha)(\underline{\xi}, \underline{\xi}')$$

for all $\underline{\xi}, \underline{\xi}' \in T_x M$. Thus, we conclude that

$$Tor' = Tor + \delta^{1,1}(\alpha). \tag{5}$$

This implies that the section τ of $H_F^{0,2}$ defined above is independent of the choice of ω and therefore only depends on the G-structure F.

Definition 2.1. Let $\pi : F \to M$ be a G-structure.

1. The vector bundle $H_F^{0,2}$ is also called *intrinsic torsion bundle* of F.

2. The section τ of $H_F^{0,2}$ is called *intrinsic torsion* of F.

3. F is called *torsion free* or *1-flat* if its intrinsic torsion τ vanishes.

The following proposition is then immediate from (5).

Proposition 2.2. *Let* $\pi : F \to M$ *be a G-structure and let τ be its intrinsic torsion.*

1. *If σ is any section of $\Lambda^2 T^* M \otimes TM$ such that $pr(\sigma) = \tau$, then there is a connection on F whose torsion section Tor equals σ.*

2. *There is a torsion-free connection on F if and only if F is torsion free.*

3. *If F is torsion-free, then there is a one-to-one correspondence between torsion-free connections on F and sections of $\mathfrak{g}_F^{(1)}$. In particular, if $\mathfrak{g}^{(1)} = 0$, then the torsion-free connection on F is unique.*

Example 2.3. **1.** Let $G = O(p, q) \subseteq Gl(V)$ with $V = \mathbb{R}^n$ and $n = p + q$. A G-structure on M^n is equivalent to a pseudo-Riemannian metric on M of signature (p, q). One can show that $\delta^{1,1} : V^* \otimes \mathfrak{o}(p, q) \to \Lambda^2 V^* \otimes V$ is an isomorphism. Thus, $\mathfrak{g}^{(1)} = 0$ and $H^{0,2}(\mathfrak{g}) = 0$. Then Proposition 1.2 implies that there is a unique torsion-free connection on such a G-structure. Of course, this reproves precisely the existence and uniqueness of the Levi-Civita connection of a (pseudo-)Riemannian metric (see [15]).

2. Suppose $n = 2m$ and let $G = Gl(m, \mathbb{C}) \subseteq Gl(n, \mathbb{R})$. A G-structure on M^n is equivalent to an almost complex structure on M. A calculation shows that $H^{0,2}(\mathfrak{gl}(m, \mathbb{C})) = \{\phi \in \Lambda^2(\mathbb{C}^n)^* \otimes_{\mathbb{R}} \mathbb{C}^n \mid \phi(ix, y) = -i\phi(x, y)\}$. Moreover, the intrinsic torsion is given by the Nijenhuis tensor. It is well known that the vanishing of this tensor, i.e. the torsion freeness of the G-structure, is equivalent to the integrability of the almost complex structure (see [15]).

3. Suppose $n = 2m$ and let $G = Sp(m) \subseteq Gl(n, \mathbb{R})$. A G-structure on M^n is equivalent to an almost symplectic structure, i.e.. a 2-form ω on M satisfying $\omega^m \neq 0$. One can show that $H^{0,2}(\mathfrak{sp}(m)) = \Lambda^3 \mathbb{R}^n$ and that the intrinsic torsion is represented by the 3-form $d\omega$. Thus, the G-structure is torsion free if and only if the almost symplectic 2-form ω is symplectic.

From these examples it should become evident that for many naturally arising G-structures the vanishing of the intrinsic torsion implies, in some sense, the "most natural integrability condition" of the underlying geometric structure.

Suppose now that F is torsion free and let ω be a torsion free connection on F, i.e.

$$d\theta + \omega \wedge \theta = 0.$$

Exterior differentiation yields the *first Bianchi identity*

$$\Omega \wedge \theta = 0 \tag{6}$$

where $\Omega = d\omega + \omega \wedge \omega$ is the *curvature 2-form* of ω. It follows that there is a section R of $\Lambda^2 T^*M \otimes \mathfrak{g}_F$ called the *curvature* of ω which satisfies

$$\Omega(\underline{\xi}_u, \underline{\xi}'_u) = uR(\underline{\xi}_u, \underline{\xi}'_u)u^{-1}$$

and (6) implies $\delta^{1,2}(R) = 0$. Therefore, R is a section of $\mathcal{Z}_F^{1,2}$ and induces a section $\rho = pr(R)$ of $H_F^{1,2}$.

Now let ω' be another torsion free connection on F, i.e. ω and ω' are related by (4) with a section α of $T^*M \otimes \mathfrak{g}_F$ satisfying $\delta^{1,1}(\alpha) = 0$. If we denote the curvature 2-form of ω' by Ω' and the curvature by R', then a calculation shows that

$$(\Omega - \Omega')(\xi_u, \xi'_u) = u\big((d\alpha + \alpha \wedge \alpha)(\underline{\xi}_u, \underline{\xi}'_u)\big)u^{-1}$$

for all $\xi_u, \xi'_u \in T_u F$ and hence

$$R' = R + d\alpha + \alpha \wedge \alpha.$$

It is now straightforward to verify that the map

$$\phi : TM \longrightarrow \mathfrak{g}^{(1)}, \qquad X \longmapsto \nabla_X \alpha + \alpha(X)\alpha \tag{7}$$

is well defined and satisfies

$$\delta^{2,1}(\phi) = d\alpha + \alpha \wedge \alpha \tag{8}$$

and thus the section $\rho = pr(R)$ of $H_F^{1,2}$ is independent of the choice of the torsion-free connection.

Definition 2.4. Let F be a torsion-free G-structure on M.

1. The section ρ of $H^{1,2}$ defined above is called the *intrinsic curvature* of F. Moreover, if $\rho \equiv 0$, then F is called *2-flat*.

2. F is called *locally flat* if there exists a torsion-free connection on F whose curvature vanishes.

Evidently, local flatness implies 2-flatness. The converse is not true in general, as the following proposition illustrates.

Proposition 2.5. *Let F be a torsion-free G-structure on M and let ρ be its intrinsic curvature. Let R be a section of $\mathcal{Z}_F^{1,2}$ such that $\rho = pr(R)$. Then for each $x \in M$ there is a torsion-free connection on M whose curvature at x equals R_x. In particular, if F is 2-flat, then for each $x \in M$ there is a torsion free connection on F whose curvature at x vanishes.*

Proof. It is easy to see that we can choose a section α of $T^*M \otimes \mathfrak{g}_F^{(1)}$ such that $\alpha_x = 0$ and $(\nabla_X \alpha)_x = -\phi_x(X)$ for all $X \in T_x^*M$. Then the statement follows from (7) and (8) ∎

3. The Spencer complex for tensor representations

We begin our discussion with the following

Proposition 3.1. *Let V_i $(i = 1, 2)$ be vector spaces over $\mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ of dimension at least 3, and let $V = V_1 \otimes V_2$. Let $\mathfrak{g}$ be the image of the natural tensor representation $\rho : \mathbb{F} \oplus \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \to \mathfrak{gl}(V)$, and let $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{sl}(V)$, i.e. $\mathfrak{h} \cong \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$. Then*

1. *$\mathfrak{g}^{(1)} \cong V^*$, with an explicit isomorphism given by $\rho \in V^* \mapsto \alpha^\rho \in \mathfrak{g}^{(1)}$ with*

$$\left. \begin{array}{c} \alpha^\rho = \alpha_1^\rho + \alpha_2^\rho \\[2mm] \alpha_1^\rho(e_1 \otimes u_1) e_2 = \rho(e_2 \otimes u_1) e_1 \\[2mm] \alpha_2^\rho(e_1 \otimes u_1) u_2 = \rho(e_1 \otimes u_2) u_1. \end{array} \right\}$$

2. *$\mathcal{Z}^{1,2}(\mathfrak{g}) \cong V^* \otimes V^*$, with an explicit isomorphism given by $\tau \in V^* \otimes V^* \mapsto \phi^\tau \in \mathcal{Z}^{1,2}$ with*

$$\left. \begin{array}{c} \phi^\tau = \phi_1^\tau + \phi_2^\tau \\[2mm] \phi_1^\tau(e_1 \otimes u_1, e_2 \otimes u_2) e_3 = \tau(e_1, u_1, e_3, u_2) e_2 - \tau(e_2, u_2, e_3, u_1) e_1 \\[2mm] \phi_2^\tau(e_1 \otimes u_1, e_2 \otimes u_2) u_3 = \tau(e_1, u_1, e_2, u_3) u_2 - \tau(e_2, u_2, e_1, u_3) u_1. \end{array} \right\}$$

3. *$H^{1,2}(\mathfrak{g}) = 0$ (indeed, every G-structure F with $\mathfrak{g}_F \cong \mathfrak{g}$ is locally flat).*

4. *$\mathfrak{h}^{(1)} = 0$.*

5. *$H^{1,2}(\mathfrak{h}) \cong \mathcal{Z}^{1,2}(\mathfrak{h}) \cong S^2 V^*$, with an explicit isomorphism given by the restriction of τ in statement 2 to $S^2 V^*$.*

Proof. We shall begin by showing statement 2, as the proof of statement 1 works analogously. We denote elements of V_1 and V_2 by e_i and u_i, respectively. Let $\phi : \Lambda^2 V \to \mathfrak{g}$ be an element of $\mathcal{Z}^{1,2}(\mathfrak{g})$. We use the decomposition $\phi = A + B$ into the $\mathfrak{gl}(V_1)$- and $\mathfrak{gl}(V_2)$-component. For given elements $e_i \in V_1$ and $u_i \in V_2$ $(i = 1, 2, 3)$ we shall abbreviate $\phi(e_i \otimes u_i, e_j \otimes u_j) = A_k + B_k$ where (i, j, k) is an even permutation of $(1, 2, 3)$, $A_k \in \mathfrak{gl}(V_1)$ and $B_k \in \mathfrak{gl}(V_2)$. Then the first Bianchi identity for ϕ reads

$$\begin{aligned} (A_1 e_1) \otimes u_1 + (A_2 e_2) \otimes u_2 + (A_3 e_3) \otimes u_3 \\ + e_1 \otimes (B_1 u_1) + e_2 \otimes (B_2 u_2) + e_3 \otimes (B_3 u_3) = 0. \end{aligned} \tag{9}$$

Let us choose the e_i's and u_i's linearly independent. Then (9) implies that $A_3 e_3 \in \text{span}(e_1, e_2, e_3)$, and hence we get

$$\begin{aligned} A(e_1 &\otimes u_1, e_2 \otimes u_2) e_3 \\ &= \lambda(e_3, e_1, u_1, e_2, u_2) e_3 + \tau(e_2, e_1, u_1, e_3, u_2) e_2 - \tau(e_1, e_2, u_2, e_3, u_1) e_1 \end{aligned}$$

for some real-valued functions λ and τ. From here it is straightforward to verify that λ and τ are independent of the first slot and linear in the remaining entries. In other words, we get

$$
\begin{aligned}
A(e_1 \otimes u_1, e_2 \otimes u_2)\, e_3 \\
= \lambda(e_1, u_1, e_2, u_2)\, e_3 + \tau(e_1, u_1, e_3, u_2)\, e_2 - \tau(e_2, u_2, e_3, u_1)\, e_1
\end{aligned}
\tag{10}
$$

for all $e_i \in V_1$ and $u_i \in V_2$, where $\lambda \in \Lambda^2 V^*$ and $\tau \in V^* \otimes V^*$. Analogously, we find that there is a $\mu \in \Lambda^2 V^*$ and a $\sigma \in V^* \otimes V^*$ such that, for all $e_i \in V_1$ and $u_i \in V_2$,

$$
\begin{aligned}
B(e_1 \otimes u_1, e_2 \otimes u_2)\, u_3 \\
= \mu(e_1, u_1, e_2, u_2)\, u_3 + \sigma(e_1, u_1, e_2, u_3)\, u_2 - \sigma(e_2, u_2, e_1, u_3)\, u_1.
\end{aligned}
\tag{11}
$$

Substituting (10) and (11) into (9), we conclude that $\lambda + \mu = 0$ and $\tau = \sigma$. Finally, note that there is a 1-dimensional kernel of the tensor representation which can be used to achieve $\lambda = \mu = 0$. Then it is evident that $\phi = \phi^\tau$ from above.

To show statement 3, one calculates that the boundary map $\delta^{1,1} : V^* \otimes \mathfrak{g}^{(1)} \to \mathcal{Z}^{1,2}(\mathfrak{g})$ is precisely the combination of the isomorphisms of both spaces with $V^* \otimes V^*$ given in assertions 1 and 2.

Finally, for statements 4 and 5, note that $\alpha^\rho \in \mathfrak{h}^{(1)}$ if and only if $\rho = 0$, and $\phi^\tau \in \mathcal{Z}^{1,2}(\mathfrak{h})$ if and only if $\tau \in S^2 V^* \subseteq V^* \otimes V^*$ ∎

Lemma 3.2. *Let W be a finite-dimensional vector space and $\mathfrak{g} \subseteq \mathfrak{gl}(W)$ an irreducibly acting Lie subalgebra. Let $W = W_1 \oplus W_2$ with $W_1, W_2 \neq 0$ be a decomposition which induces a decomposition*

$$
\mathfrak{gl}(W) = W_{11} \oplus W_{21} \oplus W_{12} \oplus W_{22}
\tag{12}
$$

with $W_{ij} = W_i^* \otimes W_j$ $(i, j = 1, 2)$. If $\mathfrak{h} \subseteq \mathfrak{gl}(W_1) \cong W_{11}$ is an irreducibly acting Lie subalgebra, and if $\mathfrak{h} \oplus W_{21} \subseteq \mathfrak{g}$, then $\mathfrak{sl}(W) \subseteq \mathfrak{g}$.

Proof. We denote the projections given by (12) by $\pi_{ij} : \mathfrak{gl}(W) \to W_{ij}$, and write for short $A_{ij} = \pi_{ij}(A)$ for any $A \in \mathfrak{gl}(W)$. From the irreducibility of $\mathfrak{g}$ it follows that there is an $A \in \mathfrak{g}$ with $A_{12} \neq 0$. Then for any $B_{11} \in \mathfrak{h}$, we calculate that $[B_{11}, A]_{12} = B_{11} \cdot A_{12}$. Thus, the image of π_{12} is $\mathfrak{h}$-invariant, and hence, using the irreducibility of both $\mathfrak{h}$ and $\mathfrak{g}$, it follows that π_{12} is surjective.

Now, let $B_{21} \in W_{21}$ and $A \in \mathfrak{g}$ be given. We then calcutate that

$$
[B_{21}, A]_{11} = A_{12} B_{21}, \qquad [B_{21}, A]_{12} = 0, \qquad [B_{21}, A]_{22} = B_{21} A_{12}.
$$

From the surjectivity of π_{12} and $W_{21} \subseteq \mathfrak{g}$ it now follows that A_{12} and B_{21} can be chosen arbitrarily, and from there we conclude that $\mathfrak{sl}(W) \subseteq \mathfrak{g}$ as claimed ∎

Theorem 3.3. *Let V_i $(i = 1, 2)$ be two vector spaces over $\mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ with $\dim V_i \geq 3$, and let $V = V_1 \otimes V_2$. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2) \subseteq \mathfrak{gl}(V)$ be an irreducibly acting Lie subalgebra. If $\mathfrak{g}$ occurs as the holonomy algebra of a torsion-free affine connection, then one of the following must hold:*

1. $\mathfrak{g} = \mathbb{F} \oplus \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$.

2. $\mathfrak{g} = \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$.

3. $\mathfrak{g} = \mathfrak{so}(g_1, V_1) \oplus \mathfrak{so}(g_2, V_2)$ where $g_i \in S^2 V_i^*$ $(i = 1, 2)$ are non-degenerate.

4. $\mathfrak{g} = \mathfrak{sp}(\omega_1, V_1) \oplus \mathfrak{sp}(\omega_2, V_2)$ where $\omega_i \in \Lambda^2 V_i^*$ $(i = 1, 2)$ are non-degenerate.

Proof. Let us write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i \subseteq \mathfrak{gl}(V_i)$ $(i = 1, 2)$. If $\mathfrak{g}$ occurs as the holonomy algebra of a torsion-free connection, then by the Ambrose-Singer Holonomy Theorem [2] and Proposition 3.1 it follows that there is a subspace $T \subseteq V^* \otimes V^*$ such that

$$\mathfrak{g} = \langle \{ im(\phi^\tau) | \ \tau \in T \} \rangle \qquad (13)$$

with $\phi^\tau : \Lambda^2 V \to \mathfrak{gl}(V)$. Obviously, $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}_i$ is for $i = 1, 2$.

Let us first discuss $\mathfrak{g}_1$. We decompose a given $\tau \in V^* \otimes V^* \cong V_1^* \otimes V_1^* \otimes V_2^* \otimes V_2^*$ as $\tau = \tau^+ + \tau^-$ with $\tau^+ \in V_1^* \otimes V_1^* \otimes S^2 V_2^*$ and $\tau^- \in V_1^* \otimes V_1^* \otimes \Lambda^2 V_2^*$. That is,

$$\tau^\pm(e_1, e_2, u_1, u_2) = \frac{1}{2} \big(\tau(e_1, u_1, e_2, u_2) \pm \tau(e_1, u_2, e_2, u_1) \big).$$

It follows that

$$\big(\phi_1^\tau(e_1 \otimes u_1, e_2 \otimes u_2) \pm \phi_1^\tau(e_1 \otimes u_2, e_2 \otimes u_1) \big) e_3$$
$$= 2 \big(\tau^\pm(e_1, e_3, u_1, u_2) e_2 \mp \tau^\pm(e_2, e_3, u_1, u_2) e_1 \big).$$

If we denote by $\mathfrak{g}_1^\pm \subseteq V_1 \otimes V_1^*$ the Lie algebra generated by all endomorphisms of V_1 given by

$$e_3 \longmapsto \tau^\pm(e_1, e_3, u_1, u_2) e_2 \mp \tau^\pm(e_2, e_3, u_1, u_2) e_1, \qquad (14)$$

with $\tau \in T$, then (13) implies that $\mathfrak{g}_1$ is generated by $\mathfrak{g}_1^+ + \mathfrak{g}_1^-$.

Let us regard $\tau^\pm$ as maps

$$\tau^+ : S^2 V_2 \to V_1^* \otimes V_1^* \qquad \text{and} \qquad \tau^- : \Lambda^2 V_2 \to V_1^* \otimes V_1^*$$

and regard an element $\alpha \in V_1^* \otimes V_1^*$ as a map $\alpha : V_1 \to V_1^*$, using contraction in the first entry. Under these conventions, it follows from (14) that

$$\mathfrak{g}_1^\pm = \Big\langle \big\{ e_1 \otimes \alpha(e_2) \mp e_2 \otimes \alpha(e_1) \big| \ e_i \in V_1 \ \text{and} \ \alpha \in im(\tau^\pm) \big\} \Big\rangle. \qquad (15)$$

Now, in order to determine $\mathfrak{g}_1^\pm$, there are several cases to be considered.

1. The case $\tau^- \notin \Lambda^2 V_1^* \otimes \Lambda^2 V_2^*$. In this case, we may choose $s_2 \in \Lambda^2 V_2$ such that $\alpha = \tau^-(s_2) \in V_1^* \otimes V_1^*$ is not skew-symmetric, i.e. the symmetric 2-tensor $\alpha^+(e_1, e_2) = \alpha(e_1, e_2) + \alpha(e_2, e_1)$ is not zero. Now, let $e_1, e_2 \in V_1$ be such that $\alpha^+(e_1, e_2) \neq 0$. Then we calculate

$$[e_1 \otimes \alpha(e_1), e_2 \otimes \alpha(e_2)]$$
$$= \alpha^+(e_1, e_2) e_1 \otimes \alpha(e_2) - \alpha(e_2, e_1)(e_1 \otimes \alpha(e_2) + e_2 \otimes \alpha(e_1))$$

and therefore, by (15), $e_1 \otimes \alpha(e_2) \in \mathfrak{g}_1^-$ for all such e_1 and e_2. From there it follows that $V_1 \otimes \alpha(V_1) \subseteq \mathfrak{g}_1^- \subseteq \mathfrak{g}_1$. Let $e_1 \in V_1$ such that $\alpha(e_1) \neq 0$, and apply Lemma 3.2 to a decomposition $V_1 = W_1 \oplus W_2$ with $W_2 = (\alpha(e_1))^\perp$. It follows that $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$.

2. The case $\tau^- \in \Lambda^2 V_1^* \otimes \Lambda^2 V_2^*$. In this case, it follows that $\mathfrak{g}_1^- \subseteq \mathfrak{sl}(V_1)$. Let $0 \neq \omega \in im(\tau^-) \subseteq \Lambda^2 V_1^*$, and write $V_1 = W_1 \oplus W_2$, where W_2 is the null space of ω. Note that $\omega|_{W_1}$ is non-degenerate. It is then easy to verify that the Lie algebra generated by all elements of the form (15) for this particular α is $\mathfrak{sp}(\omega|_{W_1}) \oplus W_{21} \subseteq \mathfrak{gl}(V_1)$ using the decomposition as in (12) and where $\mathfrak{sp}(\omega|_{W_1}) \subseteq \mathfrak{gl}(W_1)$ is the Lie algebra of endomorphisms which leave $\omega|_{W_1}$ invariant.

But then Lemma 3.2 implies that either $W_2 = 0$ or $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}$. In the first case, ω is non-degenerate and the elements of the form (15) generate $\mathfrak{sp}(\omega)$. If there is a (non-degenerate) $\omega' \in im(\tau^-)$ independent of ω, then $\mathfrak{sp}(\omega') \neq \mathfrak{sp}(\omega)$, and since both are maximal subalgebras of $\mathfrak{sl}(V_1)$, it follows that in this case again, $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$.

Finally, it is well known that there are no non-zero elements of $\mathfrak{sp}(\omega)$ of the form $e_1 \otimes \alpha(e_2) - e_2 \otimes \alpha(e_1)$ with $\alpha : V_1 \to V_1^*$. Hence, because of (15), we must have that either $\tau^+ = 0$ or $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$.

Summarizing, we have the following possibilities:

(a) $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$, or

(b) $\tau^+ = 0$, and $\tau^-(e_1, e_2, u_1, u_2) = \omega_1(e_1, e_2)\omega_2(u_1, u_2)$ for some $\omega_i \in \Lambda^2 V_i^*$, where ω_1 is non-degenerate and $\omega_2 \neq 0$. In this case, $\mathfrak{g}_1 = \mathfrak{sp}(\omega_1)$, or

(c) $\tau^- = 0$ and $\mathfrak{g}_1^- = 0$.

3. The case $\tau^- = 0$ and $\tau^+ \in S^2 V_1^* \otimes S^2 V_2^*$. In this case, it follows that $\mathfrak{g}_1 \subseteq \mathfrak{sl}(V_1)$. The investigation of this case is completely analogous to the previous one. As a result, we get the following possibilities:

(a) $\mathfrak{g}_1 = \mathfrak{sl}(V_1)$, or

(b) $\tau(e_1, e_2, u_1, u_2) = g_1(e_1, e_2)g_2(u_1, u_2)$ for some $g_i \in S^2 V_i^*$ $(i = 1, 2)$, where g_1 is non-degenerate and $g_2 \neq 0$. In this case, $\mathfrak{g}_1 = \mathfrak{so}(g_1)$.

4. The case $\tau^- = 0$ and $\tau^+ \notin S^2 V_1^* \otimes S^2 V_2^*$. In this case, we may choose $s_2 \in S^2 V_2$ such that $\alpha = \tau^+(s_2) \in V_1^* \otimes V_1^*$ is not symmetric, i.e. the 2-form $\alpha^-(e_1, e_2) = \alpha(e_1, e_2) - \alpha(e_2, e_1)$ is not zero. Without loss of generality we assume that s_2 was chosen such that α^- has maximal rank.

(a) Suppose that $rank(\alpha^-) > 1$. In this case, we can pick elements $e_2, e_3 \in V_1$ such that $\alpha^-(e_1, e_2) = \alpha^-(e_1, e_3) = 0$ and $\alpha^-(e_2, e_3) = 1$. Then we calculate

$$\left[e_1 \otimes \alpha(e_2) - e_2 \otimes \alpha(e_1), e_1 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_1) \right]$$
$$= -e_1 \otimes \alpha(e_1) - \alpha(e_3, e_1)(e_1 \otimes \alpha(e_2) - e_2 \otimes \alpha(e_1))$$
$$+ \alpha(e_1, e_2)(e_1 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_1)) - \alpha(e_1, e_1)(e_2 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_2))$$

and therefore, by (15),

$$e_1 \otimes \alpha(e_1) \in \mathfrak{g}_1^+ \tag{16}$$

for all e_1 with $\alpha^-(e_1, \_) \neq 0$. But then (15) and (16) imply that in this case $V_1 \otimes \alpha(V_1) \subseteq \mathfrak{g}_1$, and then, applying Lemma 3.2 to a decomposition $V_1 = W_1 \oplus W_2$ with $W_2 = (\alpha(e_1))^\perp$, we conclude that $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$.

   (b) Suppose that $rank(\alpha^-) = 1$ for any choice of $s_2 \in S^2 V_2$. Let $V' \subseteq V_1$ be the anihilator of α^-, let $e_2 \in V_1$ such that $\alpha^-(e_1, e_2) = 1$, and let $e_3 \in V'$. Then we calculate

$$\big[e_1 \otimes \alpha(e_2) - e_2 \otimes \alpha(e_1), e_1 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_1)\big]$$
$$= -e_1 \otimes \alpha(e_3) - \alpha(e_3, e_1)\big(e_1 \otimes \alpha(e_2) - e_2 \otimes \alpha(e_1)\big)$$
$$+ \alpha(e_1, e_2)\big(e_1 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_1)\big) - \alpha(e_1, e_1)\big(e_2 \otimes \alpha(e_3) - e_3 \otimes \alpha(e_2)\big)$$

and therefore, using (15), we conlude that

$$(V_1 \otimes \alpha(V')) \cup (V' \otimes \alpha(V_1)) \subseteq \mathfrak{g}_1^+. \tag{17}$$

   (i) $s_2 \in S^2 V_2$ can be chosen such that $\alpha(V') \neq 0$. Pick $e_1 \in V'$ such that $\alpha(e_1) \neq 0$. Then applying Lemma 3.2 to the decomposition $V_1 = W_1 \oplus W_2$ with $W_2 = (\alpha(e_1))^\perp$ and using (17), it follows that $\mathfrak{sl}(V_1) \subseteq \mathfrak{g}_1$.

   (ii) For any choice of $s_2 \in S^2(V_2)$, $\alpha(V'_{s_2}) = 0$, where V'_{s_2} is the anihilator of α^- with $\alpha = \tau(s_2) \neq 0$. Let $Gr(n-2, V_1)$ with $n = \dim V_1$ denote the space of planes of codimension 2 in V_1, and let $P \subseteq Gr(n-2, V_1)$ be the set of all subspaces $V' = V'_{s_2}$ which can occur. If $V'_{s_2} + V'_{s'_2} = V_1$, then it would follow that $rank(\tau(s_2) + \tau(s'_2))^- > 1$ which was excluded. Thus, any two $V', V'' \in P$ are not transverse. It follows easily that either

   • there is a $V_0 \subseteq V_1$ of codimension 1 such that $P \subseteq \{V' \in Gr(n-2, V_1) \mid V' \subseteq V_0\}$

or

   • there is a $V_0 \subseteq V_1$ of codimension 3 such that $P = \{V' \in Gr(n-2, V_1) \mid V_0 \subseteq V'\}$.

But in either case, V_0 would be invariant, which is impossible except if the latter holds and $\dim V_1 = 3$. However, in that case, it is easy to check that any Lie subalgebra of $\mathfrak{gl}(V_1)$ for which all 1-dimensional subspaces occur as V' must contain $\mathfrak{sl}(V_1)$.

   The Theorem now follows from combining all these cases and performing an analogous investigation for $\mathfrak{g}_2$ ∎

   Corollary 3.4. *Let $V = V_1 \otimes V_2$, and $\mathfrak{g} \subseteq \mathfrak{gl}(V_1) \oplus \mathfrak{gl}(V_2) \subseteq \mathfrak{gl}(V)$ be as in Theorem 3.3, suppose that $\mathfrak{g}$ occurs as the holonomy algebra of a torsion-free affine connection on a manifold M, but $\mathfrak{g}$ does not contain $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$. Then the connection is locally equivalent to one of the Grassmannians*

$$SO(p, q)/(SO(p_1, q_1)SO(p_2, q_2)) \quad or \quad Sp(n+m)/(Sp(n)Sp(m))$$

with $p = p_1 + p_2$ and $q = q_1 + q_2$.

   Proof. In the proof of Theorem 3.3 we gave an explicit description of the curvature for the cases where $\mathfrak{g}$ does not contain $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$. From this description it is clear that the curvature is invariant under the holonomy, and hence the connection is locally symmetric. Moreover, the curvature coincides with that of one of the symmetric spaces listed above, and this in turn suffices to conclude that M is locally equivalent to one of these spaces (see [15]) ∎

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