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Distribution Approximations for Nonlinear Functionals of Weakly Correlated Random Processes

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In memory to Prof. Dr. P. Günther

Abstract. In this paper nonlinear functionals of weakly correlated processes with correlation length $\varepsilon > 0$ are investigated. Expansions of moments and distribution densities of nonlinear functionals with respect to ε up to terms of order $o(\varepsilon)$ are considered. For the case of a single nonlinear functional a shorter proof than in [8] is given. The results are applied to eigenvalues of random matrices which are obtained by application of the Ritz method to random differential operators. Using the expansion formulas as to ε approximations of the density functions of the matrix eigenvalues can be found. In addition to [7] not only first order approximations (exact up to terms of order $O(\varepsilon)$) but also second order approximations (exact up to terms of order $o(\varepsilon)$) are investigated. These approximations are compared with estimations from Monte-Carlo simulation.

Keywords: *Random functions, weakly correlated processes, random matrix eigenvalue problems*

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1. Problem

The paper has been inspired by eigenvalue problems for ordinary differential operators with random functions as coefficients. These coefficients are assumed to be weakly correlated processes $f_{\varepsilon j} = f_{j\varepsilon}(x,\omega)$, $x \in \mathcal{D} \subset \mathbb{R}$ (cf. Section 2). Using the Ritz method an approximate solution of the given eigenvalue problem of a differential operator follows from an eigenvalue problem for a random matrix. The matrix elements are perturbated by random variables which are linear functionals

$$\int_{\mathcal{D}} F_{ij}(x) f_{j\varepsilon}(x,\omega)\, dx$$

of the coefficient processes $f_{j\varepsilon}$ with non-random functions F_{ij}. Applying perturbation theory the eigenvalues of the random matrix are expanded with respect to the linear functionals written above. Then, from these expansions approximations of the distribution densities of the eigenvalues can be given.

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In the general case, a function $d(\mathbf{y})$, $\mathbf{y} = (y_1, \dots, y_n)$ is assumed to have the following properties:

(P1) d can be represented by

$$d = d_0 + \sum_{a=1}^{n} d_a\, y_a + \sum_{a,b=1}^{n} d_{ab}\, y_a y_b + \sum_{a,b,c=1}^{n} d_{abc}\, y_a y_b y_c + \left(\sum_{a=1}^{n} y_a^2\right)^{\alpha} \cdot h(y_1, \dots, y_n) \qquad (1)$$

where $\alpha > \frac{3}{2}$ and the function h is bounded on $\mathbf{B}_\delta(0) = \{\mathbf{y} \in \mathbb{R}^n : ||\mathbf{y}|| \leq \delta\}$.

(P2) All moments of $d(\omega) = d\big(r_{1\varepsilon}(\omega), \dots, r_{n\varepsilon}(\omega)\big)$ exist, i.e. $\mathbf{E}|d^k| \leq c_k < \infty$ and $r_{i\varepsilon}$ $(i = 1, \dots, n)$ are linear functionals

$$r_{i\varepsilon}(\omega) = \int_{\mathcal{D}} F_i(x) f_\varepsilon(x, \omega)\, dx$$

of a weakly correlated process f_ε with continuous sample functions a.s. and $\mathbf{E}\{|f_\varepsilon|^p\} \leq c_p < \infty$ $(p \in \mathbb{N})$, where the functions F_i are supposed to be bounded, integrable and square integrable on an interval $\mathcal{D} \subset \mathbb{R}$.

In the case of random matrices it is deduced by perturbation results that the random eigenvalues can be represented in form (1) (cf. Section 5). Especially, for functions $d \in C^4(\mathbf{B}_\delta(0))$ the representation (1) follows by means of the Taylor expansion.

The aim of the present paper is the calculation of approximations for the distribution density of the random variable $d(\omega) = d\big(r_{1\varepsilon}(\omega), \dots, r_{n\varepsilon}(\omega)\big)$. In a first step expansions of moments of d are given with respect to the correlation length ε of a weakly correlated process $f_\varepsilon = f_\varepsilon(x, \omega)$. Then, by means of the Gram-Charlier series the expansion of the distribution density of d as to ε can be obtained using the expansion of the moments.

In order to deal with the random variable d it is necessary to investigate expansions of moments of the linear functionals $r_{i\varepsilon}$ as to the correlation length ε. These results are given in Section 2. The results of this paper can be generalized to the case of n functions $d_1, \dots, d_n$ and linear functionals $r_{i\varepsilon}$ of the form

$$r_{i\varepsilon}(\omega) = \sum_{j=1}^{l} r_{ij\varepsilon}(\omega) = \sum_{j=1}^{l} \int_{\mathcal{D}} F_{ij}(x) f_{j\varepsilon}(x, \omega)\, dx$$

where $(f_{1\varepsilon}, \dots, f_{l\varepsilon})$ is a weakly correlated vector process.

2. Calculation of moments of linear functionals

The concept of weakly correlated processes is based on the idea that these processes have no distant effect. The values of the process at two points do not correlate when the distance of these points exceeds a certain quantity $\varepsilon > 0$. This quantity ε is referred to as the *correlation length* of the random process and it is assumed to be sufficiently small.

The definition of a weakly correlated process implies $\mathbf{E}f_\varepsilon = 0$. The correlation function of f_ε can be written as

$$\mathbf{E}\{f_\varepsilon(x)f_\varepsilon(y)\} = \begin{cases} R_\varepsilon(x,y) & \text{if } |x-y| \leq \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

The theory of weakly correlated processes can be found in [8] and the notations used follow this literature.

The statistical characteristics of weakly correlated processes can be described by so-called *intensities* where, for instance, the intensity $a_{f,2}$ can be written as

$$a_{f,2}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon}^{+\varepsilon} \mathbf{E}\{f_\varepsilon(x)f_\varepsilon(x+z)\}\, dz.$$

Similarly, $a_{f,3}$ and $a_{f,4}$ are defined as integrals as to third and fourth moments, respectively. It should be noted that the intensities are independent of the correlation length ε.

The terms ${}^2A_1(\{i,j\})$, ${}^3A_2(\{i,j,l\})$, ${}^4\underline{A}_3(\{i,j,l,m\})$ denote one-dimensional integrals over the domain $\mathcal{D}$ using the intensities of the process f_ε:

$$\begin{aligned} {}^2A_1(\{i,j\}) &= \int_D F_i(x)\, F_j(x)\, a_{f,2}(x)\, dx \\ {}^3A_2(\{i,j,l\}) &= \int_D F_i(x)\, F_j(x)\, F_l(x)\, a_{f,3}(x)\, dx \\ {}^4\underline{A}_3(\{i,j,l,m\}) &= \int_D F_i(x)\, F_j(x)\, F_l(x)\, F_m(x)\, a_{f,4}(x)\, dx. \end{aligned}$$

Furthermore, we will use terms ${}^2A_2(\{i,j\})$ which contain additionally boundary terms. The expansion for moments of random variables of the form

$$r_{i\varepsilon}(\omega) = \int_{\mathcal{D}} F_i(x) f_\varepsilon(x,\omega)\, dx \qquad (x \in \mathcal{D} \subset \mathbb{R})$$

with respect to the correlation length ε is given by the following theorem (cf. [8]). For its formulation let J_1, J_2, J_3, J_4 denote all possible, non-equivalent decompositions of the set $\{i_1, \ldots, i_k\}$ in the following way:

$$\begin{aligned} \text{for } k \text{ even:}\quad J_1 &= \Big\{\{s_1,t_1\},\ldots,\{s_{\frac{k}{2}},t_{\frac{k}{2}}\}\Big\} \\ J_2 &= \Big\{\{s_1,t_1,r_1\},\{s_2,t_2,r_2\},\{s_3,t_3\},\ldots,\{s_{\frac{k}{2}-1},t_{\frac{k}{2}-1}\}\Big\} \\ J_3 &= \Big\{\{s_1,t_1,r_1,u_1\},\{s_2,t_2\},\ldots,\{s_{\frac{k}{2}-1},t_{\frac{k}{2}-1}\}\Big\} \\ \text{for } k \text{ odd:}\quad J_4 &= \Big\{\{s_1,t_1,r_1\},\{s_2,t_2\},,\ldots,\{s_{\frac{k-1}{2}},t_{\frac{k-1}{2}}\}\Big\}. \end{aligned}$$

Two decompositions are said to be *equivalent*, if they only distinguish by a permutation of the sets and a permutation of the elements in the sets.

Theorem 1. *Let f_ε denote a weakly correlated process with continuous sample functions a.s. and $\mathbf{E}\{|f_\varepsilon|^p\} \le c_p < \infty$ $(p \in \mathbb{N})$. The functions F_i are supposed to be bounded, integrable and square integrable on $\mathcal{D}$. Then the moments $\mathbf{E}\{\prod_{p=1}^{k} r_{i_p\varepsilon}\}$ have the expansion*

$$\mathbf{E}\Big\{\prod_{p=1}^{k} r_{i_p\varepsilon}\Big\} = \begin{cases} G_0(i_1,\dots,i_k)\,\varepsilon^{\frac{k}{2}} + G_2(i_1,\dots,i_k)\,\varepsilon^{\frac{k}{2}+1} + o\big(\varepsilon^{\frac{k}{2}+1}\big) & \text{if } k \text{ even} \\ G_1(i_1,\dots,i_k)\,\varepsilon^{\frac{k+1}{2}} + O\big(\varepsilon^{\frac{k+3}{2}}\big) & \text{if } k \text{ odd} \end{cases}$$

where the terms G_0, G_1, G_2 are given by

$$G_0(i_1,\dots,i_k) = \sum_{J_1} \prod_{q=1}^{\frac{k}{2}} {}^2A_1(\{s_q,t_q\}) \tag{3}$$

$$G_1(i_1,\dots,i_k) = \sum_{J_4} {}^3A_2(\{s_1,t_1,r_1\}) \prod_{q=2}^{\frac{k-1}{2}} {}^2A_1(\{s_q,t_q\}) \tag{4}$$

$$\begin{aligned} G_2(i_1,\dots,i_k) = {} & \sum_{J_1} \sum_{j=1}^{\frac{k}{2}} {}^2A_2(\{s_j,t_j\}) \prod_{j\neq q=1}^{\frac{k}{2}} {}^2A_1(\{s_q,t_q\}) \\ & + \sum_{J_2} \prod_{q=1}^{2} + {}^3A_2(\{s_q,t_q,r_q\}) \prod_{q=3}^{\frac{k}{2}-1} {}^2A_1(\{s_q,t_q\}) \\ & + \sum_{J_3} {}^4\underline{A}_3(\{s_1,t_1,r_1,u_1\}) \prod_{q=2}^{\frac{k}{2}-1} {}^2A_1(\{s_q,t_q\}). \end{aligned} \tag{5}$$

This expansion of the moments of random linear functionals with respect to the correlation length ε is used essentially in the next section.

3. Calculation of the moments of nonlinear functionals

The random variable $d(\omega) = d\big(r_{1\varepsilon}(\omega),\dots,r_{n\varepsilon}(\omega)\big)$ possesses the properties (P1) and (P2) given in Section 1. Now, the expansion of the moments $\mathbf{E}\{(d(\omega) - d_0)^k\}$ with respect to the correlation length ε is determined.

The random variable $r_{0\varepsilon}$ is defined by

$$r_{0\varepsilon}(\omega) = \sum_{a=1}^{n} d_a r_{a\varepsilon}(\omega) = \int_{\mathcal{D}} F_0(x)\, f_\varepsilon(x,\omega)\, dx$$

with

$$F_0(x) = \sum_{a=1}^{n} d_a F_a(x).$$

Using the representation (1) of d, the random variable $d(r_{1\varepsilon}(\omega),\dots,r_{n\varepsilon}(\omega))$ can be written as

$$\begin{aligned} d(\omega) &= d_0 + r_{0\varepsilon}(\omega) + \sum_{a,b=1}^{n} d_{ab}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) \\ &\quad + \sum_{a,b,c=1}^{n} d_{abc}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) r_{c\varepsilon}(\omega) + \tilde{h}(r_{1\varepsilon},\dots,r_{n\varepsilon}) \end{aligned} \tag{6}$$

where $\tilde{h}(r_{1\varepsilon},\dots,r_{n\varepsilon})$ denotes the residual term given by (1).

To simplify the following investigations the random variable $\tilde{d}$ is considered resulting from d by the linear transformation

$$\tilde{d}(\omega) = \frac{c}{\sqrt{\varepsilon}}\,(d(\omega) - d_0) \qquad \text{where} \qquad c = \frac{1}{\sqrt{{}^2A_1(\{0,0\})}}. \tag{7}$$

The k-th moments of $\tilde{d}$ are "standardized" with respect to the lowest order in the expansions with respect to ε. In particular, we have

$$\mathbf{E}\{\tilde{d}(\omega)\} = 0 + O(\sqrt{\varepsilon}) \qquad \text{and} \qquad \mathbf{D}^2\{\tilde{d}(\omega)\} = 1 + O(\varepsilon).$$

These properties simplify the following expansions.

The central moment of order k of d can be easily determined by

$$\mathbf{E}\{(d(\omega) - d_0)^k\} = \frac{1}{c^k} \cdot \mathbf{E}\{\tilde{d}^k(\omega)\} \cdot \sqrt{\varepsilon}^k$$

from the moments of $\tilde{d}$. Now the moments of $\tilde{d}$ are investigated.

From equalities (6) and (7) the representation of the random variable $\tilde{d}$ in terms of the linear functionals $r_{i\varepsilon}$ is given by

$$\begin{aligned} \tilde{d}(\omega) &= \frac{1}{\sqrt{\varepsilon}}\Bigg[c \cdot r_{0\varepsilon}(\omega) + c \cdot \sum_{a,b=1}^{n} d_{ab}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) \\ &\quad + c \cdot \sum_{a,b,c=1}^{n} d_{abc}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) r_{c\varepsilon}(\omega) + c \cdot \tilde{h}(r_{1\varepsilon},\dots,r_{n\varepsilon})\Bigg] \\ &= \frac{1}{\sqrt{\varepsilon}}\left[u(\omega) + v(\omega) + w(\omega) + \tilde{\tilde{h}}(\omega)\right] \end{aligned} \tag{8}$$

where the abbreviations

$$\left.\begin{aligned} u(\omega) &= c\, r_{0\varepsilon}(\omega) \\ v(\omega) &= c \sum_{a,b=1}^{n} d_{ab}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) \\ w(\omega) &= c \sum_{a,b,c=1}^{n} d_{abc}\, r_{a\varepsilon}(\omega) r_{b\varepsilon}(\omega) r_{c\varepsilon}(\omega) \\ \tilde{\tilde{h}}(\omega) &= c\, \tilde{h}(r_{1\varepsilon},\dots,r_{n\varepsilon}) \end{aligned}\right\} \tag{9}$$

for the terms of homogenous order in the random variables $r_{i\varepsilon}$ have been introduced. The terms u, v, w have the order 1, 2, 3 with respect to the $r_{i\varepsilon}$, respectively. The residual term $\tilde{\tilde{h}}$ contains terms of higher order than 3.

Using equality (8) the terms $\{\tilde{d}_g(\omega)\}^k$ $(k \in \mathbb{N})$ are expanded with respect to terms of homogeneous order in the random linear functionals $r_{i\varepsilon}$ $(i \in \{0, 1, \ldots, n\})$. From

$$\{\tilde{d}(\omega)\}^k = \frac{1}{\sqrt{\varepsilon}^k}\left(u + v + w + \tilde{\tilde{h}}\right)^k$$

it follows

$$\{\tilde{d}(\omega)\}^k = \frac{1}{\sqrt{\varepsilon}^k}\left[u^k + \binom{k}{1}u^{k-1}v + \binom{k}{1}u^{k-1}w + \binom{k}{2}u^{k-2}v^2 + \ldots\right] \tag{10}$$

if homogeneous terms are written up to the order $k+2$. Terms of an order higher than $k+2$ need not to be considered, since their moments (standardized by $1/\sqrt{\varepsilon^k}$) are of order $o(\varepsilon)$ which follows immediately from Theorem 1. For the expectation of (10) we have

$$\begin{aligned}\mathbf{E}\{\tilde{d}^k\} = \frac{1}{\sqrt{\varepsilon}^k}\Bigg[&\mathbf{E}\{u^k\} + \binom{k}{1}\mathbf{E}\{u^{k-1}v\} \\ &+\binom{k}{1}\mathbf{E}\{u^{k-1}w\} + \binom{k}{2}\mathbf{E}\{u^{k-2}v^2\}\Bigg] + o(\varepsilon).\end{aligned} \tag{11}$$

By means of Theorem 1 the expectations $\mathbf{E}\{u^k\}, \mathbf{E}\{u^{k-1}v\}, \mathbf{E}\{u^{k-1}w\}, \mathbf{E}\{u^{k-2}v^2\}$ can be expanded with respect to the correlation length ε. For abbreviation the terms

$$\left.\begin{aligned}{}^2A_1(u,u) &= c^2 \cdot {}^2A_1(\{0,0\}) \\ {}^2A_2(u,u) &= c^2 \cdot {}^2A_2(\{0,0\}) \\ {}^4\underline{A}_3(u,u,u,u) &= c^4 \cdot {}^4\underline{A}_3(\{0,0,0,0\}) \\ {}^3A_2(u,u,u)) &= c^3 \cdot {}^3A_2(\{0,0,0\})\end{aligned}\right\}$$

are introduced. From the definition of c

$${}^2A_1(u,u) = 1 \tag{12}$$

follows. Furthermore, e_k $(k \in \mathbb{Z})$ is defined to be the number of all non-equivalent decompositions of k elements in pairs (the number of pairs in J_1):

$$e_k = \begin{cases}\dfrac{k!}{2^{\frac{k}{2}} \cdot \left(\dfrac{k}{2}\right)!} & \text{if } k \text{ even} \\ 0 & \text{if } k \text{ odd or } k < 0.\end{cases} \tag{13}$$

Then the k-th moment $\mathbf{E}\{u^k\} = \mathbf{E}\{(c \cdot r_{0\varepsilon})^k\}$ has the expansion

$$\frac{\mathbf{E}\{u^k\}}{\sqrt{\varepsilon}^k} = \begin{cases} e_k + \left[\binom{k}{2} e_{k-2}\,{}^2A_2(u,u) + \binom{k}{4} e_{k-4}\,{}^4\underline{A}_3(u,u,u,u) \right. \\ \qquad \left. + \frac{1}{2}\binom{k}{3}\binom{k-3}{3} e_{k-6}\left({}^3A_2(u,u,u)\right)^2\right] \cdot \varepsilon + o(\varepsilon) & \text{if } k \text{ even} \\ \binom{k}{3} e_{k-3}\,{}^3A_2(u,u,u) \cdot \sqrt{\varepsilon} + O(\sqrt{\varepsilon^3}) & \text{if } k \text{ odd.} \end{cases}$$

In order to expand $\mathbf{E}\{u^{k-1}v\}$ the terms

$$\begin{aligned} {}^2A_1(v) &= c \cdot \sum_{a,b=1}^{n} d_{ab}\,{}^2A_1(\{a,b\}) \\ {}^{22}A_{11}(v,u,u) &= c^3 \cdot \sum_{a,b=1}^{n} d_{ab}\,{}^2A_1(\{0,a\})\,{}^2A_1(\{0,b\}) \\ {}^3A_2(v,u) &= c^2 \cdot \sum_{a,b=1}^{n} d_{ab}\,{}^3A_2(\{a,b,0\}) \\ {}^{32}A_{21}(v,u,u,u) &= c^4 \cdot \sum_{a,b=1}^{n} d_{ab}\left[{}^3A_2(\{a,0,0\})^2A_1(\{0,b\}) + {}^3A_2(\{b,0,0\})^2A_1(\{0,a\})\right] \end{aligned}$$

are introduced. With (9) it follows

$$\mathbf{E}\{u^{k-1}v\} = \mathbf{E}\left\{c^k \cdot r_{0\varepsilon}^{k-1} \cdot \sum_{a,b=1}^{n} d_{ab}\, r_{a\varepsilon} r_{b\varepsilon}\right\}.$$

Then this moment of order $k+1$ possesses the expansion

$$\frac{\mathbf{E}\{u^{k-1}v\}}{\sqrt{\varepsilon}^k} =$$

$$\begin{cases} \left.\begin{aligned} &\left[(k-1)\, e_{k-2}\,{}^3A_2(v,u) + \binom{k-1}{2}(k-3)\, e_{k-4}\,{}^{32}A_{21}(v,u,u,u)\right. \\ &+ \binom{k-1}{3}(k-4)(k-5)\, e_{k-6}\,{}^3A_2(u,u,u)^{22}A_{11}(v,u,u) \\ &\left. + \binom{k-1}{3} e_{k-4}\,{}^3A_2(u,u,u)^2A_1(v)\right] \cdot \varepsilon + o(\varepsilon) \end{aligned}\right\} & \text{if } k \text{ even} \\ \left[e_{k-1}\,{}^2A_1(v) + (k-1)(k-2)\, e_{k-3}\,{}^{22}A_{11}(v,u,u)\right] \cdot \sqrt{\varepsilon} + O(\sqrt{\varepsilon^3}) & \text{if } k \text{ odd.} \end{cases}$$

In order to expand $\mathrm{E}\{u^{k-1}w\}$ the additional terms

$$
\begin{aligned}
{}^{22}A_{11}(w,u) &= c^2 \cdot \sum_{a,b,c=1}^{n} d_{abc}\Big[{}^2A_1(\{a,0\})^2A_1(\{b,c\}) \\
&\qquad + {}^2A_1(\{b,0\})^2A_1(\{a,b\}) + {}^2A_1(\{c,0\})^2A_1(\{a,b\})\Big] \\
{}^{222}A_{111}(w,u,u,u) &= c^4 \cdot \sum_{a,b,c=1}^{n} d_{abc}\, {}^2A_1(\{a,0\})^2A_1(\{b,0\})^2A_1(\{c,0\})
\end{aligned}
$$

are introduced and it is

$$
\mathrm{E}\{u^{k-1}w\} = \mathbf{E}\left\{ c^k \cdot r_{0\varepsilon}^{k-1} \cdot \sum_{a;b,c=1}^{n} d_{abc}\, r_{a\varepsilon} r_{b\varepsilon} r_{c\varepsilon} \right\}
$$

using the abbreviations (9). This moment leads to a term of order $k+2$:

$$
\frac{\mathrm{E}\{u^{k-1}w\}}{\sqrt{\varepsilon}^k} = \begin{cases} \Big[(k-1)\, e_{k-2}\, {}^{22}A_{11}(w,u) & \\ \quad + 3!\binom{k-1}{3} e_{k-4}\, {}^{222}A_{111}(w,u,u,u)\Big] \cdot \varepsilon + o(\varepsilon) & \text{if } k \text{ even} \\ O(\sqrt{\varepsilon^3}) & \text{if } k \text{ odd.} \end{cases}
$$

Finally, it is defined

$$
\begin{aligned}
{}^{22}A_{11}(v,v) &= c^2 \cdot \sum_{a,b,c,d=1}^{n} d_{ab}\, d_{cd}\Big[{}^2A_1(\{a,c\})\, {}^2A_1(\{b,d\}) \\
&\qquad + {}^2A_1(\{a,d\})\, {}^2A_1(\{b,c\})\Big] \\
{}^{222}A_{111}(v,v,u,u) &= c^4 \cdot \sum_{a,b,c,d=1}^{n} d_{ab}\, d_{cd}\Big[{}^2A_1(\{a,0\})\, {}^2A_1(\{c,0\})\, {}^2A_1(\{b,d\}) \\
&\qquad + {}^2A_1(\{a,0\})\, {}^2A_1(\{d,0\})\, {}^2A_1(\{b,c\}) \\
&\qquad + {}^2A_1(\{b,0\})\, {}^2A_1(\{c,0\})\, {}^2A_1(\{a,d\}) \\
&\qquad + {}^2A_1(\{b,0\})\, {}^2A_1(\{d,0\})\, {}^2A_1(\{a,c\})\Big]
\end{aligned}
$$

in order to determine the moment $\mathrm{E}\{u^{k-1}v^2\}$ of order $k+2$. The expansion of

$$
\mathrm{E}\{u^{k-2}v^2\} = \mathbf{E}\left\{ c^k \cdot r_{0\varepsilon}^{k-2} \cdot \left(\sum_{a,b=1}^{n} d_{ab}\, r_{a\varepsilon} r_{b\varepsilon} \right)^2 \right\}
$$

can be written as

$$\frac{\mathbf{E}\{u^{k-2}v^2\}}{\sqrt{\varepsilon}^k} = \begin{cases} \left.\begin{aligned} &\Big[(k-2)(k-3)e_{k-4}\left(2\cdot {}^2A_1(v)\,{}^{22}A_{11}(v,u,u)\right. \\ &\left. + {}^{222}A_{111}(v,v,u,u)\right) + e_{k-2}\left(({}^2A_1(v))^2 + {}^{22}A_{11}(v,v)\right) \\ &+ 4!\binom{k-2}{4} e_{k-6}\left({}^{22}A_{11}(v,u,u)\right)^2\Big]\varepsilon + o(\varepsilon) \end{aligned}\right\} & \text{if } k \text{ even} \\ O(\sqrt{\varepsilon^3}) & \text{if } k \text{ odd.} \end{cases}$$

Substituting these expansions into equality (11) the expansion of the k-th moment of $\{\tilde{d}(\omega)\}$ up to terms of order $o(\varepsilon)$ can be obtained by elementary transformations and by using definition (13). The following theorem summarizes the result.

Theorem 2. *The moments* $\mathbf{E}\{\tilde{d}^k\}$ $(k \in \mathbb{N}_0)$ *have the expansion*

$$\mathbf{E}\{\tilde{d}^k\} = \begin{cases} e_k + \frac{k!}{2^{\frac{k}{2}}}\left\{\frac{1}{\left(\frac{k-2}{2}\right)!}R_{2,d} + \frac{1}{\left(\frac{k-4}{2}\right)!}R_{4,d} + \frac{1}{\left(\frac{k-6}{2}\right)!}R_{6,d}\right\}\cdot\varepsilon + o(\varepsilon) & \textit{if } k \textit{ even} \\ \frac{k!}{2^{\frac{k-1}{2}}}\left\{\frac{1}{\left(\frac{k-1}{2}\right)!}R_{1,d} + \frac{1}{\left(\frac{k-3}{2}\right)!}R_{3,d}\right\}\cdot\sqrt{\varepsilon} + o(\varepsilon) & \textit{if } k \textit{ odd} \end{cases}$$

where the coefficients $R_{i,d}$ $(i = 1,2,3,4,6)$ *are given by*

$$\begin{aligned} R_{2,d} &= {}^2A_2(u,u) + 2\ {}^3A_2(v,u) + ({}^2A_1(v))^2 + {}^{22}A_{11}(v,v) + 2\ {}^{22}A_{11}(w,u) \\ R_{4,d} &= \frac{1}{6}\ {}^4\underline{A}_3(u,u,u,u) + 2\ {}^{32}A_{21}(v,u,u,u) + \frac{2}{3}\ {}^3A_2(u,u,u)\,{}^2A_1(v) \\ &\quad + 4\ {}^{22}A_{11}(v,u,u)\,{}^2A_1(v) + 2\ {}^{222}A_{111}(v,v,u,u) + 4\ {}^{222}A_{111}(w,u,u,u) \\ R_{6,d} &= \frac{1}{9}\left({}^3A_2(u,u,u)\right)^2 + \frac{4}{3}\ {}^3A_2(u,u,u)\,{}^{22}A_{11}(v,u,u) + 4\ \left({}^{22}A_{11}(v,u,u)\right)^2 \\ R_{1,d} &= {}^2A_1(v) \\ R_{3,d} &= \frac{1}{3}\ {}^3A_2(u,u,u) + 2\ {}^{22}A_{11}(v,u,u). \end{aligned}$$

The relation between the moments of $\tilde{d}$ *and* $d - d_0$ *is given by*

$$\mathbf{E}\{(d(\omega) - d_0)^k\} = \frac{1}{c^k}\cdot\mathbf{E}\{\tilde{d}^k(\omega)\}\cdot\sqrt{\varepsilon^k}$$

with $c = \frac{1}{\sqrt{{}^2A_1(\{0,0\})}}$.

Remark 1. The $R_{i,d}$ are independent of ε, but they are also independent of the order k of the moment considered. This simplifies practical calculations considerably.

Remark 2. The expansion of $\mathbf{E}\{\tilde{d}\}$ does not contain the term $1/\left(\frac{k-i}{2}\right)!\cdot R_{i,d}$ in case of $k-i<0$. This follows from the calculations which lead to Theorem 2. Therefore, in the following considerations there is defined $\frac{1}{j!} = 0$ for $j < 0$.

Theorem 2 gives the expansions of the moments of any order using only the constants $R_{i,d}$ for $i = 1,2,3,4,6$. This enables us to simplify the representation for the density function, too.

4. Calculation of the distribution density

Let $\tilde{p}$ denote the distribution density function of $\tilde{d}$ and p that of $d - d_0$. The expansion of $\tilde{p}$ with respect to the correlation length ε is investigated. Then the representation of p can be found using the relation

$$p(u) = \tilde{p}\left(\frac{c}{\sqrt{\varepsilon}}\, u\right) \cdot \frac{c}{\sqrt{\varepsilon}}. \tag{14}$$

The distribution density of $\tilde{p}$ is assumed to have the form (Gram-Charlier series)

$$\tilde{p}(\tilde{u}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{u}^2\right) \cdot \sum_{m=0}^{\infty} (-1)^m \frac{c_m}{m!} H_m(\tilde{u}) \tag{15}$$

where H_m defined by

$$\begin{aligned} H_m(\tilde{u}) &= (-1)^m \exp\left(\frac{1}{2}\tilde{u}^2\right) \cdot \frac{d^m}{d\tilde{u}^m} \exp\left(-\frac{1}{2}\tilde{u}^2\right) \\ &= \sum_{k=0}^{[\frac{m}{2}]} (-1)^k \frac{m!}{2^k k!(m-2k)!}\, \tilde{u}^{m-2k} \end{aligned} \tag{16}$$

denotes the Chebyshev-Hermite polynomial of order m and the c_m are real coefficients which are to be determined. Using a property of Chebyshev-Hermite polynomials, namely

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\tilde{u}^2\right) H_p(\tilde{u})\, H_q(\tilde{u})\, d\tilde{u} = \sqrt{2\pi}\, p!\, \delta_{pq},$$

from equality (15) this can be done by

$$\begin{aligned} c_m &= (-1)^m \int_{-\infty}^{+\infty} \tilde{p}(\tilde{u})\, H_m(\tilde{u})\, d\tilde{u} \\ &= \sum_{k=0}^{[\frac{m}{2}]} (-1)^{m+k} \frac{m!}{2^k k!(m-2k)!} \int_{-\infty}^{+\infty} \tilde{p}(\tilde{u})\, \tilde{u}^{m-2k}\, d\tilde{u} \\ &= \sum_{k=0}^{[\frac{m}{2}]} (-1)^{m+k} \frac{m!}{2^k k!(m-2k)!}\, \mathbf{E}\{\tilde{d}^{m-2k}\} \end{aligned} \tag{17}$$

where

$$\int_{-\infty}^{+\infty} \tilde{p}(\tilde{u})\, \tilde{u}^{m-2k}\, d\tilde{u} = \mathbf{E}\{\tilde{d}^{m-2k}\}$$

is applied. By means of Theorem 2 the expectations $\mathbf{E}\{\tilde{d}^{m-2k}\}$ $(k \in \{0,1,\ldots,[\frac{m}{2}]\})$ can be expanded with respect to the correlation length ε up to terms of order $o(\varepsilon)$. Substituting these formulas into equality (17) the expansions of the coefficients c_m are obtained. The cases "m even" and "m odd" will be considered separately.

1. *The index m is assumed to be even.* Then

$$c_m = \sum_{k=0}^{\frac{m}{2}} (-1)^{m+k} \frac{m!}{2^k k!(m-2k)!} \left\{ e_{m-2k} + \frac{(m-2k)!}{2^{\frac{m-2k}{2}}} \sum_{l=1}^{3} \frac{R_{2l,d}}{\left(\frac{(m-2k)-2l}{2}\right)!} \cdot \varepsilon + o(\varepsilon) \right\}$$
$$= \sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{m!}{2^k k!(m-2k)!} e_{m-2k} + \sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{m!}{2^{\frac{m}{2}} k!} \sum_{l=1}^{3} \frac{R_{2l,d}}{\left(\frac{(m-2k)-2l}{2}\right)!} \cdot \varepsilon + o(\varepsilon)$$

and hence

$$c_m = \delta_{0m} + \frac{m!}{2^{\frac{m}{2}}} \sum_{l=1}^{3} R_{2l,d} \sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{1}{k!\left(\left(\frac{m}{2}-l\right)-k\right)!} \cdot \varepsilon + o(\varepsilon)$$

where (13) and the relation

$$\sum_{k=0}^{z} (-1)^k \frac{1}{k!(z-k)!} = \frac{1}{z!} \sum_{k=0}^{z} (-1)^k \binom{z}{k} = \frac{1}{z!}(1-1)^z = \begin{cases} 0 & \text{if } o < z \in \mathbb{N} \\ 1 & \text{if } z = 0 \end{cases}$$

with $z = \frac{m}{2}$ has been used. In the same way, setting $z = \frac{m}{2} - l$ for $l = 1,2,3$, the equality

$$\sum_{k=0}^{\frac{m}{2}} (-1)^k \frac{1}{\left(\left(\frac{m}{2}-l\right)-k\right)!\,k!} = \frac{1}{\left(\frac{m}{2}-l\right)!} \sum_{k=0}^{\frac{m}{2}} (-1)^k \binom{\frac{m}{2}-l}{k} = \delta_{2l\,m} \tag{18}$$

can be derived considering Remark 2 of Theorem 2. Thus

$$c_m = \delta_{0m} + \frac{m!}{2^{\frac{m}{2}}} \left[\delta_{2m} R_{2,d} + \delta_{4m} R_{4,d} + \delta_{6m} R_{6,d} \right] \cdot \varepsilon + o(\varepsilon). \tag{19}$$

2. *The index m is assumed to be odd.* In this case

$$c_m = \sum_{k=0}^{\frac{m-1}{2}} (-1)^{m+k} \frac{m!}{2^k k!(m-2k)!} \left\{ \frac{(m-2k)!}{2^{\frac{m-2k-1}{2}}} \sum_{l=1}^{2} \frac{R_{(2l-1),d}}{\left(\frac{(m-2k)-(2l-1)}{2}\right)!} \cdot \sqrt{\varepsilon} + o(\varepsilon) \right\}$$

and hence

$$c_m = (-1) \frac{m!}{2^{\frac{m-1}{2}}} \sum_{k=0}^{\frac{m-1}{2}} (-1)^k \left[\frac{1}{k!\left(\frac{m-1}{2}-k\right)!} R_{1,d} + \frac{1}{k!\left(\frac{m-3}{2}-k\right)!} R_{3,d} \right] \cdot \sqrt{\varepsilon} + o(\varepsilon).$$

Applying equality (18) with the same considerations as in case of m even it follows

$$c_m = (-1) \frac{m!}{2^{\frac{m-1}{2}}} \left[\delta_{1m} R_{1,d} + \delta_{3m} R_{3,d} \right] \cdot \sqrt{\varepsilon} + o(\varepsilon). \tag{20}$$

Using these terms of the coefficients c_m the expansion of the distribution density $\tilde{p}$ is summarized in the following Theorem 3.

Theorem 3. *The expansion of the distribution density $\tilde{p}$ can be explicitly given up to terms of order $o(\varepsilon)$ by*

$$\tilde{p}(\tilde{u}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{u}^2\right) \left[1 + \left(R_{1,d} H_1(\tilde{u}) + \frac{1}{2} R_{3,d} H_3(\tilde{u})\right)\sqrt{\varepsilon} + \left(\frac{1}{2} R_{2,d} H_2(\tilde{u}) + \frac{1}{4} R_{4,d} H_4(\tilde{u}) + \frac{1}{8} R_{6,d} H_6(\tilde{u})\right)\varepsilon + o(\varepsilon)\right]$$

where the $R_{i,d}$ $(i = 1,2,3,4,6)$ have been defined in Theorem 2.

Proof. The equalities (19) and (20) leads to the relations

$$\left.\begin{aligned} c_0 &= 1 \\ c_2 &= R_{2,d}\,\varepsilon + o(\varepsilon) \\ c_4 &= 90\, R_{6,d}\,\varepsilon + o(\varepsilon) \\ c_m &= o(\varepsilon) \quad (8 \leq m \text{ even}) \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} c_1 &= -1\, R_{1,d}\sqrt{\varepsilon} + o(\varepsilon) \\ c_3 &= -3\, R_{3,d}\sqrt{\varepsilon} + o(\varepsilon) \\ c_m &= o(\varepsilon) \quad (5 \leq m \text{ odd}). \end{aligned}\right\}$$

Substituting these expansions of the coefficients into the Gram-Charlier series (15) the proof is complete ∎

Remark 1. It should be noted that although only the coefficients c_m for $m = 0,1,2,3,4,6$ are used explicitly, the expansion of the density is exact up to terms of order $o(\varepsilon)$.

Remark 2. Theorem 3 includes the asymptotic result for nonlinear functionals that for $\varepsilon \to 0$ the random variable

$$\tilde{d}(\omega) = \frac{1}{\sqrt{{}^2A_1(\{0,0\})\,\varepsilon}}(d(\omega) - d_0) = \frac{1}{\sqrt{\varepsilon}}\left(d(r_{1\varepsilon}, \ldots, r_{n\varepsilon}) - d_0\right)$$

converges in distribution to a Gaussian random variable with mean zero and variance one. Because of the linear dependence on $\tilde{d}$ the random variable d converges to a Gaussian variable, too.

Remark 3. In [8] a similiar proposition for vector-valued functionals d is given. The proof of Theorem 3 for the special case of a single-valued functional is much shorter than in [8].

Applying the expansion calculated approximations of the distribution density function can be given for small ε. The approximations $\tilde{p}_0$, $\tilde{p}_1$ and $\tilde{p}_2$ are exact up to terms $o(1)$, $o(\sqrt{\varepsilon})$ and $o(\varepsilon)$, respectively:

$$\tilde{p}_0(\tilde{u}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\tilde{u}^2\right) \tag{21}$$

$$\tilde{p}_1(\tilde{u}) = \tilde{p}_0(\tilde{u}) \left[1 + \left(R_{1,d} H_1(\tilde{u}) + \frac{1}{2} R_{3,d} H_3(\tilde{u})\right)\sqrt{\varepsilon}\right] \tag{22}$$

$$\tilde{p}_2(\tilde{u}) = \tilde{p}_0(\tilde{u}) \left[1 + \left(R_{1,d} H_1(\tilde{u}) + \frac{1}{2} R_{3,d} H_3(\tilde{u})\right)\sqrt{\varepsilon} + \left(\frac{1}{2} R_{2,d} H_2(\tilde{u}) + \frac{1}{4} R_{4,d} H_4(\tilde{u}) + \frac{1}{8} R_{6,d} H_6(\tilde{u})\right)\varepsilon\right]. \tag{23}$$

From equality (14) the expansion of the distribution density of $d - d_0$ can be obtained as

$$p(u) = \frac{c}{\sqrt{2\pi \cdot \varepsilon}} \exp\left(-\frac{c^2}{2 \cdot \varepsilon} u\right) \left[1 + \left(R_{1,d} H_1 \left(\frac{c}{\sqrt{\varepsilon}} u\right) + \frac{1}{2} R_{3,d} H_3 \left(\frac{c}{\sqrt{\varepsilon}} u\right)\right) \sqrt{\varepsilon} \right.$$
$$\left. + \left(\frac{1}{2} R_{2,d} H_2 \left(\frac{c}{\sqrt{\varepsilon}} u\right) + \frac{1}{4} R_{4,d} H_4 \left(\frac{c}{\sqrt{\varepsilon}} u\right) + \frac{1}{8} R_{6,d} H_6 \left(\frac{c}{\sqrt{\varepsilon}} u\right)\right) \varepsilon + o(\varepsilon)\right].$$

According to equalities (21) - (23) approximations p_0, p_1 and p_2 of the density p can be written. These approximations are exact up to terms of order $o(1), o(\sqrt{\varepsilon})$ and $o(\varepsilon)$, respectively.

In applications to eigenvalue problems usually first order approximations p_0 have been considered only. Hence, random variables d have been approximated by Gaussian variables with mean d_0 where d_0 is the value of the averaged problem (cf. Remark 2 of Theorem 2).

Depending on the practical problem considered approximations of higher order should be used, especially, if the correlation length ε is not close to zero. The approximations of higher order give the deviation from the Gaussian distribution. The higher numerical effort to calculate the approximations p_1 or p_2 is worth-while because the results being obtained can be confirmed by simulation very well. In particular, for the mean of d derived by using the approximations p_1 or p_2 large deviations from the value d_0 have been observed. This problem is denoted as the averaging problem: the deviation of the mean of d from the value d_0.

In the next section an application is presented and the importance of considering approximations of higher order is shown.

5. Application

In this section we calculate the approximations p_0 and p_2 of the first eigenvalue of a random symmetric matrix eigenvalue problem and compare the results with those of the Monte-Carlo simulation.

Consider the eigenvalue problem of a random ordinary differential operator

$$\left.\begin{aligned} (fu'')'' &= -\lambda u'' \\ u(0) = u(1) = u''(0) &= u''(1) = 0 \end{aligned}\right\} \tag{24}$$

where $0 \leq x \leq 1$ and $f = f(x, \omega)$ is assumed to be a random process with $f(\cdot, \omega) \geq c(\omega) > 0$ almost sure. The buckling problem of a simply supported bar is described by (24) with $f = EI$ where EI denotes the bending stiffness, E being the modulus of elasticity and I the moment of inertia of the cross-sectional area. Let the bar possess always a circular cross-sectional area with random radius

$$r(x, \omega) = r_0 + \bar{r}_\varepsilon(x, \omega) \tag{25}$$

where $\overline{r}_\varepsilon$ is a weakly correlated process. Hence, $I(x,\omega) = \frac{1}{4}\pi r^4(x,\omega)$, and $\overline{I} = I - \mathbf{E}I$ and $\overline{f} = f - \mathbf{E}f$ are weakly correlated processes, too. The modulus of elasticity E is assumed to be non-random. Now (24) is replaced by the Ritz equations

$$\big(A + B(\omega)\big)nu = {}^n\lambda^n u \tag{26}$$

with

$$a_{ij} = \int_0^1 \mathbf{E}\{f(x)\}\,\phi_i''(x)\,\phi_j''(x)\,dx \qquad \text{and} \qquad b_{ij}(\omega) = \int_0^1 \overline{f}(x,\omega)\,\phi_i''(x)\,\phi_j''(x)\,dx$$

for $1 \le i,j \le n$. The functions ϕ_i are assumed to form a base of polynomials (cf.[6]):

$$\left.\begin{aligned} \phi_1(x) &= x - 2x^3 + x^4 \\ \phi_2(x) &= 7x - 10x^3 + x^5 \\ \phi_i(x) &= x^i(1-x)^3 \quad (i \ge 3). \end{aligned}\right\}$$

The averaged problem associated with (26) possesses n simple eigenvalues, denoted by ${}^n\mu_i$ $(i = 1,\dots,n)$ and the eigenvectors ${}^n u_{0i} = ({}^n u_{0i1},\dots,{}^n u_{0in})^T$ $(i = 1,\dots,n)$ which can be calculated by deterministic methods. The random variables $b_{ij}(\omega)$ are linear functionals of the weakly correlated process f.

Using perturbation results the eigenvalues can be represented by the random variables b_{ij} $(i,j = 1,\dots,n)$, the eigenvalues and the eigenvectors of the averaged problem (cf. [7]). The g-th eigenvalue of (26) can be written as

$${}^n\lambda_g(\omega) = {}^n\mu_g + \hat{b}_{gg} - \sum_{\substack{i=1\\ i\neq g}}^{n} \frac{\hat{b}_{gi}^2}{{}^n\mu_{ig}} + \sum_{\substack{i,j=1\\ i\neq g}}^{n} \frac{\hat{b}_{gi}\hat{b}_{ij}\hat{b}_{jg}}{{}^n\mu_{ig}\,{}^n\mu_{jg}} - \hat{b}_{gg}\sum_{\substack{i=1\\ i\neq g}}^{n} \frac{\hat{b}_{gi}^2}{{}^n\mu_{ig}^2} + \dots \tag{27}$$

where the $\hat{b}_{ij}$ defined by

$$\hat{b}_{ij}(\omega) = \sum_{k,l=1}^{n} {}^n u_{0ik}\,{}^n u_{0jl}\,b_{kl}(\omega) \qquad (i,j = 1,\dots,n)$$

are as the b_{ij} linear functionals of the weakly correlated process $\overline{f}$, too, and ${}^n\mu_{ig} = {}^n\mu_i - {}^n\mu_g$. It is obvious that the eigenvalues satisfy properties (P1) and (P2) when we define $r_{1\varepsilon} = b_{11}(\omega),\dots,r_{n^2\varepsilon} = b_{nn}(\omega)$.

To compare our approximations with simulation results the process r_ε is defined for $x \in [\frac{i}{N},\frac{i+1}{N}]$ $(i = 1,\dots,N-1)$ by

$$\overline{r}_\varepsilon(\omega) = \xi_i(\omega) + p_i(x)\big(\xi_{i+1}(\omega) - \xi_i(\omega)\big) \tag{28}$$

with

$$p_i(x) = 6(Nx - i)^5 - 15(Nx - i)^4 + 10(Nx - i)^3.$$

The ξ_i denote Gaussian variables, where $\xi_0, \ldots, \xi_N$ are independent with $\mathrm{E}\xi_i = 0$ and $\mathrm{D}^2\xi_i = \sigma^2 = \beta^2 r_0^2$ $(0 \leq \beta \leq 0.2)$. With this choice of p_i the trajectories of r_ε are two times continuous differentiable in (0,1) and r_ε is a weakly correlated process with correlation length $\varepsilon = \frac{2}{N}$. The process r_ε is non-stationary. The same is valid for the process $\overline{f}$, too. With respect to $\overline{f}$ the intensities are to be determined. Because of the instationarity of $\overline{f}$ the calculations are complicated. But considering the intensities it becomes obvious that they are approximately constant for small ε and so the "means" will be used for further calculations. The intensities only depend on the variance $\sigma^2 = \beta^2 r_0^2$ of the ξ_i, i.e. on the parameter β, but not on ε.

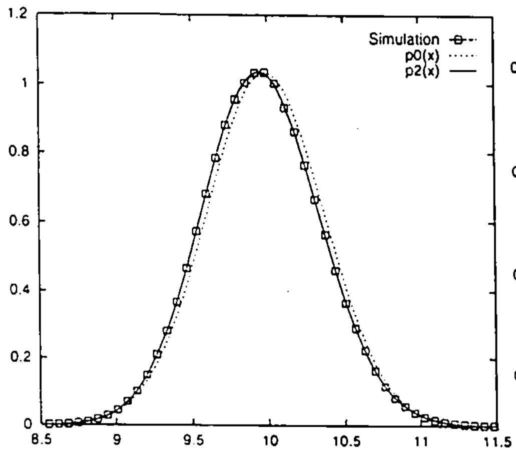

Fig. 1: $\varepsilon = 0.05$, $\beta = 0.05$

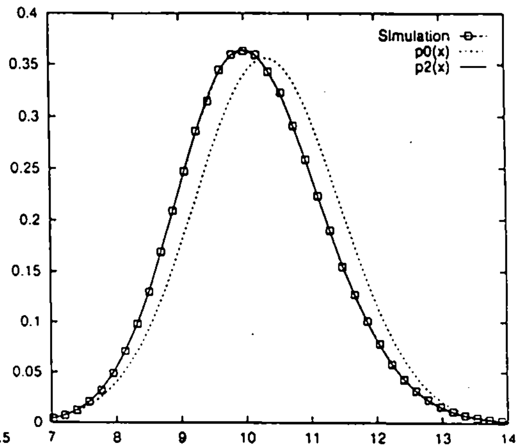

Fig. 2: $\varepsilon = 0.1$, $\beta = 0.1$

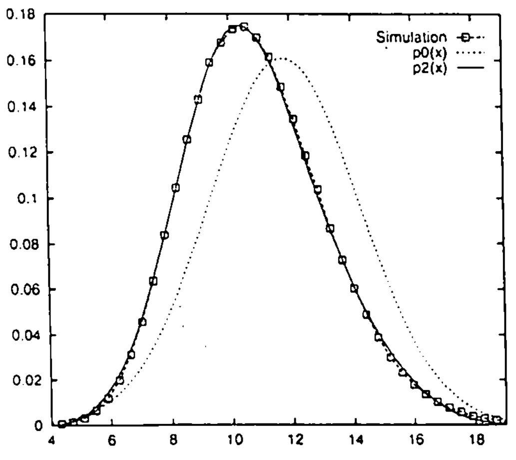

Fig. 3: $\varepsilon = 0.1$, $\beta = 0.2$

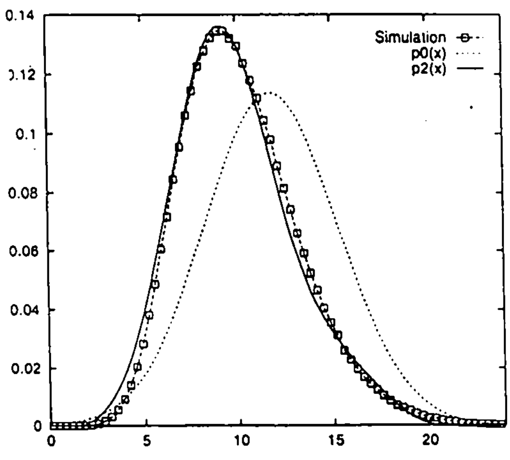

Fig. 4: $\varepsilon = 0.2$, $\beta = 0.2$

Some results are presented which are obtained for the first (smallest) eigenvalue λ_1. The smallest eigenvalue is the relevant one, since the probability that the bar fails can be computed by using it. For simplification the "normalized" eigenvalue $\overline{\lambda}_1 = \frac{4}{E\pi r_0^4}\lambda_1$ is considered.

For $n = 4$ (the dimension of matrix problem), for several values of β (as a measure of the variance of the process r_ε) and for several values of the correlation length ε, the obtained density approximations p_0 and p_2 are plotted. These approximations are compared with the results getting from Monte-Carlo simulation.

From the illustrations it is obvious that the simulation results confirm the approximations of second order which correspond very well with the simulated densities.

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