Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 16 (1997), No.1, 145–164

A Use of Ideal Decomposition in the Computer Algebra of Tensor Expressions

B. Fiedler

Dedicated to Professor Paul Günther

Abstract. Let I be a left ideal of a group ring $\mathbb{C}[G]$ of a finite group G, for which a decomposition $I = \bigoplus_{k=1}^{m} I_k$ into minimal left ideals I_k is given. We present an algorithm, which determines a decomposition of the left ideal $I \cdot a$, $a \in \mathbb{C}[G]$, into minimal left ideals and a corresponding set of primitive orthogonal idempotents by means of a computer. The algorithm is motivated by the computer algebra of tensor expressions. Several aspects of the connection between left ideals of the group ring $\mathbb{C}[\mathcal{S}_r]$ of a symmetric group $\mathcal{S}_r$, their decomposition and the reduction of tensor expressions are discussed.

Keywords: *Group rings, ideal decompositions, primitive orthogonal idempotents, Young symmetrizers, the regular representation of the $\mathcal{S}_r$, invariant irreducible subspaces, computer-aided tensor calculations, Ricci calculus*

AMS subject classification: Primary 20 C 05, secondary 20 C 30, 20 C 40, 53 - 04

1. Introduction

Investigations in differential geometry, tensor analysis and general relativity theory require often very extensive conversions of tensor expressions according to the rules of the Ricci calculus. There are many efforts to develop computer programs which can do such calculations by means of symbolic computation. Examples of such programs are the Mathematica packages MathTensor [4] and Ricci [13], the Maple package GRTensor [16] and the REDUCE package REDTEN [5].

A fundamental and unsolved problem of the manipulation of tensor expressions by a computer algebra system is the effective determination of a normal form for tensor expressions. Let us consider sums

$$\tau = \sum_{i=1}^{n} \alpha_i T_{(i)} \tag{1.1}$$

with real or complex coefficients α_i, where the $T_{(i)}$ are products of certain tensor coordinates such as

$$A_{iabc}\, A^{a}{}_{jkd}\, B^{bd}{}_{e}\, C^{ec}\,. \tag{1.2}$$

B. Fiedler: Universität Leipzig, Mathematisches Institut, Augustuspl. 10/11, D – 04109 Leipzig, Germany

ISSN 0232-2064 / $ 2.50

Free indices and contractions are allowed. If the tensors $A, B, C, \ldots$ possibly possess symmetries relating to permutations of indices and/or fulfil linear identities, then there is a possibility to express some of the $T_{(i)}$ in (1.1) by the others. This is a hard problem[1] for a tensor manipulating system. We need an efficient algorithm to detect such transformability and to carry out transformations in a defined way.

It is well-known that the determination of normal forms of tensor expressions is connected with the representation theory of the symmetric group $\mathcal{S}_r$. Littlewood made use of the Richardson-Littlewood rule and plethysms to find out the types of concomitants of a set of ground forms, the coefficents of which are coordinates of symmetric tensors (appendix of [14]). Applying the same methods, Fulling, King, Wybourne and Cummins [6] have calculated lists of normal form terms of polynomials of the Riemann curvature tensor and its derivatives by means of a program package Schur [23].

Stimulated by [6], we have worked out a way to reduce tensor expressions (1.1) to a sum over a subset $\{T_{(i_k)} \mid k = 1, \ldots, m\}$ of linearly independent $T_{(i)}$, appearing in (1.1), with the help of group ring methods. In this paper we restrict ourselves to expressions (1.1), in which the $T_{(i)}$ do not have any contractions[2]. Neglecting a possibly existing product structure of the $T_{(i)}$, we consider sums

$$\tau_{\alpha_1 \ldots \alpha_r} = \sum_{p \in P} \beta_p \, T_{\alpha_{p(1)} \ldots \alpha_{p(r)}} \quad , \quad P \subseteq \mathcal{S}_r \; , \; \beta_p \in \mathbb{C} \; , \tag{1.3}$$

which run over a certain permutation set $P \subseteq \mathcal{S}_r$. The tensor T can be associated with group ring elements T_b, which lie in a certain left ideal $\mathbb{C}[\mathcal{S}_r] \cdot a$ of the group ring $\mathbb{C}[\mathcal{S}_r]$, if T possesses a tensor symmetry and/or fulfils linear identities. If this ideal is known, then identities for the reduction of (1.3) can be obtained from the solutions of a linear equation system

$$\sum_{p' \in \mathcal{S}_r} a(p^{-1} \circ p') x_{p'} = 0 \quad , \quad p \in \mathcal{S}_r \; , \tag{1.4}$$

the coefficient matrix of which is derived from the generating element a of $\mathbb{C}[\mathcal{S}_r] \cdot a$.

Two constructions are important for an efficient handling of (1.4). We decompose $\mathbb{C}[\mathcal{S}_r] \cdot a$ into minimal left ideals by an algorithm, which is practicable by a computer. The decomposition allows us to change to the smaller equation systems of type (1.4), which belong to the minimal left ideals. Further, a fast construction of bases of the minimal left ideals by means of Young tableaux makes it possible for us to find quickly linearly independent equations of (1.4).

The decomposition of $\mathbb{C}[\mathcal{S}_r] \cdot a$ into minimal left ideals yields us a decomposition of the tensor T into parts with special symmetries.

Recently, Ilyin and Kryukov have published a program for tensor simplification

---

[1] Even different names of indices lead to trouble. For instance, the two expressions $T_{abc} T^{c}{}_{de} T^{e}{}_{f}{}^{a} T^{bdf}$ and $T_{abc} T_{de}{}^{a} T_{f}{}^{eb} T^{cfd}$ are equal, which becomes visible, if we rename the indices according to the rule $a \to c$, $b \to f$, $c \to d$, $d \to e$, $e \to a$, $f \to b$ and raise or lower suitable indices. The determination of such transformations is non-trivial.

[2] In a forthcoming paper we will treat the case of contractions.

called ATENSOR [7], which bases on the connection between tensor expressions and the group ring $\mathbb{C}[\mathcal{S}_r]$, too. But they do not use ideals and ideal decompositions. They consider a subspace $\mathcal{K}$ of the group ring which corresponds to a given set of linear identities being valid within a set of tensor expressions and construct a basis of $\mathcal{K}$ by means of Gaussian eliminations which can be used for the simplification of tensor expressions.

2. Tensors and left ideals of $\mathbb{C}[\mathcal{S}_r]$

In our considerations we make use of the following connection between tensors and elements of the group ring of a symmetric group. We denote by $\mathbb{C}[\mathcal{S}_r]$ the group ring of the symmetric group $\mathcal{S}_r$ over the field of complex numbers $\mathbb{C}$, which we identify with the set $\mathcal{FS}_r$ of all complex-valued functions on $\mathcal{S}_r$. Further let $\mathcal{T}_rV$ be the space of all complex-valued, covariant tensors of order r on the vector space V over a field $\mathbb{K}$. We suppose $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The tensors $T \in \mathcal{T}_rV$ are multilinear mappings of the r-fold cartesian product of V onto $\mathbb{C}$,

$$T : \underbrace{V \times V \times \ldots \times V}_{r\ factors} \to \mathbb{C} \quad , \quad (v_1, \ldots v_r) \mapsto T(v_1, \ldots v_r) \ .$$

Definition 2.1. Any tensor $T \in \mathcal{T}_rV$ and any subset $b := \{v_1, \ldots, v_r\} \subset V$ of r vectors from V induce a function $T_b \in \mathcal{FS}_r$ according to the rule

$$T_b(p) \ := \ T(v_{p(1)}, \ldots v_{p(r)}) \quad , \quad p \in \mathcal{S}_r \ , \qquad (2.1)$$

which we identify with the group ring element $\sum_{p \in \mathcal{S}_r} T_b(p) p \in \mathbb{C}[\mathcal{S}_r]$. For this group ring element we use the notation T_b, too.

The question whether the full group ring $\mathbb{C}[\mathcal{S}_r]$ may be generated by elements of the kind T_b is settled by the following

Lemma 2.1. *Let $b = \{v_1, \ldots, v_r\} \subset V$ be a fixed vector set and let*

$$\mathcal{F}_b\mathcal{S}_r := \{f \in \mathcal{FS}_r \mid \exists T \in \mathcal{T}_rV : f = T_b\}$$

be the set of all functions from $\mathcal{FS}_r$ which are induced by b and arbitrary tensors $T \in \mathcal{T}_rV$. Obviously, $\mathcal{F}_b\mathcal{S}_r$ is a linear subspace of $\mathcal{FS}_r$. If $\dim V \geq r$, then there exists such a subset $b = \{v_1, \ldots, v_r\} \subset V$ that $\mathcal{F}_b\mathcal{S}_r = \mathcal{FS}_r$.

Proof. In the case $\dim V \geq r$ we can choose a set $b = \{e_1, \ldots e_r\}$ of basis vectors of V and assign to every permutation $q \in \mathcal{S}_r$ a tensor $T_q \in \mathcal{T}_rV$ with the property

$$\begin{aligned} T_q(e_{q(1)}, e_{q(2)}, \ldots, e_{q(r)}) \ &= \ 1 \ , \\ T_q(e_{i_1}, e_{i_2}, \ldots, e_{i_2}) \ &= \ 0 \quad \text{in all other cases.} \end{aligned}$$

These tensors T_q fulfil

$$(T_q)_b(p) \ = \ \begin{cases} 1 & if \quad p = q \\ 0 & if \quad p \neq q \end{cases} \ ,$$

such that the functions $(T_q)_b$, $q \in \mathcal{S}_r$, form a basis of $\mathcal{FS}_r$ ∎

Definition 2.2. We use the following two operations.

1. Let $f = \sum_{p \in \mathcal{S}_r} f(p)p \in \mathbb{C}[\mathcal{S}_r]$ and $T \in \mathcal{T}_r V$. Then we denote by $fT \in \mathcal{T}_r V$ that tensor, the coordinates of which are obtained from the coordinates of T by

$$(fT)_{\alpha_1 \alpha_2 \ldots \alpha_r} \;:=\; \sum_{p \in \mathcal{S}_r} f(p)\, T_{\alpha_{p(1)} \alpha_{p(2)} \ldots \alpha_{p(r)}} \;. \tag{2.2}$$

2. We denote by $* : \mathbb{C}[\mathcal{S}_r] \to \mathbb{C}[\mathcal{S}_r]$ the mapping

$$f = \sum_{p \in \mathcal{S}_r} f(p)p \;\mapsto\; f^* := \sum_{p \in \mathcal{S}_r} f(p)p^{-1} \;. \tag{2.3}$$

Many of our calculations are based on

Lemma 2.2. *Let $f = \sum_{p \in \mathcal{S}_r} f(p)p \in \mathbb{C}[\mathcal{S}_r]$, $T \in \mathcal{T}_r V$ and $b = \{v_1, v_2, \ldots, v_r\} \subset V$ a set of r vectors from V. Then there holds true*

$$(fT)_b \;=\; \textstyle\sum_{p \in \mathcal{S}_r} f(p)\, T_b \cdot p^{-1} \;=\; T_b \cdot f^* \;. \tag{2.4}$$

Proof. Equation (2.4) follows from the calculation

$$\begin{aligned}
(fT)_b \;&=\; \sum_{p \in \mathcal{S}_r} (fT)(v_{p(1)}, v_{p(2)}, \ldots, v_{p(r)})\, p \\
&=\; \sum_{p,p' \in \mathcal{S}_r} v^{\alpha_1}_{p(1)} v^{\alpha_2}_{p(2)} \cdots v^{\alpha_r}_{p(r)} f(p') T_{\alpha_{p'(1)} \alpha_{p'(2)} \cdots \alpha_{p'(r)}}\, p \\
&=\; \sum_{p,p' \in \mathcal{S}_r} v^{\alpha_{p'(1)}}_{p \circ p'(1)} v^{\alpha_{p'(2)}}_{p \circ p'(2)} \cdots v^{\alpha_{p'(r)}}_{p \circ p'(r)} f(p') T_{\alpha_{p'(1)} \alpha_{p'(2)} \cdots \alpha_{p'(r)}}\, p \\
&=\; \sum_{p,p' \in \mathcal{S}_r} f(p') T_b(p \circ p')\, p \;=\; \sum_{p',p'' \in \mathcal{S}_r} f(p') T_b(p'')\, p'' \circ {p'}^{-1} \\
&=\; \sum_{p' \in \mathcal{S}_r} f(p') T_b \cdot {p'}^{-1} \;=\; T_b \cdot f^* \;\blacksquare
\end{aligned}$$

Now we consider tensors with certain symmetries.

Definition 2.3. We call a pair (C, ε) a *tensor symmetry*, if $C \subseteq \mathcal{S}_r$ is a subgroup of the symmetric group $\mathcal{S}_r$ and $\varepsilon : C \to S^1$ is a homomorphism of C onto a finite subgroup of the group of the unimodular numbers $S^1 := \{z \in \mathbb{C} \,|\, |z| = 1\}$. We say that a tensor $T \in \mathcal{T}_r V$ possesses the symmetry (C, ε), if

$$\forall c \in C \;:\; cT = \varepsilon(c) T \;. \tag{2.5}$$

If we form the group ring element

$$\varepsilon \;:=\; \sum_{c \in C} \varepsilon(c) c \quad \in \mathbb{C}[\mathcal{S}_r] \;, \tag{2.6}$$

then a simple calculation shows that $\varepsilon \cdot \varepsilon = |C|\, \varepsilon$ with the cardinal number $|C|$ of C. Thus ε is essentially idempotent. Further it can be seen easily that the 1-dimensional complex vector space $U := \{z\varepsilon \mid z \in \mathbb{C}\}$ is invarinat under the action $\alpha_c(u) := c \cdot u$ of C on U and that the function $1/\varepsilon$ is the character of the representation $\alpha : C \rightarrow GL(U)$.

Because of (2.4) equation (2.5) turns into $T_b \cdot c^{-1} = \varepsilon(c)T_b$ for every vector set $b = \{v_1, ..., v_r\} \subset V$. That means that every T_b of a tensor T with the tensor symmetry (C, ε) is an element of the subspace

$$W \ := \ \{f \in \mathbb{C}[\mathcal{S}_r] \mid \forall c \in C : f \cdot c^{-1} = \varepsilon(c)f\} \tag{2.7}$$

of $\mathbb{C}[\mathcal{S}_r]$.

Proposition 2.1. *Let* (C, ε) *be a tensor symmetry for tensors from* $\mathcal{T}_r V$. *Then the vector space* W *according to* (2.7) *fulfils*

$$W \ = \ \mathbb{C}[\mathcal{S}_r] \cdot \varepsilon \ .$$

Proof. First we show $\varepsilon \cdot c^{-1} = \varepsilon(c)\varepsilon$ for $c \in C$ by

$$\varepsilon \cdot c^{-1} = \sum_{c' \in C} \varepsilon(c')c' \cdot c^{-1} = \sum_{c'' \in C} \varepsilon(c'' \cdot c)c'' = \sum_{c'' \in C} \varepsilon(c)\varepsilon(c'')c'' = \varepsilon(c)\varepsilon.$$

Thus every $f = g \cdot \varepsilon \in \mathbb{C}[\mathcal{S}_r] \cdot \varepsilon$, where $g \in \mathbb{C}[\mathcal{S}_r]$, satisfies $f \cdot c^{-1} = \varepsilon(c)f$, such that $\mathbb{C}[\mathcal{S}_r] \cdot \varepsilon \subseteq W$.

On the other hand, there is valid $f = \varepsilon(c^{-1})f \cdot c^{-1}$ for every $f \in W$. The sum over all $c \in C$ yields

$$|C|f \ = \ \sum_{c \in C} \varepsilon(c^{-1})f \cdot c^{-1} \ = \ f \cdot \Big(\sum_{c \in C} \varepsilon(c)c\Big) \ = \ f \cdot \varepsilon \ ,$$

i.e. $f = \frac{1}{|C|} f \cdot \varepsilon \in \mathbb{C}[\mathcal{S}_r] \cdot \varepsilon$ ∎

A similar proposition holds true for the T_b of tensors T, which satisfy certain linear identities. Let $u_1, u_2, ..., u_m \in \mathbb{C}[\mathcal{S}_r]$ be given group ring elements and let $T \in \mathcal{T}_r V$ be a tensor, which meets the m linear identities

$$u_j T \ = \ 0 \ \ , \ \ j = 1, 2, ..., m \, . \tag{2.8}$$

On account of (2.4) relation (2.8) is equivalent to

$$\forall b = \{v_1, v_2, ..., v_r\} \subset V : \quad T_b \cdot u_j^* = 0 \quad , \ \ j = 1, 2, ..., m \ \ . \tag{2.9}$$

More generally, we consider the set of all $f \in \mathbb{C}[\mathcal{S}_r]$, which satisfy (2.9).

Proposition 2.2. *Let* $u_1, u_2, ..., u_m \in \mathbb{C}[\mathcal{S}_r]$ *be given group ring elements and let*

$$J \ := \ \{f \in \mathbb{C}[\mathcal{S}_r] \mid f \cdot u_j^* = 0 \ , \ j = 1, 2, ..., m\} \ \ . \tag{2.10}$$

Then there holds true:

1. *J is a left ideal of the group ring $\mathbb{C}[\mathcal{S}_r]$.*

2. *There exists one and only one right ideal K of $\mathbb{C}[\mathcal{S}_r]$, with the following two properties:*
   - (a) *An $u \in \mathbb{C}[\mathcal{S}_r]$ lies in K if and only if $f \cdot u = 0$ for all $f \in J$.*
   - (b) *An $f \in \mathbb{C}[\mathcal{S}_r]$ lies in J if and only if $f \cdot u = 0$ for all $u \in K$.*

The proof is trivial. If T is a tensor, for which linear identities hold true simultaneously with a tensor symmetrie, then its T_b are contained in the intersection $W \cap J$ of two left ideals W, J of type (2.7), (2.10).

An example of such a tensor is the Riemann curvature tensor. For this tensor characterizing left ideals are known. If R_{ijkl} and $\nabla_m R_{ijkl}$ are the coordinates of the curvature tensor and its first covariant derivative, then the corresponding group ring elements R_b, $(\nabla R)_{\tilde{b}}$, $b, \tilde{b} \subset V$, lie in the left ideals

$$R_b \in \mathbb{C}[\mathcal{S}_4] \cdot y^{\lambda_1} \quad , \quad (\nabla R)_{\tilde{b}} \in \mathbb{C}[\mathcal{S}_5] \cdot y^{\lambda_2} ,$$

where y^{λ_1}, y^{λ_2} denote the Young symmetrizers[1] of the Young tableaux

$$T^{\lambda_1} : \begin{array}{l} 1\,3 \\ 2\,4 \end{array} \quad , \quad T^{\lambda_2} : \begin{array}{l} 1\,3\,5 \\ 2\,4 \end{array} \quad .$$

The proof, given in [6][2], needs the symmetry properties of R_{ijkl} and the Bianchi identities.

In contrast to the left ideals (2.7) we do not know no general method to construct a generating idempotent for a left ideal (2.10) at the moment. If we are able to determine a generating element of the characterizing left ideal W, J or $W \cap J$ of a given tensor $T \in \mathcal{T}_r V$, then the tensor T may be handled within the scope of the following line of action.

We return to our main concern and consider tensor expressions, which are complex linear combinations of certain isomers of a tensor $T \in \mathcal{T}_r V$,

$$\tau_{\alpha_1 \ldots \alpha_r} = \sum_{p \in P} \beta_p T_{\alpha_{p(1)} \cdots \alpha_{p(r)}} \quad , \quad \beta_p \in \mathbb{C} \quad , \quad P \subseteq \mathcal{S}_r \quad , \tag{2.11}$$

where the sum runs over a subset P of the symmetric group $\mathcal{S}_r$. We assume that all T_b, belonging to T, lie in a left ideal $I := \mathbb{C}[\mathcal{S}_r] \cdot a$ with known generating group ring element a.

Lemma 2.3. *A relation* (2.11) *exists between $\tau, T \in \mathcal{T}_r V$ if and only if there holds true with the identity permutation id*

$$\forall b = \{v_1, \ldots v_r\} \subset V : \quad \tau_b(id) = \sum_{p \in P} \beta_p T_b(p) \quad . \tag{2.12}$$

Proof. (2.11) is equivalent to

$$\forall b = \{v_1, \ldots v_r\} \subset V : \quad \tau_{\alpha_1 \ldots \alpha_r} v_1^{\alpha_1} \ldots v_r^{\alpha_r} = \sum_{p \in P} \beta_p T_{\alpha_{p(1)} \ldots \alpha_{p(r)}} v_1^{\alpha_1} \ldots v_r^{\alpha_r}$$

which can be written as (2.12) ∎

---

[1] The definition of a Young symmetrizer gives (2.17).

[2] Moreover, the above statements are extended in [6] to the higher derivatives of the curvature tensor by means of the Ricci identity.

The elements of the left ideal I are characterized by linear identities, the knowledge of which could be used to simplify (2.12) by eliminating suitable terms $T_b(p)$.

A set of complex numbers $\{x_p \mid p \in \mathcal{S}_r\}$ determines a linear identity for all elements of I if

$$\forall f \in I: \quad \sum_{p \in \mathcal{S}_r} x_p f(p) \; = \; 0\,. \tag{2.13}$$

If we know a non-trivial identity (2.13) with $x_p = 0$ for all $p \in \mathcal{S}_r \setminus P$, we can eliminate a term $T_b(p)$ in (2.12) by means of it and get reduced variants of (2.12), (2.11)

$$\tau_b(id) = \sum_{p \in \tilde{P}} \tilde{\beta}_p T_b(p) \quad , \quad \tau_{\alpha_1 \dots \alpha_r} = \sum_{p \in \tilde{P}} \tilde{\beta}_p T_{\alpha_{p(1)} \dots \alpha_{p(r)}} \quad , \quad \tilde{P} \subset P\,.$$

Since every $f \in I = \mathbb{C}[\mathcal{S}_r] \cdot a$ can be written as $f = g \cdot a = \sum_{p,p' \in \mathcal{S}_r} g(p) a(p') p \circ p'$ with a $g \in \mathbb{C}[\mathcal{S}_r]$, we obtain from (2.13)

$$\forall g \in \mathbb{C}[\mathcal{S}_r]: \quad \sum_{p \in \mathcal{S}_r} \Big(\sum_{p' \in \mathcal{S}_r} a(p^{-1} \circ p') x_{p'} \Big) g(p) = 0\,,$$

which yields the homogeneous linear equation system

$$\sum_{p' \in \mathcal{S}_r} a(p^{-1} \circ p') x_{p'} \; = \; 0 \quad , \quad p \in \mathcal{S}_r \tag{2.14}$$

for the numbers x_p that describe the linear identities of I.

The set $\{p \cdot a \mid p \in \mathcal{S}_r\}$ generates the left ideal $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ and can be reduced to a basis of I. Because

$$p \cdot a \; = \; \sum_{p' \in \mathcal{S}_r} a(p') p \circ p' \; = \; \sum_{p'' \in \mathcal{S}_r} a(p^{-1} \circ p'') p''\,, \tag{2.15}$$

we see that the rank of the coefficient matrix $A := [a(p^{-1} \circ p')]_{p,p' \in \mathcal{S}_r}$ of (2.14) is equal to the dimension of I,

$$\operatorname{rank} A \; = \; \dim I\,. \tag{2.16}$$

Further, if $\{q \cdot a \mid q \in Q\}$ is a basis of I, then on the strength of (2.15) the rows of (2.14) with $p = q \in Q$ are a system of $\operatorname{rank} A$ linearly independent rows. Thus, the knowledge of such a basis allows us to write down immediately a set of $\operatorname{rank} A$ linearly independent rows of (2.14) without carrying out the Gaussian algorithm.

In general, the equation system (2.14) is very large since it has a $r! \times r!$ coefficient matrix. But, if we only search for solutions of (2.14) with $x_p = 0$ for $p \notin P$ and proceed to a known set of $\operatorname{rank} A$ linearly independent rows the system (2.14) is reduced to a much smaller subsystem (system (3) in Figure 1[1)]). However, we get a far greater reduction of

---

[1)] In a forthcoming paper we will give an efficient algorithm for Gaussian elimination in system (3) of Figure 1 and for simplifying expressions (2.11) by means of the solutions of system (4), Figure 1.

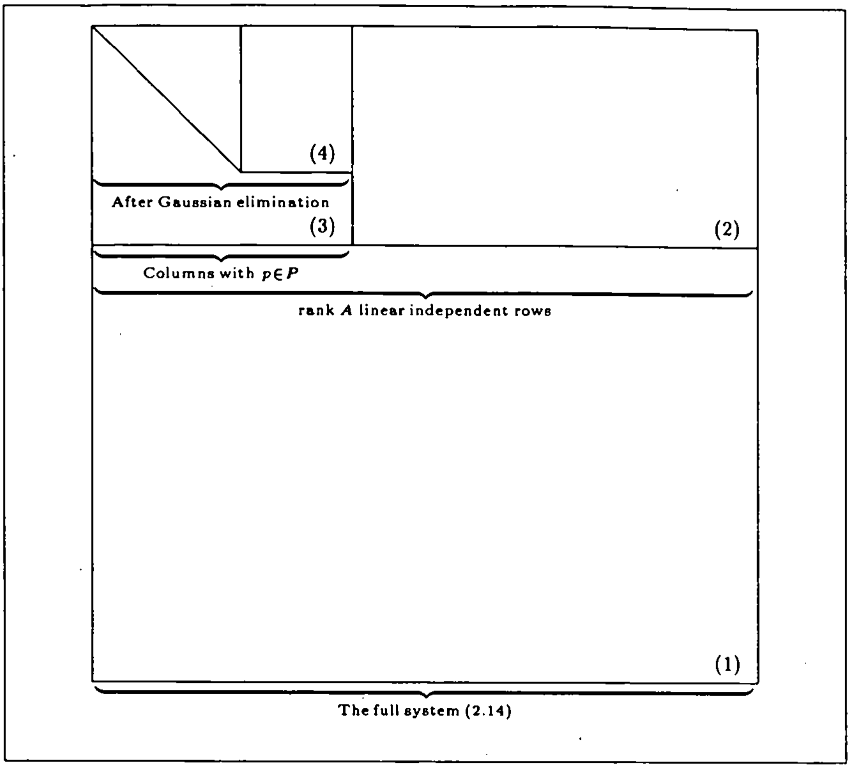

Figure 1. The reduction of system for the x_p.

(2.14), if we decompose the ideal I in a direct sum $I = \bigoplus_{k=1}^{m} I_k$ of minimal left ideals I_k and consider the linear equation systems of type (2.14) which belong to the I_k. Then the rank of the coefficient matrices A_k of these systems fulfils rank $A_k = \dim I_k < \dim I$.

To determine a decomposition $I = \bigoplus_{k=1}^{m} I_k$ for $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ in minimal left ideals we use the fact that such a decomposition of the full group ring $\mathbb{C}[\mathcal{S}_r]$ can be obtained by means of Young symmetrizers[1] which may be defined as follows. We assign to every partition[2]

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash r \ , \quad \lambda_i \in \mathbb{N} \ , \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0 \ , \quad \sum_{i=1}^{k} \lambda_i = r$$

of a natural number $r \in \mathbb{N}$ a so-called Young frame, that means a diagram of k rows of boxes, where the i-th row contains λ_i boxes (Figure 2). Then a Young tableau T_l^{λ} of

---

[1] About Young symmetrizers see, e.g., [22, 14, 1, 2, 15, 17, 9, 8, 6, 10] and the concentrated description in [20: Volume II].

[2] We write $\lambda \vdash r$, if λ is a partition of $r \in \mathbb{N}$.

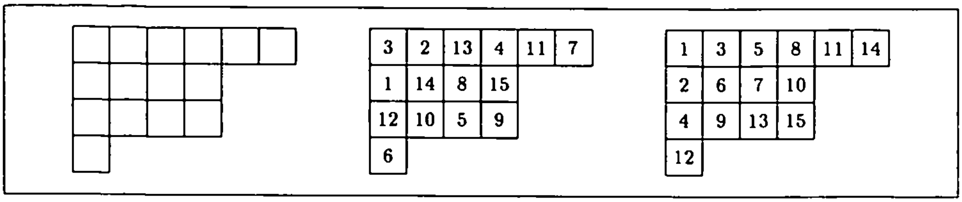

Figure 2. The Young frame and examples of a Young tableau and a standard tableau for $\lambda = (6\ 4^2\ 1)$.

λ is a Young frame, which is filled with the numbers $1, 2, ..., r$, and a standard tableau is a Young tableau, in which the numbers of every row and column form increasing sequences. The Young tableaux of λ are numbered by $l = 1, 2, ..., r!$.

If a fixed Young tableau T_l^λ is given, we denote by $\mathcal{H}_l^\lambda$ the group of all permutations, which only permute the numbers within the rows of T_l^λ (horizontal permutations), and by $\mathcal{V}_l^\lambda$ the group of all permutations, which only permute the numbers within the colums of T_l^λ (vertical permutations). The group ring element

$$y_l^\lambda \ := \ \sum_{q \in \mathcal{V}_l^\lambda} \sum_{p \in \mathcal{H}_l^\lambda} \chi(q) p \circ q \quad \in \mathbb{C}[\mathcal{S}_r] \tag{2.17}$$

is called the Young symmetrizer corresponding to T_l^λ.

Every Young symmetrizer is essentially idempotent and generates a minimal left ideal $\mathbb{C}[\mathcal{S}_r] \cdot y_l^\lambda$ of $\mathbb{C}[\mathcal{S}_r]$. Therefore it differs from a primitive idempotent e_l^λ only by a factor $\mu \in \mathbb{C}$, i.e. $y_l^\lambda = \mu e_l^\lambda$ [1: pp. 99 and 55].

All irreducible representations of the symmetric group $\mathcal{S}_r$ are obtained up to equivalence, if one chooses exactly one Young tableau T_l^λ to every partition $\lambda \vdash r$ of $r \in \mathbb{N}$ and considers as representative of a class of equivalent representations the representation

$$\alpha_l^\lambda : \mathcal{S}_r \to GL(\mathbb{C}[\mathcal{S}_r] \cdot y_l^\lambda) \quad , \quad (\alpha_l^\lambda)_p : f \mapsto p \cdot f \quad , \quad p \in \mathcal{S}_r, f \in \mathbb{C}[\mathcal{S}_r] \cdot y_l^\lambda \,. \tag{2.18}$$

Two representations $\alpha_l^\lambda, \alpha_{l'}^{\lambda'}$ are equivalent if and only if $\lambda = \lambda'$. (See [20: Volume II].) For our purpose we need

Theorem 2.1. *Let $(T_l^\lambda)_{1 \le l \le \tilde{l}_\lambda}$ be the sequence of all standard tableaux, which belong to a partition $\lambda \vdash r$ of a natural number $r \in \mathbb{N}$. Then there is valid*

$$\mathbb{C}[\mathcal{S}_r] = \bigoplus_{\lambda \vdash r} \bigoplus_{l=1}^{\tilde{l}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y_l^\lambda \quad , \quad \tilde{l}_\lambda = \dim \mathbb{C}[\mathcal{S}_r] \cdot y_l^\lambda \ , \tag{2.19}$$

where the sum runs only over Young symmetrizers y_l^λ of standard tableaux T_l^λ.

Equation (2.19) gives a decomposition of $\mathbb{C}[\mathcal{S}_r]$ into invariant irreducible subspaces of the regular representation

$$\alpha : \mathcal{S}_r \to GL(\mathbb{C}[\mathcal{S}_r]) \quad , \quad \alpha_p : f \mapsto p \cdot f \quad , \quad p \in \mathcal{S}_r, f \in \mathbb{C}[\mathcal{S}_r]$$

of the $\mathcal{S}_r$. A complete proof of Theorem 2.1 is given in [1: Chapter IV / §4 and §6]. A partial proof containing the most important proof ideas can be found in [9: Vol. I / pp. 73, 74]. The dimensions $\tilde{l}_\lambda$ can be calculated from the partitions $\lambda \vdash r$ by means of the hook length formula ([1: p. 101], [9: p. 81] or [6]).

Littlewood [14: p. 76] and Boerner [1: p. 103] have pointed out that in general the Young symmetrizers of standard tableaux are not orthogonal in pairs. For example, there holds true $y^\lambda_{(2)} \cdot y^\lambda_{(1)} = 0$ but $y^\lambda_{(1)} \cdot y^\lambda_{(2)} \neq 0$ for the Young symmetrizers which belong to the standard tableaux

$$T^\lambda_{(1)} : \begin{array}{l} 1\,2\,3 \\ 4\,5 \end{array} \quad , \quad T^\lambda_{(2)} : \begin{array}{l} 1\,3\,5 \\ 2\,4 \end{array}$$

of $\lambda = (3\ 2) \vdash 5$ (see [14: p. 76] and [1: p. 106]). The Young symmetrizers of standard tableaux define the minimal left ideals, which occur in the decomposition (2.19), but they do not give simultaneously a system of orthogonal primitive idempotents corresponding to (2.19).

Theorem 2.1 yields the non-direct sum

$$I \;=\; \mathbb{C}[\mathcal{S}_r] \cdot a \;=\; \sum_{\lambda \vdash r} \sum_{l=1}^{\tilde{l}_\lambda} \mathbb{C}[\mathcal{S}_r] \cdot y^\lambda_l \cdot a \tag{2.20}$$

for the left ideal I. In the Sections 3 and 4 we will determine a decomposition of I in a direct sum of minimal left ideals from (2.20). It is remarkable that the methods of these sections work even in a group ring $\mathbb{C}[G]$ of an arbitrary finite group G.

3. Construction of an idempotent for a minimal left ideal

We consider the group ring $\mathbb{C}[G]$ of a finite group G over the field $\mathbb{C}$ of complex numbers.

Lemma 3.1. *Let $a \in \mathbb{C}[G]$, $a \neq 0$, be an arbitrary group ring element and $e \in \mathbb{C}[G]$ a primitive idempotent of $\mathbb{C}[G]$. If $e \cdot a \neq 0$, then the left ideal $W := \mathbb{C}[G] \cdot e \cdot a$ is equivalent*[1] *to the left ideal $I := \mathbb{C}[G] \cdot e$ and minimal like I.*

Proof. The kernel $\ker = \{x \in I \mid x \cdot a = 0\}$ of the linear map $\phi : x \mapsto x \cdot a, x \in I$, is a left subideal of I. Since $e \in I$ is mapped onto $e \cdot a \neq 0$ and I is minimal, we obtain $\ker = \{0\}$, such that the map $\phi : I \to W$ has to be an isomorphism.

If $V \subset W$ is a proper left subideal of W, then $\phi^{-1}(V) \subset I$ is a proper left subideal of the minimal left ideal I. Consequently, we obtain $\phi^{-1}(V) = \{0\}$ and $V = \{0\}$, i.e. W is minimal, too ∎

---

[1] Two left ideals $I, W \subset \mathbb{C}[G]$ are called equivalent, if there exists an isomorphism $\phi : I \to W$ of the vector spaces I, W, which commutes with the left multiplication of $\mathbb{C}[G]$, that means $\phi(g \cdot f) = g \cdot \phi(f)$ for all $g \in G$ and all $f \in I$ [1: p. 52]. If I, W are equivalent, then the representations $f \mapsto g \cdot f$ and $w \mapsto g \cdot w$ of G over I, W are equivalent. Here we assume $g \in G, f \in I, w \in W$.

Due to Lemma 3.1 the left ideals $\mathbb{C}[\mathcal{S}_r]\cdot y_t^\lambda$ and $\mathbb{C}[\mathcal{S}_r]\cdot y_t^\lambda\cdot a$, considered in Section 2, are equivalent minimal left ideals, if $y_t^\lambda \cdot a \neq 0$. Now we show a possibility to construct a generating idempotent for a left ideal W according to Lemma 3.1.

Proposition 3.1. *Let $a \in \mathbb{C}[G]$, $a \neq 0$, be a group ring element and $e \in \mathbb{C}[G]$ be a primitive idempotent with $e \cdot a \neq 0$. Then there exists a group element $g \in G$, such that*

$$e \cdot a \cdot g \cdot e \;\neq\; 0 \,. \tag{3.1}$$

Moreover, the group ring element $b := g\cdot e\cdot a$, formed with this g, is essentially idempotent and generates the left ideal $W = \mathbb{C}[G] \cdot e \cdot a$.

Proof. [1] The left ideal $W = \mathbb{C}[G] \cdot e \cdot a$ possesses a generating idempotent f [1: p. 54], which can be written as $f = x \cdot e \cdot a$ with a certain $x \in \mathbb{C}[G]$ and the generating element $e \cdot a$ of W. Now, the relation

$$e \cdot a \cdot x \cdot e \;\neq\; 0 \tag{3.2}$$

follows from $f = f \cdot f = x \cdot e \cdot a \cdot x \cdot e \cdot a$. But then an element $g \in G$ has to exist which satisfies (3.2) with $x = g$, since otherwise the left-hand side of (3.2) would vanish for every $x \in \mathbb{C}[G]$.

As e is a primitive idempotent, we get

$$e \cdot a \cdot g \cdot e = \mu e$$

with a complex number $\mu \in \mathbb{C}$ [1: p. 56] and $\mu \neq 0$ on account of (3.1). Consequently, $b := g \cdot e \cdot a$ is essentially idempotent, because

$$b \cdot b = g \cdot (e \cdot a \cdot g \cdot e) \cdot a = \mu b \,,$$

and b generates W, since $\mathbb{C}[G] \cdot g = \mathbb{C}[G]$ ∎

By Proposition 3.1 it is possible to construct a generating idempotent for every minimal left ideal $\mathbb{C}[\mathcal{S}_r] \cdot y_t^\lambda \cdot a$ in (2.20) with $y_t^\lambda \cdot a \neq 0$.

The determination of the group element g for the forming of the essentially idempotent element b can be done by a computer program, which tests the validity of condition (3.1) for the finitely many group elements $g \in G$ one after another. The search stops if the first $g \in G$ is found which fulfils (3.1). We have realized such an algorithm for symmetric groups $\mathcal{S}_r$ and the corresponding group rings $\mathbb{C}[\mathcal{S}_r]$. Though symmetric groups have a very large cardinality $|\mathcal{S}_r| = r!$ in general, all examples, treated by this algorithm, claim a small number of search steps to reach a permutation $p \in \mathcal{S}_r$ which satisfies (3.1).

---

[1] Parts of the proof of Proposition 3.1 are similar to a proof of a proposition on regular group rings in [21] which is reproduced in [18: p. 68]. However, the proof in [18: p. 68] does not contain idempotent constructions on the basis of the minimality of certain left ideals, in contrast to the proof of Proposition 3.1.

4. Construction of orthogonal idempotents for a decomposition of a left ideal

Let I be a left ideal of a group ring $\mathbb{C}[G]$ of a finite group G, for which a decomposition $I = \bigoplus_{k=1}^{m} I_k$ in minimal left ideals I_k is given. Further we assume that we know a generating idempotent e_k for every I_k. The multiplication of I from the right by a group ring element $a \in \mathbb{C}[G]$, $a \neq 0$, yields a left ideal $J = I \cdot a$ which does not keep a direct sum of minimal left ideals no longer. In general we have only $J = \sum_{k=1}^{m} I_k \cdot a$.

Now we will describe a method to construct a decomposition of J in a direct sum of minimal left ideals. This method even allows to determine a system of primitive orthogonal idempotents f_l from the e_k which corresponds to the decomposition of J.

Lemma 4.1. *Let $I = \mathbb{C}[G] \cdot e$ be a left ideal of a group ring $\mathbb{C}[G]$ of a finite group G, generated by an idempotent $e \in \mathbb{C}[G]$. Then there holds true:*

1. *The group ring element $f := e - x \cdot e + e \cdot x \cdot e$ is an idempotent*[1] *with*

$$f \cdot e = f \quad , \quad e \cdot f = e \tag{4.1}$$

for every $x \in \mathbb{C}[G]$. Especially, f generates the left ideal I, too.

2. *Let f be an idempotent which fulfils* (4.1). *Then there exists an $x \in \mathbb{C}[G]$, such that $f = e - x \cdot e + e \cdot x \cdot e$.*

Proof. Ad 1.: Since e is an idempotent we obtain $f \cdot e = f$. Further $e \cdot f = e$ follows immediately from $-e \cdot x \cdot e + e \cdot e \cdot x \cdot e = 0$. Now the idempotent property of f is confirmed by

$$f \cdot f = (e - x \cdot e + e \cdot x \cdot e) \cdot f = e - x \cdot e + e \cdot x \cdot e = f \ .$$

Ad 2.: From $f \cdot e = f$ there follows $f \in I$ and consequently $f - e \in I$. Therefore we can write $f - e = -y \cdot e$ with a certain $y \in \mathbb{C}[G]$. Then $e \cdot f = e$ yields $e \cdot y \cdot e = 0$, such that $f = e - y \cdot e + e \cdot y \cdot e$ is correct ∎

Corollary 4.1. [2] *Let $e \in \mathbb{C}[G]$ be an idempotent. Then the following assertions hold true for all $x \in \mathbb{C}[G]$:*

1. *$n := x \cdot e - e \cdot x \cdot e$ is nilpotent, i.e. $n \cdot n = 0$.*
2. *$u := id - n$ is an invertible element or a unit of $\mathbb{C}[G]$ with the inverse $u^{-1} = id + n$, where id denotes the identity element of G.*
3. *The idempotent $f = e - x \cdot e + e \cdot x \cdot e$ in accordance with Lemma* 4.1 *fulfils $f = u \cdot e \cdot u^{-1}$.*

---

[1] The idea to produce a new idempotent f from a given idempotent e in this way was taken out of [18: p. 137]. However, in [18] the forming of new idempotents is carried out only by means of group elements $x = g \in G$.

[2] This remarkable property is mentioned in [18: p. 138], too. According to [18], first Zalesskii becomes aware of it.

Proof. Ad 1.: $n \cdot n = 0$ follows from $e \cdot (x \cdot e - e \cdot x \cdot e) = 0$.

Ad 2.: $u \cdot u^{-1} = id$ and $u^{-1} \cdot u = id$ result from $n \cdot n = 0$.

Ad 3.: By consideration of $e \cdot e = e$ and $e \cdot (x \cdot e - e \cdot x \cdot e) = 0$ the assertion can be easily checked:

$$\begin{aligned} u \cdot e \cdot u^{-1} &= (id - x \cdot e + e \cdot x \cdot e) \cdot e \cdot (id + x \cdot e - e \cdot x \cdot e) \\ &= (e - x \cdot e + e \cdot x \cdot e) \cdot (id + x \cdot e - e \cdot x \cdot e) \\ &= f + (id - x + e \cdot x) \cdot e \cdot (x \cdot e - e \cdot x \cdot e) \quad = \quad f \blacksquare \end{aligned}$$

The next proposition is the heart of our procedure to produce orthogonal idempotents from given non-orthogonal idempotents.

Proposition 4.1. *Let $I = \mathbb{C}[G] \cdot e$ and $\tilde{I} = \mathbb{C}[G] \cdot \tilde{e}$ be two left ideals of the group ring $\mathbb{C}[G]$, generated by the idempotents e and $\tilde{e}$. We assume that I is minimal, which involves that e is primitive. Further we require $e \cdot \tilde{e} \neq e$. Then there holds true:*

1. *A group element $g \in G$ can be found, such that*
$$e \cdot (id - \tilde{e}) \cdot g \cdot e \;\neq\; 0 \;. \tag{4.2}$$
Moreover, a complex number $\lambda \in \mathbb{C}$ belonging to that g is available, such that $f := e - x \cdot e + e \cdot x \cdot e$ with $x := \lambda(id - \tilde{e}) \cdot g$ is a generating idempotent of I which satisfies $\tilde{e} \cdot f = 0$.
2. *For a given idempotent f according to Statement 1 a group element $\tilde{g} \in G$ exists, such that*
$$f \cdot (id - \tilde{e}) \cdot \tilde{g} \cdot f \;\neq\; 0 \;. \tag{4.3}$$
Besides, a complex number $\tilde{\lambda} \in \mathbb{C}$ can be choosed, such that $\tilde{f} := \tilde{e} - \tilde{x} \cdot \tilde{e}$ with $\tilde{x} := \tilde{\lambda}(id - \tilde{e}) \cdot \tilde{g} \cdot f$ is a generating idempotent of $\tilde{I}$ which fulfils $f \cdot \tilde{f} = \tilde{f} \cdot f = 0$.

Proof. From $e \cdot \tilde{e} \neq e$ we obtain $e \cdot (id - \tilde{e}) \neq 0$. By Proposition 3.1 there is a $g \in G$, such that $e \cdot (id - \tilde{e}) \cdot g \cdot e \neq 0$. Thus (4.2) is proved. Since e is primitive, a relation

$$e \cdot (id - \tilde{e}) \cdot g \cdot e \;=\; \mu e \tag{4.4}$$

is valid with a complex number $\mu \in \mathbb{C}$ [1: p. 56], and $\mu \neq 0$ on account of (4.2). Now, if f is an idempotent according to Statement 1 of Proposition 4.1 which generates I by Lemma 4.1, we get

$$\begin{aligned} \tilde{e} \cdot f &= \tilde{e} \cdot e - \tilde{e} \cdot x \cdot e + \tilde{e} \cdot e \cdot x \cdot e \\ &= \tilde{e} \cdot e + \lambda \mu \tilde{e} \cdot e \quad , \end{aligned}$$

considering $\tilde{e} \cdot x = 0$ and (4.4). Then $\lambda = -1/\mu$ leads to $\tilde{e} \cdot f = 0$.

As f generates I, too, there follows $f \cdot \tilde{e} \neq f$, because else $I \subseteq \tilde{I}$ and consequently $e \cdot \tilde{e} = e$ would apply. Now the existence of a $\tilde{g} \in G$, which satisfies (4.3), arises from the use of Statement 1 of Proposition 4.1 to the idempotents $f, \tilde{e}$. We change to a new idempotent $\tilde{f} := \tilde{e} - \tilde{x} \cdot \tilde{e} + \tilde{e} \cdot \tilde{x} \cdot \tilde{e}$ of $\tilde{I}$, where $\tilde{x} := \tilde{\lambda}(id - \tilde{e}) \cdot \tilde{g} \cdot f$. Then $\tilde{e} \cdot \tilde{x} = 0$ yields $\tilde{e} \cdot \tilde{x} \cdot \tilde{e} = 0$. Since f is primitive as generating idempotent of the minimal left ideal I, (4.3) results in $f \cdot (id - \tilde{e}) \cdot \tilde{g} \cdot f = \tilde{\mu} f$ with $0 \neq \tilde{\mu} \in \mathbb{C}$. Thus we get for $\tilde{f} = \tilde{e} - \tilde{x} \cdot \tilde{e}$

$$f \cdot \tilde{f} = f \cdot \tilde{e} - \tilde{\lambda} f \cdot (id - \tilde{e}) \cdot \tilde{g} \cdot f \cdot \tilde{e} = (1 - \tilde{\lambda}\tilde{\mu}) f \cdot \tilde{e} \quad ,$$

and the choice $\tilde{\lambda} = 1/\tilde{\mu}$ gives $f \cdot \tilde{f} = 0$. The relation $\tilde{f} \cdot f = 0$ simply follows from $\tilde{e} \cdot f = 0 \blacksquare$

The determination of group elements $g, \tilde{g} \in G$, which satisfy (4.2), (4.3), can be carried out by a computer program in the way that was described in the end of Section 3. However, the program should check, whether $\tilde{e} \cdot e = 0$ or $f \cdot \tilde{e} = 0$, before the search for g or $\tilde{g}$ starts. If one of these cases arises, we can simply put $\lambda = 0$ or $\tilde{\lambda} = 0$ without searching for g or $\tilde{g}$.

We remark that the group ring element $\tilde{x} = \tilde{\lambda}(id - \tilde{e}) \cdot \tilde{g} \cdot f$ with $\tilde{\lambda} = 1/\tilde{\mu}$, used in the prove above, is likewise an idempotent, because

$$\tilde{x} \cdot \tilde{x} \;=\; \tilde{\lambda}^2(id - \tilde{e}) \cdot \tilde{g} \cdot (f \cdot (id - \tilde{e}) \cdot \tilde{g} \cdot f) \;=\; \tilde{\lambda}^2 \tilde{\mu}(id - \tilde{e}) \cdot \tilde{g} \cdot f \;=\; \tilde{x} \;.$$

Theorem 4.1. *Let I be a left ideal of $\mathbb{C}[G]$, for which a decomposition $I = \bigoplus_{k=1}^{m} I_k$ into minimal left ideals I_k is given. Further, let $a \in \mathbb{C}[G]$, $a \neq 0$, be a group ring element with $I \cdot a \neq \{0\}$. We assume that a primitive generating idempotent e_k is known for every I_k. The system of the e_k is allowed to be non-orthogonal. Then we can select a subset $\{e_{k_1}, e_{k_2}, \ldots, e_{k_n}\}$ from the set $\{e_k \mid e_k \cdot a \neq 0\}$, such that the left ideal $J := I \cdot a$ is the direct sum $J = \bigoplus_{l=1}^{n} J_{k_l}$ of the left ideals $J_{k_l} := I_{k_l} \cdot a = \mathbb{C}[G] \cdot e_{k_l} \cdot a$ belonging to the e_{k_l}. Moreover, we can construct primitive generating idempotents h_{k_l} of the J_{k_l} from the e_{k_l} and a, which are even orthogonal.*

Proof. Because $I \cdot a \neq \{0\}$, we have $\{e_k \mid e_k \cdot a \neq 0\} \neq \emptyset$. We choose for k_1 the smallest k with $e_k \cdot a \neq 0$. According to Proposition 3.1 we can determine a primitive generating idempotent $\hat{f}_{k_1}$ of $J_{k_1} := I_{k_1} \cdot a = \mathbb{C}[G] \cdot e_{k_1} \cdot a$ from e_{k_1} and a. In the following, we use the symbols $\tilde{J}_1 := J_{k_1}$ and $\tilde{f}_1 := \hat{f}_{k_1}$ for J_{k_1} and $\hat{f}_{k_1}$.

Now, we search for the smallest k that fulfils the tree conditions

$$k > k_1 \quad , \qquad e_k \cdot a \neq 0 \quad , \qquad e_k \cdot a \cdot \tilde{f}_1 \neq e_k \cdot a \;. \tag{4.5}$$

If such a k does not exist, then there follows $e_k \cdot a \cdot \tilde{f}_1 = e_k \cdot a$ and consequently $I_k \cdot a \subseteq \tilde{J}_1$ for every $k > k_1$ with $e_k \cdot a \neq 0$. In this case we simply finish with $J = \tilde{J}_1$.

If, however, a smallest k can be found, which satisfies (4.5), we call it k_2. Then we have $e_{k_2} \cdot a \notin \tilde{J}_1$, but $e_{k_2} \cdot a \in J_{k_2} := I_{k_2} \cdot a$, such that $\tilde{J}_1 \cap J_{k_2} = \{0\}$, since J_{k_2} is minimal. Thus, the sum of $\tilde{J}_1$ and J_{k_2} is direct. We denote it by $\tilde{J}_2 := \tilde{J}_1 \oplus J_{k_2}$.

According to Proposition 3.1 we form a primitive generating idempotent f_{k_2} of the minimal left ideal J_{k_2} from e_{k_2} and a. f_{k_2} has to fulfil $f_{k_2} \cdot \tilde{f}_1 \neq f_{k_2}$ as well as $e_{k_2} \cdot a$, because otherwise there would be $f_{k_2} \in \tilde{J}_1$ and $J_{k_2} \subseteq \tilde{J}_1$. Now, using Proposition 4.1, we produce new generating idempotents $\check{f}_1, \hat{f}_{k_2}$ from the generating idempotents $\tilde{f}_1, f_{k_2}$ of the left ideals $\tilde{J}_1, J_{k_2}$, which are orthogonal, i.e. $\check{f}_1 \cdot \hat{f}_{k_2} = \hat{f}_{k_2} \cdot \check{f}_1 = 0$. Then $\tilde{f}_2 := \check{f}_1 + \hat{f}_{k_2}$ is a generating idempotent of the left ideal $\tilde{J}_2$.

Next, we search for the smallest k, which satisfies

$$k > k_2 \quad , \qquad e_k \cdot a \neq 0 \quad , \qquad e_k \cdot a \cdot \tilde{f}_2 \neq e_k \cdot a \;. \tag{4.6}$$

If such a k can not be found, then there holds true $I_k \cdot a \subseteq \tilde{J}_2$ for every $k \geq k_2$ with $e_k \cdot a \neq 0$ and even $I_k \cdot a \subseteq \tilde{J}_1$ for every $k < k_2$ mit $e_k \cdot a \neq 0$. This yields $J = \tilde{J}_2$.

If, however, a smallest k is available, for which (4.6) is valid, we call it k_3 and consider the left ideal $J_{k_3} := I_{k_3} \cdot a$. The minimality of J_{k_3} and the relation $e_{k_3} \cdot a \notin \tilde{J}_2$ lead to $\tilde{J}_2 \cap J_{k_3} = \{0\}$, such that we get a direct sum $\tilde{J}_3 := \tilde{J}_2 \oplus J_{k_3}$. Proposition 3.1

provides us a primitive generating idempotent f_{k_3} of J_{k_3}, which is determinable from e_{k_3}, a. f_{k_3} fulfils $f_{k_3} \cdot \tilde{f}_2 \neq f_{k_3}$, such that we can change from the idempotents $\tilde{f}_2, f_{k_3}$ of the left ideals $\tilde{J}_2, J_{k_3}$ to the orthogonal idempotents $\check{f}_2, \hat{f}_{k_3}$ by means of Proposition 4.1. Besides we obtain a generating idempotent $\tilde{f}_3 := \check{f}_2 + \hat{f}_{k_3}$ of the left ideal $\tilde{J}_3$.

We continue this procedure until it terminates after a certain k_n. The result is a finite increasing sequence of left ideals

$$\tilde{J}_1 \subseteq \tilde{J}_2 \subseteq \tilde{J}_3 \subseteq \ldots \subseteq \tilde{J}_n .$$

For $l \geq 2$, every of these left ideals is a direct sum $\tilde{J}_l = \tilde{J}_{l-1} \oplus J_{k_l}$ of its predecessor and a minimal left ideal $J_{k_l} := I_{k_l} \cdot a$. Furthermore, we know a generating idempotent $\tilde{f}_l = \check{f}_{l-1} + \hat{f}_{k_l}$ of every $\tilde{J}_l$, $l \geq 2$, which consists of orthogonal generating idempotents $\check{f}_{l-1}, \hat{f}_{k_l}$ of $\tilde{J}_{l-1}, J_{k_l}$.

Since there holds true $I_k \cdot a \subseteq \tilde{J}_n$ for all $k \geq k_n$ with $e_k \cdot a \neq 0$ and even $I_k \cdot a \subseteq \tilde{J}_{n-1}$ for all $k < k_n$ with $e_k \cdot a \neq 0$, we have $J = \tilde{J}_n$. Thus, we obtain a decomposition of J into a direct sum of minimal left ideals J_{k_l},

$$J = \tilde{J}_n = \tilde{J}_{n-1} \oplus J_{k_n} = \tilde{J}_{n-2} \oplus J_{k_{n-1}} \oplus J_{k_n} = \ldots = \bigoplus_{l=1}^{n} J_{k_l} .$$

We take from Statement 2 of Proposition 4.1 that the idempotents $\check{f}_l$ of the left ideals $\tilde{J}_l$ possess the form $\check{f}_l = \tilde{f}_l - x_l \cdot \tilde{f}_l = (id - x_l) \cdot \tilde{f}_l$ with a certain group ring element $x_l \in \mathbb{C}[G]$. With it, the following calculation leads to a decomposition of the generating idempotent $\tilde{f}_n$ of $J = \tilde{J}_n$:

$$\begin{aligned}
\tilde{f}_n &= \check{f}_{n-1} + \hat{f}_{k_n} \\
&= (id - x_{n-1}) \cdot \tilde{f}_{n-1} + \hat{f}_{k_n} \\
&= (id - x_{n-1}) \cdot (\check{f}_{n-2} + \hat{f}_{k_{n-1}}) + \hat{f}_{k_n} \\
&= (id - x_{n-1}) \cdot (id - x_{n-2}) \cdot \tilde{f}_{n-2} + (id - x_{n-1}) \cdot \hat{f}_{k_{n-1}} + \hat{f}_{k_n} \\
&\;\vdots \\
&= \sum_{l=1}^{n-1} (id - x_{n-1}) \cdot (id - x_{n-2}) \cdot \ldots \cdot (id - x_l) \cdot \hat{f}_{k_l} + \hat{f}_{k_n} .
\end{aligned} \tag{4.7}$$

Formula (4.7) presents a decomposition of $\tilde{f}_n$, the summands of which fulfil

$$h_{k_l} := (id - x_{n-1}) \cdot (id - x_{n-2}) \cdot \ldots \cdot (id - x_l) \cdot \hat{f}_{k_l} \in J_{k_l} \quad , \quad h_{k_n} := \hat{f}_{k_n} \in J_{k_n} . \tag{4.8}$$

Therefore, $\tilde{f}_n = \sum_{l=1}^{n} h_{k_l}$ is the decomposition of $\tilde{f}_n$ corresponding to the direct sum $J = \bigoplus_{l=1}^{n} J_{k_l}$ and the h_{k_l} are orthogonal generating idempotents of the minimal left ideals J_{k_l} [1: p. 55] ∎

From a remark after the proof of Proposition 4.1 it follows that every x_l, appearing in (4.8), is an idempotent, which lies in J_{k_l}. Then, every factor $(id - x_l)$ is an idempotent of $\mathbb{C}[G]$, too.

5. A fast basis construction

We have pointed out after equation (2.16) that a set of rank A linearly independent rows of the linear equation system (2.14) can be stated, if a basis $\{q \cdot a \mid q \in Q\}$ of the left ideal $I = \mathbb{C}[\mathcal{S}_r] \cdot a$ is known. Such a basis can be determined by means of Young tableaux.

Let $(T_l^\lambda)_{l \geq 1}$ be the finite sequence of all standard tableaux of a given partition $\lambda \vdash r$ of a natural number $r \in \mathbb{N}$, provided with a fixed numbering, and let $T_{l_0}^\lambda$ be a selected member of these sequence. We introduce a permutation subset

$$P_{l_0}^\lambda := \{t \in \mathcal{S}_r \mid t \circ T_{l_0}^\lambda \text{ is a standard tableau of } \lambda\} ,$$

where $t \circ T_{l_0}^\lambda$ denotes the Young tableau, which arises from $T_{l_0}^\lambda$ by permuting the number entries of $T_{l_0}^\lambda$ according to the permutation $t \in \mathcal{S}_r$.

Every tableau $t \circ T_{l_0}^\lambda$, $t \in P_{l_0}^\lambda$, occurs exactly once in $(T_l^\lambda)_{l \geq 1}$. If $t[l_0]$ stands for the index $l = t[l_0]$ of $t \circ T_{l_0}^\lambda$ in $(T_l^\lambda)_{l \geq 1}$, we can write

$$\forall t \in P_{l_0}^\lambda : \quad T_{t[l_0]}^\lambda = t \circ T_{l_0}^\lambda .$$

Proposition 5.1. *Let $\lambda \vdash r$ be a partition of $r \in \mathbb{N}$ and T^λ a fixed Young tableau of λ which is transformed by a permutation $s_0 \in \mathcal{S}_r$ into a standard tableau $T_{l_0}^\lambda$ of λ, i.e. $T_{l_0}^\lambda = s_0 \circ T^\lambda$. Now, if y^λ is the Young symmetrizer of T^λ, then*

$$\{t \cdot y^\lambda \mid t \in P_{l_0}^\lambda \circ s_0\} \tag{5.1}$$

is a basis of the minimal left ideal $I^\lambda := \mathbb{C}[\mathcal{S}_r] \cdot y^\lambda$.

Proof. [1] Since $|P_{l_0}^\lambda| = \dim I^\lambda$ (Theorem 2.1), it is sufficient to proof the linear independence of the group ring elements contained in the set (5.1).

If two Young tableaux $T_{(1)}^\lambda$, $T_{(2)}^\lambda$ of λ satisfy $T_{(2)}^\lambda = s \circ T_{(1)}^\lambda$ with $s \in \mathcal{S}_r$, then their Young symmetrizers are connected by $y_{(2)}^\lambda = s \cdot y_{(1)}^\lambda \cdot s^{-1}$ or $s \cdot y_{(1)}^\lambda = y_{(2)}^\lambda \cdot s$. Using the decomposition $t = t' \circ s_0$, $t' \in P_{l_0}^\lambda$, for $t \in P_{l_0}^\lambda \circ s_0$, we can write

$$(t' \circ s_0) \cdot y^\lambda = (t' \cdot y_{l_0}^\lambda) \cdot s_0 = y_{t'[l_0]}^\lambda \cdot (t' \circ s_0) .$$

Consequently, a relation

$$\sum_{t \in P_{l_0}^\lambda \circ s_0} \gamma_t \, t \cdot y^\lambda = 0 \quad , \quad \gamma_t \in \mathbb{C} ,$$

can be converted into

$$\sum_{t' \in P_{l_0}^\lambda} \gamma_{t' \circ s_0} \, y_{t'[l_0]}^\lambda \cdot (t' \circ s_0) \;=\; 0 . \tag{5.2}$$

---

[1] An other proof of the statement of Proposition 5.1 with $s_0 = id$ is given in [1: p. 105].

We use the usual order-relation for Young tableaux of the same partition $\lambda \vdash r$. A tableau $T^{\lambda}_{(2)}$ is regarded as greater than a tableau $T^{\lambda}_{(1)}$, if the simultaneous run through the rows of both tableaux from left to right and from top to bottom reaches earlier in $T^{\lambda}_{(2)}$ a number, which is greater than the number on the corresponding place in $T^{\lambda}_{(1)}$. Further, the following multiplication rule holds true (see [9: Vol.I / p. 73] or [1: p. 101]). If T^{λ}_{l} , $T^{\lambda}_{l'}$ are two standard tableaux of $\lambda \vdash r$, then their Young symmetrizers fulfil

$$y^{\lambda}_{l} \cdot y^{\lambda}_{l'} \;=\; \begin{cases} \mu_{\lambda} y^{\lambda}_{l} \;,\; 0 \neq \mu_{\lambda} \in \mathbb{C} & \text{if } T^{\lambda}_{l} = T^{\lambda}_{l'} \\ 0 & \text{if } T^{\lambda}_{l} > T^{\lambda}_{l'} \end{cases} \;. \tag{5.3}$$

Now, let $T^{\lambda}_{l_1}$ be the greatest standard tableau of λ in accordance with the above order-relation and let $t'_1 \in P^{\lambda}_{l_0}$ be the permutation with $t'_1[l_0] = l_1$. Because of (5.3), the multiplication of (5.2) with $y^{\lambda}_{l_1}$ from the left yields

$$\gamma_{t'_1 \circ s_0}\, \mu_{\lambda}\, y^{\lambda}_{l_1} \cdot (t'_1 \circ s_0) \;=\; 0 \,,$$

and consequently $\gamma_{t'_1 \circ s_0} = 0$. After that, the left multiplication of (5.2) with the Young symmetrizer $y^{\lambda}_{l_2}$ of the second greatest standard tableau $T^{\lambda}_{l_2}$ results in

$$\gamma_{t'_2 \circ s_0}\, \mu_{\lambda}\, y^{\lambda}_{l_2} \cdot (t'_2 \circ s_0) \;=\; 0 \,, \tag{5.4}$$

where $t'_2 \in P^{\lambda}_{l_0}$ is the permutation with $t'_2[l_0] = l_2$. A possibly non-vanishing product $y^{\lambda}_{l_2} \cdot y^{\lambda}_{l_1}$ can not appear in (5.4), since $y^{\lambda}_{l_1}$ does not occur no longer in (5.2) on account of $\gamma_{t'_1 \circ s_0} = 0$. Thus we get $\gamma_{t'_2 \circ s_0} = 0$ from (5.4). If we continue this procedure for all standard tableaux T^{λ}_{l} of λ in decreasing order, we obtain $\gamma_{t' \circ s_0} = 0$ for all $t' \in P^{\lambda}_{l_0}$, i.e. the set (5.1) is a basis of I^{λ} ∎

Corollary 5.1. *Let be given the situation of Proposition* 5.1 *and let* $a \in \mathbb{C}[\mathcal{S}_r]$ *be a group ring element with* $y^{\lambda} \cdot a \neq 0$. *Then*

$$\{t \cdot y^{\lambda} \cdot a \mid t \in P^{\lambda}_{l_0} \circ s_0\} \tag{5.5}$$

is a basis of the minimal left ideal $W^{\lambda} := \mathbb{C}[\mathcal{S}_r] \cdot y^{\lambda} \cdot a$.

Proof. According to Lemma 3.1 the left ideals $I^{\lambda} = \mathbb{C}[\mathcal{S}_r] \cdot y^{\lambda}$ and $W^{\lambda} = I^{\lambda} \cdot a$ are equivalent by means of the linear map $\phi : x \mapsto x \cdot a$, $x \in I^{\lambda}$. Thus, (5.5) is a basis of W^{λ} as the image of the basis (5.1) of I^{λ} under ϕ ∎

6. Concluding remarks

Now, we see the following way to reduce tensor expressions (2.11) for a tensor $T \in \mathcal{T}_r V$, all T_b of which are contained in a left ideal $J = \mathbb{C}[\mathcal{S}_r] \cdot a$.

We start with the sum

$$\mathbb{C}[\mathcal{S}_r] \;=\; \bigoplus_{\lambda\vdash r}\bigoplus_{l=1}^{l_\lambda} \mathbb{C}[\mathcal{S}_r]\cdot y_l^\lambda\,,$$

where the y_l^λ run through all Young symmetrizers of the standard tableaux of all partitions $\lambda \vdash r$, and construct by means of Theorem 4.1 a subset

$$Y \subseteq \{y_l^\lambda \mid \lambda\vdash r\,,\, l=1,\ldots,l_\lambda\}\,,$$

such that

$$J \;=\; \mathbb{C}[\mathcal{S}_r]\cdot a \;=\; \bigoplus_{y\in Y}\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a\,. \tag{6.1}$$

Theorem 4.1 yields us orthogonal primitive idempotents, denoted by h_y , $y \in Y$, which generate the minimal left ideals $\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a$ in (6.1), i.e.

$$h_y\cdot h_{y'} = 0 \;\;,\;\; \text{if}\;\; y\neq y'\,,\; y,y'\in Y\,,$$

and

$$\forall y\in Y:\;\; \mathbb{C}[\mathcal{S}_r]\cdot y\cdot a \;=\; \mathbb{C}[\mathcal{S}_r]\cdot h_y\,.$$

The sum $h := \sum_{y\in Y} h_y$ is a generating idempotent of J. With it, we obtain for the group ring elements $T_b \in J$

$$T_b \;=\; T_b\cdot h \;=\; \sum_{y\in Y} T_b\cdot h_y \;=\; \sum_{y\in Y}(h_y^*T)_b\,.$$

The h_y^*T are tensors, which develop from T by a symmetrization rule given by $h_y^* \in \mathbb{C}[\mathcal{S}_r]$.

Now, equation (2.12) turns into

$$\tau_b(id) \;=\; \sum_{p\in P}\beta_p\,T_b(p) \;=\; \sum_{y\in Y}\sum_{p\in P}\beta_p(h_y^*T)_b(p)\,. \tag{6.2}$$

The sums $\sum_{p\in P}\beta_p(h_y^*T)_b(p)$ are independent of each other and can be reduced separately with the help of suitable identities (2.13) of the minimal left ideals $\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a$. The linear equation system (2.14) for the complex numbers $x_p \in \mathbb{C}$, which define identities (2.13) of $\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a$, has to be determined from a generating element of $\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a$. According to Section 2, every generating element of $\mathbb{C}[\mathcal{S}_r]\cdot y\cdot a$ is allowed for that purpose. If we choose $y\cdot a$ for this, i.e. if we use the linear equation system

$$\sum_{q\in\mathcal{S}_r}(y\cdot a)(p^{-1}\circ q)\,x_q \;=\; 0 \;\;,\;\; p\in\mathcal{S}_r\,, \tag{6.3}$$

to calculate the needed identities (2.13), then we can apply the quick way of finding out a maximal set of linearly independent rows of (6.3), described in Section 2, since Proposition 5.1 and Corollary 5.1 give us a basis $\{p \cdot y \cdot a \mid p \in Q\}$ of $\mathbb{C}[\mathcal{S}_r] \cdot y \cdot a$.

The construction of the decomposition (6.1) can be improved, if we know for every given partition $\lambda \vdash r$ the number of Young symmetrizers y_t^λ of λ which are contained in Y. This is synonymous with the knowledge of the multiplicity of equivalent left ideals in the decomposition (6.1) which are characterized by the partition $\lambda \vdash r$.

In simple cases these multiplicities can be calculated by scalar products of characters of certain representations of the $\mathcal{S}_r$. If the tensor T is the tensor product of other tensors, the determination of the multiplicities leads to the application of the Richardson-Littlewood rule and of plethysms. The use of these tools we will describe in a forthcoming paper.

We have realized a Mathematica package, called PERMS, to carry out all calculations described above. The heart of the handling of plethysms in PERMS is a very useful formula from [19]. Furthermore, PERMS contains a whole string of algorithms for the investigation of permutation groups from [3]. Other programs concerning the representation theory of the symmetric group are Schur [23] and SYMMETRICA [11, 12]. At present, we are working on a improvement of PERMS by replacing the tools for the calculation with group ring elements by procedures written in C/C++.

Acknowledgements. I wish to express my sincerest thanks to Prof. G. Eisenreich and Prof. J. Stückrad for checks of my work and valuable suggestions.

I am greatly indebted to the late Professor Paul Günther. He was my doctoral supervisor and promoted my work during many years by countless very helpful discussions and stimulating encouragement. His recent passing is a deep personal loss.

References

[1] Boerner, H.: *Darstellungen von Gruppen* (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen: Vol. 74). Berlin - Göttingen - Heidelberg: Springer-Verlag, 1955.

[2] Boerner, H.: *Representations of Groups* (2nd revised Ed.). Amsterdam: North-Holland Publishing Company, 1970.

[3] Butler, G.: *Fundamental Algorithms for Permutation Groups* (Lecture Notes in Computer Science: Vol. 559). Berlin - Heidelberg - New York: Springer-Verlag, 1991.

[4] Christensen, S. and L. Parker: *MathTensor: A System for Doing Tensor Analysis by Computer*. Reading - Menlo Park - New York et al.: Addison-Wesley, 1994.

[5] Dyer, C. and J. Harper: *Tensor Algebra with REDTEN. A User Manual* (1.0 Ed.). Toronto: Department of Astronomy, Scarborough College, University of Toronto, November 1994.

[6] Fulling, S., King, R., Wybourne, B. and C. Cummins: *Normal forms for tensor polynomials: I. The Riemann tensor*. Class. Quantum Grav. 9 (1992), 1151 - 1197.

[7] Ilyin, V. and A. Kryukov: *ATENSOR - REDUCE program for tensor simplification*. Computer Physics Communications 96 (1996), 36.

[8] James, G. D. and A. Kerber: *The Representation Theory of the Symmetric Group* (Encyclopedia of Mathematics and its Applications: Vol. 16). Reading - London - Amsterdam et al.: Addison-Wesley, 1981.

[9] Kerber, A.: *Representations of Permutation Groups* (Lecture Notes in Mathematics: Vol. 240, 495). Berlin - Heidelberg - New York: Springer-Verlag, 1971, 1975.

[10] Kerber, A.: *Algebraic combinatorics via finite group actions.* Mannheim - Wien - Zürich: BI-Wiss.-Verl., 1991.

[11] Kerber, A. and A. Kohnert: *SYMMETRICA 1.0.* Bayreuth: Lehrstuhl II für Mathematik, Department of Mathematics, University of Bayreuth, 1995.

[12] Kerber, A., Kohnert, A. and A. Lascoux: *SYMMETRICA, an object oriented computer-algebra system for the symmetric group.* J. Symbolic Computation 14 (1992), 195 - 203.

[13] Lee, J. M., Lear, D. and J. Roth: *Ricci. A Mathematica package for doing tensor calculations in differential geometry. User's Manual. Version 1.2.* Seattle: Department of Mathematics, University of Washington, ©1992 - 1995.

[14] Littlewood, D.: *The Theory of Group Characters and Matrix Representations of Groups* (2nd Ed.). Oxford: Clarendon Press, 1950.

[15] Macdonald, I.: *Symmetric Functions and Hall Polynomials.* Oxford: Clarendon Press, 1979.

[16] Musgrave, P., Pollney, D. and K. Lake: *GRTensor II Release 1.50 for Maple V Release 3 and 4.* Kingston, Ontario, Canada: Department of Physics, Queen's University at Kingston, 1996. Manual. Part A. Introduction and overview.

[17] Naimark, M. and A. Štern: *Theory of Group Representations* (Grundlehren der Mathematischen Wissenschaften: Vol. 246). Berlin - Heidelberg - New York: Springer-Verlag, 1982.

[18] Passman, D.: *The Algebraic Structure of Group Rings* (Pure and Applied Mathematics). New York - London - Sydney - Toronto: John Wiley & Sons, 1977. First edition: 1940.

[19] Sänger, F.: *Plethysmen von irreduziblen Darstellungen symmetrischer Gruppen.* Dissertation. Aachen: Rheinisch-Westfälische Technische Hochschule, Mathematisch-Naturwissenschaftliche Fakultät, 1980.

[20] van der Waerden, B.: *Algebra.* 9th, 6th Ed., Vol. I, II. Berlin - Heidelberg - New York et al.: Springer-Verlag, 1993.

[21] von Neumann, J.: *On regular rings.* Proc.Nat.Acad.Sci.USA 22 (1936), 707 - 713.

[22] Weyl, H.: *The Classical Groups, their Invariants and Representations.* Princeton, New Jersey: Princeton University Press, 1939.

[23] Wybourne, B. and S. Christensen: *Schur 5.1 (January 6, 1994).* © 1993 - 1994. Program package.